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COMBINED EFFECTS FOR A CLASS OF FRACTIONAL VARIATIONAL INEQUALITIES

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Abstract. In this paper, we study the existence of a nonnegative weak solution to the following nonlocal variational inequality:

$$\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u(-\Delta)^{\frac{s}{2}}(v-u)dx + \int_{\mathbb{R}^N} (1+\lambda M(x))u(v-u)dx \geq \int_{\mathbb{R}^N} f(u)(v-u)dx,$$

for all $v \in \mathbb{K}$, where $s \in (0,1)$ and M is a continuous steep potential well on $\mathbb{R}^N$. Using penalization techniques from del Pino and Felmer, as well as from Bensoussan and Lions, we establish the existence of nonnegative weak solutions. These solutions localize near the potential well $\operatorname{Int}(M^{-1}(0))$.

Keywords: fractional variational inequality, variational methods, critical nonlinearity.

Mathematics Subject Classification: 35A15, 35J86, 49J40.

1. INTRODUCTION

In this work, we focus our attention on the following nonlocal variational inequality, for $u \in \mathbb{K}$,

$$\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u(-\Delta)^{\frac{s}{2}}(v-u)dx + \int_{\mathbb{R}^N} (1+\lambda M(x))u(v-u)dx \geq \int_{\mathbb{R}^N} f(u)(v-u)dx, \forall v \in \mathbb{K}, \tag{1.1}$$

where $s \in (0,1)$, $\lambda > 0$, the function $M : \mathbb{R}^N \to \mathbb{R}$ is a continuous steep potential well, that is,

(M_1) $M(x) \geq 0$ for all $x \in \mathbb{R}^N$,
(M_2) there exists an open, connected and bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary, such that $\Omega := \operatorname{Int}(M^{-1}(\{0\})) \neq \emptyset$.

We define

$$\mathbb{K} = \{v \in E : v \geq \varphi \text{ a.e. in } \Omega\},$$

where

$$E = \left\{ u \in H^s(\mathbb{R}^N) : \int\limits_{\mathbb{R}^N} M(x)|u|^2 dx < \infty \right\},$$

and $\varphi \in H^s(\mathbb{R}^N)$, $\varphi^+ \neq 0$, $\operatorname{supp}(\varphi^+) \subset \Omega$. Moreover, for the nonlinearity f, we require that

$$f(t) = \mu t^{q-1} + t^{2_s^*-1},$$

for any $t > 0$, with $2 < q < 2_s^*$, $\mu > 0$ and f vanishes in $(-\infty, 0)$.

Variational inequalities are well-known in the applied mathematics literature. They originated in Italy with Fichera's work in 1963 on elasticity problems and Stampacchia's work in 1964 within the framework of potential theory in connection with capacity [33]. The classical physical application of a variational inequality is the problem of determining the equilibrium position that a membrane reaches in the presence of a rigid body. Its simplest mathematical formulation is to seek for minimizer of the energy functional

$$I(u) = \int\limits_{\Omega} |\nabla u|^2 dx$$

among all functions u satisfying $u \geq \varphi$ in Ω, for a given smooth obstacle φ. If u is a minimizer of I subject to the boundary conditions $u = 0$ on $\partial\Omega$, then u is a solution of the Euler–Lagrange equation

$$\begin{cases} u \geq \varphi & \text{in } \Omega, \\ \Delta u = 0 & \text{in } \{u > \varphi\}, \\ -\Delta u \geq 0 & \text{in } \Omega. \end{cases}$$

Many extensions of this problem have been considered in the literature, particularly for taking into account nonlinear elastic reactions of the membrane, non commutative effects and nonlocal interactions [1, 2, 8, 9, 18, 23, 24, 30–32]. Fractional problems with obstacle type constrain were first considered by Silvestre in [35]. Since then, many obstacle type problems with various nonlocal operators have been considered in the last years, see for example [8, 15, 20, 26, 34, 36, 39] and the references therein.

Recently, Alves *et al.* [1] pointed out that there are three methods for researching variational inequalities: the nonsmooth critical point theory [12, 13, 27, 29], the minimax principles [19, 40, 41] and the penalization method [7, 30, 31]. They also combined penalization method and the mountain pass theorem to get the existence of positive solution for the variational inequality:

$$\begin{cases} u \in \mathbb{K}, \\ \int\limits_{\mathbb{R}^N} \nabla u \nabla (v-u) dx + \int\limits_{\mathbb{R}^N} (1 + \lambda V(x)) u (v-u) dx \geq \int\limits_{\mathbb{R}^N} f(u)(v-u) dx, \quad \forall v \in \mathbb{K}, \end{cases} \tag{1.2}$$

where V is a continuous function that satisfies the following conditions:

(V_1) $V(x) \geq 0$, for all $x \in \mathbb{R}^N$,
(V_2) $\Omega = \mathrm{Int}(V^{-1}(\{0\})) \neq \emptyset$ is a connected and bounded set of $\mathbb{R}^N$ with smooth boundary,
(V_3) there is $M_0 > 0$ such that the set $\mathcal{L} = \{x \in \mathbb{R}^N : V(x) \leq M_0\}$ is nonempty with finite measure,

and f is a suitable continuous function. We also mention the recent work by Deng *et al.* [15], where the existence of a positive weak solution for a class of fractional Kirchhoff variational inequality was considered under (V_1)–(V_3).

The elliptic problem with critical growth like

$$(-\Delta)^s u + \lambda V(x)u = \mu u^{q-1} + u^{2_s^*-1} \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

where $s \in (0, 1]$, has been extensively studied by many authors. Recently, for $s = 1$, Alves and Barros [3] considered the existence and multiplicity for (1.3), assuming that V satisfies (V_1)–(V_3). Based on the important research in [3], Alves *et al.* [1, 2] solved (1.2) by using the penalization method under different conditions on V. Moreover, as for a class of problems which λ is large enough to get existence result in (1.3), we refer to [6, 11, 17] and references therein.

In the nonlocal case, that is $s \in (0, 1)$, Yan and Liu [42] considered the multiplicity and concentration of solutions for (1.3), when there is a general nonlinearity without the critical term in it and V satisfying (V_1)–(V_3). He and Zou [22] studied the existence and concentration result of (1.3) when $V(x)$ is replaced by $1+\lambda V(x)$ with V satisfying (V_1)–(V_2) and f is a general nonlinearity. Moreover, if we consider the impact of λ in different fractional equations the readers can see [10, 21, 28, 37] and references therein.

In this paper, mainly motivated by [2, 4], we intend to prove the existence of positive solution for problem (1.1) by dropping condition (V_3). More precisely, our main result in this paper can be stated as follows:

Theorem 1.1. *Suppose that M satisfies (M_1)–(M_2). Then there exist $\lambda_* > 0$ and $\mu_* > 0$ such that problem (1.1) has at least one positive weak solution u_λ for $\lambda \geq \lambda_*$, $\mu \geq \mu_*$. Furthermore, for any sequence $\lambda_n \to +\infty$, there exists a subsequence, still denoted by $\{\lambda_n\}$, such that $\{u_{\lambda_n}\}$ converges strongly in $H^s(\mathbb{R}^N)$ to a function u with $u = 0$ a.e. in Ω^c, where u is a solution of the variational inequality*

$$\begin{aligned}&\iint\limits_Q \frac{(u(x)-u(y))((v-u)(x)-(v-u)(y))}{|x-y|^{N+2s}}dxdy + \int\limits_\Omega u(v-u)dx \\ &\geq \int\limits_\Omega (\mu|u|^{q-2} + |u|^{2^*-2})u(v-u)dx,\end{aligned}$$

for every $v \in \tilde{\mathbb{K}}$, where

$$\tilde{\mathbb{K}} = \{v \in H_0^s(\Omega) : v \geq \varphi \ \textit{a.e. in } \Omega\},$$

and $Q = \mathbb{R}^{2N} \setminus (\Omega^c \times \Omega^c)$.

