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LOCAL PROPERTIES OF GRAPHS THAT INDUCE GLOBAL CYCLE PROPERTIES

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Abstract. A graph G is locally Hamiltonian if $G[N(v)]$ is Hamiltonian for every vertex $v \in V(G)$. In this note, we prove that every locally Hamiltonian graph with maximum degree at least $|V(G)| - 7$ is weakly pancyclic. Moreover, we show that any connected graph G with $\Delta(G) \leq 7$ and $\delta(G[N(v)]) \geq 3$ for every $v \in V(G)$, is fully cycle extendable. These findings improve some known results by Tang and Vumar.

Keywords: fully cycle extendability, weakly pancyclicity, locally connected.

Mathematics Subject Classification: 05C38, 05C45.

1. INTRODUCTION

We consider finite, simple and undirected graphs, for notations and definitions not included here we refer the reader to [3]. We begin by defining the global cycle properties that we shall consider. The order (number of vertices) of a graph G is denoted by $n(G)$. A graph is *Hamiltonian* if it has a cycle of length $n(G)$. If a graph G has cycles of every length from 3 up to $n(G)$, then G is *pancyclic*. An even stronger property is full cycle extendability, introduced by Hendry [7]. A cycle C in a graph G is *extendable* if there exists another cycle C' in G that contains all the vertices of C plus a single new vertex. A graph G is *cycle extendable* if every non-Hamiltonian cycle of G is extendable. If, in addition, every vertex of G lies on a 3-cycle, then G is *fully cycle extendable*. A graph is *traceable* if it has a path containing all its vertices. Such a path is called a *Hamilton path*. We say a graph is *Hamilton-connected* if it has a Hamiltonian path between every pair of vertices.

Actually, we are also interested in global cycle properties of non-Hamiltonian graphs. We recall that the girth, denoted by $g(G)$, is the length of the shortest cycle, while the circumference, denoted by $c(G)$, is the length of the longest cycle in a graph G. A graph G is called *weakly pancyclic* if G has a cycle of every length between $g(G)$ and $c(G)$.

By a local property of a graph we mean a property that is shared by the subgraphs induced by the open neighborhoods of the vertices. The open (respectively, closed) neighborhood of a vertex $v \in V(G)$ is denoted by $N(v)$ (respectively, $N[v]$, which represents the set $N(v) \cup \{v\}$). If $X \subseteq V(G)$, the subgraph induced by X is denoted by $G[X]$. For a given graph property $\mathcal{P}$, we call a graph G *locally* $\mathcal{P}$ if $G[N(v)]$ has property $\mathcal{P}$ for every $v \in V(G)$.

Skupień [10] introduced the concept of a locally Hamiltonian graph, defining a graph G to be locally Hamiltonian if $G[N(v)]$ is Hamiltonian for every $v \in V(G)$. The properties of locally connected/traceable/Hamiltonian graphs have been extensively studied, for example [1, 2, 4, 10, 13]. Tang and Vumar [11] considered the properties of locally Hamilton-connected graphs.

It is well known that the Hamilton Cycle Problem (the problem of deciding whether a graph is Hamiltonian) is NP-complete, even for claw-free graphs. The following well-known theorem of Oberly and Sumner [9], demonstrates the strength of the local connectivity property.

Theorem 1.1 ([9])**.** *If G is a connected, locally connected, claw-free graph of order at least* 3*, then G is Hamiltonian.*

The following conjecture is proposed by Ryjáček (see [12]).

Conjecture 1.2. Every locally connected graph is weakly pancyclic.

In [1], the authors studied global cycle properties of connected, locally traceable and locally Hamiltonian graphs, proposing the following two weaker conjectures.

Conjecture 1.3. Every locally traceable graph is weakly pancyclic.

Conjecture 1.4. Every locally Hamiltonian graph is weakly pancyclic.

Ryjáček's conjecture seems very difficult to settle. Nonetheless, some progress has been made for graphs with small maximum degree. The minimum and maximum degree of a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively.

Theorem 1.5 ([1])**.** *If G is a locally connected graph with $\Delta(G) \le 5$, then G is weakly pancyclic.*

Theorem 1.6 ([6])**.** *Let G be a connected, locally connected graph with $n(G) \ge 3$, $\Delta(G) \le 5$ and $\Delta(G) - \delta(G) \le 1$. Then G is fully cycle extendable.*

Recently, Gordon *et al.* [5] extended the aforementioned result to the case $\Delta = 5$ and $\delta = 3$.

Theorem 1.7 ([5])**.** *If G is a connected, locally connected graph with $\Delta(G) = 5$ and $\delta(G) = 3$, then G is fully cycle extendable.*

Irzhavski [8] proved the following theorem regarding locally connected graphs.

