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AUGMENTING GRAPHS TO PARTITION THEIR VERTICES INTO A TOTAL DOMINATING SET AND AN INDEPENDENT DOMINATING SET

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Abstract. A graph G whose vertex set can be partitioned into a total dominating set and an independent dominating set is called a TI-graph. There exist infinite families of graphs that are not TI-graphs. We define the TI-augmentation number $\mathrm{ti}(G)$ of a graph G to be the minimum number of edges that must be added to G to ensure that the resulting graph is a TI-graph. We show that every tree T of order $n \geq 5$ satisfies $\mathrm{ti}(T) \leq \frac{1}{5}n$. We prove that if G is a bipartite graph of order n with minimum degree $\delta(G) \geq 3$, then $\mathrm{ti}(G) \leq \frac{1}{4}n$, and if G is a cubic graph of order n, then $\mathrm{ti}(G) \leq \frac{1}{3}n$. We conjecture that $\mathrm{ti}(G) \leq \frac{1}{6}n$ for all graphs G of order n with $\delta(G) \geq 3$, and show that there exist connected graphs G of sufficiently large order n with $\delta(G) \geq 3$ such that $\mathrm{ti}(T) \geq (\frac{1}{6} - \varepsilon)n$ for any given $\varepsilon > 0$.

Keywords: total domination, independent domination, vertex partitions.

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1. INTRODUCTION

A *dominating set* in a graph G is a set D of vertices of G such that every vertex in $V(G) \setminus D$ has a neighbor in D, where two vertices are neighbors if they are adjacent. Thus, a set D is a dominating set of G if every vertex in $V(G) \setminus D$ is adjacent to at least one vertex in D. An *independent dominating set*, abbreviated ID-*set*, in G is a dominating set in G that is also an independent set. If G is an isolate-free graph, then a *total dominating set*, abbreviated a TD-set, in G is a set D of vertices of G such that every vertex in $V(G)$ has a neighbor in D. Thus, a set D is a TD-set of G if every vertex is adjacent to at least one vertex in D. A thorough treatment of domination in graphs and its variants can be found in the books [12–14, 22].

A classic 1962 result by Ore [23] shows that the vertex set of any isolate-free graph can be partitioned into two dominating sets. However, this result does not necessarily extend to other types of domination. A natural problem is to consider which graphs can be partitioned into two specific types of dominating sets. Such problems have been studied in [1, 2, 4–6, 8, 15–21, 24, 25] and elsewhere.

In 2019 Delgado, Desormeaux, and Haynes [4] initiated a study of graphs whose vertex set can be partitioned into a TD-set and an ID-set. We refer to such a partition of the vertices of a graph G as a *TI-partition* of G. If G has a TI-partition, then we say that G is a *TI-graph*. We remark that if G is a TI-graph, then every TD-set of G contains at least two vertices from every component of G and every ID-set of G contains at least one vertex from every component of G, implying that every component of G has order at least 3. Subsequent to the introductory paper in [4], TI-graphs have been studied, for example, by the authors in [9–11].

In this paper we consider the question of how many edges must be added to a graph G to ensure that the resulting graph is a TI-graph. We define the TI-*augmentation number* of a graph G to be the minimum number of edges that must be added to G to ensure that the vertex set $V(G)$ can be partitioned into a TD-set and an ID-set. We denote this minimum number by $\mathrm{ti}(G)$. It is clear that $\mathrm{ti}(G)$ can only exist for graphs with at least three vertices.

1.1. GRAPH THEORY NOTATION AND TERMINOLOGY

For graph theory notation and terminology, we generally follow [14]. Specifically, let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and of order $n(G) = |V(G)|$ and size $m(G) = |E(G)|$. Two vertices u and v of G are *adjacent* if $uv \in E(G)$. Two adjacent vertices are called *neighbors*. The *open neighborhood* $N_G(v)$ of a vertex v in G is the set of neighbors of v, while the *closed neighborhood* of v is the set $N_G[v] = \{v\} \cup N_G(v)$. We denote the degree of v in G by $\deg_G(v)$, and so $\deg_G(v) = |N_G(v)|$. The maximum and minimum degree among the vertices of G is denoted by $\Delta(G)$ and $\delta(G)$, respectively. If $X \subseteq V(G)$ and $v \in V(G)$, then we denote the number of neighbors of v in G that belong to the set X by $\deg_X(v)$, and so $\deg_X(v) = |N_G(v) \cap X|$. In particular, if $X = V(G)$, then $\deg_X(v) = \deg_G(v)$.

An *isolated vertex* in G is a vertex of degree 0 in G. An *isolate-free graph* is a graph which contains no isolated vertex. A *leaf* is a vertex of degree 1 and its unique neighbor is called a *support vertex*. A support vertex adjacent to exactly one leaf is called a *weak support vertex* and a support vertex adjacent to two or more leaves is called a *strong support vertex*.