Remark 1.2. Compared with the previous results, Theorem 1.1 can be regarded as an extension of Theorem 1.1 in [2] from the local to nonlocal case. Secondly, we note that one of the main challenges is associated with fact that we need to prove that, if $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, the limit below must hold

$$\int\limits_{\mathbb{R}^N} f(u_n)u_n dx \to \int\limits_{\mathbb{R}^N} f(u)u dx. \tag{1.4}$$

This limit is not easy to be proved because there is no compact embedding of $H^1(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$ for $p \in (2, 2^*)$ and hence the energy functional does not satisfy the (PS) condition at any level c.

As we see in [1, 15], condition (V_3) is used to get a compactness control when λ is large enough, and consequently, it is possible to show that (1.4) holds. In this work we do not consider condition (V_3), making it more challenging to prove that (1.4) is satisfied and, as a result, to verify that the energy functional meets the (PS) condition. To overcome this difficulty, we use the penalization method found in del Pino and Felmer [14] combined with the penalization method due to Bensoussan and Lions [7], which helps us to transform an inequality into an equality.

On the other hand, it should be noted that in many works it is assumed that $V(x) \geq V_0 > 0$, that is, V is bounded away from 0 (see [5, 11, 14, 17, 22, 39] and the references therein). This is not the case in our situation, since by condition (M_2) the potential V is not bounded away from 0.

The remainder part of this paper is organized as follows. In Section 2, we present the variational setting of the problem, and we show some preliminary results. In Section 3, we study the penalized problem. In Section 4, we deal with the modified inequality. Finally, in Section 5, we present the main part proof of Theorem 1.1.

Throughout the paper, we will use the following notation. The letter C changes from line to line. By $B_r(x)$, we understand the ball in $\mathbb{R}^N$ centered at $x \in \mathbb{R}^N$ with radius r. The symbol Γ^c means $\mathbb{R}^N \setminus \Gamma$, where $\Gamma \subset \mathbb{R}^N$. The notation $|\cdot|_r$ refers to the norm in $L^r(\mathbb{R}^N)$ and $|u|_{r(M)} := (\int_M |u|^r dx)^{\frac{1}{r}}$, where $M \subset \mathbb{R}^N$ and $u \in L^r(\mathbb{R}^N)$.

2. ABSTRACT SETTING AND PRELIMINARY RESULTS

We now collect some preliminary results for the fractional Laplacian. A complete introduction to fractional Sobolev space can be found in [5]. For $s \in (0, 1)$, we define the fractional Sobolev space $H^s(\mathbb{R}^N)$ as

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : [u] < \infty \right\},$$

where

$$[u]^2 = \iint\limits_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy$$

is the so-called Gagliardo norm of u. It is well-known that $H^s(\mathbb{R}^N)$ is a Hilbert space, under the norm

$$\|u\|^2 = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy + \int_{\mathbb{R}^N} |u|^2 dx.$$

From now on, we work on the Hilbert space

$$E_\lambda = \left\{ u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} \lambda M(x) u^2 dx < \infty \right\},$$

endowed with the inner product

$$\langle u, v \rangle_\lambda = \iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dxdy + \int_{\mathbb{R}^N} (1+\lambda M(x)) uv dx.$$

Then, the norm related to the inner product is given by

$$\|u\|_\lambda^2 = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy + \int_{\mathbb{R}^N} (1+\lambda M(x)) |u|^2 dx.$$

By (M_1), for any $u \in E_\lambda$, we get

$$\|u\| \le \|u\|_\lambda.$$

This means the embedding $E_\lambda \hookrightarrow L^r(\mathbb{R}^N)$ is continuous for any $r \in [2, 2_s^*]$ and the embedding $E_\lambda \hookrightarrow L^r_{loc}(\mathbb{R}^N)$ is compact for any $r \in [1, 2_s^*)$. The following Sobolev inequality can be found in [5]: for all $u \in E_\lambda$, we have

$$[u]^2 \ge S \left(\int_{\mathbb{R}^N} |u|^{2_s^*} dx \right)^{\frac{2}{2_s^*}}.$$

3. THE PENALIZED PROBLEM

To establish the existence of positive solutions, we modify this inequality adapting some arguments explored in [14]. More precisely, we fix $k = \frac{2q}{q-2} > 2$ and $a > 0$ such that $f(a) = \frac{a}{k}$, and we consider the following function:

$$\tilde{f}(t) = \begin{cases} f(t), & t \le a, \\ \frac{t}{k}, & t \ge a. \end{cases}$$

Let $\Lambda \subset \mathbb{R}^N$ be a bounded, connected set such that $\overline{\Omega} \subset \Lambda$. We introduce the penalized nonlinearity $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ by setting

$$g(x,t) = \chi_\Lambda(x) f(t) + (1 - \chi_\Lambda(x)) \tilde{f}(t).$$

Define $G(x,t) = \int_0^t g(x,\tau) d\tau$.

From the definition of f, it follows that g verifies the following properties:

(g_1) $\lim\limits_{t\to 0} \dfrac{g(x,t)}{t} = 0$ uniformly in $x \in \mathbb{R}^N$,
(g_2) $g(x,t) \leq \mu|t|^{q-1} + |t|^{2_s^*-1}$ for any $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$,
(g_3) $0 < qG(x,t) \leq tg(x,t)$ for any $x \in \Lambda$ and $t > 0$,
(g_4) $0 < 2G(x,t) \leq tg(x,t) \leq \frac{1}{k}(1+\lambda M(x))t^2$ for each $x \in \Lambda^c$ and $t > 0$,
(g_5) $g(x,t) = 0$ for all $x \in \mathbb{R}^N$ and $t \leq 0$.

Now we study the following modified inequality: for $u \in \mathbb{K}$,

$$\int\limits_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u(-\Delta)^{\frac{s}{2}}(v-u)dx + \int\limits_{\mathbb{R}^N} (1+\lambda M(x))u(v-u)dx \geq \int\limits_{\mathbb{R}^N} g(x,u)(v-u)dx, \forall v \in \mathbb{K}. \tag{3.1}$$

As the method in [2, 7], we get the penalized problem of (3.1),

$$(-\Delta)^s u + (1+\lambda M(x))u - \frac{1}{\varepsilon}(\varphi - u)^+ \chi_\Omega = g(x,u) \quad \text{in } \mathbb{R}^N, \tag{3.2}$$

where $\varepsilon > 0$ is the penalization parameter. Let $\langle P(u), v\rangle = -\int_\Omega (\varphi - u)^+ v dx$. We know that P is the penalty operator and $\frac{1}{\varepsilon}\int_\Omega (\varphi - u)^+ v dx$ is the penalization term. The associated energy functional to problem (3.2) is

$$J_{\lambda,\epsilon}(u) = \frac{1}{2}\|u\|_\lambda^2 + \frac{1}{2\epsilon}\int\limits_\Omega [(\varphi - u)^+]^2\, dx - \int\limits_{\mathbb{R}^N} G(x,u)\, dx.$$

It is easy to check that $J_{\lambda,\varepsilon} \in C^1(E_\lambda, \mathbb{R})$ and its differential is defined as

$$J'_{\lambda,\varepsilon}(u)v = \langle u, v\rangle_\lambda - \frac{1}{\varepsilon}\int\limits_\Omega (\varphi - u)^+ v dx - \int\limits_{\mathbb{R}^N} g(x,u) v dx,$$

for any $v \in E_\lambda$. Let us note that if $u \in \mathbb{K}$ satisfies

$$\langle u, v-u\rangle_\lambda \geq \int\limits_{\mathbb{R}^N} g(x,u)(v-u)dx \quad \text{for all } v \in \mathbb{K}, \tag{3.3}$$

and

$$|u| \leq a \quad \text{for all } x \in \Lambda^c, \tag{3.4}$$

then u is a solution to (1.1).

