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SPECTRAL ANALYSIS OF INFINITE MARCHENKO–SLAVIN MATRICES

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Abstract. This work tackles the problem of spectral characterization of a class of infinite matrices arising from the modeling of small oscillations in a system of interacting particles. The class of matrices under discussion corresponds to the infinite Marchenko–Slavin class. The spectral functions of these matrices are completely characterized, and an algorithm is provided for the reconstruction of the matrix from its spectral function. The techniques used in this work are based on recent results for the spectral characterization of infinite band symmetric matrices with so-called degenerations.

Keywords: inverse spectral problem, band symmetric matrices, spectral measure.

Mathematics Subject Classification: 34K29, 47B39, 70F17.

1. INTRODUCTION

It follows from Lagrangian mechanics that the mechanical characteristics of a system of interacting particles, when the potential energy depends only on each particle's spatial position, is given by a finite Hermitian block matrix [10, Chap. 8]. This matrix encodes the potential energy in the approximation obtained when each particle is very close to the equilibrium state. The relevance of this matrix is that its spectral function determines the movement of each particle. The dimension of the matrix's blocks corresponds to the degrees of freedom of the particles. For instance, in the theory of small oscillations, any linear arrangement of particles interacting only between immediate neighbors and moving only along the line of interaction is mechanically equivalent to a string of masses and springs. In this case the matrix of the system's mechanical characteristics is a finite scalar Jacobi matrix [11]. Still with one degree of freedom, but with interactions occurring not only between immediate neighbors, the mechanical matrix is a scalar Hermitian band matrix. Intricate arrangements of interacting particles with n degrees of freedom correspond to Hermitian matrices with $n \times n$ block entries.

To illustrate the main ideas of the theory of small oscillations, let us delve into the details of the simplest mechanical system referred above, viz. a linear mass-spring system of N particles which is modeled by an $N \times N$ Jacobi matrix. In this case the (k, l) entry of the matrix encodes, among other mechanical properties, the interaction between the k-th and l-th particles. Thus, the one-degree of freedom movement of the k-th particle, when the l-th particle is excited by imposing to it the velocity v, is given by the function (cf. [10, Eqs. 1.11 and 8.14])

$$q_l(k,t) = v \sum_{j=1}^{N} \sqrt{\frac{m_l}{m_k}} E_j(k,l) \frac{\sin \sqrt{\lambda_j}\, t}{\sqrt{\lambda_j}}, \tag{1.1}$$

where m_k and m_l are the masses of the k-th and l-th particles, respectively, and $E_j(k,l)$ is the (k,l) entry of the orthogonal projector onto the eigenspace corresponding to the positive eigenvalue λ_j. The function (1.1) is the solution to the direct mechanical problem which is the result of solving the direct spectral problem.

Due to the fact that Jacobi matrices are simple selfadjoint operators, the spectral function is completely determined by the scalar measure μ given by its projection onto the first element of the canonical basis (since it is a cyclic vector), i.e.

$$\mu(t) = \sum_{j<t} E_j(1,1).$$

This corresponds to the fact that the Jacobi matrix can be reconstructed by this scalar measure via an orthonormalizing procedure. Moreover, by observing the small oscillations of the *first* particle of the system, one determines the mechanical characteristics of it and therefore the oscillations of all the interacting particles close to the equilibrium state. Indeed (see [10, Lem. 1]),

$$\lim_{T\to\infty} \frac{2}{T} \int_0^T \frac{q_1(1,t)}{v} \sqrt{\lambda_j} \sin \sqrt{\lambda_j} t dt = E_j(1,1). \tag{1.2}$$

This formula provides a connection between the inverse spectral problem and the inverse mechanical problem.

In the more general setting of arbitrary block Hermitian matrices, which model a system of interacting particles having one or more degrees of freedom, Marchenko and Slavin tackled the inverse problem by characterizing in [10] a class of block Hermitian matrices corresponding to systems of interacting particles for which the observation of a subset of particles is sufficient for unambiguously find their mechanical parameters. Solving this problem not only amounts to finding the spectral function and the generating space of the corresponding operator, but requires, firstly, establishing that the movement of the above mentioned subset of particles determines the corresponding projections of the spectral functions and, secondly, finding an algorithm for the reconstruction of the matrix from the spectral function. The class of matrices given by Marchenko and Slavin are the result of using the theory of extendable sets which is introduced in [10, Chap. 12]. This theory connects the structure of a block Hermitian matrix with the theory of unitary invariants of the spectral measure (see [3, Sec. 7.4]). In [10, Chap. 9], the generalization of (1.2) is given, while a reconstruction algorithm is provided in [10, Chap. 14].

In this work, we study direct and inverse spectral problems for the infinite dimensional generalization of the Marchenko–Slavin matrices (see in [10, Chap. 15] the definition of the class for finite matrices). We restrict ourselves to infinite matrices with scalar entries and focus on developing a general approach suitable for a generalization to infinite matrices with matrix entries (block matrices). It is worth mentioning that the theory of extendable sets is inherently finite and, thus, for the infinite case, one has to rely on other techniques which are rather build up on the theory developed in [9].

The theory pertaining to the class of infinite matrices studied in [9], denoted in this work by $\widetilde{\mathfrak{M}}$ and illustrated in Figure 3, is based on the results of [7] where a linear multidimensional interpolation problem is treated (see Section 4). Actually, the finite and infinite classes of matrices given in [8] and [9], respectively, were tailor-made to fit the structure of the solutions to that interpolation problem. Indeed, the vector polynomials arising from the so-called "degenerations" of the matrix (see Figure 3) generate all the solutions to the above mentioned linear multidimensional interpolation problem [7, Sec. 4]. Noteworthily, the degenerations of the matrix are interpreted as *inner boundary conditions* [8, Sec. 2] of the associated difference equation and are relevant in the theory of such equations.

The infinite dimensional Marchenko–Slavin class, introduced in Definition 2.2, illustrated in Figure 1 and denoted by $\mathfrak{M}$, contains properly the class $\widetilde{\mathfrak{M}}$. However, it is proven in Theorem 6.6, that every matrix in $\mathfrak{M}$ is unitary equivalent to a matrix in $\widetilde{\mathfrak{M}}$. This equivalence between $\mathfrak{M}$ and $\widetilde{\mathfrak{M}}$ has implications in both the linear multidimensional interpolation problem and the modeling of systems of interacting particles. Indeed, a system of the more general type modeled by a matrix in $\mathfrak{M}$ can always be reduced to a system whose mechanical characteristics are given by a simpler matrix in $\widetilde{\mathfrak{M}}$.

Proving the equivalence of the classes $\mathfrak{M}$ and $\widetilde{\mathfrak{M}}$ requires firstly a complete characterization of the spectral functions of a given matrix in $\mathfrak{M}$ (see Definition 5.8) and, secondly, the application of the reconstruction algorithm developed in [9]. For finding the spectral functions of the infinite matrix, one needs to characterize the spectral functions of finite submatrices of the matrix (see Section 2) and recur to a limit process. This involves solving a problem related to the *truncated* moment problem which can be studied by means of finite matrices (see Theorem 5.3).

2. THE CLASS $\mathfrak{M}$ OF SEMI-INFINITE MATRICES

Let $l_2(\mathbb{N})$ be the Hilbert space of square summable sequences with entries in $\mathbb{C}$ and $\{\delta_k\}_{k\in\mathbb{N}}$ be the canonical orthonormal basis of it. Along the text, a sequence $\{u_k\}_{k\in\mathbb{N}}$ is identified with $u=\sum_{k=1}^{\infty} u_k\delta_k$. Also, we consider square summable sequences $\{u_k\}_{k\in G}$, where $G\subset\mathbb{N}$, and denote the corresponding space by $l_2(G)$.

Definition 2.1. For $F \subset G$ and any $u \in l_2(G)$, define:

(i) $\Pi_{G\to F} u \coloneqq \{u_k\}_{k\in G}$, where

$$u_k = \begin{cases} u_k = u_k, & k \in F, \\ u_k = 0, & k \in G \setminus F, \end{cases}$$

(ii) $\widetilde{\Pi}_{G\to F} u \coloneqq \{u_k\}_{k\in F}$.

Note that $\Pi_{G\to F}$ is an orthogonal projector while $\widetilde{\Pi}_{G\to F}$ is not.

Let us agree on the following terminology. The nonzero entries of a matrix to the right of which there are only zero entries on the same row are called *row-edge entries*. Similarly, the nonzero entries of a matrix for which there are only zero entries on the same column are called *column-edge entries*.

The following matrices form the class of infinite Marchenko–Slavin matrices (see in [10] the finite analogue of this class).

Definition 2.2. The matrix $M = \{m_{jk}\}_{j,k\in\mathbb{N}}$ is in the class $\mathfrak{M}$ when $m_{jk} = \overline{m}_{kj}$ for any $j,k \in \mathbb{N}$ and the following conditions are met.

(1) There is $l \in \mathbb{N}$ such that there is one *row-edge entry* on each of the columns $l+1, l+2, \dots$. Denote either by n_M or n the minimal l with this property.
(2) There are numbers $j,k \in \mathbb{N}$ with $k > j$ such that

$$m_{jk},\ m_{j+1k+1},\ m_{j+2k+2}, \dots$$

are simultaneously row-edge and column-edge entries. Denote by j_0, k_0 the smallest j,k having this property.

The condition (2) of Definition 2.2 gives the "tail" of the matrix. An illustration of a matrix in $\mathfrak{M}$ is given in Figure 1, where we agree on considering only one row-edge entry in the columns $n+1, n+2, \dots$. In what follows, we always assume this convention for any matrix in $\mathfrak{M}$.

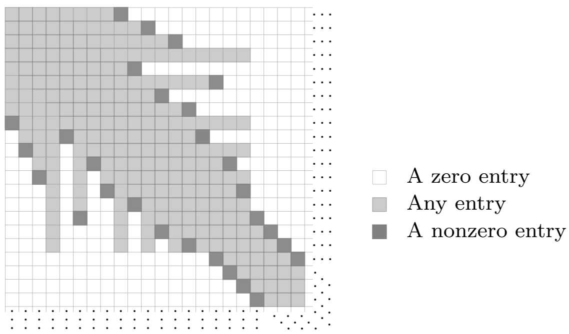

Fig. 1. An example of an element of $\mathfrak{M}$

According to [2, Sec. 47, Thm. 4], given an orthonormal basis in $l_2(\mathbb{N})$, one can construct uniquely from the matrix M a closed symmetric operator, which is denoted by $\mathcal{M}$, in such a way that M is its matrix representation with respect to the given orthonormal basis (see [2, Sec. 47]).