Theorem 1.8 ([8])**.** *The Hamilton Cycle Problem for locally connected graphs with minimum degree* 2 *and maximum degree* 5 *is NP-complete. Moreover, for any integers $D \ge 6$ and $2 \le d \le D$, the Hamilton Cycle Problem is NP-complete in the class of locally connected graphs with minimum degree d and maximum degree D.*

For $k \geq 3$, the magwheel M_k is the graph obtained from the wheel W_k by adding, for each edge e on the rim of W_k, a vertex v_e and joining it to the two ends of the edge e. Magwheels are examples of connected, non-Hamiltonian locally connected graphs with $\delta = 2$.

Theorem 1.9 ([1]).

(1) *Suppose G is a connected, locally traceable graph with $n(G) \geq 3$ and $\Delta(G) \leq 5$. Then G is fully cycle extendable if and only if $G \notin \{M_3, M_4, M_5\}$.*
(2) *Let G be a connected, locally Hamiltonian graph with $n(G) \geq 3$ and $\Delta(G) \leq 6$. Then G is fully cycle extendable.*
(3) *If G is a connected, locally Hamiltonian graph of order n with $\Delta(G) \geq n - 5$, then G is weakly pancyclic.*

In [1], the authors proposed some open problems. One of them is the question: Is every connected locally Hamiltonian graph with $\Delta = 7$ Hamiltonian? Tang and Vumar addressed this problem in [11] and partially improved Theorem 1.9(2) and (3) with the following results.

Theorem 1.10 ([11]).

(1) *Let G be a connected, locally Hamilton-connected graph of order $n \geq 3$ with $\Delta(G) \leq 7$. Then G is fully cycle extendable.*
(2) *If G is a connected, locally Hamiltonian graph of order n with $\Delta(G) \geq n - 6$, then G is weakly pancyclic.*

Obviously, if G a connected, locally Hamilton-connected graph of order at least 4, then it does not contain a cut-vertex. Furthermore, G also does not contain a cut-set of order two, otherwise, there is no Hamiltonian path between the two vertices in the cut-set. Thus, if G is Hamilton-connected, then it is 3-connected. Then the following holds directly.

Observation 1.11. *If G is a connected, locally Hamilton-connected graph of order greater than 4, then $\delta(G[N(v)]) \geq 3$ for every $v \in V(G)$.*

If G is a connected graph with $\delta(G[N(v)]) \geq 3$ for every $v \in V(G)$, then G has order at least 5 and $\delta(G) \geq 4$. If for some vertex $v \in V(G)$, there exists a pair of vertices $x, y \in N(v)$ such that x and y belong to different components of $G[N(v)]$, then each component containing x or y in $G[N(v)]$ has at least 4 vertices, since $d_{G[N(v)]}(x) \geq 3$ and $d_{G[N(v)]}(y) \geq 3$. Thus, $d_G(v) \geq 8$. Therefore, the following result follows immediately.

Observation 1.12. *If G is a connected graph with $\Delta(G) \leq 7$ and $\delta(G[N(v)]) \geq 3$ for every $v \in V(G)$, then G is locally connected.*

Motivated by these observations and Irzhavski's Theorem, we improve Theorem 1.10(1) by the following conclusion.

Theorem 1.13. *Let G be a connected graph with $\Delta(G) \leq 7$. If $\delta(G[N(v)]) \geq 3$ for every $v \in V(G)$, then G is fully cycle extendable.*

Note that, if G consists of two K_5's that share exactly one common vertex, then $\Delta(G) = 8$ and $\delta(G[N(v)]) \geq 3$ for every $v \in V(G)$, but G is not cycle extendable. This implies that the bound on $\Delta(G)$ in Theorem 1.13 cannot be increased.

We also improve Theorem 1.10(2) by the following conclusion.

Theorem 1.14. *If G is a connected, locally Hamiltonian graph of order n with $\Delta(G) \geq n - 7$, then G is weakly pancyclic.*

2. PROOF OF THEOREM 1.13

For two disjoint subsets $X, Y \subset V(G)$, we define

$$E(X, Y) = \{uv \in E(G) : u \in X, v \in Y\}.$$

Let $C = v_0v_1v_2 \dots v_{t-1}v_0$ be a t-cycle in a graph G. If $i \neq j$ and $\{i, j\} \subseteq \{0, 1, \dots, t-1\}$, then $v_i\overrightarrow{C}v_j$ and $v_i\overleftarrow{C}v_j$ denote the paths $v_iv_{i+1} \dots v_j$ and $v_iv_{i-1} \dots v_j$ (subscripts expressed modulo t), respectively. For a given non-extendable cycle C of G, a vertex of G is called a *cycle vertex* if it lies on C, and an *off-cycle vertex* if it is in $V(G)\backslash V(C)$. A cycle vertex that is adjacent to an off-cycle vertex will be called an *attachment vertex*.