Firstly, we will show that the functional $J_{\lambda,\epsilon}(u)$ satisfies the mountain pass geometry [38].

Lemma 3.1.

(i) *There exist constant $r_\mu, \rho_\mu > 0$, with $r_\mu, \rho_\mu \to 0$ as $\mu \to +\infty$, independent of λ and ϵ, such that*
$$J_{\lambda,\epsilon}(u) \geq \rho_\mu \quad \textit{for} \quad \|u\|_\lambda = r_\mu.$$

(ii) *There is $e \in E_\lambda$ with $\|e\|_\lambda > r_\mu$ and $J_{\lambda,\epsilon}(e) < 0$.*

Proof. (i) From (g_2), for any fixed $\beta > 0$, there exists $C_\beta > 0$, such that

$$|g(x,t)| \le \mu\beta|t|^{q-1} + C_\beta|t|^{2_s^*-1} \quad \text{for all } x \in \mathbb{R}^N, t \in \mathbb{R}, \tag{3.5}$$

and

$$|G(x,t)| \le \frac{\mu\beta}{q}|t|^q + \frac{C_\beta}{2_s^*}|t|^{2_s^*} \quad \text{for all } x \in \mathbb{R}^N, t \in \mathbb{R}. \tag{3.6}$$

Then, by the Sobolev embedding, we have

$$J_{\lambda,\epsilon}(u) \ge \frac{1}{2}\|u\|_\lambda^2 - \frac{C_1\mu}{q}\|u\|_\lambda^q - \frac{C_2}{2_s^*}\|u\|_\lambda^{2_s^*}.$$

As $2 < q < 2_s^*$, if we choose $r_\mu > 0$ satisfying

$$r_\mu < \min\left\{\left(\frac{3\cdot 2_s^*}{8C_2}\right)^{\frac{1}{2_s^*-2}}, \left(\frac{3q}{8C_1\mu}\right)^{\frac{1}{q-2}}\right\},$$

we will get

$$J_{\lambda,\epsilon}(u) \ge \frac{1}{8}r_\mu^2 := \rho_\mu \quad \text{for } \|u\|_\lambda = r_\mu.$$

(ii) As $\varphi^+ \in E_\lambda$ and $\operatorname{supp}(\varphi^+) \subset \Omega$, we have

$$J_{\lambda,\epsilon}(\varphi^+) \le \frac{1}{2}\|\varphi^+\|_\lambda^2 = \frac{1}{2}\|\varphi^+\|^2.$$

Choose $\|\varphi^+\|$ small enough such that

$$\frac{1}{2}\|\varphi^+\|^2 \le \rho_\mu.$$

Besides,

$$\int\limits_{\mathbb{R}^N} G(x,t\varphi^+)dx = \frac{\mu t^q}{q}\int\limits_\Omega |\varphi^+|^q dx - \frac{t^{2_s^*}}{2_s^*}\int\limits_\Omega |\varphi^+|^{2_s^*}dx \quad \text{for } t \in \mathbb{R}^N.$$

and

$$\int\limits_\Omega [(\varphi - t\varphi^+)^+]^2 dx = 0$$

for $t \ge 1$. Thus, we have

$$J_{\lambda,\epsilon}(t\varphi^+) = \frac{t^2}{2}\|\varphi^+\|^2 - \frac{t^q\mu}{q}\int\limits_\Omega |\varphi^+|^q dx - \frac{t^{2_s^*}}{2_s^*}\int\limits_\Omega |\varphi^+|^{2_s^*}dx,$$

which implies that $J_{\lambda,\epsilon}(t\varphi^+) \to -\infty$ as $t \to +\infty$. Therefore, as t_0 is large enough with $e = (1+t_0)\varphi^+$,

$$J_{\lambda,\epsilon}(e) < 0 \quad \text{and} \quad \|e\|_\lambda = \|e\| > r_\mu. \qquad \square$$

From [38, Theorem 1.15], there exists a $(PS)_{c_{\lambda,\epsilon}}$ sequence $\{u_n\} \subset E_\lambda$ such that $J_{\lambda,\epsilon}(u_n) \to c_{\lambda,\epsilon}$ and $J'_{\lambda,\epsilon}(u_n) \to 0$, where $c_{\lambda,\epsilon}$ is the mountain pass level characterized by

$$c_{\lambda,\epsilon} = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_{\lambda,\epsilon}(\gamma(t))$$

and

$$\Gamma = \{\gamma \in C([0,1], E_\lambda) :\ \gamma(0) = \varphi^+ \text{ and } \gamma(1) = e\}.$$

Lemma 3.2. *There are $\tau > 0$ and $\mu_* = \mu_*(\tau) > 0$ such that*

$$c_{\lambda,\epsilon} < \frac{q-2}{2q} S^{\frac{N}{2s}} - \tau \quad \textit{for all } \lambda > 0, \epsilon > 0 \textit{ and } \mu \geq \mu_*.$$

Proof. Firstly, we fix the following path $\gamma \in \Gamma$:

$$\gamma : [0,1] \ni t \longmapsto \gamma(t) = (1 + tt_0)\varphi^+ \in E_\lambda.$$

Since $(1 + tt_0)\varphi^+ \geq \varphi$, we get

$$\int_\Omega [(\varphi - (1 + tt_0)\varphi^+)^+]^2 dx = 0,$$

and

$$\begin{aligned} J_{\lambda,\epsilon}(\gamma(t)) &= \frac{(1+tt_0)^2}{2}\|\varphi^+\|^2 - \frac{\mu(1+tt_0)^q}{q}\int_\Omega |\varphi^+|^q dx - \frac{(1+tt_0)^{2_s^*}}{2_s^*}\int_\Omega |\varphi^+|^{2_s^*} dx \\ &\leq \frac{(1+tt_0)^2}{2}\|\varphi^+\|^2 - \frac{\mu(1+tt_0)^q}{q}\int_\Omega |\varphi^+|^q dx. \end{aligned}$$

Taking $A = \frac{1}{2}\|\varphi^+\|^2$ and $B = \frac{1}{q}\int_{\mathbb{R}^N} |\varphi^+|^q dx$, we then consider the following function:

$$l_\mu(t) = At^2 - \mu B t^q \quad \text{for all } t \in [0, \infty).$$

The maximum point of l_μ is $t_{\max} = (\frac{2A}{\mu B q})^{\frac{1}{q-2}} > 0$, so

$$\begin{aligned} c_{\lambda,\epsilon} &\leq \max_{t\geq 0} J_{\lambda,\epsilon}(\gamma(t)) \\ &\leq \max_{t\geq 0}\left\{\frac{(1+tt_0)^2}{2}\|\varphi^+\|^2 - \frac{\mu(1+tt_0)^q}{q}\int_\Omega |\varphi^+|^q dx\right\} \\ &= \max_{t\geq 0} l_\mu(t) = l_\mu(t_{\max}) < \frac{q-2}{2q} S^{\frac{N}{2s}} - \tau, \end{aligned}$$

for $\lambda > 0, \epsilon > 0$ and $\mu \geq \mu_*$, where μ_* is sufficiently large. $\square$

Lemma 3.3. *The $(PS)_{c_{\lambda,\epsilon}}$ sequence $\{u_n\}$ of $J_{\lambda,\epsilon}$ is bounded in E_λ.*

Proof. For any sequence $\{u_n\} \subset E_\lambda$, we get

$$[(\varphi - u_n)^+]^2 + (\varphi - u_n)^+ u_n \geq (\varphi - u_n)^+ \varphi.$$