For a set $S \subseteq V(G)$, the subgraph induced by S is denoted by $G[S]$. Further, the subgraph of G obtained from G by deleting all vertices in S and all edges incident with vertices in S is denoted by $G - S$; that is, $G - S = G[V(G) \setminus S]$. For $k \geq 1$ an integer, we let $[k]$ denote the set $\{1, \ldots, k\}$ and we let $[k]_0 = [k] \cup \{0\} = \{0, 1, \ldots, k\}$.

We use P_n, C_n, and K_n to denote a *path*, a *cycle*, and a *complete graph*, respectively, on n vertices. For $1 \leq r \leq s$, we denote a *complete bipartite graph* with partite sets of cardinalities r and s, respectively, by $K_{r,s}$. A *star* is a complete bipartite graph $K_{1,s}$ for some $s \geq 1$. For $1 \leq r \leq s$ a *double star* $S(r, s)$ is a tree with exactly two (adjacent) vertices that are not leaves, with one of the vertices having r leaf neighbors and the other s leaf neighbors. A *cubic graph* is graph in which every vertex has degree 3.

The *corona* $G \circ K_1$ of a graph G is the graph obtained from G by adding for each vertex $v \in V$ a new vertex v' and the edge vv'.

The *vertex independence number* $\alpha(G)$ of G is the cardinality of a maximum independent set of G, and an independent set in G of cardinality $\alpha(G)$ is called an α-set of G.

A *rooted tree* $T = T_r$ distinguishes one vertex r called the *root* and the tree T_r is said to be *rooted at vertex* r. For each vertex $v \neq r$ of T_r, the *parent* of v is the neighbor of v on the unique (r, v)-path, while a *child* of v is any other neighbor of v. The root r does not have a parent in T_r and all its neighbors are its children. A *descendant* of v is a vertex x such that the unique (r, x)-path contains v. Thus, every child of v is a descendant of v. Let $C(v)$ and $D(v)$ denote the set of children and descendants, respectively, of v, and let $D[v] = D(v) \cup \{v\}$. The *maximal subtree* T_v rooted at v is the subgraph induced by $D[v]$.

2. MOTIVATION AND KNOWN RESULTS

The decision problem to determine whether a given graph is a TI-graph is shown in [11] to be NP-complete, when restricted to bipartite graphs. Moreover, the authors in [11] provide a constructive characterization of TI-graphs and show that all such graphs can be constructed starting from two base graphs and applying a sequence of eleven operations. Although their proof yields an algorithm for this decision problem to decide if the vertex set of a given graph can be partitioned into a TD-set and an ID-set, their algorithm is far from polynomial time complexity.

The paths and cycles having a TI-partition were determined in [4].

Proposition 2.1 ([4])**.** *The following hold.*

(a) *A cycle C_n is a TI-graph if and only if* $n \equiv 0 \pmod 3$.
(b) *A nontrivial path P_n is a TI-graph if and only if* $n \equiv 1 \pmod 3$.

A constructive characterization of TI-trees is given in [4], as well as a characterization of the TI-graphs of diameter 2.

Proposition 2.2 ([4])**.** *A graph G of diameter 2 is a TI-graph if and only if G has a maximal independent set that is not the open neighborhood of some vertex.*

Constructions that yield infinite families of graphs that are TI-graphs, as well as constructions that yield infinite families of graphs that are not TI-graphs, are given in [9]. Moreover, they study regular graphs that are TI-graphs and prove, for example, that all toroidal graphs are TI-graphs. In [10], the graphs G such that at least one of G and its complement $\overline{G}$ is a TI-graph are characterized.

3. CYCLES, PATHS AND STARS

In this section we determine the TI-augmentation number $\mathrm{ti}(G)$ when G is a cycle, a path, or a star.

Proposition 3.1. *For $n \geq 3$,* $\mathrm{ti}(C_n) = 0$ *if* $n \equiv 0 \pmod 3$ *and* $\mathrm{ti}(C_n) = 1$ *otherwise.*

Proof. Let $G = C_n$ be the cycle $v_1v_2 \dots v_nv_1$ for $n \geq 3$. It follows from Proposition 2.1(a) that $\mathrm{ti}(C_n) = 0$ if and only if $n \equiv 0 \pmod 3$. Thus, if $n \not\equiv 0 \pmod 3$, then $\mathrm{ti}(C_n) \geq 1$. We now consider the graph $G + v_1v_3$, where $n \geq 4$ and $n \not\equiv 0 \pmod 3$.