The following basic results on non-extendable cycles will be used frequently.

Lemma 2.1 ([1]). *Let $C = v_0v_1 \dots v_{t-1}v_0$ be a non-extendable cycle in a graph G. Suppose v_i and v_j $(i < j)$ are two attachment vertices of C that are adjacent to a common off-cycle vertex. Then the following holds.* (*All subscripts are expressed modulo t.*)

(1) $j \neq i + 1$.
(2) *Neither $v_{i+1}v_{j+1}$ nor $v_{i-1}v_{j-1}$ is in $E(G)$.*
(3) *If $v_{i-1}v_{i+1} \in E(G)$, then neither $v_{j-1}v_i$ nor $v_{j+1}v_i$ is in $E(G)$.*
(4) *If $j = i + 2$, then v_{i+1} does not have two neighbors v_k, v_{k+1} on the path $v_{i+2} \dots v_i$.*

Proof of Theorem 1.13. Suppose G is a connected graph with $\Delta(G) \leq 7$ and $\delta(G[N(v)]) \geq 3$ for every $v \in V(G)$. Then, for every $v \in V(G)$, each neighbor of v in G has at least three neighbors and at most three non-neighbors in $N(v)$. Then every vertex lies on a 3-cycle. Thus, to show that G is fully cycle extendable, it suffices to show that every cycle is extendable.

Assume, to the contrary, that there exists a non-Hamiltionian cycle $C = v_0v_1 \dots v_{t-1}v_0$ that is not extendable. Since G is connected, there exists a vertex on C, say v_0, that has an off-cycle neighbor. Since v_1 has at least three neighbors in $N(v_0)$ and v_1 is not adjacent to any off-cycle neighbor of v_0 (by Lemma 2.1(1)), we have that v_1 is adjacent to at least three vertices in $N_G(v_0) \cap V(C)$. Thus, v_0 has at least 4 neighbors on C. Furthermore, $|N(v_0)| \geq 6$, otherwise, the unique off-cycle neighbor of v_0 has at most two neighbors in $N_G(v_0)$. Now the following claim holds.

Claim 2.2. Let $v_i \in N(v_0)$ with $1 < i < t - 1$. If v_0 and v_i have a common off-cycle neighbor x and $v_iv_1 \in E(G)$, then $v_{i+1}v_0 \in E(G)$ and $v_{t-1}v_1 \notin E(G)$.

Proof of Claim 2.2. Suppose, to the contrary, that $v_{i+1}v_0 \notin E(G)$. By Lemma 2.1(2), we have $v_{i+1}v_1 \notin E(G)$. By Lemma 2.1(3), we conclude that $v_{i+1}v_{i-1} \notin E(G)$. By Lemma 2.1(1), we get $v_{i-1}x, v_{i+1}x, v_{t-1}x \notin E(G)$. The vertices v_0, v_1, v_{i-1}, x are non-neighbours of v_{i+1} in $N(v_i)$. Since v_{i+1} has at least three neighbors in $N_G(v_i)$, we have $d_G(v_i) = 7$ and v_{i+1} has exactly three neighbors and three non-neighbors in $N_G(v_i)$. This implies that $i - 1 = 1$ and $i = 2$.