Since $q > 2$, it follows that

$$\frac{1}{2\varepsilon}\int\limits_{\Omega}[(\varphi - u_n)^+]^2 dx + \frac{1}{q\varepsilon}\int\limits_{\Omega}(\varphi - u_n)^+ u_n dx \geq \frac{1}{q\varepsilon}\int\limits_{\Omega}(\varphi - u_n)^+ \varphi dx.$$

If $\{u_n\}$ is a $(PS)_{c_{\lambda,\epsilon}}$ sequence of $J_{\lambda,\epsilon}$, then from (g_3), (g_4), the Hölder inequality and the Sobolev embedding, we obtain

$$\begin{aligned}
J_{\lambda,\epsilon}(u_n) - \frac{1}{q}\langle J'_{\lambda,\epsilon}(u_n), u_n\rangle &\geq \left(\frac{1}{2} - \frac{1}{q}\right)\|u_n\|_\lambda^2 + \frac{1}{q\varepsilon}\int\limits_{\Omega}(\varphi - u_n)^+\varphi dx - \frac{1}{2k}\|u_n\|_\lambda^2 \\
&\geq \frac{1}{2}\left(\frac{1}{2} - \frac{1}{q}\right)\|u_n\|_\lambda^2 - \frac{1}{q\varepsilon}\int\limits_{\Omega}(|\varphi| + |u_n|)|\varphi| dx \\
&\geq \frac{q-2}{4q}\|u_n\|_\lambda^2 - \frac{1}{q\varepsilon}|\varphi|_2^2 - \frac{1}{q\varepsilon}|\varphi|_2\|u_n\|_\lambda,
\end{aligned}$$

which yields

$$\frac{q-2}{4q}\|u_n\|_\lambda^2 \leq c_{\lambda,\epsilon} + o_n(1) + \frac{1}{q\varepsilon}|\varphi|_2^2 + (o_n(1) + \frac{1}{q\varepsilon}|\varphi|_2)\|u_n\|_\lambda.$$

Thus, $\{u_n\}$ is bounded in E_λ. □

The following lemma can be found in [1, Lemma 3.4] and [39, Lemma 2.2].

Lemma 3.4. *$\{u_n^+\}$ is also a $(PS)_{c_{\lambda,\epsilon}}$ sequence of $J_{\lambda,\epsilon}$, where $u_n^+ = \max\{u_n, 0\}$.*

Remark 3.5. From Lemmas 3.2 and 3.4, we have

$$c_{\lambda,\epsilon} \in \left[0, \frac{q-2}{2q}S^{\frac{N}{2s}} - \tau\right) \quad \text{for all } \lambda > 0, \epsilon > 0 \text{ and } \mu \geq \mu_*.$$

Due to the presence of the critical Sobolev exponent, we will use the well-known concentration-compactness principle to get our compact result.

Lemma 3.6. *Let $c \in \left[0, \frac{q-2}{2q}S^{\frac{N}{2s}} - \tau\right)$. Then $J_{\lambda,\epsilon}$ satisfies the Palais–Smale condition at the level c.*

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence of $J_{\lambda,\epsilon}$, then $J_{\lambda,\epsilon}(u_n) \to c$ and $J'_{\lambda,\epsilon}(u_n) \to 0$. From Lemmas 3.3 and 3.4, the sequence $\{u_n\}$ is bounded and nonnegative. Consequently, we may assume that:

$$\begin{aligned}
&u_n \rightharpoonup u \quad \text{weakly in } E_\lambda, \\
&u_n \to u \quad \text{strongly in } L^r_{loc}(\mathbb{R}^N) \text{ for any } r \in [1, 2_s^*), \\
&u_n \to u \quad \text{for a.e. } x \in \mathbb{R}^N.
\end{aligned} \tag{3.7}$$

Next, we divide the proof into four steps.

Step 1. We show $\{u_n\}$ is tight. That means for any $\xi > 0$, there exists $R = R_\xi > 0$ such that $\Lambda \subset B_{\frac{R}{2}}$ and

$$\limsup_{n\to\infty} \int_{B_R^c}\int_{\mathbb{R}^N} \frac{(u_n(x)-u_n(y))^2}{|x-y|^{N+2s}}dxdy + \int_{B_R^c} (1+\lambda M(x))|u_n|^2 dx \leq \xi. \tag{3.8}$$

Let $\psi_R \in C^\infty(\mathbb{R}^N, [0,1])$ be such that $0 \leq \psi_R \leq 1$, $\psi_R = 0$ in $B_{\frac{R}{2}}$, $\psi_R = 1$ in B_R^c and $|\nabla \psi_R| \leq \frac{C}{R}$ for some $C > 0$ independent of R. From $\langle J'_{\lambda,\epsilon}(u_n), \psi_R u_n\rangle = o_n(1)$ and $\Omega \subset \Lambda$, we get

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))(u_n(x)\psi_R(x)-u_n(y)\psi_R(y))}{|x-y|^{N+2s}}dxdy + \int_{\mathbb{R}^N} (1+\lambda M(x))\psi_R|u_n|^2 dx \\
&= \frac{1}{\varepsilon}\int_{\Omega} (\varphi - u_n)^+\psi_R u_n dx + \int_{\mathbb{R}^N} g(x,u_n)\psi_R u_n dx + o_n(1) \\
&= \int_{\mathbb{R}^N} g(x,u_n)\psi_R u_n dx + o_n(1).
\end{aligned}$$

Since

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))(u_n(x)\psi_R(x)-u_n(y)\psi_R(y))}{|x-y|^{N+2s}}dxdy \\
&= \iint_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))^2\psi_R(x)}{|x-y|^{N+2s}}dxdy \\
&\quad + \iint_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))(\psi_R(x)-\psi_R(y))u_n(y)}{|x-y|^{N+2s}}dxdy,
\end{aligned}$$

from (g_4), we have

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))^2\psi_R(x)}{|x-y|^{N+2s}}dxdy + \int_{\mathbb{R}^N} (1+\lambda M(x))\psi_R|u_n|^2 dx \\
&\leq -\iint_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))(\psi_R(x)-\psi_R(y))u_n(y)}{|x-y|^{N+2s}}dxdy \\
&\quad + \int_{\mathbb{R}^N} \frac{1+\lambda M(x)}{k}\psi_R|u_n|^2 dx.
\end{aligned} \tag{3.9}$$

Moreover, by the Hölder inequality, we obtain

$$\begin{aligned}
&\left| \iint\limits_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))(\psi_R(x)-\psi_R(y))u_n(y)}{|x-y|^{N+2s}} dxdy \right| \\
&\le \left(\iint\limits_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))^2}{|x-y|^{N+2s}} dxdy \right)^{\frac{1}{2}} \left(\iint\limits_{\mathbb{R}^{2N}} \frac{(\psi_R(x)-\psi_R(y))^2 u_n^2(y)}{|x-y|^{N+2s}} \right)^{\frac{1}{2}} \\
&\le C \left(\iint\limits_{\mathbb{R}^{2N}} \frac{(\psi_R(x)-\psi_R(y))^2 u_n^2(y)}{|x-y|^{N+2s}} \right)^{\frac{1}{2}}.
\end{aligned}$$

Then, from [4, Lemma 3.2], we get

$$\limsup_{R\to\infty} \limsup_{n\to\infty} \iint\limits_{\mathbb{R}^{2N}} \frac{(\psi_R(x)-\psi_R(y))^2 u_n^2(y)}{|x-y|^{N+2s}} = 0.$$

Combining this with (3.9), we can see for any $\xi > 0$, as R is large enough, (3.8) is satisfied.