Suppose firstly that $n \equiv 1 \pmod 3$ where $n \geq 4$. If $n = 4$, then $G + v_1v_3$ is the diamond graph $K_4 - e$ and the sets $\{v_1, v_3\}$ and $\{v_2, v_4\}$ partition the vertex set into an TD-set and an ID-set, respectively. Hence, in this case, $\mathrm{ti}(G) \leq 1$ and so $\mathrm{ti}(G) = 1$. Henceforth, we may assume that $n \geq 7$. In this case, the subgraph $G' = G - \{v_1, v_2, v_3\}$ is a path $v_4v_5 \dots v_n$ of order $n' = n - 3 \geq 4$ with $n' \equiv 1 \pmod 3$. Thus, by Proposition 2.1(b), there exists a TI-partition $\{I', S'\}$ of the vertices of G' where I' is an ID-set and S' is a TD-set of G'. Necessarily, $\{v_4, v_n\} \subseteq I'$. It follows that $\{I, S\}$, where $I = I' \cup \{v_2\}$ and $S = S' \cup \{v_1, v_3\}$, is a TI-partition of $G + v_1v_3$ and so the $G + v_1v_3$ is a TI-graph. Thus, $\mathrm{ti}(G) \leq 1$ and so $\mathrm{ti}(G) = 1$.

Suppose next that $n \equiv 2 \pmod 3$ where $n \geq 5$. In this case,

$$I = \{v_i \colon i \equiv 2 \pmod 3 \text{ and } i \in [n]\}$$

and $S = V(G) \setminus I$ form a partition $\{I, S\}$ of $G + v_1v_3$ into an ID-set and TD-set, respectively. Therefore, $G + v_1v_3$ is a TI-graph. Thus, $\mathrm{ti}(G) \leq 1$ and so $\mathrm{ti}(G) = 1$. □

We next determine the TI-augmentation number of a path.

Proposition 3.2. *For $n \geq 3$,* $\mathrm{ti}(P_n) = 0$ *if* $n \equiv 1 \pmod 3$ *and* $\mathrm{ti}(P_n) = 1$ *otherwise.*

Proof. Let $G = P_n$ be the path $v_1v_2 \dots v_n$ for $n \geq 3$. It follows from Proposition 2.1(b) that $\mathrm{ti}(P_n) = 0$ if and only if $n \equiv 1 \pmod 3$. Thus, if $n \not\equiv 1 \pmod 3$, then $\mathrm{ti}(P_n) \geq 1$. If $n \equiv 0 \pmod 3$, then $G + v_1v_n = C_n$, which is a TI-graph by Proposition 2.1(a). Therefore, in this case, $\mathrm{ti}(G) \leq 1$ and so $\mathrm{ti}(G) = 1$, as claimed. Hence, we may assume that $n \equiv 2 \pmod 3$ where $n \geq 5$. Here we consider the graph $G' = G + v_2v_{n-1}$. In this case,

$$I = \{v_1, v_n\} \cup \{v_i \colon i \equiv 0 \pmod 3 \text{ and } 3 \leq i \leq n-2\} \quad \text{and} \quad S = V(P_n) \setminus I$$

form a partition $\{I, S\}$ of G' into an ID-set I and TD-set S. Therefore, G' is a TI-graph and so $\mathrm{ti}(G) \leq 1$, implying that $\mathrm{ti}(G) = 1$ for $n \equiv 2 \pmod 3$. □

Proposition 3.3. *For $n \geq 3$, if T is a star $K_{1,n-1}$, then* $\mathrm{ti}(T) = 1$.

Proof. Let T be a star $K_{1,n-1}$ where $n \geq 3$. Since T is not a TI-graph, we note that $\mathrm{ti}(T) \geq 1$. Let v be the central vertex of T (of degree $n-1$) and let v_1 and v_2 be two arbitrary leaves in T. Let T' be the graph obtained from T by adding the edge v_1v_2, and so $T' = T + v_1v_2$. Let $I' = V(T) \setminus \{v, v_1\}$ and let $S' = \{v, v_1\}$. The set I' and S' form a partition $\{I', S'\}$ of T' into an ID-set I' and TD-set S'. Thus, T' is a TI-graph, and so $\mathrm{ti}(T) \leq 1$. Consequently, $\mathrm{ti}(T) = 1$. □

4. TREES

In this section, we determine upper bounds on the TI-augmentation number of a tree. If T is a tree of order $n \geq 3$ with exactly one support vertex, then T is the star $K_{1,n-1}$, and so by Proposition 3.3 we have $\mathrm{ti}(T) = 1$. Next we consider trees with more than one support vertex.

Theorem 4.1. *If T is a tree with $s \geq 2$ support vertices, then* $\mathrm{ti}(T) \leq \frac{1}{2}s$.

Proof. Let T be a tree with $s \geq 2$ support vertices. We proceed by induction on s.