As $|N(v_0)| \leq 7$ and $v_1x, v_{t-1}x, v_{t-1}v_1 \notin E(G)$ (by Lemma 2.1(1) and (2)), we get that v_1 and x have at least two common neighbors in $N(v_0)$. Obviously, $v_2 \in N(v_1) \cap N(x) \cap N(v_0)$. Since v_1 and x have no common off-cycle neighbour, there exists a vertex $v_j \in (N(v_0) \cap N(v_1) \cap N(x)) \setminus \{v_2\}$. Since $v_{t-1}v_1, v_3v_0 \notin E(G)$, we have $3 < j < t - 1$. By Lemma 2.1(3), $v_{j-1}v_{j+1} \notin E(G)$. By Lemma 2.1 (4), $v_1v_{j-1}, v_1v_{j+1} \notin E(G)$. By Lemma 2.1(1), $xv_{j-1}, xv_{j+1} \notin E(G)$. Since each of v_{j-1} and v_{j+1} has at least three neighbors in $N(v_j)$, we have $v_{j-1}v_0, v_{j+1}v_0 \in E(G)$. By our assumption $v_0v_3 \notin E(G)$, we have $j - 1 \geq 4$. Since x has at least three neighbors in $N(v_0)$ and $xv_1, xv_{j-1}, xv_{j+1}, xv_{t-1} \notin E(G)$, we have $j + 1 = t - 1$ and $j = t - 2$. By Lemma 2.1(2), $v_{j-1}v_{t-1} \notin E(G)$. Since $v_1v_{j-1} \notin E(G)$, we have $I_0 = \{x, v_1, v_{t-1}, v_{j-1}\}$ is an independent set of $G[N(v_0)]$. Then each vertex in I_0 is adjacent to each vertex in $N(v_0) - I_0$. Hence, $\{v_{t-1}v_2, v_{j-1}v_2\} \subseteq E(G)$. Considering the neighbors of x in $G[N(v_2)]$, we have $xv_{j-1}, xv_{t-1}, xv_1, xv_3 \notin E(G)$. Since x has at least three neighbors in $N(v_2)$, we have $j - 1 = 3$, contradicting that $j - 1 \geq 4$. This proves that $v_{i+1}v_0 \in E(G)$. Furthermore, by Lemma 2.1(3), $v_{t-1}v_1 \notin E(G)$. □

If x is the off-cycle neighbour of v_0, then $v_1x \notin E(G)$, by Lemma 2.1(1). Since $|N(v_0)| \leq 7$ and $\delta(G[N(v_0)]) \geq 3$, we infer that v_1 and x have a common neighbour v_j in $N(v_0)$, such that $1 < j < t-1$ and hence, by Claim 2.2, $v_1v_{t-1} \notin E(G)$. This implies that v_1 and x have at least two common neighbours v_j and v_k in $N(v_0)$. We may assume that $1 < j < k < t - 1$. By Claim 2.2, $v_{j+1}, v_{k+1} \in N(v_0)$. By Lemma 2.1(1), we have $v_{j+1}, v_{k+1}, v_{t-1}, v_1$ are non-neighbours of x in $N(v_0)$, which implies that $v_{k+1} = v_{t-1}$, i.e. $k = t - 2$. Thus, v_{t-1}, v_{j+1} and v_1 are three distinct non-neighbours of x in $N(v_0)$, and hence every vertex in $N(v_0) - \{v_{j+1}, v_{t-1}, v_1\}$ is a neighbour of x. By Lemma 2.1 (2), we have $v_1v_{j+1} \notin E(G)$. Since v_1 has at least three neighbours in $N(v_0)$, there is a vertex $v_i \in N(v_0) \cup N(v_1)$ such that $i \notin \{j, k\}$. Thus, $v_i \in N(x)$. By Claim 2.2, $v_0v_{i+1} \in E(G)$. Hence, $v_1, v_{i+1}, v_{j+1}, v_{t-1}$ are four distinct non-neighbours of x in $N(v_0)$, a contradiction. This completes the proof of Theorem 1.13. □

3. PROOF OF THEOREM 1.14

The following results on locally Hamiltonian graphs and non-extendable cycles are quite useful.

Lemma 3.1 ([1])**.** *If G is a locally Hamiltonian graph and $uv \in E(G)$, then $|N(u) \cap N(v)| \geq 2$.*

Lemma 3.2 ([1])**.** *If G is a connected locally Hamiltonian graph of order n with maximum degree Δ, then G has cycles of length k for every k such that $3 \leq k \leq \min\{\Delta + 2, n\}$.*

Observation 3.3. *Suppose a graph G contains an $(n-r)$-cycle $C = v_0v_1 \dots v_{n-r-1}v_0$, but G does not contain an $(n-r-1)$-cycle. If $x \in V(G) - V(C)$ and $xv_i \in E(G)$, then the following hold.*

(1) $xv_{i+3}, xv_{i-3} \notin E(G)$.
(2) *If $y \in V(G) - V(C)$ and $xy \in E(G)$, then $yv_{i+4}, yv_{i-4} \notin E(G)$.*

Inspired by the work of Tang and Vumar [11], we give the following proof.

Proof of Theorem 1.14. Suppose G is a connected, locally Hamiltonian graph of order n with $\Delta(G) \geq n-7$. To prove that G is weakly pancyclic, by Theorem 1.10(2), we only need to consider the case $\Delta = n-7$. If $n \leq 13$, then $\Delta = n-7 \leq 6$, and the result holds from Theorem 1.9(2). Now we assume $n \geq 14$. Note that by Lemma 3.2, G has cycles of length k for every k with $3 \leq k \leq n-5$. It suffices to show that if G has an $(n-i)$-cycle, then G also has an $(n-i-1)$-cycle for $i = 0,1,2,3$, respectively. Here, we only prove that if G has an $(n-3)$-cycle, then G also has an $(n-4)$-cycle. The other three cases ($i = 0,1,2$) can be proved by a similar approach, and we omit the proofs.

