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EXTENDED SYMMETRY OF THE WITTEN–DIJKGRAAF–VERLINDE–VERLINDE EQUATION OF MONGE–AMPERE TYPE

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Abstract. We construct the Lie algebra of extended symmetry group for the Monge–Ampere type Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equation. This algebra includes novel generators that are unobtainable within the framework of the classical Lie approach and correspond to non-point group transformation of dependent and independent variables. The expansion of symmetry is achieved by introducing new variables through second-order derivatives of the dependent variable. By integrating the Lie equations, we derive transformations that enable the generation of new solutions to the Witten–Dijkgraaf–Verlinde–Verlinde equation from a known one. These transformations yield formulas for obtaining new solutions in implicit form and Bäcklund-type transformations for the nonlinear associativity equations. We also demonstrate that, in the case under study, introducing a substitution of variables via third-order derivatives, as previously used in the literature, does not yield generators of non-point transformations. Instead, this approach produces only the Lie groups of classical point transformations. Furthermore, we perform a group reduction of partial differential equations in two independent variables to a system of ordinary differential equations. This reduction leads to the explicit solution of the fully nonlinear differential equation. Notably, the symmetry group of non-point transformations expands significantly when this method is applied to the second-order differential equation, resulting in a corresponding infinite-dimensional Lie algebra. Finally, we show that auxiliary variables can be systematically derived within the framework of a generalized approach to symmetry reduction of differential equations.

Keywords: non-point symmetries, Witten–Dijkgraaf–Verlinde–Verlinde equation, symmetry group, transformations, Lie algebra.

Mathematics Subject Classification: 35B06, 35A22.

1. INTRODUCTION

At the beginning, we shall introduce the necessary notions and explanations related to the Lie transformation group which serves as the foundation for the group-theoretical analysis of differential equations. Let V be an open set in $\mathbb{R}^n$ and Δ be an interval of $\mathbb{R}$ symmetric with respect to zero. A local one-parameter Lie group G_1 of transformations of the space $\mathbb{R}^n$ is a family of transformations $f : V \times \Delta \to \mathbb{R}^n$, which possess the following properties:

1. $f(x, 0) = x$ for any $x \in V$,
2. $f(f(x, \epsilon), \lambda) = f(x, \epsilon + \lambda)$ for any $\epsilon, \lambda, \epsilon + \lambda \in \Delta, x \in V$,
3. if $\epsilon \in U(0, \delta)$ and $f(x, \epsilon) = x$ for any $x \in V$, then $\epsilon = 0$,
4. $f \in C^\infty(V \times \Delta)$,

where ϵ is the group parameter.

Each transformation $f_\epsilon \in G_1$, $\epsilon \in \Delta$ is written in the coordinate form

$$f_\epsilon : {x'}^i = f^i(x, \epsilon), \quad i = 1, \dots, n. \tag{1.1}$$

The vector $\xi(x)$ with components

$$\xi^i(x) = \frac{\partial f^i(x, \epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}, \quad i = 1, \dots, n$$

is the tangent vector to the curve given by (1.1) at the point x and is called the tangent vector field of the group. The infinitesimal generator connected with the tangent vector $\xi : V \to \mathbb{R}^n$ is in the form

$$\mathbb{X} = \sum_{i=1}^{n} \xi^i(x) \frac{\partial}{\partial x_i}.$$

A one-parameter group can be entirely reconstituted if we know the vector field $\xi(x)$ by solving the corresponding Lie equations with initial conditions

$$\frac{df^i(x, \epsilon)}{d\epsilon} = \xi^i(f^1, \dots, f^n), \quad f^i(x_1, \dots, x_n, 0) = x_i.$$

An infinitely differentiable function $F(x)$ is an invariant of the Lie transformation group (1.1) if

$$F(f(x, \epsilon)) \equiv F(x).$$

It is well known that a function $F(x)$ is an invariant if and only if it satisfies the partial differential equation

$$\sum_{i=1}^{n} \xi^i(x) \frac{\partial F(x)}{\partial x_i} = 0.$$

One of the most important applications of Lie group theory is its use in creating algorithms for constructing solutions of differential equations. In this context, let us distinguish between the m dependent variables $u = (u^1, \dots, u^m)$ and the n independent variables $x = (x_1, \dots, x_n)$, where $u = u(x)$ and $m \geq 2$. In this case, the group transformations for the dependent and independent variables, as well as the infinitesimal generator of the Lie transformation group, take the following form:

$${x'}^i = f^i(x, u, \epsilon), \quad i = 1, \dots, n, \tag{1.2}$$

$$u'^j = g^j(x,u,\epsilon), \quad j = 1,\dots,m, \tag{1.3}$$
$$\mathbb{X} = \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta^j(x,u)\frac{\partial}{\partial u^j},$$

where

$$\xi^i(x) = \frac{\partial f^i(x,u,\epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}, \quad \eta^j(x,u) = \frac{\partial g^j(x,u,\epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}.$$

Transformations (1.2), (1.3) are called point transformations (unlike contact transformations, in which the transformed variables x', u' also depend on the first order derivatives), and the group G is called a group of point transformations. Another commonly employed term is geometric symmetry transformations. One can define transformations of partial derivatives of $u(x)$ under the action of point transformation group (1.2), (1.3), which are regarded as a change of variables and write down the k-th prolongation of the infinitesimal generator

$$\mathbb{X}^{(k)} = \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta^j(x,u)\frac{\partial}{\partial u^j} + \zeta_i^{(1)j}(x,u,u_1)\frac{\partial}{\partial u_i^j}$$
$$+ \dots + \zeta_{i_1 i_1 \dots i_k}^{(k)j}(x,u,u_1,\dots,u_k)\frac{\partial}{\partial u_{i_1 i_1 \dots i_k}^j},$$

where

$$u_i^j = \frac{\partial u^j}{\partial x_i}, \quad u_{i_1 i_2}^j = \frac{\partial u_{i_1}^j}{\partial x_2}, \quad \dots, \quad u_{i_1 i_2 \dots i_n}^j = \frac{\partial u_{i_1 i_2 \dots i_{n-1}}^j}{\partial x_n},$$
$$\zeta_i^{(1)j} = D_i \eta^j - u_l^j (D_i \xi_l), \tag{1.4}$$
$$\zeta_{i_1 i_1 \dots i_k}^{(k)j} = D_{i_k}\zeta_{i_1 i_1 \dots i_{k-1}}^{(k-1)j} - u_{i_1 i_1 \dots i_{k-1} l}^j (D_{i_k}\xi_l),$$
$$D_i = \frac{\partial}{\partial x_i} + u_i^j \frac{\partial}{\partial u^j} + u_{il}^j \frac{\partial}{\partial u_l^j} + \dots + u_{i i_1 i_2 \dots i_n}^j \frac{\partial}{\partial u_{i_1 i_2 \dots i_n}^j} + \dots,$$

where $i_\sigma = 1,\dots,n$, $\sigma = 1,\dots,k$, $k = 2,3,\dots$, the summation convention over repeated indices is used in the above formulas, (see [2, 20, 22] for more details).

The associative equation introduced by E. Witten, R. Dijkgraaf, H. Verlinde, and E. Verlinde (WDVV) [5, 28] (necessary physical motivations are given in [7]) can be reduced to a Monge–Ampere type equation ([7] see also [13]). This equation is mainly used in differential geometry, e.g. in Minkowski's or Weyl's problems in differential geometry of surfaces and in supersymmetric quantum mechanics. In two-dimensional topological field theory, the WDVV equation is called the associativity equation [7, 13, 14]. Such equations can be described by a fully nonlinear third-order differential equation of the Monge–Ampere type given in the form [4, 7, 14, 15]

$$f_{xxx} f_{yyy} - f_{xxy} f_{xyy} - 1 = 0, \tag{1.5}$$

where x and y are independent variables, and $f = f(x,y)$ is an unknown function. Note that the term “fully nonlinear” specifies partial differential equations that are nonlinear in the highest-order derivatives of a prospective solution.

The generator of Lie group of point symmetry of equation (1.5) is of the form

$$\mathbb{X} = \xi_1(x,y,f)\frac{\partial}{\partial x} + \xi_2(x,y,f)\frac{\partial}{\partial y} + \eta(x,y,f)\frac{\partial}{\partial f}.$$

The classical Lie symmetry group is 10–parameter and the corresponding Lie algebra is 10–dimensional. The basic elements of this algebra have the form [4]:

$$\begin{aligned}
&\mathbb{X}_1 = \frac{\partial}{\partial f}, \quad \mathbb{X}_2 = \frac{\partial}{\partial y}, \quad \mathbb{X}_3 = \frac{\partial}{\partial x}, \quad \mathbb{X}_4 = x\frac{\partial}{\partial f},\\
&\mathbb{X}_5 = y\frac{\partial}{\partial f}, \quad \mathbb{X}_6 = xy\frac{\partial}{\partial f}, \quad \mathbb{X}_7 = \frac{1}{2}x^2\frac{\partial}{\partial f}, \quad \mathbb{X}_8 = \frac{1}{2}y^2\frac{\partial}{\partial f},\\
&\mathbb{X}_9 = y\frac{\partial}{\partial y} + \frac{3}{2}f\frac{\partial}{\partial f}, \quad \mathbb{X}_{10} = x\frac{\partial}{\partial x} + \frac{3}{2}f\frac{\partial}{\partial f}.
\end{aligned}$$

Each of the obtained generators generates a one-parameter transformation Lie group [2]. These transformations contain both independent and dependent variables. It turns out that we can extend the class of transformations to such transformations that contain the second-order derivatives. We show that these transformations will also form a Lie group.