The analysis of the spectral function of operator $\mathcal{M}$ is carried out by means of the auxiliary operator $\mathcal{M}_N := \widetilde{\Pi}_{\mathbb{N}\to G_N}\mathcal{M}\restriction_{l_2(G_N)}$, where $G_s := \{1,\dots,s\}$ and $N > n$. Note that $\mathcal{M}_N$ can be identified with the operator whose matrix representation is the finite dimensional submatrix corresponding to the $N \times N$ upper-left corner of M which is denoted by M_N. Note that, since $N > n$, any matrix M_N determines a set $K \subset G_N$ such that for each $k \in K$ the k-th row has no row-edge entry of M inside M_N in the columns $n+1,\dots,N$, and the cardinality of K is always equal to n (see Definition 2.2 (1)).

3. SPECTRAL FUNCTIONS OF SUBMATRICES

For a matrix $M \in \mathfrak{M}$, consider the finite submatrix M_N. To study the spectral properties of the operator $\mathcal{M}_N$, one recurs to the equation

$$M_N\psi = z\psi \tag{3.1}$$

for $\psi \in l_2(G_N)$. To solve this equation, one starts by finding the solution to

$$\Pi_{G_N\to K^\perp}(M_N - zI)\psi = 0, \tag{3.2}$$

where $K^\perp$ is the complement in G_N of the set K given at the end of the previous section.

The solution to (3.2) is constructed recursively as follows: one gives the first n elements of the sequence ψ and finds ψ_{n+1} from the first row equation of (3.2) (see (3.3) in Example 3.1 below). It is possible to solve this equation due to the fact that the row-edge entry, which is on the $n+1$-th column, is not zero. Having found ψ_{n+1}, one finds the subsequent elements of the solution using the row equations where there are row-edge entries (that is, row equations corresponding to $K^\perp$).

Example 3.1. If one has the matrix M_7 corresponding to Figure 2, then the difference equations corresponding to the rows are

$$m_{11}\psi_1 + m_{12}\psi_2 + m_{13}\psi_3 + \boldsymbol{m}_{14}\psi_4 = z\psi_1, \tag{3.3}$$

$$m_{21}\psi_1 + m_{22}\psi_2 + m_{23}\psi_3 + m_{24}\psi_4 + m_{25}\psi_5 + \boldsymbol{m}_{26}\psi_6 = z\psi_2,$$

$$m_{31}\psi_1 + m_{32}\psi_2 + m_{33}\psi_3 + m_{34}\psi_4 + m_{35}\psi_5 + m_{36}\psi_6 = z\psi_3, \tag{3.4}$$

$$m_{41}\psi_1 + m_{42}\psi_2 + m_{43}\psi_3 + m_{44}\psi_4 + \boldsymbol{m}_{45}\psi_5 = z\psi_4,$$

$$m_{52}\psi_2 + m_{53}\psi_3 + m_{54}\psi_4 + m_{55}\psi_5 + m_{56}\psi_6 = z\psi_5, \tag{3.5}$$

$$m_{62}\psi_2 + m_{63}\psi_3 + m_{65}\psi_5 + m_{66}\psi_6 + \boldsymbol{m}_{67}\psi_7 = z\psi_6,$$

$$m_{76}\psi_6 + m_{77}\psi_7 = z\psi_7, \tag{3.6}$$

where the row-edge entries have been denoted in bold typeface. Note that in (3.4), (3.5) and (3.6) there are no row-edge entries. In this case, the system (3.1), without equations (3.4), (3.5), (3.6) (i.e., the system corresponding to (3.2)) and with boundary conditions given by the numbers ψ_1, ψ_2, ψ_3, has the following solution:

$$\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\\ \psi_5\\ \psi_6\\ \psi_7\end{pmatrix} = \begin{pmatrix} \psi_1\\ \psi_2\\ \psi_3\\ \boldsymbol{m}_{14}^{-1}(z\psi_1 - m_{11}\psi_1 - m_{12}\psi_2 - m_{13}\psi_3)\\ \boldsymbol{m}_{45}^{-1}(z\psi_4 - m_{41}\psi_1 - m_{42}\psi_2 - m_{43}\psi_3 - m_{44}\psi_4)\\ \boldsymbol{m}_{26}^{-1}(z\psi_2 - m_{21}\psi_1 - m_{22}\psi_2 - m_{23}\psi_3 - m_{24}\psi_4 - m_{25}\psi_5)\\ \boldsymbol{m}_{67}^{-1}(z\psi_6 - m_{62}\psi_2 - m_{63}\psi_3 - m_{65}\psi_5 - m_{66}\psi_6)\end{pmatrix}.$$

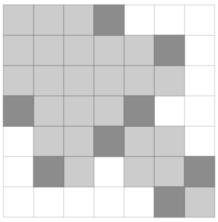

Fig. 2. A submatrix of $\mathfrak{M}$ with $N = 7$ and $n = 3$

As was already mentioned, the first n elements of the sequence ψ play the role of boundary conditions for the difference equation. For assigning these conditions, we introduce an $n \times n$ upper triangular, invertible matrix $\mathcal{T} = \{t_{ij}\}_{i,j=1}^{n}$ which will be called *boundary matrix*. Let $\psi_{\mathcal{T}}^{j}(z)$ $(j \in G_n)$ be an N-dimensional vector solution to (3.2) such that $\left\langle \delta_i, \psi_{\mathcal{T}}^{j}(z) \right\rangle = t_{ji}$ for $i \in G_n$. Thus,

$$\widetilde{\Pi}_{G_N \to G_n} \Psi_{\mathcal{T}}(z) = \mathcal{T}^*, \tag{3.7}$$

where $\Psi_{\mathcal{T}}(z)$ is the $N \times n$ matrix whose columns are the solutions $\psi_{\mathcal{T}}^{1}(z), \dots, \psi_{\mathcal{T}}^{n}(z)$. We may use the notation:

$$\Psi_{\mathcal{T}}(z) = \begin{pmatrix}\psi_{\mathcal{T}}^{1}(z) & \psi_{\mathcal{T}}^{2}(z) & \dots & \psi_{\mathcal{T}}^{n}(z)\end{pmatrix}. \tag{3.8}$$

Lemma 3.2. *For any $z \in \mathbb{C}$, the N-dimensional vector $\eta(z)$ is a solution to* (3.2) *if and only if $\eta(z) = \Psi_{\mathcal{T}}(z)C$, where C is an n-dimensional vector. If C_1 and C_2 are two linearly independent vectors in $l_2(G_n)$, then $\Psi_{\mathcal{T}}(z)C_1$ and $\Psi_{\mathcal{T}}(z)C_2$ are linearly independent solutions to* (3.2).

Proof. The fact that $\Psi_{\mathcal{T}}(z)C$ is a solution to (3.2) is straightforward. Let η be a solution to (3.2) for a fixed z. Since $\mathcal{T}$ is a triangular invertible matrix, there is a vector C_η such that $\Psi_{\mathcal{T}}(z)C_\eta$ has the same first n entries as η. Therefore, $\eta - \Psi_{\mathcal{T}}(z)C_\eta$ is the trivial solution (see the paragraph above Example 3.1). For proving the second part, one uses that $\operatorname{null}\Psi(z) = 0$ which again follows from the invertibility of $\mathcal{T}$. $\square$

Proposition 3.3. *Let M_N be a submatrix of $M \in \mathfrak{M}$. Define*

$$\Theta_{\mathfrak{T}}(z) := \widetilde{\Pi}_{G_N \to K}(M_N - zI)\Psi_{\mathfrak{T}}(z). \tag{3.9}$$

The spectrum of M_N is given by the zeros of the polynomial $\det \Theta_{\mathfrak{T}}(z)$*. Moreover, if $\lambda \in \operatorname{spec}(M_N)$, then there is a nonzero vector C^λ such that the corresponding normalized eigenvector φ^λ is given by*

$$\varphi^\lambda = \Psi_{\mathfrak{T}}(\lambda)C^\lambda. \tag{3.10}$$

Proof. If λ is an eigenvalue and φ^λ is the corresponding eigenvector of M_N, then (3.1) and (3.2) are satisfied for $z = \lambda$ and $\psi = \varphi^\lambda$. Due to Lemma 3.2 there is a nonzero vector C^λ such that $\varphi^\lambda = \Psi_{\mathfrak{T}}(\lambda)C^\lambda$. Also,

$$\begin{aligned} 0 &= \widetilde{\Pi}_{G_N \to K}(M_N - \lambda I)\Psi_{\mathfrak{T}}(\lambda)C^\lambda \\ &= \Theta_{\mathfrak{T}}(\lambda)C^\lambda. \end{aligned} \tag{3.11}$$

Hence, the homogeneous linear system (3.11) has a nontrivial solution if and only if $\det \Theta_{\mathfrak{T}}(\lambda) = 0$. The normalization of the vector φ^λ can clearly be achieved by modifying C^λ appropriately. ☐

Corollary 3.4. *The multiplicity of the eigenvalues of M_N is no greater than n. If $\{\lambda_k\}_{k=1}^N$ is the spectrum of the matrix enumerated so that the multiplicity of eigenvalues is taken into account, then there is a finite sequence of vectors, $\{\varphi^{\lambda_k}\}_{k=1}^N$, being an orthonormal basis of $l_2(G_N)$.*

Proof. It follows from (3.10) that the dimension of the eigenspaces is at most n since C^λ is n-dimensional. The second assertion of the corollary is a consequence of M_N being selfadjoint. ☐

Henceforth, the spectrum of M_N is written as $\{\lambda_k\}_{k=1}^N$, where the eigenvalues are enumerated so that the multiplicity of them is taken into account, and the corresponding normalized eigenvectors are denoted by $\varphi^{\lambda_1}, \dots, \varphi^{\lambda_N}$. Also, $C_{\mathfrak{T}}^{\lambda_k}$ is the nonzero n-dimensional vector determined by φ^{λ_k} through (3.10).