Now suppose that G has an $(n-3)$-cycle $C = v_0v_1v_2 \dots v_{n-4}v_0$, but G has no $(n-4)$-cycles. Then C has no short chords (edges of the form v_iv_{i+2}) for any $0 \leq i \leq n-4$, where the subscripts are taken modulo $n-3$. Let $V(G) \setminus V(C) = \{x, y, z\}$.

Claim 3.4. Each vertex not on C has degree less than $n-7$.

Proof of Claim 3.4. Suppose, to the contrary, that there exists a vertex $x \in V(G) \setminus V(C)$ such that $d_G(x) = n-7$. By Observation 3.3(1), we have that $n-3 = |V(C)| \geq 2(n-7-2)$. Thus, $14 \leq n \leq 15$. We will distinguish the following two cases based on the value of n.

Case A. $n = 14$.

Then $d(x) = n-7 = 7$ and the following two facts hold.

Fact 3.5. *$N(x)$ does not contain three consecutive vertices of C.*

Proof of Fact 3.5. Without loss of generality, suppose that $\{xv_0, xv_1, xv_2\} \subseteq E(G)$. By Observation 3.3(1), we have $E(\{x\}, \{v_3, v_4, v_5, v_8, v_9, v_{10}\}) = \emptyset$. As $d(x) = 7$, we get that $N(x) = \{v_0, v_1, v_2, v_6, v_7, y, z\}$. By Observation 3.3(2), we have $\{yv_0, yv_2, yv_6, yv_7\} \cap E(G) = \emptyset$. Since $G[N(x)]$ is Hamiltonian and $d_{G[N(x)]}(y) \geq 2$, we have $\{yv_1, yz\} \subseteq E(G)$. But then $xzyv_1\overleftarrow{C}v_6x$ is an $(n-4)$-cycle of G, a contradiction. □

Fact 3.6. *$N(x)$ does not contain two consecutive vertices of C.*

Proof of Fact 3.6. Without loss of generality, suppose that $\{xv_0, xv_1\} \subseteq E(G)$. By Observation 3.3(1), we have $E(\{x\}, \{v_3, v_4, v_8, v_9\}) = \emptyset$. By Fact 3.5, $E(\{x\}, \{v_2, v_{10}\}) = \emptyset$. Since $d(x) = 7$, we infer that $N(x) = \{v_0, v_1, v_5, v_6, v_7, y, z\}$, which contradicts Fact 3.5. □

Without loss of generality, suppose that $xv_0 \in E(G)$. By Fact 3.6, we have $E(\{x\},\{v_1,v_{10}\}) = \emptyset$. By Observation 3.3(1), we have $E(\{x\},\{v_3,v_8\}) = \emptyset$. If $xv_6 \in E(G)$, then by Fact 3.6, we have $E(\{x\},\{v_5,v_7\}) = \emptyset$. By Observation 3.3(1), we have $xv_9 \notin E(G)$. Then $d(x) \leq 6$, a contradiction. This implies that $xv_6 \notin E(G)$. By symmetry, $xv_5 \notin E(G)$. As $d(x) = 7$, we infer that $N(x) = \{v_0, v_2, v_4, v_7, v_9, y, z\}$, which contradicts Observation 3.3(1).

Case B. $n = 15$.

Then $d(x) = n - 7 = 8$ and the following two facts hold.

Fact 3.7. *$N(x)$ does not contain three consecutive vertices of C.*

Proof of Fact 3.7. Without loss of generality, suppose that $\{xv_0, xv_1, xv_2\} \subseteq E(G)$. By Observation 3.3(1), we have $E(\{x\},\{v_3,v_4,v_5,v_9,v_{10},v_{11}\}) = \emptyset$. As $d(x) = 8$, we infer that $N(x) = \{v_0,v_1,v_2,v_6,v_7,v_8,y,z\}$. By Observation 3.3(2), we have $E(\{y,z\},\{v_0,v_2,v_6,v_8\}) = \emptyset$. Since $G[N(x)]$ is Hamiltonian and $d_{G[N(x)]}(y) \geq 2$, we have y is adjacent to at least one of v_1 and v_7. By symmetry, suppose that $yv_1 \in E(G)$. Then $yz \notin E(G)$, otherwise $xzyv_1 \overleftarrow{C} v_6 x$ is an $(n-4)$-cycle of G, a contradiction. Thus, $N_{G[N(x)]}(y) = N_{G[N(x)]}(z) = \{v_1, v_7\}$, which contradicts that $G[N(x)]$ is Hamiltonian. ☐