We apply the classical Lie theory approach to a system of equations. The system is obtained from the equation under study by introducing auxiliary variables through the second-order derivatives of the dependent variable. We shall construct special transformations similar to Bäcklund transformations for equation (1.5) by using its particular solution. Generally, the new symmetry can be also used to construct the solutions of equation (1.5) by virtue of the group-invariant solution method. The original group is made up of transformations only $x' = x'(x,y,f,\epsilon)$, $y' = y'(x,y,f,\epsilon)$, $f' = f'(x,y,f,\epsilon)$. After the expansion, the transformations will also include second-order derivatives.

The purpose of this paper is to construct the Lie algebra of the symmetry group which contains novel generators unobtainable in the framework of the classical Lie approach and corresponding to the non-point group transformation. For this, we introduce an auxiliary variables a, b, c. We also show their application for finding solutions of eqaution (1.5) and transformations of the Bäcklund type.

In Section 2, we find the finite-dimensional Lie algebras of extended Lie symmetry groups for associativity equations (1.5), (2.5) and two-dimensional Chazy equation (2.13). The symmetry group of non-point transformations expands significantly when this method is applied to second-order differential equation (2.15). In this case, the corresponding Lie algebra becomes infinite-dimensional.

We also show that introducing new variables through third-order derivatives, as in [14, 15], or through first-order derivatives does not yield new generators for non-point transformations of equations (1.5), (2.5), and (2.13). Instead, it only produces generators of the Lie group of classical point transformations.

In Section 3, we obtain the generalized auto-transformations for equation (1.5) which are similar to classical Bäcklund transformations and contain second-order derivatives. We construct these transformations by using the group transformations generated by infinitesimal generator $\mathbb{X}_6 + \mathbb{X}_2$ and $\mathbb{X}_8$ from (2.4). These operators are generators of the classical symmetry group of system (2.2) however they cannot be obtained by applying the classical method to equation (1.5). Such transformations can be constructed by using any one-dimensional Lie subalgebra containing linear combinations of operators $\mathbb{X}_6$ and $\mathbb{X}_8$ from (2.4) for equation (1.5), and nontrivial linear combinations of generators (2.12) including $\mathbb{X}_9$ for equation (2.5). These transformations provide formulas for deriving new solutions in implicit form from known ones. The first set of formulas is derived from the solution presented in [4], while the second set originates from solution (4.8) in Section 4.

Moreover, the generators of the Lie group of non-point transformations $\mathbb{X}_6$, $\mathbb{X}_8$, and $\mathbb{X}_9$ can be used to find solutions of the nonlinear associativity equations via the symmetry reduction method, as shown in Section 4. We use the generator $\mathbb{X}_6$ from (2.4) to reduce equation (1.5) to the system of ordinary differential equations and find explicit solution (4.8) to fully nonlinear differential equation (1.5).

We also show that auxiliary variables can be derived within the framework of the generalized approach of symmetry reduction of partial differential equations. This approach makes use of differential invariants. Variables (2.1) are obtained by using the second-order differential invariants of the three-parameter classical symmetry Lie group of point transformations. By making use of the first-order differential invariants of a one-parameter group of conditional symmetry we derive the well-known Bäcklund transformation obtained in [6] through a different technique.

2. SYMMETRY OF WITTEN–DIJKGRAAF–VERLINDE–VERLINDE EQUATION

We introduce a non-point change of variables for the second-order derivatives

$$\begin{cases} f_{xx} = a, \\ f_{xy} = b, \\ f_{yy} = c. \end{cases} \tag{2.1}$$

Using substitution (2.1) we get a system of three differential equations that represent equation (1.5) itself and the consistency conditions:

$$\begin{cases} a_y = b_x, \\ b_y = c_x, \\ a_x c_y - a_y c_x - 1 = 0, \end{cases} \tag{2.2}$$

where x, y are independent variables and a, b, c are dependent variables. One can extend the classes of symmetries admitted by differential equation (1.5) beyond point symmetries to include non-point symmetries by considering system (2.2) associated with the given WDVV equation.

The group transformations and the infinitesimal generator of the Lie group in this case are of the form

$$\begin{aligned} x' &= x'(x, y, a, b, c, \epsilon), \\ y' &= y'(x, y, a, b, c, \epsilon), \\ a' &= a'(x, y, a, b, c, \epsilon), \\ b' &= b'(x, y, a, b, c, \epsilon), \\ c' &= c'(x, y, a, b, c, \epsilon), \end{aligned}$$

$$\begin{aligned} \mathbb{X} = {} & \xi_1(x, y, a, b, c)\frac{\partial}{\partial x} + \xi_2(x, y, a, b, c)\frac{\partial}{\partial y} \\ & + \eta^1(x, y, a, b, c)\frac{\partial}{\partial a} + \eta^2(x, y, a, b, c)\frac{\partial}{\partial b} + \eta^3(x, y, a, b, c)\frac{\partial}{\partial c}. \end{aligned} \tag{2.3}$$

The infinitesimal criterion of invariance [2, 20] is used to study the symmetry of the first-order partial differential equations (2.2). We need the first prolongation of the infinitesimal generator

$$\mathbb{X}^{(1)} = \mathbb{X} + \zeta_1^1\frac{\partial}{\partial a_x} + \zeta_2^1\frac{\partial}{\partial a_y} + \zeta_1^2\frac{\partial}{\partial b_x} + \zeta_2^2\frac{\partial}{\partial b_y} + \zeta_1^3\frac{\partial}{\partial c_x} + \zeta_2^3\frac{\partial}{\partial c_y},$$

where

$$\begin{aligned} \zeta_1^1 &= D_x\eta^1 - a_x D_x\xi_1 - a_y D_x\xi_2, \\ \zeta_2^1 &= D_y\eta^1 - a_x D_y\xi_1 - a_y D_y\xi_2, \\ \zeta_1^2 &= D_x\eta^2 - b_x D_x\xi_1 - b_y D_x\xi_2, \\ \zeta_2^2 &= D_y\eta^2 - b_x D_y\xi_1 - b_y D_y\xi_2, \\ \zeta_1^3 &= D_x\eta^3 - c_x D_x\xi_1 - c_y D_x\xi_2, \\ \zeta_2^3 &= D_y\eta^3 - c_x D_y\xi_1 - c_y D_y\xi_2. \end{aligned}$$

The coefficients ζ_j^i are derived from prolongation formulas (1.4) taking into account that $x_1 = x$, $x_2 = y$, $u^1 = a$, $u^2 = b$, $u^3 = c$, and $D_1 = D_x$, $D_2 = D_y$.

Theorem 2.1. *The system of equations* (2.2) *is invariant with respect to the 9–parameter Lie group of point transformations of independent variables x, y and dependent variables a, b, c. The basis of its Lie algebra consists of the vector fields:*

$$\begin{aligned} & \mathbb{X}_1 = \frac{\partial}{\partial a}, \quad \mathbb{X}_2 = \frac{\partial}{\partial c}, \quad \mathbb{X}_3 = \frac{\partial}{\partial x}, \quad \mathbb{X}_4 = \frac{\partial}{\partial y}, \quad \mathbb{X}_5 = \frac{\partial}{\partial b}, \\ & \mathbb{X}_6 = -\frac{1}{2}c\frac{\partial}{\partial x} - \frac{1}{2}b\frac{\partial}{\partial y} + y\frac{\partial}{\partial a} + \frac{3}{2}x\frac{\partial}{\partial b}, \\ & \mathbb{X}_7 = \frac{1}{4}x\frac{\partial}{\partial x} + \frac{3}{4}y\frac{\partial}{\partial y} + a\frac{\partial}{\partial a} + \frac{1}{2}b\frac{\partial}{\partial b}, \\ & \mathbb{X}_8 = -\frac{1}{2}b\frac{\partial}{\partial x} - \frac{1}{2}a\frac{\partial}{\partial y} + \frac{3}{2}y\frac{\partial}{\partial b} + x\frac{\partial}{\partial c}, \\ & \mathbb{X}_9 = \frac{3}{4}x\frac{\partial}{\partial x} + \frac{1}{4}y\frac{\partial}{\partial y} + \frac{1}{2}b\frac{\partial}{\partial b} + c\frac{\partial}{\partial c}. \end{aligned} \tag{2.4}$$

Proof. We are looking for the generator of the Lie symmetry group in the form (2.3) and apply Lie's invariance criterion to system under consideration (2.2). We must apply the prolongated generator to each of the three equations, i.e.