Suppose that $s = 2$. Thus, T is a tree consisting of two support vertices u and v such that every nonleaf in $V(T) \setminus \{u, v\}$ is a vertex of degree 2 on the unique (u, v)-path connecting u and v. If u and v are adjacent, then T is the double star. In this case, the set I of leaves of T and $S = \{u, v\}$ form a partition $\{I, S\}$ of T into an ID-set I and TD-set S. Thus, T is a TI-graph, and so $\mathrm{ti}(T) = 0$. Hence, assume that u and v are not adjacent, since otherwise the result holds. Let $P\colon v_1v_2 \dots v_k$ be obtained from the (u, v)-path in T by deleting the vertices u and v. Since u and v are not adjacent, we note that $k \geq 1$. Further, every vertex on P has degree 2 in T. Let L be the set of leaf vertices of T. We now consider the tree $T^* = T + uv$. If $k \pmod 3 \in \{0, 2\}$, then let

$$I = L \cup \{v_i \colon i \equiv 2 \,(\mathrm{mod}\ 3) \text{ and } i \in [k]\} \quad \text{and} \quad S = V(T^*) \setminus I.$$

If $k \equiv 1 \,(\mathrm{mod}\ 3)$, then let

$$I = L \cup \{v_i \colon i \equiv 1 \,(\mathrm{mod}\ 3) \text{ and } i \in [k]\} \quad \text{and} \quad S = V(T^*) \setminus I.$$

In both cases, $\{I, S\}$ is a partition of $V(T^*)$ into an ID-set I and TD-set S, and so T^* is a TI-graph. Hence, $\mathrm{ti}(T) \leq 1 = \frac{1}{2}s$. This establishes the base case when $s = 2$.

Let $s \geq 3$ and suppose that for all trees T' with s' support vertices where $2 \leq s' < s$ that $\mathrm{ti}(T') \leq \frac{s}{2}$. For the remainder of the proof, our goal is to construct a TI-graph T^* by adding zero or more edges to T. We show that T^* is a TI-graph by giving a black-white coloring, called a TI-*coloring*, of the vertices such that the black vertices form an ID-set of T^* and the white vertices form a TD-set of T^*. We note that such a black-white TI-coloring of T^* will color the leaves of T^* black and the support vertices white.

To simplify the discussion in what follows, we need only consider a "pruned version" of the tree T where each support vertex is adjacent to exactly one leaf. The *pruned tree* $\hat{T}$ of T is the subtree of T formed by removing leaves from T such that every support vertex of T is a weak support vertex in $\hat{T}$, that is, $\hat{T}$ is formed by removing every leaf neighbor except one from each support vertex of T. We note that T and $\hat{T}$ have the same number of support vertices. Hence, we will construct our proposed TI-graph T^* by adding zero or more edges to $\hat{T}$. As remarked earlier, in our proposed black-white TI-coloring of T^*, the support vertices are colored white and leaves are colored black. Such a black-white TI-coloring of T^* can be extended to a TI-coloring of T by coloring the pruned leaves black. To further simplify the discussion, we therefore simply refer to the pruned tree $\hat{T}$ as T.

To aid in the presentation, we introduce a few more definitions. A *limb* of a pruned tree T is a maximal path starting at a leaf and not containing a vertex of degree 3 or more in T. A vertex of degree at least 3 in T that is adjacent to an endvertex of a limb is called a *branch vertex*. A limb on k vertices whose endvertex x is adjacent to a branch vertex v is said to be a limb of length k attached at v and x is called an *attachment vertex* of the limb. A branch vertex all of whose neighbors, except possibly one, are attachment vertices is called a *peripheral branch vertex*. We say that a limb is *bad* if its length is congruent to $1 \pmod 3$ and *good* otherwise.

Let v be a branch vertex and label the vertices of a limb attached at v and having length ℓ by $v_1 v_2 \dots v_\ell$, where v_1 is a leaf in T and v_ℓ is the attachment vertex of the limb (adjacent to v in T). We define a *standard limb coloring* as follows: Color the vertices in the set $\{v_i \colon i \equiv 1 \pmod 3 \text{ and } i \in [\ell]\}$ black and the vertices in the set $\{v_i \colon i \pmod 3 \in \{0,2\} \text{ and } i \in [\ell]\}$ white.

We continue the proof with the following claims.

Claim 4.2. If $s = 3$, then $\mathrm{ti}(T) \le \frac{1}{2}s$.

Proof. Assume that $s = 3$. It follows that the pruned tree $T = \hat{T}$ is a tree having a unique branch vertex v of degree 3, that is, all the other vertices of T have degree 1 or 2 and are on the limbs of T.