Fact 3.8. *$N(x)$ does not contain two consecutive vertices of C.*

Proof of Fact 3.8. Without loss of generality, suppose that $\{xv_0, xv_1\} \subseteq E(G)$. By Observation 3.3(1), we have $E(\{x\},\{v_3,v_4,v_9,v_{10}\}) = \emptyset$. By Fact 3.7, $E(\{x\},\{v_2,v_{11}\}) = \emptyset$. As $d(x) = 8$, we infer that $N(x) = \{v_0,v_1,v_5,v_6,v_7,v_8,y,z\}$, which contradicts Fact 3.7. ☐

Without loss of generality, suppose that $xv_0 \in E(G)$. By Fact 3.8, $E(\{x\},\{v_1,v_{11}\}) = \emptyset$. By Observation 3.3(1), $E(\{x\},\{v_3,v_9\}) = \emptyset$. As $d(x) = 8$, we have that $N(x) = \{v_0,v_2,v_4,v_6,v_8,v_{10},y,z\}$. By Observation 3.3(2), we have $E(\{y\},\{v_0,v_2,v_4,v_6,v_8,v_{10}\}) = \emptyset$. Thus, z is the only neighbor of y in $G[N(x)]$, contradicting that G is locally Hamiltonian. This proves Claim 3.4. ☐

By Claim 3.4, there exists a vertex of degree $n-7$ on C, say, $d(v_0) = n-7$. We have the following observation. The proof of the observation is straightforward.

Observation 3.9.

(1) *If $v_i \in N(v_1) \cap N(v_2)$ for $v_i \in \{v_5, \ldots, v_{n-5}\}$, then neither v_{i-1} nor v_{i-2} is a neighbor of v_0.*

(2) *If $v_j \in N(v_{n-5}) \cap N(v_{n-4})$ for $v_j \in \{v_2, \ldots, v_{n-8}\}$, then neither v_{j+1} nor v_{j+2} is a neighbor of v_0.*

Now we distinguish the following two cases.

Case 1. $\{x,y,z\} \cap N(v_1) \cap N(v_2) \neq \emptyset$ or $\{x,y,z\} \cap N(v_{n-4}) \cap N(v_{n-5}) \neq \emptyset$.

Without loss of generality, suppose that $\{x,y,z\}\cap N(v_1)\cap N(v_2)\neq\emptyset$ and let $\{x\}\subseteq\{x,y,z\}\cap N(v_1)\cap N(v_2)$. By Observation 3.3(1), $x\notin N(v_{n-4})\cap N(v_{n-5})$. Since G has no $(n-4)$-cycles, we have $E(\{v_0\},\{v_{n-5},v_{n-6},v_2\})=\emptyset$ and $v_{n-5}v_{n-8}\notin E(G)$.

Firstly, suppose that $\{x,y,z\}\cap N(v_{n-4})\cap N(v_{n-5})=\{y,z\}$. By Observation 3.3(1), we have that $y,z\notin N(v_1)\cap N(v_2)$. Since G has no $(n-4)$-cycles, we have

$$\{v_3,v_4,v_5,v_6,v_{n-4},v_{n-5}\}\cap N(v_1)\cap N(v_2)=\emptyset.$$

Thus, by Lemma 3.1, there exists a vertex $v_i\in\{v_7,v_8,\dots,v_{n-6}\}$ such that $v_i\in N(v_1)\cap N(v_2)$. By Observation 3.9(1), $E(\{v_0\},\{v_{i-1},v_{i-2}\})=\emptyset$. Since $7\leq i\leq n-6$, $v_{i-1},v_{i-2},v_{n-5},v_{n-6}$ are four distinct vertices. Recall that $d(v_0)=n-7$. Thus,

$$V(G)\setminus N[v_0]=\{v_2,v_3,v_{i-1},v_{i-2},v_{n-5},v_{n-6}\}.$$

Then $v_0v_{i-3}\overleftarrow{C}v_2xv_1v_i\overrightarrow{C}v_0$ is an $(n-4)$-cycle of G, a contradiction.