Using the notation introduced in (3.8), one defines

$$\Lambda := \operatorname{diag}\{\lambda_1, \dots, \lambda_N\} \quad \text{and} \quad \Phi := \begin{pmatrix} \varphi^{\lambda_1} & \cdots & \varphi^{\lambda_N} \end{pmatrix}.$$

Thus,

$$\Phi^* M_N \Phi = \Lambda, \quad \text{and} \quad \Phi\Phi^* = \Phi^*\Phi = I.$$

Let us also define

$$\Phi_0 := \widetilde{\Pi}_{G_N \to G_n}\Phi.$$

Lemma 3.5. *Let $I_n := \widetilde{\Pi}_{G_N \to G_n} I \upharpoonright_{l_2(G_N)}$. The following decomposition takes place*

$$I_n = \mathfrak{T}^* \sum_{k \in G_N} C_{\mathfrak{T}}^{\lambda_k} (C_{\mathfrak{T}}^{\lambda_k})^* \mathfrak{T}.$$

Proof. One verifies, on the basis of the unitary properties of Φ, that

$$\Phi_0\Phi_0^* = I_n.$$

Note, however that $\Phi_0^*\Phi_0 \neq I_n$. If $\varphi_0^{\lambda_k} := \widetilde{\Pi}_{G_N\to G_n}\varphi^{\lambda_k} = \Phi_0\delta_k$, then

$$I_n = \Phi_0\Phi_0^* = \Phi_0\left(\sum_{k\in G_N}\delta_k\delta_k^*\right)\Phi_0^* = \sum_{k\in G_N}\Phi_0\delta_k\delta_k^*\Phi_0^* = \sum_{k\in G_N}\varphi_0^{\lambda_k}(\varphi_0^{\lambda_k})^*. \tag{3.12}$$

Now, it follows from (3.7) and (3.10) that

$$\mathcal{T}^* C_{\mathcal{T}}^{\lambda_k} = \varphi_0^{\lambda_k}. \tag{3.13}$$

Thus, combining (3.12) and (3.13), one has

$$I_n = \sum_{k\in G_N}\varphi_0^{\lambda_k}(\varphi_0^{\lambda_k})^* = \sum_{k\in G_N}\mathcal{T}^*C_{\mathcal{T}}^{\lambda_k}(C_{\mathcal{T}}^{\lambda_k})^*\mathcal{T} = \mathcal{T}^*\sum_{k\in G_N}C_{\mathcal{T}}^{\lambda_k}(C_{\mathcal{T}}^{\lambda_k})^*\mathcal{T}. \qquad \square$$

The next assertion is an immediate consequence of the previous result.

Corollary 3.6. *The following property of the vectors $C_{\mathcal{T}}^{\lambda_k}$, $k = 1,\dots,N$ takes place*

$$\begin{pmatrix}C_{\mathcal{T}}^{\lambda_1} & \dots & C_{\mathcal{T}}^{\lambda_N}\end{pmatrix}^*\widetilde{\delta}_j \neq 0 \quad \textit{for all} \quad j = 1,\dots,n,$$

where $\widetilde{\delta}_j = \widetilde{\Pi}_{\mathbb{N}\to G_n}\delta_j$.

Proof. If there is $j_0 \in G_n$ such that

$$\begin{pmatrix}C_{\mathcal{T}}^{\lambda_1} & \dots & C_{\mathcal{T}}^{\lambda_N}\end{pmatrix}^*\widetilde{\delta}_{j_0} = 0,$$

then the matrix $C_{\mathcal{T}}^{\lambda_k}(C_{\mathcal{T}}^{\lambda_k})^*$ would have a zero row (the j_0-th one) for all $k \in G_N$, which contradicts Lemma 3.5 in view of the properties of $\mathcal{T}$. $\square$

Definition 3.7. Define the matrix-valued function

$$\sigma_N^{\mathcal{T}}(t) := \sum_{\lambda_k<t}C_{\mathcal{T}}^{\lambda_k}(C_{\mathcal{T}}^{\lambda_k})^*, \quad t\in\mathbb{R},$$

and the space of n-dimensional vector valued functions $L_2(\mathbb{R},\sigma_N^{\mathcal{T}})$ with inner product

$$\langle \boldsymbol{f},\boldsymbol{g}\rangle_{L_2(\mathbb{R},\sigma_N^{\mathcal{T}})} := \int_{\mathbb{R}}\left\langle \boldsymbol{f}(t), d\sigma_N^{\mathcal{T}}(t)\boldsymbol{g}(t)\right\rangle_{l_2(G_n)}.$$

Note that $\sigma_N^{\mathcal{T}}$ has N points of growth so $L_2(\mathbb{R},\sigma_N^{\mathcal{T}})$ is an N-dimensional space and the inner product can be expressed as follows

$$\langle \boldsymbol{f},\boldsymbol{g}\rangle_{L_2(\mathbb{R},\sigma_N^{\mathcal{T}})} = \sum_{k\in G_N}\boldsymbol{f}(\lambda_k)^*C_{\mathcal{T}}^{\lambda_k}(C_{\mathcal{T}}^{\lambda_k})^*\boldsymbol{g}(\lambda_k). \tag{3.14}$$

Definition 3.8. Define the n-dimensional vector polynomials:

$$\boldsymbol{p}_k(z) := \Psi_{\mathfrak{J}}(z)^* \delta_k, \quad k \in G_N, \tag{3.15}$$
$$\boldsymbol{q}_j(z) := \Theta_{\mathfrak{J}}(z)^* \delta_j, \quad j \in G_n, \tag{3.16}$$

where $\Psi_{\mathfrak{J}}(z)$ and $\Theta_{\mathfrak{J}}(z)$ are given in (3.8) and (3.9), respectively.

Remark 3.9. It follows from Definition 3.8 that

$$\widetilde{\Pi}_{G_N \to K}(M_N - zI) \begin{pmatrix} \boldsymbol{p}_1 & \dots & \boldsymbol{p}_N \end{pmatrix}^* = \begin{pmatrix} \boldsymbol{q}_1 & \dots & \boldsymbol{q}_n \end{pmatrix}^* \tag{3.17}$$

and

$$\widetilde{\Pi}_{G_N \to K^\perp}(M_N - zI) \begin{pmatrix} \boldsymbol{p}_1 & \dots & \boldsymbol{p}_N \end{pmatrix}^* = 0. \tag{3.18}$$

Note that, with the help of the characteristic function for a subset G of $\mathbb{N}$ given by

$$\chi_G(k) := \begin{cases} 1 & \text{if } k \in G, \\ 0 & \text{if } k \in G^{\complement}, \end{cases}$$

the expressions (3.17) and (3.18) can be written as follows

$$z\boldsymbol{p}_k(z) = \sum_{i \in G_N} m_{ki}\, \boldsymbol{p}_i(z) + \chi_K(k)\, \boldsymbol{q}_{\gamma(k)}, \tag{3.19}$$

where m_{ki} are the entries of M_N and $\gamma(k)$ is such that k is the $\gamma(k)$-th element of K.

For Example 3.1, the expression (3.17) gives the following equalities:

$$\begin{aligned}
\boldsymbol{q}_1(z) &= (m_{33} - z)\boldsymbol{p}_3 + m_{31}\boldsymbol{p}_1 + m_{32}\boldsymbol{p}_2 + m_{34}\boldsymbol{p}_4 + m_{35}\boldsymbol{p}_5 + m_{36}\boldsymbol{p}_6, \\
\boldsymbol{q}_2(z) &= (m_{55} - z)\boldsymbol{p}_5 + m_{52}\boldsymbol{p}_2 + m_{53}\boldsymbol{p}_3 + m_{54}\boldsymbol{p}_4 + m_{56}\boldsymbol{p}_6, \\
\boldsymbol{q}_3(z) &= (m_{77} - z)\boldsymbol{p}_7 + m_{76}\boldsymbol{p}_6
\end{aligned}$$

and (3.18) can be solved recursively as follows:

$$\begin{aligned}
&\boldsymbol{p}_1(z) = t_{11}\widetilde{\delta}_1, \\
&\boldsymbol{p}_2(z) = t_{12}\widetilde{\delta}_1 + t_{22}\widetilde{\delta}_2, \\
&\boldsymbol{p}_3(z) = t_{13}\widetilde{\delta}_1 + t_{23}\widetilde{\delta}_2 + t_{33}\widetilde{\delta}_3, \\
&\boldsymbol{p}_4(z) = \boldsymbol{m}_{14}^{-1}\left[(z - m_{11})\boldsymbol{p}_1(z) - m_{12}\boldsymbol{p}_2(z) - m_{13}\boldsymbol{p}_3(z)\right], \\
&\boldsymbol{p}_5(z) = \boldsymbol{m}_{45}^{-1}\left[(z - m_{44})\boldsymbol{p}_4(z) - m_{41}\boldsymbol{p}_1(\zeta) - m_{42}\boldsymbol{p}_2(z) - m_{43}\boldsymbol{p}_3(z)\right], \\
&\boldsymbol{p}_6(z) = \boldsymbol{m}_{26}^{-1}\left[(z - m_{22})\boldsymbol{p}_2(z) - m_{21}\boldsymbol{p}_1(\zeta) - m_{23}\boldsymbol{p}_3(z) - m_{24}\boldsymbol{p}_4(z) - m_{25}\boldsymbol{p}_5(z)\right], \\
&\boldsymbol{p}_7(z) = \boldsymbol{m}_{67}^{-1}\left[(z - m_{66})\boldsymbol{p}_6(z) - m_{62}\boldsymbol{p}_2(\zeta) - m_{63}\boldsymbol{p}_3(z) - m_{65}\boldsymbol{p}_5(z)\right],
\end{aligned}$$

where $\widetilde{\delta}_j$ is given in Corollary 3.6.