$$\begin{aligned}
&\mathbb{X}^{(1)}(a_y - b_x)\Big|_{a_y=b_x, b_y=c_x, a_xc_y-a_yc_x-1=0} \\
&= (\zeta_2^1 - \zeta_1^2)\Big|_{a_y=b_x, b_y=c_x, a_xc_y-a_yc_x-1=0} \equiv 0, \\
&\mathbb{X}^{(1)}(b_y - c_x)\Big|_{a_y=b_x, b_y=c_x, a_xc_y-a_yc_x-1=0} \\
&= (\zeta_2^2 - \zeta_1^3)\Big|_{a_y=b_x, b_y=c_x, a_xc_y-a_yc_x-1=0} \equiv 0, \\
&\mathbb{X}^{(1)}(a_xc_y - a_yc_x - 1)\Big|_{a_y=b_x, b_y=c_x, a_xc_y-a_yc_x-1=0} \\
&= (c_y\zeta_1^1 + a_x\zeta_2^3 - c_x\zeta_2^1 - a_y\zeta_1^3 - 1)\Big|_{a_y=b_x, b_y=c_x, a_xc_y-a_yc_x-1=0} \equiv 0.
\end{aligned}$$

Upon completing this operation, we obtain three equations to determine the unknown functions $\xi_1, \xi_2, \eta^1, \eta^2, \eta^3$. Since the above equations are polynomial in the derivatives $a_x, a_y, b_x, b_y, c_x, c_y$, with coefficients depending on x, y, a, b, c but not on the derivatives, they decompose into the system of determining linear homogeneous partial differential equations

$$\begin{gathered}
(\eta^1)_b = 0, \quad (\eta^1)_c = 0, \quad (\eta^1)_x = 0, \\
(\eta^1)_{a,a} = 0, \quad (\eta^1)_{a,y} = 0, \quad (\eta^1)_{y,y} = 0, \\
(\eta^2)_a = 0, \quad (\eta^2)_b = \frac{1}{2}(\eta^1)_a + \frac{1}{2}(\eta^3)_c, \quad (\eta^2)_c = 0, \\
(\eta^2)_x = \frac{3}{2}(\eta^1)_y, \quad (\eta^2)_y = \frac{3}{2}(\eta^3)_x, \\
(\eta^3)_a = 0, \quad (\eta^3)_b = 0, \quad (\eta^3)_y = 0, \quad (\eta^3)_{c,c} = 0, \quad (\eta^3)_{c,x} = 0, \quad (\eta^3)_{x,x} = 0, \\
(\xi_1)_a = 0, \quad (\xi_1)_b = -\frac{1}{2}(\eta^3)_x, \quad (\xi_1)_c = -\frac{1}{2}(\eta^1)_y, \\
(\xi_1)_x = \frac{3}{4}(\eta^3)_c + \frac{1}{4}(\eta^1)_a, \quad (\xi_1)_y = 0, \\
(\xi_2)_a = -\frac{1}{2}(\eta^3)_x, \quad (\xi_2)_b = -\frac{1}{2}(\eta^1)_y, \quad (\xi_2)_c = 0, \quad (\xi_2)_x = 0, \\
(\xi_2)_y = \frac{1}{4}(\eta^3)_c + \frac{3}{4}(\eta^1)_a.
\end{gathered}$$

We mean that

$$(\xi_i)_x = \frac{\partial\xi_i}{\partial x}, \quad (\xi_i)_y = \frac{\partial\xi_i}{\partial y}, \quad (\xi_i)_a = \frac{\partial\xi_i}{\partial a}, \quad (\xi_i)_b = \frac{\partial\xi_i}{\partial b}, \quad (\xi_i)_c = \frac{\partial\xi_i}{\partial c}$$

and

$$(\eta^i)_x = \frac{\partial\eta^i}{\partial x}, \quad (\eta^i)_y = \frac{\partial\eta^i}{\partial y}, \quad (\eta^i)_a = \frac{\partial\eta^i}{\partial a}, \quad (\eta^i)_b = \frac{\partial\eta^i}{\partial b}, \quad (\eta^i)_c = \frac{\partial\eta^i}{\partial c}.$$

Considering the above-overdetermined system, one can see that

$$\xi_1 = \xi_1(x, b, c), \quad \xi_2 = \xi_2(y, a, b), \quad \eta^1 = \eta^1(y, a),$$
$$\eta^2 = \eta^2(x, y, b), \quad \eta^3 = \eta^3(x, c).$$

First we solve the equation $(\eta^1)_{y,y} = 0$. It gives $\eta^1 = z_1(a)y + z_2(a)$. By substituting them into the equations $(\eta^1)_{a,a} = 0$ and $(\eta^1)_{a,y} = 0$ we obtain $\eta^1 = c_1 y + c_2 a + c_3$. Likewise, the function η^3 is given by the formula $\eta^3 = c_4 x + c_5 c + c_6$. Moving on to finding the function η^2, we solve equations $(\eta^2)_b = \frac{1}{2}(\eta^1)_a + \frac{1}{2}(\eta^3)_c$, $(\eta^2)_x = \frac{3}{2}(\eta^1)_y$ and $(\eta^2)_y = \frac{3}{2}(\eta^3)_x$. This yields

$$\eta^2 = \frac{1}{2}(c_2 + c_5)b + \frac{3}{2}c_1 x + \frac{3}{2}c_4 y + c_9.$$

In a similar way we calculate ξ_1 and ξ_2. Solving the equations for each of these two functions we find

$$\xi_1 = \frac{1}{4}(c_2 + 3c_5)x - \frac{1}{2}c_1 c - \frac{1}{2}c_4 b + c_7$$

and

$$\xi_2 = \frac{1}{4}(c_5 + 3c_2)y - \frac{1}{2}c_1 b - \frac{1}{2}c_4 a + c_8.$$

The solution of the above system of determining equations gives the general form of the generator for the classical Lie symmetry group

$$\begin{aligned}\mathbb{X} = &\left(\frac{1}{4}(c_2 + 3c_5)x - \frac{1}{2}c_1 c - \frac{1}{2}c_4 b + c_7\right)\frac{\partial}{\partial x} \\ &+ \left(\frac{1}{4}(c_5 + 3c_2)y - \frac{1}{2}c_1 b - \frac{1}{2}c_4 a + c_8\right)\frac{\partial}{\partial y} + (c_1 y + c_2 a + c_3)\frac{\partial}{\partial a} \\ &+ \left(\frac{1}{2}(c_2 + c_5)b + \frac{3}{2}c_1 x + \frac{3}{2}c_4 y + c_9\right)\frac{\partial}{\partial b} + (c_4 x + c_5 c + c_6)\frac{\partial}{\partial c},\end{aligned}$$

where $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9 \in \mathbb{R}$. Then we get a 9–dimensional Lie algebra of generators, with the following basic elements:

$$\begin{aligned}&\mathbb{X}_1 = \frac{\partial}{\partial a}, \quad \mathbb{X}_2 = \frac{\partial}{\partial c}, \quad \mathbb{X}_3 = \frac{\partial}{\partial x}, \quad \mathbb{X}_4 = \frac{\partial}{\partial y}, \quad \mathbb{X}_5 = \frac{\partial}{\partial b}, \\ &\mathbb{X}_6 = -\frac{1}{2}c\frac{\partial}{\partial x} - \frac{1}{2}b\frac{\partial}{\partial y} + y\frac{\partial}{\partial a} + \frac{3}{2}x\frac{\partial}{\partial b}, \\ &\mathbb{X}_7 = \frac{1}{4}x\frac{\partial}{\partial x} + \frac{3}{4}y\frac{\partial}{\partial y} + a\frac{\partial}{\partial a} + \frac{1}{2}b\frac{\partial}{\partial b}, \\ &\mathbb{X}_8 = -\frac{1}{2}b\frac{\partial}{\partial x} - \frac{1}{2}a\frac{\partial}{\partial y} + \frac{3}{2}y\frac{\partial}{\partial b} + x\frac{\partial}{\partial c}, \\ &\mathbb{X}_9 = \frac{3}{4}x\frac{\partial}{\partial x} + \frac{1}{4}y\frac{\partial}{\partial y} + \frac{1}{2}b\frac{\partial}{\partial b} + c\frac{\partial}{\partial c}.\end{aligned}$$

□

Note that the operators $\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3, \mathbb{X}_4, \mathbb{X}_5, \mathbb{X}_7, \mathbb{X}_9$ are the same as the operators for equation (1.5) found in [4] or can easily be obtain from them through the prolongation procedure. In contrast, the operators $\mathbb{X}_6$ and $\mathbb{X}_8$ are novel. It is worth pointing out that they cannot be obtained in the framework of a classical Lie approach.

It can be shown that the above set of generators forms a Lie algebra under the operation of the commutator

$$[\mathbb{X}_i, \mathbb{X}_j] = \mathbb{X}_i \mathbb{X}_j - \mathbb{X}_j \mathbb{X}_i.$$

The commutators for all the above-mentioned operators are calculated, and we conclude that for any two generators, the relationship $[\mathbb{X}_i, \mathbb{X}_j] = C_{ij}^k \mathbb{X}_k$ holds, where C_{ij}^k are the structure constants of Lie algebra. The calculations are presented in Table 1.