We consider all possibilities of the lengths ℓ_1, ℓ_2, and ℓ_3 of the three limbs $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$, respectively, of T. To color the tree T^*, which will be determined as follows based on the length of the three limbs of T, we begin by coloring the vertices of the limbs of T using a standard coloring. Recall that the leaves of T^* that were pruned from T are already colored black. All that is left is to define T^* (by carefully adding additional edges to T) and to color the branch vertex v.

If $\ell_j \equiv 0 \pmod 3$ for $j \in [3]$, then let $T^* = T$ (add no edges) and color v black. We note that v is not a support vertex of T or T^*. Further, if $\ell_j \equiv 1 \pmod 3$ for $j \in [2]$ and $\ell_3 \equiv 2 \pmod 3$, or if $\ell_j \equiv 2 \pmod 3$ for $j \in [2]$ and $\ell_3 \equiv 1 \pmod 3$, then let $T^* = T$ and color v white. In these cases, the white and black vertices form a TI-coloring of T^*, implying that $\mathrm{ti}(T) = 0$.

If $\ell_j \equiv 1 \pmod 3$ for $j \in [3]$, then color v white and form T^* from T by adding edge vu where u is a white vertex. If $\ell_j \equiv 2 \pmod 3$ for $j \in [3]$, then color v white and form T^* from T by adding edge vu where u is a black vertex. In these cases, the white and black vertices form a TI-coloring of T^*, implying that $\mathrm{ti}(T) \le 1$.

If $\ell_j \equiv 0 \pmod 3$ for $j \in [2]$ and $\ell_3 \equiv 1 \pmod 3$, then let $\mathcal{L}_3 \colon v_1 v_2 \dots v_\ell$, where v_1 is a leaf in T and v_ℓ is the attachment vertex of the limb $\mathcal{L}_3$. We color v black and recolor v_ℓ white. Form T^* by adding the edge $vv_{\ell-1}$. Now the white and black vertices form a TI-coloring of T^*, implying that $\mathrm{ti}(T) \le 1$.

If $\ell_j \equiv 0 \pmod 3$ for $j \in [2]$ and $\ell_3 \equiv 2 \pmod 3$, then color v black and form T^* by adding an edge from the attachment vertex of $\mathcal{L}_3$ to a white vertex. Now the white and black vertices form a TI-coloring of T^*, implying that $\mathrm{ti}(T) \le 1$.

If $\ell_1 \equiv 0 \pmod 3$ and $\ell_j \equiv 1 \pmod 3$ for $j \in \{2,3\}$ or if $\ell_1 \equiv 0 \pmod 3$, $\ell_2 \equiv 1 \pmod 3$, and $\ell_3 \equiv 2 \pmod 3$, then color v white and form T^* by adding an edge from the attachment vertex of $\mathcal{L}_1$ to a black vertex. Now the white and black vertices form a TI-coloring of T^*, implying that $\mathrm{ti}(T) \le 1$.

Finally, if $\ell_1 \equiv 0 \pmod 3$ and $\ell_j \equiv 2 \pmod 3$ for $j \in \{2,3\}$, then color v black and form T^* by adding an edge between the attachment vertices of the limbs $\mathcal{L}_2$ and $\mathcal{L}_3$. Now the white and black vertices form a TI-coloring of T^*, implying that $\mathrm{ti}(T) \le 1$.

Since relabeling the limbs, if necessary, all possible combinations of limb lengths have been considered and in all cases, $\mathrm{ti}(T) \le 1 < \frac{s}{2}$ for $s = 3$, this result holds in this case, completing the proof of Claim 4.2. $\square$

By Claim 4.2, if $s = 3$, then the desired result holds. Henceforth, we may therefore assume that $s \ge 4$.

Claim 4.3. If T has two good limbs whose removal does not create a new support vertex in the resulting tree, then $\mathrm{ti}(T) \le \frac{1}{2}s$.

Proof. Assume that T has two good limbs whose removal does not create a new support vertex in the resulting tree. Select such a pair of limbs $\mathcal{L}_u$ and $\mathcal{L}_v$, where $\mathcal{L}_u$ is a good limb attached at branch vertex u and $\mathcal{L}_v$ is a good limb attached at branch vertex v, and let ℓ_u and ℓ_v be the length of the removed branch at u and v, respectively. We note that u and v may be the same vertex. We also note that $\ell_u \pmod 3 \in \{0,2\}$ and $\ell_v \pmod 3 \in \{0,2\}$.