Proposition 3.10. *The vector polynomials* $\{\boldsymbol{p}_k(z)\}_{k=1}^N$*, defined by* (3.15)*, satisfy*

$$\left\langle \boldsymbol{p}_j, \boldsymbol{p}_k \right\rangle_{L_2(\mathbb{R},\sigma_N^{\mathfrak{T}})} = \delta_{jk} \quad \textit{for} \quad j,k \in G_N,$$

where δ_{jk} *is the Kronecker symbol.*

Proof. Using (3.14) and (3.15), one obtains

$$\begin{aligned}\left\langle \boldsymbol{p}_i, \boldsymbol{p}_j \right\rangle_{L_2(\mathbb{R},\sigma_N^{\mathfrak{T}})} &= \sum_{k\in G_N} (\boldsymbol{p}_i(\lambda_k))^* C_{\mathfrak{T}}^{\lambda_k} (C_{\mathfrak{T}}^{\lambda_k})^* \boldsymbol{p}_j(\lambda_k) \\ &= \sum_{k\in G_N} \delta_i^* \Psi_{\mathfrak{T}}(\lambda_k) C_{\mathfrak{T}}^{\lambda_k} (C_{\mathfrak{T}}^{\lambda_k})^* \Psi_{\mathfrak{T}}(\lambda_k) \\ &= \sum_{k\in G_N} \delta_i^* \varphi^{\lambda_k} (\varphi^{\lambda_k})^* \delta_j = \delta_i^* \delta_j = \delta_{ij},\end{aligned}$$

where in the third equality, one recurs to (3.10). □

Corollary 3.11. *The function* $\Psi_{\mathfrak{T}}(t)$ (*see* (3.7)) *with* $t \in \mathbb{R}$ *gives rise to a map* $\Upsilon_{\mathfrak{T}}$ *by means of the expression*

$$l_2(G_N) \ni u \overset{\Upsilon_{\mathfrak{T}}}{\longmapsto} \Psi^*(t)u \in L_2(\mathbb{R}, \sigma_N^{\mathfrak{T}}).$$

This map is an isometry.

Proof. Indeed, by (3.15), $\Upsilon_{\mathfrak{T}}$ maps the canonical basis into the orthonormal basis $\{\boldsymbol{p}_k\}_{k=1}^N$ in $L_2(\mathbb{R}, \sigma_N^{\mathfrak{T}})$. □

Proposition 3.12. *The isometry* $\Upsilon_{\mathfrak{T}}$ *transforms the operator* $\mathcal{M}_N$ *into the operator of multiplication by the independent variable in* $L_2(\mathbb{R}, \sigma_N^{\mathfrak{T}})$.

Proof. Taking into account that $\mathcal{M}_N = \sum_{k\in G_N} \lambda_k \varphi^{\lambda_k} (\varphi^{\lambda_k})^*$ together with (3.10) and (3.14), one obtains

$$\begin{aligned}\langle \delta_i, \mathcal{M}_N \delta_j \rangle_{l_2(G_N)} &= \delta_i^* \sum_{k\in G_N} \lambda_k \varphi^{\lambda_k} (\varphi^{\lambda_k})^* \delta_j \\ &= \sum_{k\in G_N} \lambda_k \delta_i^* \Psi C_{\mathfrak{T}}^{\lambda_k} (C_{\mathfrak{T}}^{\lambda_k})^* \Psi^* \delta_j \\ &= \sum_{k\in G_N} \lambda_k (\boldsymbol{p}_i(\lambda_k))^* C_{\mathfrak{T}}^{\lambda_k} (C_{\mathfrak{T}}^{\lambda_k})^* \boldsymbol{p}_j(\lambda_k) \\ &= \left\langle \boldsymbol{p}_i(t), t\boldsymbol{p}_j(t) \right\rangle_{L_2(\mathbb{R},\sigma_N^{\mathfrak{T}})}.\end{aligned}$$

□

Remark 3.13. In view of the spectral theorem, Proposition 3.12 is somehow straightforward. However, the concrete realization for the boundary matrix $\mathfrak{T}$ together with the construction algorithm are relevant for the further discussion and elaborations.

Proposition 3.14. *The vector polynomials* $\{\boldsymbol{q}_j\}_{j=1}^n$*, defined in* (3.16)*, have zero norm in* $L_2(\mathbb{R}, \sigma_N^{\mathfrak{T}})$.

Proof. By (3.14) and (3.16), one has

$$\begin{aligned}\left\langle \boldsymbol{q}_j, \boldsymbol{q}_j \right\rangle_{L_2(\mathbb{R},\sigma_N^{\mathfrak{T}})} &= \sum_{k\in G_N} \boldsymbol{q}_j(\lambda_k)^* C_{\mathfrak{T}}^{\lambda_k} (C_{\mathfrak{T}}^{\lambda_k})^* \boldsymbol{q}_j(\lambda_k) \\ &= \sum_{k\in G_N} \delta_j^* \Theta_{\mathfrak{T}}(\lambda_k) C_{\mathfrak{T}}^{\lambda_k} (C_{\mathfrak{T}}^{\lambda_k})^* \Theta_{\mathfrak{T}}(\lambda_k)^* \delta_j.\end{aligned}$$

The assertion then follows after noticing that

$$\Theta_{\mathfrak{T}}(\lambda_k) C_{\mathfrak{T}}^{\lambda_k} = \widetilde{\Pi}_{G_n\to K}(M_N - \lambda_k I)\Psi_{\mathfrak{T}}(\lambda_k) C_{\mathfrak{T}}^{\lambda_k} = 0$$

due to the fact that $\Psi_{\mathfrak{T}}(\lambda_k) C_{\mathfrak{T}}^{\lambda_k} = \varphi^{\lambda_k}$. □

Theorem 3.15. *The function $\sigma_N^{\mathfrak{T}}(t)$ given in Definition 3.7 is a spectral function of $\mathcal{M}_N$, that is, a complete measure*[1] *with the following properties:*

(I) *It is a nondecreasing monotone step function which is continuous from the left.*
(II) *Each jump is a matrix of rank not greater than n.*
(III) *The sum of the ranks of all jumps equals N.*

Proof. The fact that $\sigma_N^{\mathfrak{T}}$ is a spectral function for $\mathcal{M}_N$ is a consequence of Corollary 3.11 and Proposition 3.12. Note that completeness of the spectral measure is also given by Proposition 3.12. The property (I) is an immediate outcome of Definition 3.7 since the quadratic form $\boldsymbol{f}^* C_{\mathfrak{T}}^{\lambda_k} (C_{\mathfrak{T}}^{\lambda_k})^* \boldsymbol{f}$ is the norm squared of the vector $(C_{\mathfrak{T}}^{\lambda_k})^* \boldsymbol{f}$. To prove (II) and (III), one uses Corollary 3.4 and the fact that $C_{\mathfrak{T}}^{\lambda_k} (C_{\mathfrak{T}}^{\lambda_k})^*$ is a rank one matrix since $C_{\mathfrak{T}}^{\lambda_k}$ is a nonzero vector. □

Remark 3.16. For any complete function σ satisfying (I)–(III) of Definition 3.7 and having jumps in $\mu_1, \dots, \mu_N$, there are n-dimensional vectors $C^1, \dots, C^N$ such that

$$\sigma(t) = \sum_{\mu_k < t} C^k (C^k)^*, \tag{3.20}$$

where there are no zero elements in the collection $C^1, \dots, C^N$, and it satisfies Corollary 3.6 with $C_{\mathfrak{T}}^{\lambda_k} = C^k$ for any $k = 1, \dots, N$ (see [6, Thm. 2.2] and [8, Sec. 2]). As has been shown in this section, any such collection of vectors $C^1, \dots, C^N$ and numbers $\mu_1, \dots, \mu_N$ determine uniquely by (3.20) a complete function which satisfies (I)–(III).

4. BAND DIAGONAL MATRICES WITH DEGENERATIONS

Definition 4.1. The matrix $M \in \mathfrak{M}$ (see Definition 2.2) is in the class $\widetilde{\mathfrak{M}}$ when the following conditions are met.

(a) If $(k(j), n + j)$ are the coordinates of the row-edge entries for $j \in \mathbb{N}$, then $k(j) < k(j + 1)$ for $j \in \mathbb{N}$.
(b) All the row-edge entries are simultaneously column-edge entries.

[1] Completeness of a spectral measure is given in [3, Chap. 5, Sec. 1.1]. Here it means that $\mathfrak{T}^* \sum_{k\in G_N} C_{\mathfrak{T}}^{\lambda_k} (C_{\mathfrak{T}}^{\lambda_k})^* \mathfrak{T} = I$, see [8, Lem. 2.3].

Remark 4.2. Note that item (a) in Definition 4.1 allows for the existence of rows without row-edge entries. We say that there is a *degeneration* in each such row (see Figure 3). It is straightforward to verify that the class $\widetilde{\mathfrak{M}}$ is equivalent to the class given in [9, Def. 1].

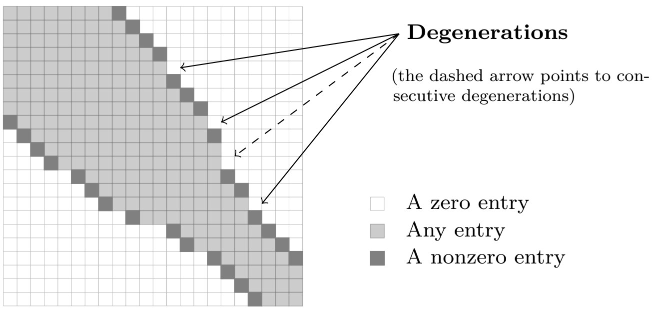

Fig. 3. The structure of a matrix in $\widetilde{\mathfrak{M}}$

In [8, Thm. 4.1] the next result is proven. It plays an important role in our further considerations.

Theorem 4.3. *If σ is a $n \times n$-matrix-valued function defined on the real line and satisfying* (I) *of Theorem 3.15, then there is an upper-triangular invertible $n \times n$-matrix $\mathfrak{T}$ and a matrix $\widetilde{M} \in \widetilde{\mathfrak{M}}$ such that $\sigma = \widetilde{\sigma}_N^{\mathfrak{T}}$. Here $\widetilde{\sigma}_N^{\mathfrak{T}}$ is the function given by Definition 3.7 for the matrix $\widetilde{M}_N$.*

Remark 4.4. The results of [8, Sec. 4] give a constructive algorithm for finding the entries of $\widetilde{M}_N$ on the basis of the function σ.