Step 2. We will show that

$$\lim_{n\to\infty} \int\limits_{B_R^c} g(x,u_n)u_n dx = \int\limits_{B_R^c} g(x,u)udx. \tag{3.10}$$

From (g_2), Step 1 and the Sobolev embedding, we obtain that for any $\xi > 0$, there exists $R = R_\xi > 0$, such that

$$\int\limits_{B_R^c} g(x,u_n)u_n dx \le C \left(\int\limits_{B_R^c} |u_n|^q dx + \int\limits_{B_R^c} |u_n|^{2_s^*} dx \right) \le C \left(\xi^{\frac{q}{2}} + \xi^{\frac{2_s^*}{2}} \right).$$

Moreover, choosing R large enough, we get

$$\int\limits_{B_R^c} g(x,u)udx \le \xi.$$

Then (3.10) is satisfied.

Step 3. We will show that

$$\lim_{n\to\infty} \int\limits_{B_R\cap\Lambda^c} g(x,u_n)u_n dx = \int\limits_{B_R\cap\Lambda^c} g(x,u)udx. \tag{3.11}$$

From (g_4), we get

$$g(x,u_n)u_n \leq \frac{(1+\lambda M(x))}{k}|u_n|^2 \quad \text{for any} \quad x \in \Lambda^c.$$

Since $B_R \cap \Lambda^c$ is bounded, by the Dominated Convergence Theorem and (3.7), we can deduce that (3.11) is satisfied.

Step 4. We will show that

$$\lim_{n\to\infty} \int\limits_{\Lambda} g(x,u_n)u_n dx = \int\limits_{\Lambda} g(x,u)udx. \tag{3.12}$$

At this point, we firstly prove $u_n \to u$ in $L^{2_s^*}(\Lambda)$. If this happens, from (g_2) and the Dominated Convergence Theorem, (3.12) is true.

From Step 1 and the concentration compactness principle, we can find an at most countable index set I, and $\{x_i\} \subset \mathbb{R}^N$, $\{\mu_i\}, \{\nu_i\} \subset (0,\infty)$ such that for any $i \in I$,

$$\nu = |u|^{2_s^*} + \Sigma_{i\in I}\nu_i\delta_{x_i}, \quad \mu \geq |(-\Delta)^{\frac{s}{2}}u|^2 + \Sigma_{i\in I}\mu_i\delta_{x_i} \quad \text{and} \quad \mu_i \geq S\nu_i^{\frac{2}{2_s^*}}.$$

Let us show that $\{x_i\} \cap \Lambda = \emptyset$. Assume by contradiction that $x_i \in \Lambda$ for some $i \in I$. For any $\rho > 0$, we define $\psi_\rho(x) = \psi(\frac{x-x_i}{\rho})$, where $\psi \in C_0^\infty(\mathbb{R}^N,[0,1])$ is such that $\psi = 1$ in $B_{\frac{1}{2}}$, $\psi = 0$ in B_1^c, and $|\nabla\psi| \leq \frac{C}{\rho}$. We suppose that $\operatorname{supp}(\psi_\rho) \subset \Lambda$ for a suitably chosen ρ. Since $\{\psi_\rho u_n\}$ is bounded in E_λ, from (g_2) and Step 1, we get

$$\begin{aligned}
&\iint\limits_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))^2\psi_\rho(x)}{|x-y|^{N+2s}}dxdy \\
&= -\iint\limits_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))(\psi_\rho(x)-\psi_\rho(y))u_n(x)}{|x-y|^{N+2s}}dxdy \\
&\quad + \frac{1}{\varepsilon}\int\limits_{\Omega}(\varphi-u_n)^+u_n\psi_\rho dx + \int\limits_{\mathbb{R}^N} g(x,u_n)u_n\psi_\rho dx \\
&\quad - \int\limits_{\mathbb{R}^N}(1+\lambda M(x))u_n^2\psi_\rho dx + o_n(1) \\
&\leq -\iint\limits_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))(\psi_\rho(x)-\psi_R(y))u_n(x)}{|x-y|^{N+2s}}dxdy \\
&\quad + \frac{1}{\varepsilon}\int\limits_{\Omega}(\varphi-u_n)^+u_n\psi_\rho dx + \mu\int\limits_{\mathbb{R}^N}|u_n|^q\psi_\rho dx \\
&\quad + \int\limits_{\mathbb{R}^N}|u_n|^{2_s^*}\psi_\rho dx + o_n(1).
\end{aligned} \tag{3.13}$$

Due to the fact that ψ_ρ has compact support and (3.7), we observe that

$$\lim_{\rho\to 0}\lim_{n\to\infty}\int\limits_{\mathbb{R}^N}\mu|u_n|^q\psi_\rho dx=\lim_{\rho\to 0}\int\limits_{B_\rho(x_i)}\mu|u|^q\psi_\rho dx=0.$$

Moreover, let

$$I_n:=\frac{1}{\varepsilon}\int\limits_{\Omega}(\varphi-u_n)^+u_n\psi_\rho dx,$$

then from the Hölder inequality, we get

$$\begin{aligned}
I_n&\leq\frac{1}{\varepsilon}\int\limits_{B_\rho(x_i)}(\varphi-u_n)^+u_n\psi_\rho dx\\
&\leq\frac{1}{\varepsilon}\int\limits_{B_\rho(x_i)}(|\varphi|+u_n)u_n\psi_\rho dx\\
&\leq\frac{1}{\varepsilon}\int\limits_{B_\rho(x_i)}(|\varphi|+u_n)u_n dx\\
&\leq\frac{1}{\varepsilon}\left(\int\limits_{B_\rho(x_i)}|\varphi|^2dx\right)^{\frac{1}{2}}\left(\int\limits_{B_\rho(x_i)}|u_n|^2dx\right)^{\frac{1}{2}}+\frac{1}{\varepsilon}\left(\int\limits_{B_\rho(x_i)}|u_n|^2dx\right)^{\frac{1}{2}}.
\end{aligned}$$

Since

$$\lim_{\rho\to 0}\lim_{n\to\infty}\int\limits_{B_\rho(x_i)}|u_n|^2dx=0,$$

we get

$$\lim_{\rho\to 0}\lim_{n\to\infty}|I_n|=0.$$

Furthermore, by applying a variable substitution in Step 1, we also have

$$\limsup_{\rho\to 0}\limsup_{n\to\infty}\iint\limits_{\mathbb{R}^{2N}}\frac{(\psi_\rho(x)-\psi_\rho(y))^2u_n^2(x)}{|x-y|^{N+2s}}=0.$$

Therefore, taking limits as $n \to \infty$ and $\rho \to 0$ in (3.13), we obtain $\nu_i \geq S^{\frac{N}{2s}}$. Using (g_3) and (g_4), we can deduce that, as $n \to \infty$,

$$\begin{aligned} c_{\lambda,\varepsilon} + o_n(1) &= J_{\lambda,\epsilon}(u_n) - \frac{1}{q}\langle J'_{\lambda,\epsilon}(u_n), u_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{q}\right)\|u_n\|_\lambda^2 + \frac{1}{2\varepsilon}\int_\Omega [(\varphi - u_n)^+]^2 dx + \frac{1}{q\varepsilon}\int_\Omega (\varphi - u_n)^+ u_n dx \\ &\quad + \frac{1}{q}\int_{\mathbb{R}^N} g(x,u_n)u_n - qG(x,u_n)dx \\ &\geq \left(\frac{1}{2} - \frac{1}{q}\right)\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx \\ &\geq \left(\frac{1}{2} - \frac{1}{q}\right) S\nu_i^{\frac{2}{2_s^*}} \geq \frac{q-2}{2q} S^{\frac{N}{2s}}, \end{aligned}$$

which gives a contradiction. This means that $u_n \to u$ in $L^{2_s^*}(\Lambda)$.