Let T' be the tree formed by deleting $\mathcal{L}_u$ and $\mathcal{L}_v$ from T and let s' be the number of support vertices in T'. By assumption, every support vertex in T' is a support vertex in T, and so $s' = s - 2 \ge 2$ noting that $s \ge 4$. Applying our inductive hypothesis to the tree T', we have $\mathrm{ti}(T') \le \frac{s'}{2}$. Let $(T')^*$ be a TI-tree obtained from T' by adding at most $\frac{s'}{2}$ edges and let $\mathcal{C}'$ be a TI-coloring of the vertices of $(T')^*$ such that the black vertices form an ID-set I' and the white vertices form a TD-set S' of $(T')^*$.

To color the limbs $\mathcal{L}_u\colon u_1,\dots,u_{\ell_u}$ and $\mathcal{L}_v\colon v_1,\dots,v_{\ell_v}$, we consider the color of u and v in $\mathcal{C}'$ and the lengths ℓ_u and ℓ_v of the limbs. Note that any additional leaf vertices adjacent to u_2 and v_2 in the unpruned tree T will be colored black in T^*.

Case 1. Both u and v are colored white in $\mathcal{C}'$. If $\ell_u \equiv 2 \pmod 3$ and $\ell_v \equiv 2 \pmod 3$, then color $\mathcal{L}_u$ and $\mathcal{L}_v$ with a standard coloring. In this case, no further edge needs to be added to form T^* since the white and black vertices of $\mathcal{C}'$ along with this standard coloring of the deleted limbs form a TI-coloring of T^*, implying that $\mathrm{ti}(T) \le 0 + \frac{s'}{2} = \frac{s-2}{2} < \frac{s}{2}$.

Hence, assume that at least one of ℓ_u and ℓ_v, say ℓ_u, is congruent to $0 \pmod 3$. To color $\mathcal{L}_u$, color the set $\{u_i\colon i \equiv 1 \pmod 3 \text{ and } i \in [\ell_u - 1]\}$ black and the set $\{u_i\colon i \equiv 0,2 \pmod 3 \text{ and } i \in [\ell_u - 1]\}$ white. Finally, color u_{ℓ_u} black. If $\ell_v \equiv 0 \pmod 3$, then color $\mathcal{L}_v$ in the same coloring applied to $\mathcal{L}_u$, else apply a standard coloring to $\mathcal{L}_v$.

Next we add edges to form T^* from $(T')^*$ as follows. If $\ell_v \equiv 0 \pmod 3$, then add the edge $u_{\ell_u-1}v_{\ell_v-1}$. If $\ell_v \equiv 2 \pmod 3$, then add the edge $u_{\ell_u-1}v_2$. Now the white and black vertices form a TI-coloring of T^*, implying that $\mathrm{ti}(T) \le 1 + \frac{s'}{2} = 1 + \frac{s-2}{2} = \frac{s}{2}$.

Case 2. Both u and v are colored black in $\mathcal{C}'$. In this case, we color both $\mathcal{L}_u$ and $\mathcal{L}_v$ with a standard coloring. If $\ell_u \equiv 0 \pmod 3$ and $\ell_v \equiv 0 \pmod 3$, then no further edge needs to be added since the white and black vertices of $\mathcal{C}'$ along with this coloring of the limbs form a TI-coloring of T^*, implying that $\mathrm{ti}(T) \le 0 + \frac{s'}{2} = \frac{s-2}{2} < \frac{s}{2}$. Otherwise, we form T^* from $(T')^*$ as follows: If $\ell_u \equiv 0 \pmod 3$ and $\ell_v \equiv 2 \pmod 3$, then add the edge $u_2 v_{\ell_v}$. If $\ell_u \equiv 2 \pmod 3$ and $\ell_v \equiv 2 \pmod 3$, then add the edge $u_{\ell_u} v_{\ell_v}$. In both cases, the white and black vertices form a TI-coloring of T^*, implying that $\mathrm{ti}(T) \le 1 + \frac{s'}{2} = 1 + \frac{s-2}{2} = \frac{s}{2}$.

Case 3. Vertex u is black and vertex v is white in $\mathcal{C}'$. Clearly, $u \neq v$. If $\ell_u \equiv 0 \pmod 3$ and $\ell_v \equiv 0 \pmod 3$, then color both limbs $\mathcal{L}_u$ and $\mathcal{L}_v$ with a standard coloring and add the edge $v_{\ell_v} u_1$ to form T^*. If $\ell_u \equiv 2 \pmod 3$ and $\ell_v \equiv 2 \pmod 3$, then color both limbs $\mathcal{L}_u$ and $\mathcal{L}_v$ with a standard coloring and add the edge $u_{\ell_u} v_{\ell_v}$ to form T^*. If $\ell_u \equiv 0 \pmod 3$ and $\ell_v \equiv 2 \pmod 3$, then color both limbs with a standard coloring and add no edges to form T^*. If $\ell_u \equiv 2 \pmod 3$ and $\ell_v \equiv 0 \pmod 3$, then color the limb $\mathcal{L}_u$ with a standard coloring and the limb $\mathcal{L}_v$ with a standard coloring except for vertex v_{ℓ_v}, which is colored black and add the edge $u_{\ell_u} v_{\ell_v - 1}$ to form T^*. In all the above cases, the white and black vertices form a TI-coloring of T^*, implying that $\mathrm{ti}(T) \le 1 + \frac{s'}{2} = 1 + \frac{s-2}{2} = \frac{s}{2}$. This concludes the proof of Claim 4.3. □