Theorem 4.5. *For any $M \in \mathfrak{M}$ there is a $\widetilde{M} \in \widetilde{\mathfrak{M}}$ such that $n_M = n_{\widetilde{M}}$ (see Definition 2.2) and M_N is unitary equivalent to $\widetilde{M}_N$ for any $N > n = n_M = n_{\widetilde{M}}$. Here we actually mean that $\mathcal{M}_N$ is unitary equivalent to $\widetilde{\mathcal{M}}_N$ (see the last paragraph of Section 2).*

Proof. For a matrix $M \in \mathfrak{M}$, consider the $n_M \times n_M$-matrix-valued function $\sigma_N^{\mathfrak{T}}$ given in Definition 3.7 with $N > n_M$ and an arbitrary boundary matrix $\mathfrak{T}$. On the basis of Theorems 3.15 and 4.3 there is an upper-triangular invertible matrix $\widetilde{\mathfrak{T}}$ and a matrix $\widetilde{M} \in \widetilde{\mathfrak{M}}$ such that $\sigma_N^{\mathfrak{T}} = \widetilde{\sigma}_N^{\widetilde{\mathfrak{T}}}$, where $\widetilde{\sigma}_N^{\widetilde{\mathfrak{T}}}$ is the spectral function of $\widetilde{M}_N$. Clearly, $n_M = n_{\widetilde{M}}$. For completing the proof, use Proposition 3.12. □

The interpolation problem for n-dimensional vector polynomials as stated in [7] consists in finding n-dimensional vector polynomials $\boldsymbol{r}(z)$ such that

$$\boldsymbol{r}(\mu_k)^* C^k (C^k)^* \boldsymbol{r}(\mu_k) = 0 \tag{4.1}$$

for $k = 1, \ldots, N$. Here $C^1, \ldots, C^N$ is a collection of nonzero n-dimensional vectors and $\mu_1, \ldots, \mu_N$ is a collection of real numbers in which each number can appear repeatedly at most n times. This problem is a generalization of the rational interpolation problem (also known as Cauchy–Jacobi problem) studied in [12] and used by [5] for the spectral analysis of CMV matrices.

In view of the second part of Remark 3.16 and Theorem 4.3, one can always consider the collection of numbers $\mu_1, \ldots, \mu_N$ appearing in Remark 3.16 to be $\lambda_1, \ldots, \lambda_N$, i.e. the spectrum of $\widetilde{M}_N$ ($\widetilde{M} \in \widetilde{\mathfrak{M}}$), and the collection of vectors $C^1, \ldots, C^N$ to be the corresponding eigenvectors $C^{\lambda_1}_{\mathfrak{T}}, \ldots, C^{\lambda_N}_{\mathfrak{T}}$ for a boundary matrix $\mathfrak{T}$. Due to the fact that $C^{\lambda_k}_{\mathfrak{T}}(C^{\lambda_k}_{\mathfrak{T}})^*$ is nonnegative, the interpolation problem is equivalent to finding vector polynomials in the equivalence class of the zero function in $L_2(\mathbb{R}, \sigma^{\mathfrak{T}}_N)$.

Definition 4.6. Let $\boldsymbol{r}(z) = (r_1(z), r_2(z), \ldots, r_n(z))^\intercal$ be an n-dimensional vector polynomial. The height of $\boldsymbol{r}(z)$ is

$$h(\boldsymbol{r}) := \max_{j \in \{1,\ldots,n\}} \{n \deg(r_j) + j - 1\},$$

where it is assumed that $\deg 0 := -\infty$ and $h(\mathbf{0}) := -\infty$. The height of the set S is defined by

$$h(S) := \min\{h(\boldsymbol{r}) : \boldsymbol{r} \in S, \boldsymbol{r} \neq 0\}.$$

Definition 4.7. Denote by $\mathbb{S}_N$ the solutions to the interpolation problem, that is, the set of all n-dimensional vector polynomials satisfying (4.1) for $k = 1, \ldots, N$. Consider $\boldsymbol{g}_1 \in \mathbb{S}_N$ such that $h(\boldsymbol{g}_1) = h(\mathbb{S}_N)$ and $\boldsymbol{g}_k \in \mathbb{S}_N \setminus \operatorname{span}\{M(\boldsymbol{g}_j)\}_{j=1}^{k-1}$ such that

$$h(\boldsymbol{g}_k) = h(\mathbb{S}_N \setminus \operatorname{span}\{\mathbb{M}(\boldsymbol{g}_j)\}_{j=1}^{k-1}),$$

where $\mathbb{M}(\boldsymbol{r}) := \{\boldsymbol{s} : \boldsymbol{s}(z) = s(z)\boldsymbol{r}(z),\ s \text{ is an arbitrary scalar polynomial}\}$. The vector polynomial $\boldsymbol{g}_k$ is the k-th generator of $\mathbb{S}_N$.

If there are two elements of $\mathbb{S}_N$ with the same height, then they differ from each other only by a multiplicative constant [7, Lem. 4.1]. On the basis of this, it is shown in [7, Sec. 4] (see also [8, Prop. 3.3]) that the generators are unique modulo a multiplicative constant and that

$$\mathbb{S}_N = \mathbb{M}(\boldsymbol{g_1}) \dotplus \ldots \dotplus \mathbb{M}(\boldsymbol{g_n}),$$

where $\dotplus$ denotes *direct sum*. Note that the generators $\boldsymbol{g}_1, \ldots, \boldsymbol{g}_n$ have different heights and the height of any element in $\mathbb{M}(\boldsymbol{g}_i)$ is $h(\boldsymbol{g}_i) + kn$ with $k \in \mathbb{N} \cup \{0\}$.

According to Proposition 3.14, $\boldsymbol{q}_j \in \mathbb{S}_N$ for all $j = 1, \ldots, n$. Moreover, the following assertion is proven in [8, Thm. 3.1].

Proposition 4.8. *Fix $N \in \mathbb{N}$. If $M \in \widetilde{\mathfrak{M}}$, then $\boldsymbol{q}_j$ is the j-th generator of $\mathbb{S}_N$ for $j = 1, \ldots, n$.*

Remark 4.9. Using the notation of Definition 4.7, a way of restating the previous proposition is to say that, for $M \in \widetilde{\mathfrak{M}}$, $\boldsymbol{q}_j = \boldsymbol{g}_j$ (modulo a multiplicative constant) for $j = 1, \ldots, n$. This implies that, when $M \in \widetilde{\mathfrak{M}}$, the heights of $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$ are in different equivalent classes of $\mathbb{Z}/n\mathbb{Z}$.

Remark 4.10. It is not true in general that, for $M \in \mathfrak{M}$, $\boldsymbol{q}_j$ is the j-th generator of $\mathbb{S}_N$. This is a remarkable difference between the classes $\mathfrak{M}$ and $\widetilde{\mathfrak{M}}$, actually $\widetilde{\mathfrak{M}}$ was somehow tailored to have the property given by Proposition 4.8.

Proposition 4.11. *Let $M \in \widetilde{\mathfrak{M}}$ and $\boldsymbol{p}_k$ and $\boldsymbol{q}_j$ be the corresponding vector polynomials given by* (3.15) *and* (3.16)*, respectively. The heights of the polynomials $\boldsymbol{p}_1, \dots, \boldsymbol{p}_N$ are increasing and cannot coincide with the heights of any polynomial in $\mathbb{S}_N$ (see* [8*, Lem. 3.2*]*).*

Remark 4.12. For a matrix $M \in \mathfrak{M}$, Proposition 4.11 does not hold. There are cases (see Example 4.14 below) where the heights of the polynomials $\boldsymbol{p}_1, \dots, \boldsymbol{p}_N$ are not increasing or coincide with the heights of elements in $\mathbb{S}_N$.

There is a relevant property for the heights of the vector polynomials considered in the hypothesis of Proposition 4.11. Indeed, it turns out that the set of numbers $h(\boldsymbol{p}_1), \dots, h(\boldsymbol{p}_N), h(\boldsymbol{q}_1) + nk_1, \dots, h(\boldsymbol{q}_n) + nk_n$ with $k_1, \dots, k_n \in \mathbb{N} \cup \{0\}$ cover all nonnegative integers. As a consequence of this and the fact that any nonzero vector polynomial of height h is decomposed in a linear combination of vector polynomials of heights $0, \dots, h$ [8, Prop. 3.1], one has the following proposition (see [8, Cor. 3.2]).

Proposition 4.13. *Fix $N \in \mathbb{N}$. Let $M \in \widetilde{\mathfrak{M}}$ and $\boldsymbol{p}_k$ and $\boldsymbol{q}_j$ be the corresponding vector polynomials given by* (3.15) *and* (3.16)*, respectively. Any vector polynomial $\boldsymbol{r}$ admits the decomposition*

$$\boldsymbol{r}(z) = \sum_{k \in G_N} a_k \boldsymbol{p}_k(z) + \sum_{j \in G_n} R_j(z) \boldsymbol{q}_j(z),$$

where $a_k \in \mathbb{C}$ and $R_j(z)$ is a scalar polynomial for any $k = 1, \dots, N$ and $j = 1, \dots, n$.

Example 4.14. Consider Example 3.1 again.

1. If $m_{25} = 0$ and $m_{35} \neq 0$, then $h(\boldsymbol{p}_5) = h(\boldsymbol{q}_1) = 6$ (see Remark 4.12) and $h(\boldsymbol{q}_2) = 9$ which is in the same equivalence class of $\mathbb{Z}/3\mathbb{Z}$ as $h(\boldsymbol{q}_1)$ (see Remark 4.10).
2. If $m_{25} \neq 0$ and $m_{35} = 0$, then $h(\boldsymbol{p}_5) = h(\boldsymbol{p}_6) = h(\boldsymbol{q}_1) = 6$, $h(\boldsymbol{p}_7) = h(\boldsymbol{q}_2) = 9$ $h(\boldsymbol{q}_3) = 12$. See Remarks 4.10 and 4.12 and note that in this case $h(\boldsymbol{q}_1)$, $h(\boldsymbol{q}_2)$ and $h(\boldsymbol{q}_3)$ all fall in the same equivalence class of $\mathbb{Z}/3\mathbb{Z}$.
3. If $m_{25} = 0$ and $m_{35} = 0$, then the set of numbers $h(\boldsymbol{p}_1), \dots, h(\boldsymbol{p}_7)$, $h(\boldsymbol{q}_1) + 3k_1$, $\dots, h(\boldsymbol{q}_3) + 3k_3$ with $k_1, \dots, k_3 \in \mathbb{N} \cup \{0\}$ cover all nonnegative integers. Here $h(\boldsymbol{q}_1)$, $h(\boldsymbol{q}_2)$ and $h(\boldsymbol{q}_3)$ fall in the three different equivalence classes of $\mathbb{Z}/3\mathbb{Z}$, but the sequence $h(\boldsymbol{p}_1), \dots, h(\boldsymbol{p}_7)$ is not increasing.