From Steps 2–4, we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} g(x,u_n)u_n dx = \int_{\mathbb{R}^N} g(x,u)u dx. \tag{3.14}$$

Since $\{u_n\}$ is a $(PS)_c$ sequence, we obtain

$$\|u_n\|_\lambda^2 = \frac{1}{\varepsilon}\int_\Omega (\varphi - u_n)^+ u_n dx + \int_{\mathbb{R}^N} g(x,u_n)u_n dx + o_n(1). \tag{3.15}$$

Moreover, by the density of $C_0^\infty(\mathbb{R}^N)$ in E_λ, we have

$$\|u\|_\lambda^2 = \frac{1}{\varepsilon}\int_\Omega (\varphi - u)^+ u dx + \int_{\mathbb{R}^N} g(x,u)u dx. \tag{3.16}$$

Hence, from (3.7), (3.14), (3.15), (3.16) and the Dominated Convergence Theorem, we infer that $\|u_n\|_\lambda \to \|u\|_\lambda$. This completes the lemma. □

Lemma 3.7. *Let $\{u_n\} \subset E_\lambda$ be a $(PS)_{c_{\lambda,\varepsilon}}$ sequence for $J_{\lambda,\varepsilon}$ with*

$$c_{\lambda,\varepsilon} \in \left[0, \frac{q-2}{2q} S^{\frac{N}{2s}} - \tau\right).$$

Then one of the following conditions holds:

(i) $u_n \to 0$ *in* E_λ,
(ii) *there exists a sequence* $(y_n) \subset \mathbb{R}^N$ *and constants* $R, \beta > 0$ *such that*

$$\liminf_{n\to\infty} \int_{B_R(y_n)} |u_n|^2 dx \geq \beta > 0.$$

Proof. Assume that (ii) does not occur. Then, for every $R > 0$, we have

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int\limits_{B_R(y)} |u_n|^2 dx = 0.$$

Therefore, since Ω is a bounded domain, we have

$$\int\limits_{\Omega} (\varphi - u_n)^+ u_n dx = o_n(1).$$

Besides, from the fractional Lions Lemma, we obtain $u_n \to 0$ in $L^r(\mathbb{R}^N)$ for all $r \in (2, 2_s^*)$. By $\langle J'_{\lambda,\varepsilon}(u_n), u_u\rangle = o_n(1)$, we get

$$\|u_n\|_\lambda^2 = \int\limits_{\mathbb{R}^N} |u_n|^{2_s^*} dx + o_n(1).$$

From Lemma 3.3, we can assume that there is $m \geq 0$ such that

$$\|u_n\|_\lambda^2 \to m \quad \text{and} \quad \int\limits_{\mathbb{R}^N} |u_n|^{2_s^*} dx \to m.$$

On the one hand, if $m = 0$, then (i) holds.

On the other hand, if $m > 0$, then by

$$[u]^2 \geq S \left(\int\limits_{\mathbb{R}^N} |u_n|^{2_s^*} dx \right)^{\frac{2}{2_s^*}},$$

we have $m \geq S^{\frac{N}{2s}}$. Moreover,

$$\begin{aligned}
c_{\lambda,\varepsilon} &= J_{\lambda,\epsilon}(u_n) - \frac{1}{q}\langle J'_{\lambda,\epsilon}(u_n), u_n\rangle + o_n(1) \\
&= \left(\frac{1}{2} - \frac{1}{q}\right) \|u_n\|_\lambda^2 + \frac{1}{2\varepsilon} \int\limits_{\Omega} [(\varphi - u_n)^+]^2 dx + \frac{1}{q\varepsilon} \int\limits_{\Omega} (\varphi - u_n)^+ u_n dx \\
&\quad + \frac{1}{q} \int\limits_{\mathbb{R}^N} g(x, u_n)u_n - qG(x, u_n) dx + o_n(1) \\
&\geq \left(\frac{1}{2} - \frac{1}{q}\right) [u]^2 + o_n(1) \geq \frac{q-2}{2q} S^{\frac{N}{2s}},
\end{aligned}$$

which leads to a contradiction. Therefore, $m = 0$, and (i) is true. □

Remark 3.8. From Lemmas 3.1, 3.2 and 3.6, there exists $\mu_* > 0$ such that for every $\mu \geq \mu_*$ and $\lambda, \varepsilon > 0$, problem (3.2) has at least one weak solution u_ε. Besides, from Lemmas 3.4 and 3.7, u_ε is positive.

4. THE MODIFIED INEQUALITY

Now, for any $v \in E_\lambda$, making the change of notations

$$\epsilon = \frac{1}{n}, \quad u_n = u_{\frac{1}{n}}, \quad J_n = J_{\lambda,\epsilon} \quad \text{and} \quad J_n(u_n) = c_n = c_{\lambda,\epsilon},$$

we get

$$\langle u_n, v\rangle_\lambda - n \int\limits_\Omega (\varphi - u_n)^+ v dx = \int\limits_{\mathbb{R}^N} g(x, u_n) v dx. \tag{4.1}$$

Lemma 4.1 ([1, Lemma 3.11]). *Let u_n satisfy (4.1), then there is $u \in E_\lambda$ such that, up to a subsequence, $u_n \rightharpoonup u$ in E_λ. Moreover, $u \in \mathbb{K}$.*

From the following lemma, we can transition from equality to inequality.

Lemma 4.2. *We have $u_n \to u$ in E_λ.*

Proof. Since $J_n'(u_n) = 0$, we have

$$\|u_n\|_\lambda^2 = \langle J_n'(u_n), u_n\rangle - n\langle P(u_n), u_n\rangle + \int\limits_{\mathbb{R}^N} g(x, u_n) u_n dx$$

and

$$-\langle u_n, u\rangle_\lambda = \langle J_n'(u_n), u\rangle + n\langle P(u_n), u\rangle - \int\limits_{\mathbb{R}^N} g(x, u_n) u dx.$$

Then, by $u_n \rightharpoonup u$ in E_λ, we get

$$\|u_n - u\|_\lambda^2 = n\langle P(u_n), u - u_n\rangle + \int\limits_{\mathbb{R}^N} g(x, u_n) u_n - g(x, u_n) u dx + o_n(1).$$

From the property of P in [1] and Lemma 4.1, we have

$$\|u_n - u\|_\lambda^2 \le \int\limits_{\mathbb{R}^3} g(x, u_n) u_n - g(x, u_n) u dx + o_n(1). \tag{4.2}$$

Moreover, from $u_n \rightharpoonup u$ in E_λ, we get $u_n \to u$ in $L^q_{loc}(\mathbb{R}^N)$ for $q \in [1, 2_s^*)$ and $u_n \to u$ a.e. in $\mathbb{R}^N$.

By Lemma 3.6, first, we can conclude that the sequence $\{u_n\}$ is bounded. Then, we choose $\psi_\rho u_n - \psi_\rho \varphi^+$ as a test function to ensure that the penalization term is negative, as shown in [2, Lemma 3.12], i.e.

$$\langle P(u_n), \psi_\rho u_n - \psi_\rho \varphi^+\rangle \le 0,$$

where ψ_ρ is from Step 4 in Lemma 3.6. Therefore, following the approach in Lemma 3.6, $u_n \to u$ in $L^{2_s^*}(\Lambda)$. From (4.2), we obtain $u_n \to u$ in E_λ. □

Remark 4.3. For any $v \in \mathbb{K}$, taking $v - u_n$ as a test function in (4.1), from Lemma 4.1, we know $u \not\equiv 0$. Besides, from Lemma 4.2, we can obtain u is a positive solution for (3.3).

5. PROOF OF THE THEOREM 1.1

This final section is devoted to proving the main result of this paper. We will show that (3.4) holds. For this purpose, we need the third strong convergence of $\{u_n\}$, which will be defined later, and use a variant of the Moser iteration scheme [25]. From Remark 4.3, let $u_n \in \mathbb{K}$ satisfy

$$\langle u_n, v - u_n \rangle_{\lambda_n} \geq \int\limits_{\mathbb{R}^N} g(x, u_n)(v - u_n)dx, \tag{5.1}$$

for all $v \in \mathbb{K}$.