Table 1
Commutator table

$[\cdot,\cdot]$	$\mathbb{X}_1$	$\mathbb{X}_2$	$\mathbb{X}_3$	$\mathbb{X}_4$	$\mathbb{X}_5$	$\mathbb{X}_6$	$\mathbb{X}_7$	$\mathbb{X}_8$	$\mathbb{X}_9$
$\mathbb{X}_1$	0	0	0	0	0	0	$\mathbb{X}_1$	$-\frac{1}{4}\mathbb{X}_4$	0
$\mathbb{X}_2$	0	0	0	0	0	$-\frac{1}{2}\mathbb{X}_3$	0	0	$\mathbb{X}_2$
$\mathbb{X}_3$	0	0	0	0	0	$\frac{3}{2}\mathbb{X}_5$	$\frac{1}{4}\mathbb{X}_3$	$\mathbb{X}_2$	$\frac{3}{4}\mathbb{X}_3$
$\mathbb{X}_4$	0	0	0	0	0	$\mathbb{X}_1$	$\frac{3}{4}\mathbb{X}_4$	$\frac{3}{2}\mathbb{X}_5$	$\frac{1}{4}\mathbb{X}_4$
$\mathbb{X}_5$	0	0	0	0	0	$-\frac{1}{2}\mathbb{X}_4$	$\frac{1}{2}\mathbb{X}_5$	$-\frac{1}{2}\mathbb{X}_3$	$\frac{1}{2}\mathbb{X}_5$
$\mathbb{X}_6$	0	$\frac{1}{2}\mathbb{X}_3$	$-\frac{3}{2}\mathbb{X}_5$	$-\mathbb{X}_1$	$\frac{1}{2}\mathbb{X}_4$	0	$\frac{1}{4}\mathbb{X}_6$	$\frac{1}{2}\mathbb{X}_7 - \frac{1}{2}\mathbb{X}_9$	$-\frac{1}{4}\mathbb{X}_6$
$\mathbb{X}_7$	-$\mathbb{X}_1$	0	$-\frac{1}{4}\mathbb{X}_3$	$-\frac{3}{4}\mathbb{X}_4$	$-\frac{1}{2}\mathbb{X}_5$	$-\frac{1}{4}\mathbb{X}_6$	0	$\frac{1}{4}\mathbb{X}_8$	0
$\mathbb{X}_8$	$\frac{1}{4}\mathbb{X}_1$	0	$-\mathbb{X}_2$	$-\frac{3}{2}\mathbb{X}_5$	$\frac{1}{2}\mathbb{X}_3$	$-\frac{1}{2}\mathbb{X}_7 + \frac{1}{2}\mathbb{X}_9$	$-\frac{1}{4}\mathbb{X}_8$	0	$\frac{1}{4}\mathbb{X}_8$
$\mathbb{X}_9$	0	-$\mathbb{X}_2$	$-\frac{3}{4}\mathbb{X}_3$	$-\frac{1}{4}\mathbb{X}_4$	$-\frac{1}{2}\mathbb{X}_5$	$\frac{1}{4}\mathbb{X}_6$	0	$-\frac{1}{4}\mathbb{X}_8$	0

The articles [7, 14, 15, 19] also consider second associativity equation

$$f_{ttt} = f_{xxt}^2 - f_{xxx} f_{xtt} \tag{2.5}$$

(see also [23, 27]).

In what follows, new independent variables are introduced via third-order derivatives. In new variables equations (1.5) and (2.5) acquire the form of systems of hydrodynamic type:

$$\begin{cases} \alpha_y = \beta_x, \\ \beta_y = \gamma_x, \\ \gamma_y = (\frac{\beta\gamma+1}{\alpha})_x, \end{cases} \tag{2.6}$$

where

$$\begin{cases} f_{xxx} = \alpha, \\ f_{xxy} = \beta, \\ f_{xyy} = \gamma, \end{cases}$$

and

$$\begin{cases} \mu_t = \nu_x, \\ \nu_t = \tau_x, \\ \tau_t = (\nu^2 - \mu\tau)_x, \end{cases} \tag{2.7}$$

where

$$\begin{cases} \mu = f_{xxx}, \\ \nu = f_{xxt}, \\ \tau = f_{xtt}, \end{cases}$$

respectively. The main advantage of this representation is the existence of an efficient theory of integrability for systems of hydrodynamic type (see, for example, [8, 24] and also articles [9–12] which discuss systems that do not possess Riemann invariants). However, the Lie algebras of the Lie symmetry groups of systems (2.6) and (2.7) do not contain any new elements aside from generators of point symmetries or their prolongations. Indeed, by applying the method used in proving Theorem 2.1 we obtain the Lie algebras of the symmetry groups for the systems (2.6) and (2.7) in the form:

$$\begin{aligned} \mathbb{X}_1 &= y\partial_y + \frac{3}{2}\alpha\partial_\alpha + \frac{1}{2}\beta\partial_\beta - \frac{1}{2}\gamma\partial_\gamma, \quad \mathbb{X}_2 = \partial_y, \\ \mathbb{X}_3 &= x\partial_x - \frac{3}{2}\alpha\partial_\alpha - \frac{1}{2}\beta\partial_\beta + \frac{1}{2}\gamma\partial_\gamma, \quad \mathbb{X}_4 = \partial_x, \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \mathbb{X}_1 &= t^2\frac{\partial}{\partial t} + xt\frac{\partial}{\partial x} + (-t\mu + 3x)\frac{\partial}{\partial \mu} + (-2t\nu - x\mu)\frac{\partial}{\partial \nu} \\ &\quad + (-3t\tau - 2x\nu)\frac{\partial}{\partial \tau}, \\ \mathbb{X}_2 &= t\frac{\partial}{\partial x} + 3\frac{\partial}{\partial \mu} - \mu\frac{\partial}{\partial \nu} - 2\nu\frac{\partial}{\partial \tau}, \\ \mathbb{X}_3 &= x\frac{\partial}{\partial x} + \mu\frac{\partial}{\partial \mu} + 2\nu\frac{\partial}{\partial \nu} + 3c\frac{\partial}{\partial \tau}, \\ \mathbb{X}_4 &= \frac{\partial}{\partial x}, \quad \mathbb{X}_5 = t\frac{\partial}{\partial t} - \mu\frac{\partial}{\partial \mu} - 2\nu\frac{\partial}{\partial \nu} - 3\tau\frac{\partial}{\partial \tau}, \quad \mathbb{X}_6 = \frac{\partial}{\partial t}, \end{aligned} \tag{2.9}$$

respectively. From (2.8) and (2.9), we conclude there are no possibilities of generating nonpoint transformations via elements of these algebras. We prove that the same property holds when the new variables are introduced using the first-order derivatives. At the same time, introducing the new variables

$$\begin{cases} u = f_{xx}, \\ v = f_{xt}, \\ w = f_{tt}, \end{cases} \tag{2.10}$$

one can obtain the system

$$\begin{cases} u_t = v_x, \\ v_t = w_x, \\ w_t = u_t^2 - u_x v_t \end{cases} \tag{2.11}$$

corresponding to (2.5). By using the infinitesimal method of invariance of differential equations we show that the Lie algebra of system (2.11) is given by basis elements

$$\begin{aligned} &\mathbb{X}_1 = \frac{\partial}{\partial w}, \quad \mathbb{X}_2 = \frac{\partial}{\partial t}, \quad \mathbb{X}_3 = \frac{\partial}{\partial v}, \quad \mathbb{X}_4 = \frac{\partial}{\partial x}, \quad \mathbb{X}_5 = \frac{\partial}{\partial u}, \\ &\mathbb{X}_6 = \frac{1}{2}t\frac{\partial}{\partial x} - \frac{3}{2}x\frac{\partial}{\partial u} + \frac{1}{2}u\frac{\partial}{\partial v} + v\frac{\partial}{\partial w}, \\ &\mathbb{X}_7 = t\frac{\partial}{\partial t} + \frac{3}{4}x\frac{\partial}{\partial x} + \frac{1}{2}u\frac{\partial}{\partial u} + \frac{1}{4}v\frac{\partial}{\partial v}, \\ &\mathbb{X}_8 = \frac{1}{4}x\frac{\partial}{\partial x} + \frac{1}{2}u\frac{\partial}{\partial u} + \frac{3}{4}v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w}, \\ &\mathbb{X}_9 = x\frac{\partial}{\partial t} + \frac{1}{2}u\frac{\partial}{\partial x} - \frac{3}{2}v\frac{\partial}{\partial u} - \frac{1}{2}w\frac{\partial}{\partial v}. \end{aligned} \tag{2.12}$$

The operator $\mathbb{X}_9$ from (2.12) generates nonpoint transformations, while the remaining operators generate point transformations for the equation (2.5). Thus, we see that among the three possibilities involving first-, second-, and third-order derivatives, only the change of variables using second-order derivatives, as described by (2.1), (2.10), results in new generators for the Lie symmetry group of nonpoint transformations for the associativity equations (1.5) and (2.5).