5. SPECTRAL FUNCTIONS FOR THE CLASS $\mathfrak{M}$

The vector polynomials given in Definition 3.8 for a matrix $M \in \mathfrak{M}$ can be used in a decomposition similar to the one given in Proposition 4.13 which was restricted to the class $\widetilde{\mathfrak{M}}$. To prove this decomposition for the general case, the following simple observation is needed.

Lemma 5.1. *Let $M \in \mathfrak{M}$ and $N > n$. Consider that n-dimensional vector polynomials $\boldsymbol{p}_k$ given by* (3.15)*. For any given $h \in \mathbb{N} \cup \{0\}$, there are $l \in \mathbb{N} \cup \{0\}$ and $m \in G_n$ (see last paragraph of Section 2) such that*

$$h(z^l \boldsymbol{p}_m(z)) = h.$$

Proof. Write $h = nl + k - 1$, where $l \in \mathbb{N} \cup \{0\}$ and $k \in G_n$. By (3.15), $h(\boldsymbol{p}_k(z)) = k - 1$ for $k \in G_n$. On the other hand, it follows from [7, Eq. 3] that

$$h(z^l \boldsymbol{p}_k(z)) = nl + k - 1 \quad \text{for any} \quad k \in G_n. \qquad \square$$

The following assertion plays an important role in this section since it allows us to expand any vector polynomial in terms of the vector polynomials given by Definition 3.8.

Theorem 5.2. *Let $M \in \mathfrak{M}$ and $N > n$. For any $l \in \mathbb{N} \cup \{0\}$ and $m \in G_n$, there are $a_k \in \mathbb{C}$ $(k \in G_N)$ and scalar polynomials R_j $(j \in G_n)$ such that*

$$z^l \boldsymbol{p}_m(z) = \sum_{k \in G_N} a_k \boldsymbol{p}_k(z) + \sum_{j \in G_n} R_j(z) \boldsymbol{q}_j(z),$$

where $\boldsymbol{p}_k$ and $\boldsymbol{q}_j$ are given by (3.15) *and* (3.16)*.*

Proof. The assertion follows trivially for $l = 0$. When $l = 1$, it is a consequence of (3.17) and (3.18). If the statement holds for l, then

$$\begin{aligned} z^{l+1} \boldsymbol{p}_m(z) &= \sum_{k \in G_N} a_k z \boldsymbol{p}_k(z) + \sum_{j \in G_n} z R_j(z) \boldsymbol{q}_j(z) \\ &= \left(\sum_{k \in K^\perp} + \sum_{k \in K} \right) a_k z \boldsymbol{p}_k(z) + \sum_{j \in G_n} z R_j(z) \boldsymbol{q}_j(z). \end{aligned}$$

Observe that, due to (3.19),

$$\sum_{k \in G_N} a_k z \boldsymbol{p}_k(z) = \sum_{k \in G_N} a_k \left(\sum_{i \in G_N} m_{ki}\, \boldsymbol{p}_i(z) + \chi_K(k)\, \boldsymbol{q}_{\gamma(k)} \right).$$

Thus,

$$\begin{aligned} &\left(\sum_{k \in K^\perp} + \sum_{k \in K} \right) a_k z \boldsymbol{p}_k(z) + \sum_{j \in G_n} z R_j(z) \boldsymbol{q}_j(z) \\ &= \sum_{k \in G_N} \tilde{a}_k \boldsymbol{p}_k(z) + \sum_{j \in G_n} \left(a_{\gamma^{-1}(j)} + z R_j(z) \right) \boldsymbol{q}_j(z). \qquad \square \end{aligned}$$

For a fixed $M \in \mathfrak{M}$ and $N > n$, consider the corresponding spectral function $\sigma_N^{\mathcal{T}}(t)$. Let $\{\boldsymbol{e}_k\}_{k=1}^n$ be the canonical basis in $\mathbb{C}^n$, i.e.

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \boldsymbol{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{5.1}$$

Define a family of vector polynomials for $k \in \mathbb{N}$ and $i = 1, \dots, n$ as follows

$$\boldsymbol{e}_{nk+i}(z) \coloneqq z^k \boldsymbol{e}_i. \tag{5.2}$$

These vector polynomials give an expression for the entries of the $(k+l)$-th matrix moment, $S_{k+l}(N)$, of the function $\sigma_N^{\mathcal{T}}$. Indeed,

$$\begin{aligned} \langle \boldsymbol{e}_i, S_{k+l}(N)\boldsymbol{e}_j \rangle &= \left\langle \boldsymbol{e}_i, \int_{\mathbb{R}} t^{k+l} d\sigma_N^{\mathcal{T}}(t)\boldsymbol{e}_j \right\rangle \\ &= \langle \boldsymbol{e}_{nk+i}(t), \boldsymbol{e}_{nl+j}(t) \rangle_{L_2(\mathbb{R}, \sigma_N^{\mathcal{T}})}. \end{aligned}$$

Theorem 5.3. *Let $M \in \mathfrak{M}$. For any $m \in \mathbb{N}$ there is $N \in \mathbb{N}$ such that*

$$\int_{\mathbb{R}} t^k \, d\sigma_N^{\mathcal{T}}(t) = \int_{\mathbb{R}} t^k \, d\sigma_{N'}^{\mathcal{T}}(t)$$

for any $N' \geq N$ and $k = 0, 1, \dots, 2m-1, 2m$, where $\sigma_N^{\mathcal{T}}$ and $\sigma_{N'}^{\mathcal{T}}$ are the spectral functions of M_N and $M_{N'}$, respectively.

Proof. Independently of the value of $N > n$, for $i, j = 1, \dots, n$, one has

$$\int_{\mathbb{R}} t^{k+l} \, d\sigma_N^{\mathcal{T}}(t) = \left\langle t^k \boldsymbol{e}_i, t^l \boldsymbol{e}_j \right\rangle_{L_2(\mathbb{R}, \sigma_N^{\mathcal{T}})} = \left\langle \mathcal{T}^{-1} t^k \boldsymbol{p}_i, \mathcal{T}^{-1} t^l \boldsymbol{p}_j \right\rangle_{L_2(\mathbb{R}, \sigma_N^{\mathcal{T}})} \tag{5.3}$$

since $\mathcal{T}\boldsymbol{e}_i = \boldsymbol{p}_i$ for $i \in G_n$ (which in turn follows from (3.7) and (3.15)).

Thus, the goal is to find a decomposition involving the vector polynomials given in Definition 3.8 for $z^l \boldsymbol{p}_i$ with $l \in \mathbb{N}$ and $i \in G_n$. For the whole infinite matrix M and $i \in G_{j_0-1}$, one obtains from (3.17) and (3.18) that

$$z\boldsymbol{p}_i(z) = \sum_{k=1}^{k_0-1} a_k^i \boldsymbol{p}_k(z) + \sum_{j=1}^{n-k_0+j_0} R_j^i(z)\boldsymbol{q}_j(z), \tag{5.4}$$

where $a_k^i \in \mathbb{C}$ ($k \in G_n$, $i \in G_{j_0-1}$) and R_j^i is a scalar polynomial ($j \in G_{n-k_0+j_0}$, $i \in G_{j_0-1}$). Also, directly from Definition 2.2, one verifies that, for any $m \in \mathbb{N} \cup \{0\}$,

$$z\boldsymbol{p}_{j_0+m}(z) = \sum_{j=1}^{k_0-1+m} m_{j_0+m,j}\boldsymbol{p}_j(z) + \boldsymbol{m}_{j_0+m,k_0+m}\boldsymbol{p}_{k_0+m}, \tag{5.5}$$

where we have used the notation of Definition 2.2 for the entries of M and denote the edge entry in bold typeface.

In the case $j_0 < n$, one recurs to (5.4) and (5.5) to find $z\boldsymbol{p}_i$ for $i \in G_{j_0-1}$ and $i \in G_n \setminus G_{j_0-1}$, respectively (in (5.5) put $m = 0, \dots, n - j_0$). If $j_0 \geq n$, then one only uses (5.4) to determine $z\boldsymbol{p}_i$ for $i \in G_n$.

As it was done in the proof of Theorem 5.2 an induction argument, which involves (5.4) and (5.5), yields

$$z^l \boldsymbol{p}_i(z) = \sum_{k=1}^{i+l(k_0-j_0)} a_k^i(l)\boldsymbol{p}_k(z) + \sum_{j=1}^{n-k_0+j_0} R_j^i(l,z)\boldsymbol{q}_j(z), \quad i \in G_n, \tag{5.6}$$

for any $l \in \mathbb{N}$, where we have indicated the dependence on l of the coefficients. If one considers the finite submatrix M_N, then (5.6) remains invariant as long as $N \geq n + l(k_0 - j_0)$. This invariance is a consequence of the fact that all equations in (5.5) correspond to rows of the matrix M on the tail (see (2) of Definition 2.2).

Now, it follows from (5.3), (5.4), (5.6), and Propositions 3.10 and 3.14 that the entries of the moment matrix $S_s(N)$, with $s \in G_{2l}$ and $N \geq n + l(k_0 - j_0)$, are given by $\mathfrak{T}$ and the coefficients $a_k^i(l)$, where $k \in G_{n+l(k_0-j_0)}$ and $i \in G_n$. $\square$

Note that Theorem 5.3 is more complex than its counterpart in the more simple setting treated in [9]. This is a consequence of the different behavior of the vector polynomials $\boldsymbol{p}_k$ and $\boldsymbol{q}_j$ with respect to heights in the classes $\mathfrak{M}$ and $\widetilde{\mathfrak{M}}$ as stated in Remark 4.12.

Having Theorem 5.3 at our disposal, the following assertion is a direct consequence of [9, Lems. 3.1, 3.2 and Prop. 3.1]. The proof is omitted here since the reasoning repeats the one used in [9], which in turn relies on generalizations of Helly's theorems (see [4, Thms. 4.3 and 4.4]) and the argumentation found in [1, Sec. 2.1].