By Claim 4.3, we may assume that T does not have a pair of good limbs whose removal does not create a new support vertex in the resulting tree, for else the result holds. Thus, if T has two or more branch vertices, then at most one of them has an attached good limb; otherwise, removing a good limb from each of these two branch vertices would suffice. Moreover, if T has a branch vertex v of degree at least 4, then at most one good limb is attached at v since any two attached good limbs would also satisfy the conditions. We deduce that either T has exactly one branch vertex, or T has at two or more branch vertices, at most one of which has a good limb attached.

Claim 4.4. If T has exactly one branch vertex, then $\mathrm{ti}(T) \le \frac{1}{2}s$.

Proof. Assume that the pruned tree $T = \hat{T}$ is a tree having a unique branch vertex v, that is, all the other vertices of T have degree 1 or 2 and are on the limbs of T. By our assumption, we have $s \ge 4$, and we infer that the vertex v has degree at least 4.

In this case, every neighbor of v, except possibly one, is an attachment vertex of a bad limb. If every limb attached at v is bad, then we color the tree T by applying a standard coloring of the limbs of T (recall that all pruned leaves will be colored black) and coloring v white. We then add the edge vy where y is any white vertex to form T^*. We note that such a vertex exists since at most one limb attached at v has length one. The white and black vertices form a TI-coloring of T^*, implying that $\mathrm{ti}(T) \le 1 < \frac{s}{2}$.

If exactly one limb $\mathcal{L}_v$ attached at v is good, then we color all the limbs of T, except $\mathcal{L}_v$, with a standard coloring and color v white. If $\ell_v \equiv 0 \pmod 3$, then color $\mathcal{L}_v$ with a standard coloring and add the edge $v_{\ell_v} x$, where x is any black vertex on one of the bad limbs, to form T^*. If $\ell_v \equiv 2 \pmod 3$, then color $\mathcal{L}_v$ with a standard coloring and let $T^* = T$. In both cases, the white and black vertices form a TI-coloring of T^*, implying that $\mathrm{ti}(T) \le 1 < \frac{s}{2}$. □

By Claims 4.3 and 4.4, we may assume that T has at least two branch vertices, at most one of which has an attached good limb, for otherwise the desired result follows. This implies that T contains a peripheral branch vertex w such that all neighbors of w except for one neighbor, say x, are attachment vertices of bad limbs. Let r be a leaf such that the (r, w)-path contains the vertex x. We now root the tree at the vertex r. Thus, each child of w in the resulting rooted tree T is an attachment vertex of a bad limb and the vertex x is the parent of w in T. We now consider the maximal subtree T_w of T rooted at w. Thus, T_w consists of w and all attached limbs of w.

Suppose that x is not a leaf in $T - V(T_w)$. In this case, we let $T' = T - V(T_w)$ and let T' have s' support vertices. Since T' is a pruned tree with at least one branch vertex and x is not a leaf in T', it follows that $2 \le s' \le s - 2$. We apply our inductive hypothesis to T' and consider $(T')^*$ with coloring $\mathcal{C}'$. We apply a standard coloring to all the limbs attached at w and color w white. If x is colored black by $\mathcal{C}'$, then we add the edge wy where y is any white vertex to form T^*. Such a vertex exists since at most one limb attached at w has length 1. If x is colored white, then we do not add any edges to form T^*. In both cases, the white and black vertices form a TI-coloring of T^*, implying that $\mathrm{ti}(T) \le \frac{1}{2}s' + 1 \le \frac{1}{2}(s-2) + 1 \le \frac{1}{2}s$.

Thus, we may assume that x is a leaf of T', for otherwise the desired result follows. With this assumption, the vertex x has degree 2 in T. Let $x_1x_2\dots x_k$ be the path in T such that the vertex $x_1 = x$, each vertex x_i for $i \in [k]$ has degree 2 in T, and x_k is adjacent to a branch vertex y of degree 3 or more in T. We note that $k \ge 1$ and such a vertex y exists since T has at least two branch vertices.

We now consider the maximal subtree T_{x_k} of T rooted at x_k and consider the tree $T'' = T - V(T_{x_k})$. Let T'' have s'' support vertices. We note that $2 \le s'' \le s - 2$. We apply our inductive hypothesis to T'' and consider $(T'')^*$ with coloring $\mathcal{C}''$. Now we color the path $x_1x_2\dots x_k$ based on the color of y in $\mathcal{C}''$ and the value of k. We apply a standard coloring to all the limbs attached at w and color w white.