We also study the equation

$$f_{ppp}f_{rrr} - f_{ppr}f_{prr} - 2f_{ppr} = 0, \tag{2.13}$$

which is derived from the Chazy equation

$$f_{xxx}f_{yyy} - f_{xxy}f_{xyy} - 2f_{xyz} = 0,$$

by symmetry reduction with by using a one-parameter group of transformations $x' = x + a$, and $z' = z - a$, where a is the group parameter, $p = x + z$ and $r = y$.

In what follows, we introduce new variables using the second-order derivatives:

$$\begin{cases} f_{pp} = m, \\ f_{pr} = n, \\ f_{rr} = q. \end{cases}$$

In the new variables equation (2.13) is written in the form

$$\begin{cases} m_r = n_p, \\ n_r = q_p, \\ m_p q_r - n_p n_r - 2n_p = 0. \end{cases} \tag{2.14}$$

System (2.14) is invariant with respect to the Lie transformation group, whose Lie algebra is given by the basis elements:

$$
\begin{aligned}
&\mathbb{X}_1 = \frac{\partial}{\partial r}, \quad \mathbb{X}_2 = \frac{\partial}{\partial m}, \quad \mathbb{X}_3 = \frac{\partial}{\partial q}, \quad \mathbb{X}_4 = \frac{\partial}{\partial p}, \quad \mathbb{X}_5 = \frac{\partial}{\partial n},\\
&\mathbb{X}_6 = -\frac{1}{2}p\frac{\partial}{\partial p} - \frac{1}{2}r\frac{\partial}{\partial r} + r\frac{\partial}{\partial n} + p\frac{\partial}{\partial q},\\
&\mathbb{X}_7 = \frac{1}{2}r\frac{\partial}{\partial r} + m\frac{\partial}{\partial m} + \frac{1}{2}n\frac{\partial}{\partial n},\\
&\mathbb{X}_8 = p\frac{\partial}{\partial p} + \frac{1}{2}r\frac{\partial}{\partial r} + \frac{1}{2}n\frac{\partial}{\partial n} + q\frac{\partial}{\partial q},\\
&\mathbb{X}_9 = (2p+q)\frac{\partial}{\partial p} + (r+n)\frac{\partial}{\partial r} - n\frac{\partial}{\partial n},\\
&\mathbb{X}_{10} = (-r - \frac{1}{2}n)\frac{\partial}{\partial p} - \frac{1}{2}m\frac{\partial}{\partial r} + m\frac{\partial}{\partial n} + (2r+n)\frac{\partial}{\partial q}.
\end{aligned}
$$

This follows by the same method as in Theorem 2.1. By integrating the Lie equations, one can construct nonpoint transformations generated by $\mathbb{X}_9$ and $\mathbb{X}_{10}$. We emphasize that introducing new variables through third-order derivatives, as done in works [14, 15], or through first-order derivatives does not yield new generators of nonpoint transformations for equation (2.13). Instead, it only produces generators of the Lie groups of the classical point transformations. By contrast, the method of introducing new variables via second-order derivatives, as employed in this article, yields two new generators of non-point transformations for third-order differential equations (1.5) and (2.13), and one generator for equation (2.5).

The symmetry group of non-point transformations expands significantly when this method is applied to the second-order differential equation

$$u_{xx} = F(u_{xy}, u_{yy}), \tag{2.15}$$

where F is a smooth function of two variables. Note, that from (2.15) one can obtain two-dimensional Boer-Finley equation and the potential form of the dispersionless Kadomtse–Petviashvili equation. Introducing the auxiliary variables

$$
\begin{cases}
a = u_{xx},\\
b = u_{xy},\\
c = u_{yy},
\end{cases}
$$

equation (2.15) results in the first-order system

$$
\begin{cases}
b_y = c_x,\\
b_x = F_b b_y + F_c c_y,
\end{cases}
\tag{2.16}
$$

where $F_b = \frac{\partial F(b,c)}{\partial b}$, $F_c = \frac{\partial F(b,c)}{\partial c}$. The generator of the Lie group of non-point transformations has the form

$$\mathbb{X} = G(b,c)\partial_x + H(b,c)\partial_y,$$

where $G(b,c)$, $H(b,c)$ are unknown functions in this case. The infinitesimal criterion of invariance of system (2.16)

$$\mathbb{X}_{(1)}(b_y - c_x)\Big|_{b_y=c_x, b_x=F_b b_y + F_c c_y} \equiv 0,$$
$$\mathbb{X}_{(1)}(b_x - F_b b_y - F_c c_y)\Big|_{b_y=c_x, b_x=F_b b_y + F_c c_y} \equiv 0$$

gives the determining equations in the form of system of two first-order linear homogeneous differential equations

$$\begin{cases} G_c = H_b, \\ H_c + G_c F_b = G_b F_c. \end{cases} \tag{2.17}$$

It follows that the corresponding Lie algebra is infinite-dimensional and therefore equation (2.15) can be linearized by the hodograph transformation for arbitrary $F(u_{xy}, u_{yy})$ (see [16, 25]).

3. CONSTRUCTION OF NON-POINT TRANSFORMATIONS

In this section, we use the property of the Lie symmetry group to generate new solutions of the system (2.2) from a known one. We show that it is possible to construct a new nonpoint transformation that generates solutions and transforms them into other solutions. This method is illustrated with a specific example related to a one-parameter subgroup with a generator of the form $\mathbb{X}_6 + \mathbb{X}_2$. Subsequently, group transformations are required. According to the first fundamental theorem of Lie, we solve the Cauchy problem for the system of ordinary differential equations

$$\begin{cases} \frac{dx'}{d\epsilon} = -\frac{1}{2}c', \\ \frac{dy'}{d\epsilon} = -\frac{1}{2}b', \\ \frac{da'}{d\epsilon} = y', \\ \frac{db'}{d\epsilon} = \frac{3}{2}x', \\ \frac{dc'}{d\epsilon} = 1, \end{cases} \qquad \begin{cases} x'|_{\epsilon=0} = x, \\ y'|_{\epsilon=0} = y, \\ a'|_{\epsilon=0} = a, \\ b'|_{\epsilon=0} = b, \\ c'|_{\epsilon=0} = c. \end{cases}$$

The solution of the above system is a one-parameter group of transformations given by the formula

$$\begin{cases} x' = x - \frac{1}{4}\epsilon^2 - \frac{1}{2}\epsilon c, \\ y' = y + \frac{1}{64}\epsilon^4 + \frac{1}{16}\epsilon^3 c - \frac{3}{8}\epsilon^2 x - \frac{1}{2}\epsilon b, \\ a' = a + \frac{1}{320}\epsilon^5 + \frac{1}{64}\epsilon^4 c - \frac{1}{8}\epsilon^3 x - \frac{1}{4}\epsilon^2 b + \epsilon y, \\ b' = b - \frac{1}{8}\epsilon^3 - \frac{3}{8}\epsilon^2 c + \frac{3}{2}\epsilon x, \\ c' = c + \epsilon. \end{cases} \tag{3.1}$$

If we have special solutions to system (2.2), then we can also find new solutions. To formulate the next theorem, we need to introduce the following relations

$$\begin{cases} a' = h(x', y'), \\ b' = g(x', y'), \\ c' = k(x', y'). \end{cases} \tag{3.2}$$

Theorem 3.1. *Let $a = h(x, y)$, $b = g(x, y)$ and $c = k(x, y)$ be a particular solution of system* (2.2). *Then $\widehat{h}(x, y)$, $\widehat{g}(x, y)$, $\widehat{k}(x, y)$, which satisfy system* (3.2), *will also be the solution of system* (2.2).

Proof. With the help of group transformations (3.1), system (3.2) can be rewritten in the form:

$$\begin{cases} a = h(x', y') - (\frac{1}{320}\epsilon^5 + \frac{1}{64}\epsilon^4(k(x', y') - \epsilon) - \frac{1}{8}\epsilon^3 x \\ \qquad - \frac{1}{4}\epsilon^2(g(x', y') - (-\frac{1}{8}\epsilon^3 - \frac{3}{8}\epsilon^2(k(x', y') - \epsilon) + \frac{3}{2}\epsilon x)) + \epsilon y), \\ b = g(x', y') - (-\frac{1}{8}\epsilon^3 - \frac{3}{8}\epsilon^2(k(x', y') - \epsilon) + \frac{3}{2}\epsilon x), \\ c = k(x', y') - \epsilon, \end{cases} \tag{3.3}$$

where x', y' are given by formula (3.1). It is a system of equations that, when solved, gives us new solutions $\widehat{h}(x, y)$, $\widehat{g}(x, y)$, $\widehat{k}(x, y)$. We examine whether these new functions also satisfy the system of equations (2.2). Let us calculate the first-order partial derivatives of a, b, c from system (3.3). Then we obtain the system of six algebraic equations with six unknowns a_x, a_y, b_x, b_y, c_x, and c_y:

$$\begin{aligned} a_x = & -\frac{1}{2}\epsilon h_{x'} c_x + h_{x'} + \frac{1}{16}\epsilon^3 h_{y'} c_x - \frac{3}{8}\epsilon^2 h_{y'} - \frac{1}{2}\epsilon h_{y'} b_x - \frac{5}{128}\epsilon^5 k_{x'} c_x \\ & + \frac{5}{64}\epsilon^4 k_{x'} + \frac{5}{1024}\epsilon^7 k_{y'} c_x - \frac{15}{512}\epsilon^6 k_{y'} - \frac{5}{128}\epsilon^5 k_{y'} b_x - \frac{1}{4}\epsilon^3 - \frac{1}{8}\epsilon^3 g_{x'} c_x \\ & + \frac{1}{4}\epsilon^2 g_{x'} + \frac{1}{64}\epsilon^5 g_{y'} c_x - \frac{3}{32}\epsilon^4 g_{y'} - \frac{1}{8}\epsilon^3 g_{y'} b_x, \\ a_y = & -\frac{1}{2}\epsilon h_{x'} c_y + \frac{1}{16}\epsilon^3 h_{y'} c_y - \frac{1}{2}\epsilon h_{y'} b_y + h_{y'} - \frac{5}{128}\epsilon^5 k_{x'} c_y + \frac{5}{1024}\epsilon^7 k_{y'} c_y \\ & - \frac{5}{128}\epsilon^5 k_{y'} b_y + \frac{5}{64}\epsilon^4 k_{y'} - \frac{1}{8}\epsilon^3 g_{x'} c_y + \frac{1}{64}\epsilon^5 g_{y'} c_y - \frac{1}{8}\epsilon^3 g_{y'} b_y \\ & + \frac{1}{4}\epsilon^2 g_{y'} - \epsilon, \end{aligned}$$

$$
\begin{aligned}
b_x &= -\frac{1}{2}\epsilon g_{x'}c_x + g_{x'} + \frac{1}{16}\epsilon^3 g_{y'}c_x - \frac{3}{8}\epsilon^2 g_{y'} - \frac{1}{2}\epsilon g_{y'}b_x - \frac{3}{16}\epsilon^3 k_{x'}c_x \\
&\quad + \frac{3}{8}\epsilon^2 k_{x'} + \frac{3}{128}\epsilon^5 k_{y'}c_x - \frac{9}{64}\epsilon^4 k_{y'} - \frac{3}{16}\epsilon^3 k_{y'}b_x - \frac{3}{2}\epsilon, \\
b_y &= -\frac{1}{2}\epsilon g_{x'}c_y + \frac{1}{16}\epsilon^3 g_{y'}c_y - \frac{1}{2}\epsilon g_{y'}b_y + g_{y'} - \frac{3}{16}\epsilon^3 k_{x'}c_y + \frac{3}{128}\epsilon^5 k_{y'}c_y \\
&\quad - \frac{3}{16}\epsilon^3 k_{y'}b_y + \frac{3}{8}\epsilon^3 k_{y'}, \\
c_x &= -\frac{1}{2}\epsilon k_{x'}c_x + k_{x'} + \frac{1}{16}\epsilon^3 k_{y'}c_x - \frac{3}{8}\epsilon^2 k_{y'} - \frac{1}{2}\epsilon k_{y'}b_x, \\
c_y &= -\frac{1}{2}\epsilon k_{x'}c_y + \frac{1}{16}\epsilon^3 k_{y'}c_y - \frac{1}{2}\epsilon k_{y'}b_y + k_{y'}.
\end{aligned}
$$

The Maple program was used to find the desired coefficients of the resulting system of algebraic equations.

By determining $a_x, a_y, b_x, b_y, c_x, c_y$ and substituting them into the system of equations (2.2) we arrive at

$$
\begin{cases}
16\epsilon^2 g_{y'} + 32\epsilon^2 k_{x'} + 32\epsilon h_{x'}k_{y'} - 32\epsilon h_{y'}k_{x'} + 64\epsilon - 64h_{y'} \\
= 24\epsilon^2 g_{y'} + 24\epsilon^2 k_{x'} + 96\epsilon - 64 g_{x'}, \\
8g_{y'} = 8k_{x'}, \\
\epsilon g_{y'} - \epsilon k_{x'} - 2h_{x'}k_{y'} + 2h_{y'}k_{x'} + 2 = 0.
\end{cases}
\tag{3.4}
$$

After applying the appropriate transformations and simplifications we obtain the required dependencies

$$
\begin{cases}
h_{y'} = g_{x'}, \\
g_{y'} = k_{x'}, \\
h_{x'}k_{y'} - h_{y'}k_{x'} - 1 = 0
\end{cases}
\tag{3.5}
$$

from (3.4). □

In the case where the system (3.3) is considered in the original variables f, x, and y, we obtain the transformations for the second-order derivatives

$$
\begin{cases}
f_{xx} = h(\widehat{x}', \widehat{y}') - (\frac{1}{320}\epsilon^5 + \frac{1}{64}\epsilon^4(k(\widehat{x}', \widehat{y}') - \epsilon) \\
\qquad - \frac{1}{8}\epsilon^3 x - \frac{1}{4}\epsilon^2(g(\widehat{x}', \widehat{y}' - (-\frac{1}{8}\epsilon^3 - \frac{3}{8}\epsilon^2(k(\widehat{x}', \widehat{y}') - \epsilon) + \frac{3}{2}\epsilon x)) + \epsilon y), \\
f_{xy} = g(\widehat{x}', \widehat{y}') - (-\frac{1}{8}\epsilon^3 - \frac{3}{8}\epsilon^2(k(\widehat{x}', \widehat{y}') - \epsilon) + \frac{3}{2}\epsilon x), \\
f_{yy} = k(\widehat{x}', \widehat{y}') - \epsilon,
\end{cases}
$$

where

$$
\begin{cases}
\widehat{x}' = x - \frac{1}{4}\epsilon^2 - \frac{1}{2}\epsilon f_{yy}, \\
\widehat{y}' = y + \frac{1}{64}\epsilon^4 + \frac{1}{16}\epsilon^3 f_{yy} - \frac{3}{8}\epsilon^2 x - \frac{1}{2}\epsilon f_{xy},
\end{cases}
$$

which represents the consistent overdetermined system of the second-order partial differential equation. These transformations can be interpreted as generalized Bäcklund auto-transformations for the WDVV equation (1.5).

Let us consider the second generator of non-point transformations

$$\mathbb{X}_8 = -\frac{1}{2}b\frac{\partial}{\partial x} - \frac{1}{2}a\frac{\partial}{\partial y} + \frac{3}{2}y\frac{\partial}{\partial b} + x\frac{\partial}{\partial c}.$$

We obtain the finite group transformations

$$\begin{cases} x' = x - \epsilon\frac{b}{2} - \epsilon^2\frac{3y}{8} + \epsilon^3\frac{a}{16}, \\ y' = y - \epsilon\frac{a}{2}, \\ a' = a, \\ b' = b + \epsilon\frac{3y}{2} - \epsilon^2\frac{3a}{8}, \\ c' = c + \epsilon x - \epsilon^2\frac{b}{4} - \epsilon^3\frac{y}{8} + \epsilon^4\frac{a}{64}, \end{cases} \tag{3.6}$$

by solving the Cauchy problem for the Lie system in this case. Relations (3.6) now yield

$$\begin{cases} a = h(x', y'), \\ b = g(x', y') - \epsilon\frac{3y}{2} + \frac{3\epsilon^2}{8}h(x', y'), \\ c = k(x', y') - \epsilon x + \frac{\epsilon^2}{4}(g(x', y') - \epsilon\frac{3y}{2} + \frac{3\epsilon^2}{8}h(x', y')) \\ \qquad + \epsilon^3\frac{y}{8} - \frac{\epsilon^4}{64}h(x', y') \end{cases} \tag{3.7}$$

analogous to (3.3). By performing calculations similar to those used for the operator $\mathbb{X}_6 + \mathbb{X}_2$, we deduce that the system of equations (3.5) is satisfied in this case as well.

We can now present formulas for constructing new solutions of the WDVV equation (1.5) from known solutions. Let us refer to solution

$$f = f_1(x, y) = \frac{2y^2}{15k}\left(\frac{x}{y} - ky\right)^{5/2}$$

from article [4].

Then we find the derivatives

$$\begin{aligned} f_{1xx} &= h_1(x, y) = \frac{1}{2k}\sqrt{\frac{x}{y} - ky}, \\ f_{1xy} &= g_1(x, y) = \frac{1}{3k}\left(\frac{x}{y} - ky\right)^{3/2} - \frac{y}{2k}\left(\frac{x}{y^2} + k\right)\sqrt{\frac{x}{y} - ky}, \\ f_{1yy} &= k_1(xy) = \frac{4}{15k}\left(\frac{x}{y} - ky\right)^{5/2} - \frac{4y}{3k}\left(\frac{x}{y^2} + k\right)\left(\frac{x}{y} - ky\right)^{3/2} \\ &\quad + \frac{y^2}{2k}\left(\frac{x}{y^2} + k\right)^2\sqrt{\frac{x}{y} - ky} + \frac{2x}{3ky}\left(\frac{x}{y} - ky\right)^{3/2}. \end{aligned}$$

From (3.7), it follows that the second-order derivatives of the solution we are looking for are implicitly defined by the equations

$$\begin{cases} f_{xx} = h_1(\widehat{x}', \widehat{y}'), \\ f_{xy} = g_1(\widehat{x}', \widehat{y}') - \epsilon\frac{3y}{2} + \frac{3\epsilon^2}{8}h_1(x', y'), \\ f_{yy} = k_1(\widehat{x}', \widehat{y}') - \epsilon x + \frac{\epsilon^2}{4}(g_1(\widehat{x}', \widehat{y}')) - \epsilon\frac{3y}{2} + \frac{3\epsilon^2}{8}h_1(\widehat{x}', \widehat{y}')) \\ \qquad + \epsilon^3\frac{y}{8} - \frac{\epsilon^4}{64}h_1(\widehat{x}', \widehat{y}'), \end{cases} \tag{3.8}$$

where

$$\begin{cases} \widehat{x}' = x - \epsilon\frac{f_{xy}}{2} - \epsilon^2\frac{3y}{8} + \epsilon^3\frac{f_{xx}}{16}, \\ \widehat{y}' = y + \frac{1}{64}\epsilon^4 + \frac{1}{16}\epsilon^3 f_{yy} - \frac{3}{8}\epsilon^2 x - \frac{1}{2}\epsilon f_{xy}. \end{cases} \tag{3.9}$$

One can obtain the solution of equation (1.5) by integrating system (3.8), (3.9).