Theorem 5.4. *Let $M \in \mathfrak{M}$ and $l \in \mathbb{N}$ and consider the spectral functions $\sigma_N^{\mathfrak{T}}$ of M_N with $N \geq n + l(k_0 - j_0)$ for a certain boundary matrix $\mathfrak{T}$ (see Section 3), then there exists a subsequence $\{\sigma_{N_j}^{\mathfrak{T}}\}_{j=1}^{\infty}$ converging pointwise to a matrix-valued function σ such that*

$$\int_{\mathbb{R}} t^k d\sigma_{N_j}^{\mathfrak{T}}(t) = \int_{\mathbb{R}} t^k d\sigma(t)$$

for any nonnegative integer $k \leq 2l$.

As a result of Theorem 5.3, the matrix moments of a spectral function of the submatrix M_N do not depend on $N \gg 1$. More specifically, by defining

$$S_k := \int_{\mathbb{R}} t^k d\sigma_N^{\mathfrak{T}}(t) \tag{5.7}$$

for any $k = 0, 1, \dots$ as long as $N \geq n + l(k_0 - j_0)$ and $k \leq 2l$, Theorem 5.4 yields that σ is a solution to the moment problem given by the sequence $\{S_k\}_{k=0}^{\infty}$ which in turn is determined by $M \in \mathfrak{M}$ and the boundary matrix $\mathfrak{T}$.

In Section 3, we considered the finite sequence of polynomials $\{\boldsymbol{p}_k\}_{k=1}^N$ associated, via Definition 3.8, with the submatrix M_N of a matrix $M \in \mathfrak{M}$. Here we consider the infinite sequence $\{\boldsymbol{p}_k\}_{k=1}^\infty$ given again by Definition 3.8. Also, we now substitute the $n \times n$-matrix-valued function $\sigma_N^{\mathfrak{T}}$ in Definition 3.7 with the $n \times n$-matrix-valued function σ obtained in Theorem 5.4.

Lemma 5.5. *The sequence of vector polynomials $\{\boldsymbol{p}_k\}_{k=1}^\infty$ is an orthonormal sequence in $L_2(\mathbb{R}, \sigma)$, where σ is the function given in Theorem 5.4.*

Proof. It is a consequence of Proposition 3.10 and Theorem 5.4 that, for N sufficiently large (depending on k and j), it is satisfied that

$$\delta_{jk} = \left\langle \boldsymbol{p}_j, \boldsymbol{p}_k \right\rangle_{L_2(\mathbb{R}, \sigma_N^{\mathfrak{T}})} = \left\langle \boldsymbol{p}_j, \boldsymbol{p}_k \right\rangle_{L_2(\mathbb{R}, \sigma)} . \qquad \square$$

Due to the fact that the space $L_2(\mathbb{R}, \sigma)$ is infinite dimensional, one has the following assertion.

Corollary 5.6. *The function σ to which the subsequence $\{\sigma_{N_j}^{\mathfrak{T}}\}_{j=1}^\infty$ converges according to Theorem 5.4 is a solution with an infinite number of growing points to the matrix moment problem given by the sequence $\{S_k\}_{k=0}^\infty$.*

Lemma 5.7. *If σ is the function given in Theorem 5.4, then each vector polynomial $\boldsymbol{q}_j$, $j \in G_{n-k_0+j_0}$, has zero norm.*

Proof. If one fixes $j \in G_{n-k_0+j_0}$ and considers $N \gg 1$, then

$$0 = \left\| \boldsymbol{q}_j \right\|^2_{L_2(\mathbb{R}, \sigma_N^{\mathfrak{T}})} = \int\limits_{\mathbb{R}} \left\langle \boldsymbol{q}_j, d\sigma_N^{\mathfrak{T}} \boldsymbol{q}_j \right\rangle$$

as a result of Proposition 3.14. Now, by Theorem 5.4 there is a subsequence $\{\sigma_{N_i}^{\mathfrak{T}}\}_{i=1}^\infty$ such that

$$0 = \int\limits_{\mathbb{R}} \left\langle \boldsymbol{q}_j, d\sigma_{N_i}^{\mathfrak{T}} \boldsymbol{q}_j \right\rangle = \int\limits_{\mathbb{R}} \left\langle \boldsymbol{q}_j, d\sigma \boldsymbol{q}_j \right\rangle = \left\| \boldsymbol{q}_j \right\|^2_{L_2(\mathbb{R}, \sigma)} . \qquad \square$$

Definition 5.8. A nondecreasing $n \times n$ matrix-valued function σ with an infinite number of growing points and finite moments is said to belong to the class $\mathfrak{S}_n$ when $\int_{\mathbb{R}} d\sigma$ is invertible. A function $\sigma \in \mathfrak{S}_n$ is called a spectral function of a matrix M in $\mathfrak{M}$ (actually we mean the spectral function of $\mathcal{M}$; see Section 2) when there exist a boundary matrix $\mathfrak{T}$ such that, in $L_2(\mathbb{R}, \sigma)$, $\{\boldsymbol{p}_k\}_{k=1}^\infty$ is an orthonormal sequence and $\boldsymbol{q}_j$ is in the equivalence class of zero for $j \in G_{n-k_0+j_0}$.

Remark 5.9. Due to Lemmas 5.5 and 5.7, and Corollary 5.6, any $M \in \mathfrak{M}$ has at least one spectral function. Indeed, the function σ given in Theorem 5.4 is a matrix valued function having an infinite number of growing points and $\int_{\mathbb{R}} d\sigma$ is invertible since

$$\int\limits_{\mathbb{R}} d\sigma = [(\mathfrak{T})^*]^{-1} \mathfrak{T}^{-1},$$

where $\mathfrak{T}$ is the boundary matrix involved in Theorem 5.4.

By Definition 5.8, one constructs an isometry $\mathcal{U}$ between $l_2(\mathbb{N})$ and the closure of the polynomials in $L_2(\mathbb{R},\sigma)$ by associating the orthonormal basis $\{\delta_k\}_{k=1}^{\infty}$ with the orthonormal system $\{\boldsymbol{p}_k\}_{k=1}^{\infty}$, i.e. $\mathcal{U}\delta_k = \boldsymbol{p}_k$ for all $k \in \mathbb{N}$.

Theorem 5.10. *Let σ be the spectral function of $M \in \mathfrak{M}$ given in Definition 5.8 and consider the isometry $\mathcal{U}$ given above. If $\mathcal{A}$ is the operator of multiplication by the independent variable defined on its maximal domain in the closure of the vector polynomials in $L_2(\mathbb{R},\sigma)$, then $\mathcal{U}^{-1}\mathcal{A}\mathcal{U}$ is a selfadjoint extension of $\mathcal{M}$.*

Proof. Note that $\mathcal{A}$, i.e. the operator of multiplication by the independent variable defined on its maximal domain in $L_2(\mathbb{R},\sigma)$, is selfadjoint. Moreover, on the basis of (3.19), one has

$$\begin{aligned}\left\langle \boldsymbol{p}_j, \mathcal{A}\boldsymbol{p}_k \right\rangle_{L_2(\mathbb{R},\sigma)} &= \int_{\mathbb{R}} \left\langle \boldsymbol{p}_j(t), d\sigma\, t\boldsymbol{p}_k(t) \right\rangle \\ &= \int_{\mathbb{R}} \left\langle \boldsymbol{p}_j(t), d\sigma \sum_{i\in G_N} m_{ki}\, \boldsymbol{p}_i(t) + \chi_K(k)\, \boldsymbol{q}_{\gamma(k)}(t) \right\rangle\end{aligned}$$

for a sufficiently large $N \in \mathbb{N}$. Thus,

$$\begin{aligned}\left\langle \boldsymbol{p}_j, \mathcal{A}\boldsymbol{p}_k \right\rangle_{L_2(\mathbb{R},\sigma)} &= \sum_{i\in G_N} m_{ki} \int_{\mathbb{R}} \left\langle \boldsymbol{p}_j(t), d\sigma\, \boldsymbol{p}_i(t) \right\rangle \\ &= \sum_{i\in G_N} m_{ki} \left\langle \delta_j, \delta_i \right\rangle = \sum_{i\in G_N} m_{ki}\delta_{ji} = m_{kj}.\end{aligned}$$

This means that the operator $\mathcal{U}\mathcal{M}\mathcal{U}^{-1}$ is a restriction of the operator of multiplication by the independent variable. □

Remark 5.11. Theorem 5.10 justifies the name given to the function of Definition 5.8. Note that in general the image of $\mathcal{U}$ is a subspace of $L_2(\mathbb{R},\sigma)$. The vector polynomials are densely contained in $L_2(\mathbb{R},\sigma)$ only when the orthonormal system $\{\boldsymbol{p}_k\}_{k=1}^{\infty}$ turns out to be complete, i.e. when $\mathcal{U}l_2(\mathbb{N}) = L_2(\mathbb{R},\sigma)$.

Remark 5.12. Taking into account the notation of Theorem 5.10, if there is only one solution to the moment problem given by the sequence $\{S_k\}_{k=0}^{\infty}$ (see (5.7)), then the operator of multiplication $\mathcal{A}$ is such that $\mathcal{U}^{-1}\mathcal{A}\mathcal{U} = \mathcal{M}$, i.e. $\mathcal{M}$ is selfadjoint. Note that if one changes the boundary matrix $\mathfrak{T}$, then one has another spectral function for the same operator. In this case the moment problem changes, but all the time there is only one solution to the problem (determinate moment problem). If there is more than one solution to the moment problem (indeterminate moment problem), then the selfadjoint extension $\mathcal{U}^{-1}\mathcal{A}\mathcal{U}$ does not necessarily correspond to a canonical selfadjoint extension[2)] of $\mathcal{M}$ [2, Appx. I].

[2)] A canonical selfadjoint extension of a symmetric nonselfadjoint operator A is a restriction of A^*

6. RECONSTRUCTION OF THE MATRIX

The goal of this section is to reconstruct a matrix from an arbitrary function $\sigma \in \mathfrak{S}_n$ (see Definition 5.8). In the previous sections, we constructed a spectral function for a given matrix in $\mathfrak{M}$. This spectral function turned out to belong to $\mathfrak{S}_n$. Now, we deal with the inverse problem whose solution provides a complete characterization of the spectral functions of the operators related to the matrices in $\mathfrak{M}$.