Next, suppose that $\{x,y,z\}\cap N(v_{n-4})\cap N(v_{n-5})=\{z\}$. Since G has no $(n-4)$-cycles, we have

$$\{v_0v_3,v_{n-5}v_{n-7},v_{n-5}v_{n-8},v_{n-5}v_{n-9},v_{n-4}v_2\}\cap E(G)=\emptyset.$$

Hence, by Lemma 3.1, there exists a vertex $v_i\in\{v_3,\dots,v_{n-10}\}$ such that $v_i\in N(v_{n-4})\cap N(v_{n-5})$. By Observation 3.9(2), $E(\{v_0\},\{v_{i+1},v_{i+2}\})=\emptyset$. It follows from $d(v_0)=n-7$ that

$$V(G)\setminus N[v_0]=\{v_2,v_3,v_{i+1},v_{i+2},v_{n-5},v_{n-6}\}.$$

Then $v_0v_{i+3}\overrightarrow{C}v_{n-5}zv_{n-4}v_i\overleftarrow{C}v_0$ is an $(n-4)$-cycle of G, a contradiction.

Lastly, suppose that $\{x,y,z\}\cap N(v_{n-4})\cap N(v_{n-5})=\emptyset$. By Lemma 3.1, there exist two distinct vertices v_i,v_j with $2\leq i<j\leq n-9$, such that

$$\{v_i,v_j\}\subseteq N(v_{n-4})\cap N(v_{n-5}).$$

By Observation 3.9(2), we have

$$\{v_2,v_{i+1},v_{i+2},v_{j+1},v_{j+2},v_{n-5},v_{n-6}\}\subseteq V(G)\setminus N[v_0].$$

Since $d(v_0)=n-7$ and $2\leq i<j\leq n-9$, we infer that $i+2=j+1$ and $j=i+1$. If $j+3\neq n-6$, then $v_{j+3}v_0\in E(G)$, but then $v_0v_{j+3}\overrightarrow{C}v_{n-4}v_j\overleftarrow{C}v_2xv_1v_0$ is an $(n-4)$-cycle of G, a contradiction. Thus, $j+3=n-6$. Then we have $\{x,y,z\}\cap N(v_1)\cap N(v_2)=\{x\}$. Otherwise, suppose that $y\in\{x,y,z\}\cap N(v_1)\cap N(v_2)$, then $v_0xv_1yv_2v_3\overrightarrow{C}v_jv_{n-5}v_{n-4}v_0$ is an $(n-4)$-cycle of G, a contradiction. Recall that

$$V(G)\setminus N[v_0]=\{v_2,v_{n-5},v_{n-6},v_{n-7},v_{n-8},v_{n-9}\}$$

and $v_{n-5}\notin N(v_1)\cap N(v_2)$. By Lemma 3.1 and Observation 3.9, there exists $v_l\in\{v_{n-6},v_{n-7}\}$ such that $v_l\in N(v_1)\cap N(v_2)$. Then $v_1v_{n-6}\overleftarrow{C}v_{n-9}v_{n-5}v_{n-10}\overleftarrow{C}v_2xv_1$ or $v_1v_{n-7}v_{n-8}v_{n-9}v_{n-5}\ v_{n-4}v_{n-10}\overleftarrow{C}v_2xv_1$ is an $(n-4)$-cycle of G, a contradiction.

Case 2. $\{x, y, z\} \cap N(v_1) \cap N(v_2) = \emptyset$ and $\{x, y, z\} \cap N(v_{n-4}) \cap N(v_{n-5}) = \emptyset$.

Since C has no short chords, $\{v_3, v_4, v_0, v_{n-4}\} \cap N(v_1) \cap N(v_2) = \emptyset$ and $\{v_0, v_1, v_{n-6}, v_{n-7}\} \cap N(v_{n-4}) \cap N(v_{n-5}) = \emptyset$. By Lemma 3.1, there exist two distinct vertices v_i, v_j with $5 \leq i < j \leq n-5$ such that $v_i, v_j \in N(v_1) \cap N(v_2)$ and there exist two distinct vertices v_k, v_l with $2 \leq k < l \leq n-8$ such that $v_k, v_l \in N(v_{n-4}) \cap N(v_{n-5})$. By Observation 3.9(1), $E(\{v_0\}, \{v_{i-1}, v_{i-2}, v_{j-1}, v_{j-2}\}) = \emptyset$ and $E(\{v_0\}, \{v_{k+1}, v_{k+2}, v_{l+1}, v_{l+2}\}) = \emptyset$. Since $5 \leq i < j \leq n-5$, we have $v_2, v_{n-5} \notin \{v_{i-1}, v_{i-2}, v_{j-1}, v_{j-2}\}$ and $i-2 < i-1 \leq j-2 < j-1$.

Subcase 2.1. $i - 1 < j - 2$.