Let us consider the second solution

$$f = f_2(x,y) = \frac{4}{3}\sqrt{\frac{2c_1x + 2c_2 - x^3}{2}}y^{3/2} + r_1y + \frac{1}{2}c_3x^2 + r_2x + r_3$$

obtained by the symmetry reduction method in Section 4. Next, we have

$$\begin{aligned} h_2(x,y) &= -\frac{y^{3/2}}{6}\frac{(4c_1 - 6x^2)^2}{(-2x^3 + 4c_1x + 4c_2)^{3/2}} - \frac{4xy^{3/2}}{\sqrt{-2x^3 + 4c_1x + 4c_2}} + c_3, \\ g_2(x,y) &= \frac{\sqrt{y}}{2}\frac{4c_1 - 6x^2}{\sqrt{-2x^3 + 4c_1x + 4c_2}}, \\ k_2(x,y) &= \frac{1}{2\sqrt{y}}\sqrt{-2x^3 + 4c_1x + 4c_2}. \end{aligned}$$

As in the previous case, (3.7) leads to

$$\begin{cases} f_{xx} = h_2(\widehat{x}', \widehat{y}'), \\ f_{xy} = g_2(\widehat{x}', \widehat{y}') - \epsilon\frac{3y}{2} + \frac{3\epsilon^2}{8}h_2(\widehat{x}', \widehat{y}'), \\ f_{yy} = k_2(\widehat{x}', \widehat{y}') - \epsilon x + \frac{\epsilon^2}{4}(g_2(\widehat{x}', \widehat{y}')) - \epsilon\frac{3y}{2} + \frac{3\epsilon^2}{8}h_2(\widehat{x}', \widehat{y}') \\ \qquad + \epsilon^3\frac{y}{8} - \frac{\epsilon^4}{64}h_2(\widehat{x}', \widehat{y}'). \end{cases} \tag{3.10}$$

The solution of equation (1.5) generated from the known solution $f = f_2(x,y)$ is given by system (3.10), (3.9).

Let us now show the application of the method of generating a solution from a known one using an example of the second-order differential equation

$$u_{xx} = \sin u_{yy}. \tag{3.11}$$

In [18] an 8-dimensional Lie algebra is used for the construction of invariant solutions of the equation

$$u_{xx} = \exp(-u_{yy}).$$

The Lie algebra of the Lie symmetry group of point transformations for equation (3.11) is 7-dimensional. The generator of the Lie group has the form

$$\mathbb{X} = (c_1 + c_4x)\partial_x + (c_2 + c_4x)\partial_y + (c_3 + 2c_4u + c_5x + c_6y + c_7xy)\partial_u,$$

where $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ are arbitrary real constants. We use the method of generating a solution from the particular solution

$$u = \frac{xy^2}{2} - \sin x \tag{3.12}$$

of equation (3.11) by virtue of generator

$$\mathbb{X} = G(b,c)\partial_x + H(b,c)\partial_y,$$

where $G(b,c) = (\frac{C}{2}b^2 - C\cos c)$, $H(b,c) = Cb\sin c$, and $C \in \mathbb{R}$, is the solution of determining equations (2.17). Then we obtain

$$\begin{cases} u_{xx} = \sin\left(x + a(\frac{C}{2}u_{xy}^2 - C\cos u_{yy})\right), \\ u_{xy} = y + a(Cu_{xy}\sin u_{yy}), \\ u_{yy} = x + a(\frac{C}{2}u_{xy}^2 - C\cos u_{yy}). \end{cases} \tag{3.13}$$

This follows by the same method as in the previous cases of the associativity equations. Formulas (3.13) represent the overdetermined and consistent system of the second-order partial differential equation. Note, that solutions of system (3.13) are not invariant solutions in the classical Lie sense, whereas the initial solution (3.12) is invariant with respect to a one-parameter Lie group of point transformations with generator $\partial_y + xy\partial_u$.

4. SYMMETRY REDUCTION AND CONSTRUCTION OF INVARIANTS, SOLUTION, AND AUXILIARY VARIABLES FOR NONLINEAR EQUATIONS (1.5) AND (4.9)

It is well known that if a differential equation is invariant under a Lie group of point transformations, special solutions, known as invariant solutions, can be found. These solutions remain invariant under certain subgroups of the complete symmetry group of the given equation. Here, we demonstrate the procedure of symmetry reduction for partial differential equations using one-parameter subgroups of non-point transformations. Consider a one-parameter subgroup generated by

$$\mathbb{X}_6 = -\frac{1}{2}c\frac{\partial}{\partial x} - \frac{1}{2}b\frac{\partial}{\partial y} + y\frac{\partial}{\partial a} + \frac{3}{2}x\frac{\partial}{\partial b}.$$

To construct the ansatz that reduces system (2.2) to a system of ordinary differential equations, we need the invariants of this group satisfying the equation $\mathbb{X}_6\omega(x,y,a,b,c) = 0$.

All functionally independent invariants can be obtained by integrating characteristic equations

$$\frac{dx}{-\frac{1}{2}c} = \frac{dy}{-\frac{1}{2}b} = \frac{da}{y} = \frac{db}{\frac{3}{2}x}. \tag{4.1}$$

We are looking for a function $s = s(x,y,a,b,c)$. It is evident that the operator $\mathbb{X}_6$ does not involve differentiation with respect to the variable c, so the first invariant can be identified as

$$\omega_0 = c.$$

We solve the first equation from the system (4.1):

$$\frac{dx}{-\frac{1}{2}c} = \frac{db}{\frac{3}{2}x}, \tag{4.2}$$

and obtain $b = -\int \frac{3x}{c}dx = -\frac{3x^2}{2c} + \omega_1$ from (4.2). Then we have the second invariant

$$\omega_1 = b + \frac{3x^2}{2c}.$$

Solving the next equation from system (4.1)

$$\frac{dy}{-\frac{1}{2}b} = \frac{db}{\frac{3}{2}x},$$

and using the dependency $x = \pm\sqrt{\frac{2c}{3}(\omega_1 - b)}$, we get

$$\frac{dy}{-\frac{1}{2}b} = \frac{db}{\frac{3}{2}\sqrt{\frac{2c}{3}(\omega_1 - b)}}.$$

Hence,

$$\begin{aligned} y &= -\frac{1}{3}\sqrt{\frac{3}{2c}}\int \frac{b}{\sqrt{\omega_1 - b}}db \\ &= -\frac{2}{9}\sqrt{\frac{3}{2c}}(b + 2\omega_1)\sqrt{\omega_1 - b} + \omega_2 = \frac{xb}{c} + \frac{x^3}{c^2} + \omega_2. \end{aligned}$$

The third invariant has the form

$$\omega_2 = y - \frac{xb}{c} - \frac{x^3}{c^2}.$$

To find the last invariant, we use variable substitution with the already found invariants i.e. we introduce the new variables:

$$\begin{cases} a^* = a, \\ b^* = b, \\ c^* = c, \\ x^* = b + \frac{3x^2}{2c}, \\ y^* = y - \frac{xb}{c} - \frac{x^3}{c^2}. \end{cases}$$

Then we are looking for a function $s = s(x^*, y^*, a^*, b^*, c^*)$. Rewriting the operator $\mathbb{X}_6$ in new variables we obtain

$$\begin{aligned} \mathbb{X}_6 s &= y\frac{\partial s}{\partial a^*} + \frac{3}{2}x\frac{\partial s}{\partial b^*} \\ &= \left(y^* + \frac{1}{c^*}\sqrt{\frac{2c^*}{3}(x^* - b^*)}\left(b^* + \frac{2}{3}(x^* - b^*)\right)\right)\frac{\partial s}{\partial a^*} \\ &\quad + \frac{3}{2}\sqrt{\frac{2c^*}{3}(x^* - b^*)}\frac{\partial s}{\partial b^*} = 0. \end{aligned}$$