In [9, Sec. 5] the inverse problem of reconstructing a matrix from its spectral function was addressed. The starting point there is a function in $\mathfrak{S}_n$ just as in this section. Thus, we use the results found there and bring up some of their details below for the reader convenience.

The following statement is a rephrasing of [9, Lem. 5.1 and Prop. 5.1].

Proposition 6.1. *Given a function $\sigma \in \mathfrak{S}_n$ there is an infinite sequence of orthonormal polynomials $\{\widetilde{\boldsymbol{p}}_k : k \in \mathbb{N}\}$ in $L_2(\mathbb{R}, \sigma)$ and a finite sequence $\{\widetilde{\boldsymbol{q}}_j : j \in G_{n_0}\}$, $n_0 < n$ such that $\left\|\widetilde{\boldsymbol{q}}_j\right\| = 0$ for any $j \in G_{n_0}$. Moreover, for any n-dimensional vector polynomial $\boldsymbol{r}$, there is $l \in \mathbb{N}$ such that*

$$\boldsymbol{r}(z) = \sum_{k \in G_l} a_k \widetilde{\boldsymbol{p}}_k(z) + \sum_{j \in G_{n_0}} R_j(z) \widetilde{\boldsymbol{q}}_j(z),$$

where each a_k is a complex number ($k \in G_l$) and each R_j is a scalar polynomial ($j \in G_{n_0}$).

The proof of this assertion if based on the Gram-Schmidt procedure taking into account the properties of a function in $\mathfrak{S}_n$. The algorithm is illustrated in the flow chart of Figure 4, where the polynomials $\boldsymbol{e}_k$ ($k \in \mathbb{N}$) are given in (5.1) and (5.2).

Remark 6.2. It follows directly from the algorithm (see Figure 4) that the heights of $\{\widetilde{\boldsymbol{p}}_k : k \in \mathbb{N}\}$ are increasing with k and cannot coincide with the heights of the vector polynomials in $\mathbb{M}(\widetilde{\boldsymbol{q}}_j)$ for any $j \in G_{n_0}$.

Remark 6.3. If σ is nondecreasing $n \times n$-matrix-valued function such that $\int_{\mathbb{R}} d\sigma$ is invertible, but it has N growing points, then $n_0 = n$ because the corresponding interpolation problem has n generators (see Proposition 4.8). In this case, the algorithm (Figure 4) yields $\|\widehat{\boldsymbol{p}}_s\| = 0$ for all $s > N$ (this is why the space $L_2(\mathbb{R}, \sigma)$ is N-dimensional). On the other hand, when $\sigma \in \mathfrak{S}_n$, the algorithm can give us zero-norm polynomials, but it should be that $n_0 < n$ since otherwise $L_2(\mathbb{R}, \sigma)$ would be finite dimensional which is impossible when there is an infinite number of growing points.

Remark 6.4. Clearly, $\{\widetilde{\boldsymbol{p}}_k : k \in \mathbb{N}\}$ is an orthonormal basis of the closure of the polynomials in $L_2(\mathbb{R}, \sigma)$. However, the polynomials might not be dense in $L_2(\mathbb{R}, \sigma)$. The density of polynomials in $L_2(\mathbb{R}, \sigma)$ is an intrinsic property of the measure.

As in the previous section, let us denote by $\mathcal{A}$ the operator of multiplication by the independent variable defined in its maximal domain in the Hilbert space $\mathfrak{H}$ being the closure of the polynomials in $L_2(\mathbb{R}, \sigma)$ ($\sigma \in \mathfrak{S}_n$). On the basis of Proposition 6.1, it is shown in [9, Thm. 5.1] that the numbers $\left\langle \widetilde{\boldsymbol{p}}_j, \mathcal{A}\widetilde{\boldsymbol{p}}_k \right\rangle$ form a matrix in the class $\widetilde{\mathfrak{M}}$.

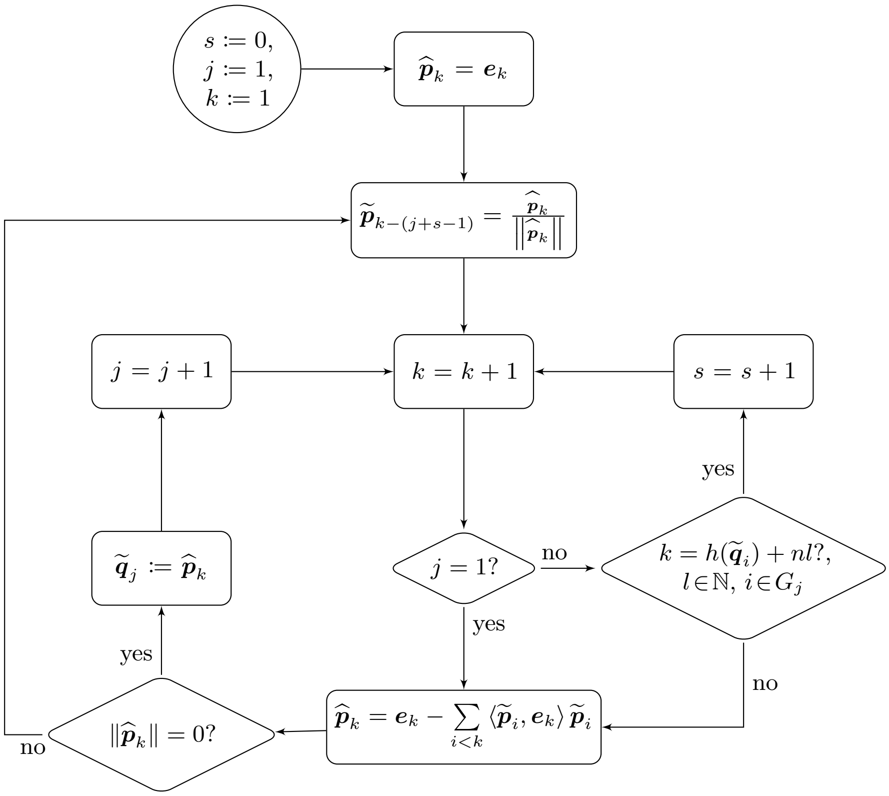

Fig. 4. Orthonormalization algorithm

If this matrix is denoted by $\widetilde{M}$, then [9, Thm. 5.1] actually says that $\mathcal{A}$ is unitary equivalent to a selfadjoint extension of $\widetilde{\mathcal{M}}$ (the operator corresponding to $\widetilde{M}$). Recall that according to [2, Sec. 47], $\widetilde{\mathcal{M}}$ is the minimal closed operator satisfying

$$\left\langle \delta_j, \widetilde{\mathcal{M}} \delta_k \right\rangle = \left\langle \widetilde{\boldsymbol{p}}_j, \mathcal{A} \widetilde{\boldsymbol{p}}_k \right\rangle \quad \text{for all} \quad k, j \in \mathbb{N},$$

where $\{\delta_k\}_{k\in\mathbb{N}}$ is the canonical basis in $l_2(\mathbb{N})$.

Having constructed the matrix $\widetilde{M}$, there is a boundary matrix $\mathfrak{T}$ such that the polynomials $\boldsymbol{p}_k$ given by (3.15) coincide with $\widetilde{\boldsymbol{p}}_k$ for $k = 1, \dots, n$. Then any of the polynomials $\boldsymbol{p}_k$ and $\boldsymbol{q}_j$ given by (3.15) and (3.16) for $k > n$ and $j = 1, \dots, n_0$ differ from $\widetilde{\boldsymbol{p}}_k$ and $\widetilde{\boldsymbol{q}}_j$, respectively, only by polynomials in the equivalence class of zero (see [9, Eqs. 5.7 and 5.8]). This implies, according to Definition 5.8, that σ is a spectral function of M with respect to $\mathfrak{T}$.

Let us summarize the above paragraph in the following statement which turns out to be a rewording of [9, Thms. 5.1 and 5.2].

Proposition 6.5. *If $\sigma \in \mathfrak{S}_n$, then there is a matrix $\widetilde{M} \in \widetilde{\mathfrak{M}}$ and a boundary matrix $\mathfrak{T}$ such that σ is the corresponding spectral function of $\widetilde{M}$ with respect to $\mathfrak{T}$.*

The above result, which solves the inverse problem posed at the beginning of this section, has a remarkable repercussion for the infinite Marchenko–Slavin class $\mathfrak{M}$. Indeed, according to Definition 5.8, the following statement is a consequence of Lemmas 5.5 and 5.7, Corollary 5.6, and Proposition 6.5.

Theorem 6.6. *If $M \in \mathfrak{M}$ and σ is a spectral function of M with respect to a boundary matrix Υ, then there is $\widetilde{M} \in \widetilde{\mathfrak{M}}$ and a boundary matrix $\widetilde{\Upsilon}$ such that σ is also a spectral function of $\widetilde{M}$ with respect to $\widetilde{\Upsilon}$.*

Due to the theory of unitary invariants for spectral functions [3, Sec. 7.4], Theorem 6.6 immediately gives the following result.

Corollary 6.7. *Any operator associated with a matrix in $\mathfrak{M}$ is unitary equivalent to an operator associated with a matrix in $\widetilde{\mathfrak{M}}$.*

Note that Corollary 6.7 not only holds for infinite matrices, but also for any finite submatrix of an arbitrary matrix in $\mathfrak{M}$, namely, for a fix $N \in \mathbb{N}$ and $M \in \mathfrak{M}$, there is a unitary map $\mathcal{V} : l_2(G_N) \to l_2(G_N)$ and a matrix $\widetilde{M} \in \widetilde{\mathfrak{M}}$ such that

$$\mathcal{V} M_N \mathcal{V}^{-1} = \widetilde{M}_N.$$

A consequence of Corollary 6.7 is that it gives a new way of constructing solutions to a linear multidimensional interpolation problem (see Section 4). Also, the fact that a matrix in the Marchenko–Slavin class is unitary equivalent to a matrix in the subclass given in Definition 4.1 implies that systems of interacting particles can always be treated by their equivalent model corresponding to a finite submatrix in $\widetilde{\mathfrak{M}}$. In the infinite dimensional setting, this equivalence is relevant in studying the properties of spaces $L_2(\mathbb{R}, \sigma)$ for an arbitrary $\sigma \in \mathfrak{S}_n$.

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