Then $v_{i-1}, v_{i-2}, v_{j-1}, v_{j-2}$ are four distinct vertices and

$$V(G) \setminus N[v_0] = \{v_2, v_{n-5}, v_{i-1}, v_{i-2}, v_{j-1}, v_{j-2}\}.$$

By Lemma 3.1 and Observation 3.9(2), we have $k = i-3$ and $l = j-3$. Then $j+1 = n-5$ or $i-4 = 2$, otherwise, $v_0 v_{j+1} \overrightarrow{C} v_{n-5} v_{i-3} \overrightarrow{C} v_j v_1 \overrightarrow{C} v_{i-4} v_0$ is an $(n-4)$-cycle of G, a contradiction.

If $j+1 = n-5$ and $i-4 = 2$, then $v_{n-5} v_{n-4} v_0 v_1 v_{n-6} \overleftarrow{C} v_3 v_{n-5}$ is an $(n-4)$-cycle of G, a contradiction. If $j+1 = n-5$ and $i-4 \neq 2$, then $v_0 v_{i-4} \overleftarrow{C} v_1 v_{n-6} \overleftarrow{C} v_{i-3} v_{n-4} v_0$ is an $(n-4)$-cycle of G, a contradiction. If $j+1 \neq n-5$ and $i-4 = 2$, then $v_0 v_{j+1} \overrightarrow{C} v_{n-4} v_{i-3} \overrightarrow{C} v_j v_1 v_0$ is an $(n-4)$-cycle of G, a contradiction.

Subcase 2.2. $i - 1 = j - 2$.

Then v_{i-2}, v_{i-1}, v_i are three distinct vertices and

$$\{v_2, v_{n-5}, v_i, v_{i-1}, v_{i-2}\} \subseteq V(G) \setminus N[v_0].$$

Since $d(v_0) = n-7$, by observation 3.9(2), we have $i-3 \leq k+1 < l+2 \leq i+1$, thus $k \in \{i-4, i-3, i-2\}$ and $l \leq i-1$. If $l = i-1$, then $v_1 v_i \overrightarrow{C} v_{n-4} v_{i-1} \overleftarrow{C} v_1$ is an $(n-4)$-cycle of G, a contradiction. Thus, $l < i-1$.

Firstly, suppose that $k = i-4$. Then

$$V(G) \setminus N[v_0] = \{v_2, v_{n-5}, v_i, v_{i-1}, v_{i-2}, v_{i-3}\}$$

and $l \in \{i-3, i-2\}$. If $l = i-3$ and $j \neq n-5$, then $v_{n-5} v_{i-3} \overrightarrow{C} v_i v_2 \overrightarrow{C} v_{i-4} v_{n-4} v_0 v_j \overrightarrow{C} v_{n-5}$ is an $(n-4)$-cycle of G. If $l = i-3$ and $j = n-5$, then $v_{n-5} v_{n-10} \overleftarrow{C} v_1 v_{n-6} \overleftarrow{C} v_{n-9} v_{n-4} v_{n-5}$ is an $(n-4)$-cycle of G. If $l = i-2$, then $v_1 v_j \overrightarrow{C} v_{n-5} v_{i-2} v_{i-1} v_i v_2 \overrightarrow{C} v_{i-4} v_{n-4} v_0 v_1$ is an $(n-4)$-cycle of G.

Next, suppose that $k = i-3$. Then $l = i-2$. If $i = n-6$ or $i = n-7$, then $v_{n-5} v_{n-8} \overleftarrow{C} v_2 v_{n-6} v_1 v_0 v_{n-4} v_{n-5}$ or $v_{n-5} v_{n-9} \overrightarrow{C} v_{n-6} v_1 \overrightarrow{C} v_{n-10} v_{n-4} v_{n-5}$ is an $(n-4)$-cycle of G, a contradiction. Thus, $i \notin \{n-6, n-7\}$. Then we have $v_{n-6} v_0 \notin E(G)$, otherwise, $v_0 v_{n-6} \overleftarrow{C} v_{i-2} v_{n-4} v_{i-3} \overleftarrow{C} v_0$ is an $(n-4)$-cycle of G, a contradiction. This proves that $G - N[v_0] = \{v_{n-6}, v_{n-5}, v_i, v_{i-1}, v_{i-2}, v_2\}$. Hence,

$v_jv_0 \in E(G)$. Thus, $v_0v_j\overrightarrow{C}v_{n-5}v_{i-2}v_{n-4}v_{i-3}\overleftarrow{C}v_2v_iv_1v_0$ is an $(n-4)$-cycle of G, a contradiction.

Lastly, suppose that $k = i - 2$. Then $l = i - 1$, a contradiction. This completes the proof of Theorem 1.14. $\square$

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