Using the method of characteristics, we obtain the equation

$$\frac{da^*}{y^* + \frac{1}{3c^*}\sqrt{\frac{2c^*}{3}(x^* - b^*)}(b^* + 2x^*)} = \frac{db^*}{\frac{3}{2}\sqrt{\frac{2c^*}{3}(x^* - b^*)}},$$

that is,

$$\begin{aligned} a^* &= \frac{2}{3}y^*\sqrt{\frac{3}{2c^*}}\int\frac{db^*}{\sqrt{x^* - b^*}} + \frac{2}{9c^*}\int(b^* + 2x^*)db^* \\ &= \frac{b^{*2}}{9c^*} + \frac{4x^*b^*}{9c^*} - \frac{4}{3}\sqrt{\frac{3}{2c^*}}y^*\sqrt{x^* - b^*} + \omega_3. \end{aligned}$$

Going back to the previous variables we have

$$a = \frac{5b^2}{9c} + \frac{8bx^2}{3c^2} - \frac{2xy}{c} + \frac{2x^4}{c^3} + \omega_3.$$

Thus, we find the last invariant

$$\omega_3 = a - \frac{5b^2}{9c} - \frac{8bx^2}{3c^2} + \frac{2xy}{c} - \frac{2x^4}{c^3}.$$

Then we form the corresponding ansatz

$$\begin{cases} p_1(c(x,y)) = b + \frac{3x^2}{2c}, \\ p_2(c(x,y)) = y - \frac{xb}{c} - \frac{x^3}{c^2}, \\ p_3(c(x,y)) = a - \frac{5b^2}{9c} - \frac{8bx^2}{3c^2} + \frac{2xy}{c} - \frac{2x^4}{c^3}, \end{cases} \tag{4.3}$$

where p_1, p_2, p_3 are unknown functions of the variable c.

We differentiate each equation of (4.3) with respect to x and y, and obtain a system of six equations. Solving this system of algebraic equations for $a_x, a_y, b_x, b_y, c_x, c_y$ and substituting the results into (2.2), we derive the reduced system of equations:

$$\begin{cases} p_1' = -\frac{p_1}{c}, \\ p_2' = -\frac{2p_2}{c}, \\ p_3' = \frac{24p_1^2}{9c^2}. \end{cases} \tag{4.4}$$

The solution of the system (4.4) is

$$\begin{cases} p_1 = \frac{c_1}{c}, \\ p_2 = \frac{c_2}{c^2}, \\ p_3 = -\frac{8c_1^2}{9c^3} + c_3. \end{cases} \tag{4.5}$$

Substituting (4.5) into (4.3) yields

$$\begin{cases} \frac{c_1}{c} = b + \frac{3x^2}{2c}, \\ \frac{c_2}{c^2} = y - \frac{xb}{c} - \frac{x^3}{c^2}, \\ -\frac{8c_1^2}{9c^3} + c_3 = a - \frac{5b^2}{9c} - \frac{8bx^2}{3c^2} + \frac{2xy}{c} - \frac{2x^4}{c^3}. \end{cases} \tag{4.6}$$

By finding the functions a, b, c from the system (4.6) and using (2.1) we derive

$$\begin{cases} c = f_{yy} = \pm\sqrt{\frac{2c_1x+2c_2-x^3}{2y}}, \\ b = f_{xy} = \pm(c_1 - \frac{3x^2}{2})\sqrt{\frac{2y}{2c_1x+2c_2-x^3}}, \\ a = f_{xx} = -\frac{8c_1^2}{9c^3} + c_3 - (-\frac{5b^2}{9c} - \frac{8bx^2}{3c^2} + \frac{2xy}{c} - \frac{2x^4}{c^3}). \end{cases} \tag{4.7}$$

This system is the consistent overdetermined system of partial differential equations. We integrate (4.7) and obtain the solution of (1.5) in the explicit form

$$f(x,y) = \pm\frac{4}{3}\sqrt{\frac{2c_1x+2c_2-x^3}{2}}y^{3/2} + r_1y + \frac{1}{2}c_3x^2 + r_2x + r_3, \tag{4.8}$$

where c_i and r_i are constants for $i = 1, 2, 3$. Using other subalgebras containing the elements $\mathbb{X}_6$ and $\mathbb{X}_8$, one can construct additional solutions of (1.5) by means of the non-point symmetry reduction method.

In a similar way, we also obtained the reduced system and solution in implicit form for equation (2.5) using the generator $\mathbb{X}_9$ from (2.12).

Note that variables in (2.1) can be obtained within the framework of the general approach to symmetry reduction of partial differential equations [3, 22, 25, 26]. In this case, we utilize differential invariants, not just the invariants of the group of point transformations, which are used for constructing classical invariant solutions.

Indeed, equation (1.5) is invariant under a 3-parameter Lie group with the basis elements of the corresponding Abelian Lie algebra $\{\mathbb{X}_1 = \frac{\partial}{\partial f}, \mathbb{X}_4 = x\frac{\partial}{\partial f}, \mathbb{X}_5 = y\frac{\partial}{\partial f}\}$. The functionally independent differential invariants of the second order of this group are $\omega_1 = x$, $\omega_2 = y$, $\omega_3 = f_{xx}$, $\omega_4 = f_{xy}$, $\omega_3 = f_{yy}$. We find the desired substitutions $\omega_3 = a(\omega_1, \omega_2)$, $\omega_4 = b(\omega_1, \omega_2)$, $\omega_5 = c(\omega_1, \omega_2)$ expressed through the obtained invariants, where a, b, c are unknown functions which reduce equation (1.5) to system (2.2).

We emphasize that the differential invariants of the group of conditional symmetry [1, 17, 21, 29] can also be used to construct auxiliary variables. We illustrate this with an example of a nonlinear equation

$$u_{tx} = [1 - k^2u_x^2]^{1/2}\sin u, \tag{4.9}$$

where $k \in \mathbb{R}$. We use the conditional symmetry generator

$$Q = \partial_u + k\cos u\partial_{v^1} + \frac{1}{k}\sqrt{1-k^2(v^2)^2}\partial_{v^2}$$

of the system of equations

$$\begin{cases} v_x^1 + v_u^1v^2 = v_t^2 + v_u^2v^1, \\ v_t^2 + v_u^2v^1 = \sqrt{1-k^2(v^2)^2}\sin u \end{cases}$$

corresponding to the original equation, where $u_t = v^1$, $u_x = v^2$, and v^1, v^2 are functions on t, x, u, rather than the generators of the symmetry group of equation (4.9) itself.

The infinitesimal criterion of conditional invariance

$$Q_{(1)}(v_x^1 + v_u^1 v^2 - v_t^2 - v_u^2 v^1)\Big|_{\begin{cases} v_x^1 = -v_u^1 v^2 + v_t^2 + v_u^2 v^1, \\ v_t^2 = -v_u^2 v^1 + \sqrt{1-k^2(v^2)^2}, \\ v_u^1 = k\cos u, v_u^2 = \frac{1}{k}\sqrt{1-k^2(v^2)^2}, \end{cases}} = 0,$$

$$Q_{(1)}(v_t^2 + v_u^2 v^1 - \sqrt{1-k^2(v^2)^2}\sin u)\Big|_{\begin{cases} v_x^1 = -v_u^1 v^2 + v_t^2 + v_u^2 v^1, \\ v_t^2 = -v_u^2 v^1 + \sqrt{1-k^2(v^2)^2}, \\ v_u^1 = k\cos u, v_u^2 = \frac{1}{k}\sqrt{1-k^2(v^2)^2} \end{cases}} = 0$$

holds in this case. Functionally independent differential invariants of the first order satisfying the conditions $Q\omega_1 = 0$, $Q\omega_2 = 0$, $Q\omega_3 = 0$, and $Q\omega_4 = 0$ are given by $\omega_1 = t$, $\omega_2 = x$, $\omega_3 = u - \arcsin(kv^2)$, $\omega_4 = v^1 - k\sin u$. By using these invariants, one can easily receive the formula for the desired variables

$$\begin{cases} u_x = \frac{1}{k}\sin(u - \varphi_1(t,x)), \\ u_t = \varphi_2(t,x) + k\sin u, \end{cases} \tag{4.10}$$

where $\varphi_1(t,x)$, $\varphi_2(t,x)$ are unknown functions (auxiliary variables). Substituting (4.10) into equation (4.9) and taking into account the consistency condition $u_{tx} = u_{xt}$, we obtain the reduced system

$$\begin{cases} \varphi_{2x} = \sin\varphi_1, \\ \varphi_2 = \varphi_{1t}. \end{cases} \tag{4.11}$$

From (4.11) it follows that φ_1 satisfies the sine-Gordon equation $\varphi_{1xt} = \sin\varphi_1$. We can now rewrite (4.10) in the following form

$$\begin{cases} u_x = \frac{1}{k}\sin(u - w), \\ u_t = w_t + k\sin u, \end{cases} \tag{4.12}$$

where $\varphi_1 = w$. Thus, (4.12) gives the Bäcklund transformation that has been obtained in [6] by another means. It maps the solution of the sine-Gordon equation into a solution of the equation (4.9).

Thus, we conclude that the suggested approach extends the applicability of the group symmetry method for finding solutions to associativity equations and constructing Bäcklund transformations for nonlinear partial differential equations.

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