Lemma 5.1. *If $\{u_n\}$ is defined as above, then, as $\lambda_n \to \infty$,*

$$\limsup_{n\to\infty} \|u_n\|^2_{\lambda_n} \leq \frac{4q\alpha_0}{q-2},$$

where $\alpha_0 := \sup\limits_{n\in\mathbb{N}} c_{\lambda_n,\varepsilon}$.

Proof. From Lemma 4.2, we have $u_m \to u$ in E_λ. Moreover, from Lemmas 3.3 and 3.4,

$$\limsup_{n\to\infty} \|u_n\|^2_{\lambda_n} \leq \limsup_{m\to\infty} \|u_m\|^2_{\lambda} \leq \frac{4q\alpha_0}{q-2}. \qquad \square$$

From Lemma 5.1, $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$. Then, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, and $u \in H^s(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^N)$.

Lemma 5.2. *Let $\{u_n\} \subset E_{\lambda_n}$ and u be defined as above. Then:*

(i) $u = 0$ *a.e. in* Ω^c,
(ii) $\|u_n - u\|_{\lambda_n} \to 0$.

Proof. (i) By the Fatou lemma, we have

$$\int\limits_{\mathbb{R}^N} M(x)u^2dx \leq \liminf_{n\to\infty} \int\limits_{\mathbb{R}^N} M(x)u_n^2dx \leq \liminf_{n\to\infty} \frac{\|u_n\|^2_{\lambda_n}}{\lambda_n} = 0,$$

which implies that $u = 0$ a.e. in Ω^c.

(ii) By $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^N)$ and (i), we show that

$$\langle u, u_n - u \rangle_{\lambda_n} = \int\limits_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u(-\Delta)^{\frac{s}{2}}(u_n - u)dx + \int\limits_{\Omega} u(u_n - u)dx = o_n(1).$$

Therefore,

$$\|u_n - u\|^2_{\lambda_n} = \langle u_n, u_n - u \rangle_{\lambda_n} - \langle u, u_n - u \rangle_{\lambda_n} = \langle u_n, u_n - u \rangle_{\lambda_n} + o_n(1).$$

Since $u_n \in \mathbb{K}$, it follows that $u \geq \varphi$ a.e.in Ω. From (5.1), we can get

$$\langle u_n, u_n - u \rangle_{\lambda_n} \leq \int\limits_{\mathbb{R}^N} g(x, u_n)(u_n - u)dx,$$

and thus,

$$\|u_n - u\|^2_{\lambda_n} \leq \int_{\mathbb{R}^N} g(x, u_n)(u_n - u)dx + o_n(1).$$

To complete this proof, we need to show that $u_n \to u$ in $L^{2^*_s}(\Lambda)$. As in Lemma 3.6, define the same ψ_ρ. We suppose that $\mathrm{supp}(\psi_\rho) \subset \Lambda$. Letting $v_n = u_n - \psi_\rho(u_n - \varphi^+)$, we get $v_n \in \mathbb{K}$. Next, taking v_n in (5.1) and proceeding as in Lemma 3.6, we obtain $\|u_n - u\|_{\lambda_n} \to 0$. □

From Lemma 5.2, we can directly obtain the following techniques.

Lemma 5.3 ([2, Proposition 3.13]). *Let $\{u_n\}$ be from Lemma 5.2. Then:*

(i) *as $\lambda_n \to \infty$ we have the following limits:*

$$\begin{aligned}
&u_n \to u \quad \textit{in } H^s(\mathbb{R}^N),\\
&\lambda_n \int_{\mathbb{R}^N} M(x)|u_n|^2 dx \to 0,\\
&\|u_n\|^2_{\lambda_n} \to \int_Q \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy + \int_\Omega |u|^2 dx,
\end{aligned}$$

where $Q := \mathbb{R}^{2N} \setminus (\Omega^c \times \Omega^c)$,

(ii) *the function u is a solution to the fractional variational inequality*

$$\begin{aligned}
&\int_Q \frac{(u(x) - u(y))((w - u)(x) - (w - u)(y))}{|x - y|^{N+2s}} dxdy + \int_\Omega u(w - u)dx\\
&\geq \int_\Omega (\mu u^{q-2} + u^{2^*_s - 2})u(w - u)dx,
\end{aligned}$$

for every $w \in \tilde{\mathbb{K}}$, where

$$\tilde{\mathbb{K}} = \{w \in H^s_0(\Omega) : w \geq \varphi \ \textit{a.e. in}\, \Omega\}.$$

Next, we present the main proof of Theorem 1.1.

Lemma 5.4. *Let $u_n \in \mathbb{K}$ be a solution to* (5.1) *for all $v_n \in \mathbb{K}$. Then there exists $C > 0$, such that*

$$|u_n|_{\infty(\Lambda^c)} \leq C|u_n|_{2^*_s(\Omega^c)}.$$

Proof. For any $L > 0$ and $n \in \mathbb{N}$, we define $u_{n,L} = \min\{u_n, L\}$. Let $d = \operatorname{dist}(\overline{\Omega}, \partial\Lambda)$ and $r, R \in (0, d)$, with $r < R$. Given $x_0 \in \overline{\Lambda}^c$, let

$$v_{n,L}(x) = \left(u_n - u_n\psi^2 u_{n,L}^{2(\beta-1)}\right)(x),$$

where $\psi \in C_0^\infty(\mathbb{R}^N, [0,1])$ is a function such that $0 \le \psi \le 1$, $|\nabla\psi| \le \frac{1}{R}$ and

$$\psi(x) = \begin{cases} 1, & \text{if } x \in B_r(x_0), \\ 0, & \text{if } x \in B_R^c(x_0), \end{cases}$$

and $\beta > 1$ will be determined later.

Step 1. We will show $v_{n,L} \in \mathbb{K}$.

Since $\Omega \subset B_R^c(x_0)$, we have for a.e. $x \in \Omega$, then

$$v_{n,L} = u_n,$$

that is, $v_{n,L} \ge \varphi$ a.e. in Ω.

Step 2. We will finish this lemma.

Taking $v_{n,L}$ in (5.1), we get

$$\langle u_n, u_n\psi^2 u_{n,L}^{2(\beta-1)}\rangle_{\lambda_n} \le \int\limits_{\mathbb{R}^N} g(x, u_n) u_n \psi^2 u_{n,L}^{2(\beta-1)} dx.$$

Let

$$\begin{aligned} A_n &:= \iint\limits_{\mathbb{R}^{2N}} \frac{(u_n(x) - u_n(y))(u_n(x)\psi^2(x)u_{n,L}^{2(\beta-1)}(x) - u_n(y)\psi^2(y)u_{n,L}^{2(\beta-1)}(y))}{|x-y|^{N+2s}} dxdy \\ &= \iint\limits_{\mathbb{R}^{2N}} \frac{(u_n(x) - u_n(y))(u_n(x)u_{n,L}^{2(\beta-1)}(x) - u_n(y)u_{n,L}^{2(\beta-1)}(y))\psi^2(x)}{|x-y|^{N+2s}} dxdy \\ &\quad + \iint\limits_{\mathbb{R}^{2N}} \frac{(u_n(x) - u_n(y))(\psi^2(x) - \psi^2(y))u_n(y)u_{n,L}^{2(\beta-1)}(y)}{|x-y|^{N+2s}} dxdy \\ &=: A_n^1 + A_n^2. \end{aligned}$$

As for A_n^2, from the Hölder inequality and $\psi \in C_0^\infty(\mathbb{R}^N)$, we get

$$\begin{aligned}
|A_n^2| &\leq \iint\limits_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)||\psi^2(x)-\psi^2(y)|u_n(y)u_{n,L}^{2(\beta-1)}(y)}{|x-y|^{N+2s}}dxdy \\
&\leq \iint\limits_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)||\psi^2(x)-\psi^2(y)|u_n^{2\beta-1}(y)}{|x-y|^{N+2s}}dxdy \\
&\leq 2\alpha_1^{2\beta-1}\left(\iint\limits_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{\frac{1}{2}}\left(\iint\limits_{\mathbb{R}^{2N}} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{\frac{1}{2}} \\
&\leq C\left(\int\limits_{B_R(x_0)}\int\limits_{B_R(x_0)} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{\frac{1}{2}} \\
&\leq C\left(\int\limits_{B_R(x_0)}\int\limits_{|x-y|\geq R} \frac{1}{|x-y|^{N+2s}}dxdy + \int\limits_{B_R(x_0)}\int\limits_{|x-y|\leq R} \frac{|x-y|^2}{|x-y|^{N+2s}}dxdy\right)^{\frac{1}{2}} \\
&\leq C,
\end{aligned}$$

where $\alpha_1 := \sup\limits_{B_R(x_0)} u_n(x)$. Moreover, as for A_n^1, performing like in [4, Lemma 5.1],

$$A_n^1 \geq C\frac{1}{\beta^2}|u_n u_{n,L}^{\beta-1}|^2_{2_s^*(B_r(x_0))}.$$

By calculation, from (g_2), we obtain for any $\delta > 0$,

$$g(x,t) \leq \mu\delta|t| + C_\delta|t|^{2_s^*-1},$$

thus

$$\begin{aligned}
C\frac{1}{\beta^2}|u_n u_{n,L}^{\beta-1}|^2_{2_s^*(B_r(x_0))} \leq C_\delta \int\limits_{\mathbb{R}^N} &\left(u_n^{2_s^*}\psi^2 u_{n,L}^{2(\beta-1)}\right)dx \\
&+ (\mu\delta - 1)\int\limits_{\mathbb{R}^N}\left(u_n^2\psi^2 u_{n,L}^{2(\beta-1)}\right)dx + |A_n^2|.
\end{aligned} \tag{5.2}$$

Fixing $0 < \delta < \frac{1}{\mu}$, from (5.2), we get

$$C\frac{1}{\beta^2}|u_n u_{n,L}^{\beta-1}|^2_{2_s^*(B_r(x_0))} \leq C_\delta \int\limits_{\mathbb{R}^N}\left(u_n^{2_s^*}\psi^2 u_{n,L}^{2(\beta-1)}\right)dx + C.$$

Therefore, by the Hölder inequality, choosing $\beta = \frac{2_s^*}{2}$, we have

$$
\begin{aligned}
&|u_n u_{n,L}^{\beta-1}|^2_{2_s^*(B_r(x_0))} \\
&\le C\beta^2 \left(\int\limits_{B_R(x_0)} \left(|u_n|^2 u_{n,L}^{2(\beta-1)} \right)^{\frac{2_s^*}{2}} dx \right)^{\frac{2}{2_s^*}} \left(\int\limits_{B_R(x_0)} (|u_n|^{2_s^*-2})^{\frac{2_s^*}{2_s^*-2}} dx \right)^{\frac{2_s^*-2}{2_s^*}} + C\beta^2.
\end{aligned} \tag{5.3}
$$

Since $u_n \to u$ in $H^s(\mathbb{R}^N)$, for any $\tau > 0$, as R small enough, we obtain

$$
\int\limits_{B_R(x_0)} |u_n|^{2_s^*} dx < \tau.
$$

Then, as $L \to \infty$, we get $u_n \in L^{\frac{(2_s^*)^2}{2}}(B_r(x_0))$. If we put $\beta = \frac{2_s^*(t-1)}{2t}$ with $t = \frac{(2_s^*)^2}{2(2_s^*-2)}$, then $\beta > 1$, $\frac{2t}{t-1} < 2_s^*$ and $u_n \in L^{\frac{2\beta t}{t-1}}(B_R(x_0))$. Returning to (5.3), we have

$$
\begin{aligned}
|u_n u_{n,L}^{\beta-1}|^2_{2_s^*(B_r(x_0)} &\le C\beta^2 + C\beta^2 \int\limits_{B_R(x_0)} u_n^{2_s^*} u_{n,L}^{2(\beta-1)} dx \\
&\le C\beta^2 + C\beta^2 \int\limits_{B_R(x_0)} u_n^{2_s^*+2\beta-2} dx \\
&\le C\beta^2 + C\beta^2 \left(\int\limits_{B_R(x_0)} u_n^{(2_s^*-2)t} dx \right)^{\frac{1}{t}} \left(\int\limits_{B_R(x_0)} u_n^{\frac{2\beta t}{t-1}} dx \right)^{\frac{t-1}{t}}.
\end{aligned} \tag{5.4}
$$

Note that $u_n \in L^{\frac{(2_s^*)^2}{2}}(B_R(x_0))$. From (5.4), we conclude

$$
|u_{n,L}|^2_{2_s^*(B_r(x_0))}) \le C\beta^2 \left(\int\limits_{B_R(x_0)} u_n^{\frac{2\beta t}{t-1}} dx \right)^{\frac{t-1}{t}}. \tag{5.5}
$$

Thus, by (5.5),

$$
\left(\int\limits_{B_r(x_0)} u_{n,L}^{2_s^*\beta} dx \right)^{\frac{2\beta}{2_s^*\beta}} \le \left(\int\limits_{B_r(x_0)} (u_n u_{n,L}^{\beta-1})^{2_s^*} dx \right)^{\frac{2\beta}{2_s^*\beta}} \le C\beta^2 \left(\int\limits_{B_R(x_0)} u_n^{\frac{2\beta t}{t-1}} dx \right)^{\frac{t-1}{t}}.
$$

As $L \to \infty$, from the Fatou lemma,

$$
|u_n|^{2\beta}_{2_s^*\beta(B_r(x_0))} \le C\beta^2 |u_n|^{2\beta}_{\frac{2\beta t}{t-1}(B_R(x_0))}.
$$

If we set $a = \frac{2_s^*(t-1)}{2t}$, $b = \frac{2t}{t-1}$, and let $\beta = a^m$, then by iterating this process, we can obtain

$$|u_n|_{a^{m+1}b(B_r(x_0))} \leq C^{\sum_{i=1}^m a^{-i}} a^{\sum_{i=1}^m ia^{-i}} |u_n|_{ab(B_R(x_0))}.$$

Letting $m \to \infty$, we obtain

$$|u_n|_{\infty(B_r(x_0))} \leq C|u_n|_{2_s^*(B_R(x_0))}.$$

Since $B_R(x_0) \subset \Omega^c$ and x_0 is arbitrary in $\overline{\Lambda}^c$, then

$$|u_n|_{\infty(\Lambda^c)} \leq C|u_n|_{2_s^*(B_R(x_0))} \leq C|u_n|_{2_s^*(\Omega^c)}. \tag{5.6}$$

□

Now, we are ready to complete the proof of Theorem 1.1.

Proof. We will show u_n satisfies (3.4). Since $u_n \to u$ in $H^s(\mathbb{R}^N)$, by Sobolev embedding, we get as $\lambda_n \to \infty$

$$0 \leq |u_n - u|_{2_s^*} \leq C\|u_n - u\| \to 0.$$

From Lemma 5.2 (i), we get

$$\int\limits_{\Omega^c} |u_n - u|^{2_s^*} dx = \int\limits_{\Omega^c} |u_n|^{2_s^*} dx \to 0.$$

Combining this with (5.6), we have for any $\kappa > 0$, there exists $n_0 \in \mathbb{N}$, for all $n \geq n_0$, $|u_n|_{\infty(\Lambda^c)} < \kappa$. Therefore, there is $\lambda^* > 0$ such that

$$u_\lambda \leq a, \forall x \in \Lambda^c \quad \text{and} \quad \lambda \geq \lambda^*.$$

Then, (3.4) holds true. Therefore, u_λ satisfies (1.1). □

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