Suppose that y is colored white in $\mathcal{C}''$. If $k \equiv 1 \pmod 3$, then color the vertices in the set $\{x_i \colon i \equiv 1 \pmod 3 \text{ and } i \in [\ell]\}$ black and the vertices in the set $\{x_i \colon i \pmod 3 \in \{0,2\} \text{ and } i \in [\ell]\}$ white, and add the edge wz where z is any white vertex to form T^*. If $k \pmod 3 \in \{0,2\}$, then color the vertices in the set $\{x_i \colon i \equiv 2 \pmod 3 \text{ and } i \in [\ell]\}$ black and the vertices in the set $\{x_i \colon i \pmod 3 \in \{0,1\} \text{ and } i \in [\ell]\}$ white, and add no edges to form T^*.

Suppose that y is colored black in $\mathcal{C}''$. If $k \equiv 1 \pmod 3$, then color the vertices in the set $\{x_i \colon i \equiv 2 \pmod 3 \text{ and } i \in [\ell]\}$ black and the vertices in the set $\{x_i \colon i \pmod 3 \in \{0,1\} \text{ and } i \in [\ell]\}$ white, and add no edges to form T^*. If $k \pmod 3 \in \{0,2\}$, then color the vertices in the set $\{x_i \colon i \equiv 1 \pmod 3 \text{ and } i \in [\ell]\}$ black and the vertices in the set $\{x_i \colon i \pmod 3 \in \{0,2\} \text{ and } i \in [\ell]\}$ white, and add the edge wx_k to form T^*.

In all cases, the white and black vertices form a TI-coloring of T^*. Further, in all cases we add at most one edge to $(T'')^*$ to form T^*. Hence, $\mathrm{ti}(T) \le \frac{1}{2}s'' + 1 \le \frac{1}{2}(s-2) + 1 \le \frac{1}{2}s$. This completes the proof of Theorem 4.1. □

We next present an upper bound on the TI-augmentation number of a tree in terms of its order and number of support vertices.

Theorem 4.5. *If T is a tree of order $n \geq 4$ with s support vertices, then one of the following properties hold.*

(a) $n = 2s$ *and* $\mathrm{ti}(T) = 0$.
(b) $n = 2s + 1$ *and* $\mathrm{ti}(T) \leq 1$.
(c) $n \geq 2s + 2$ *and* $\mathrm{ti}(T) \leq n - 2s - 1$.

Proof. Let T be a tree of order $n \geq 4$ with s support vertices. If $s = 1$, then T is the star $K_{1,n-1}$ and $n \geq 2s + 2$. In this case, by Proposition 3.3 we have $\mathrm{ti}(T) = 1 \leq n - 3 = n - 2s - 1$. Hence, we may assume that $s \geq 2$, for otherwise the desired result is immediate. Let C be the set of leaves of T, and let B be the set of support vertices of T. Thus, $|B| = s$ and $|C| \geq s$. Let $A = V(T) \setminus (A \cup B)$. We note that $n = |A| + |B| + |C| \geq 0 + s + s = 2s$.

Note that if $A = \emptyset$, then every vertex in B has at least one neighbor in B since $s \geq 2$ and T is connected. Hence, the sets $I = C$ and $S = B$ form a partition $\{I, S\}$ of T into an ID-set I and TD-set S, and so T is a TI-graph and $\mathrm{ti}(T) = 0$. If $|A| = 1$, then adding an edge from the vertex in A to a vertex in C gives a graph where C is an ID-set and $A \cup B$ is a TD-set. Thus, $\mathrm{ti}(T) \leq 1$.

If $n = 2s$, then $A = \emptyset$ and $\mathrm{ti}(T) = 0$, yielding the result of part (a).

If $n = 2s + 1$, then either $A = \emptyset$ and $|C| = s + 1$, or $|A| = 1$ and $|C| = s$. For the former, we again have that $\mathrm{ti}(T) = 0$, giving the result of part (b). For the later, $\mathrm{ti}(T) \leq 1$, and again (b) holds.

Hence, we may assume that $n \geq 2s + 2$. From our previous remarks, if $A = \emptyset$, then $\mathrm{ti}(T) = 0 < n - 2s - 1$ and (c) holds. Further, if $|A| = 1$, then $\mathrm{ti}(T) \leq 1$ and (c) holds. Therefore, we may further assume that $|A| \geq 2$, for otherwise the desired result follows.

Let T^* be obtained from T by adding an edge from each vertex in A to a vertex in C. In this case, the set $I = C$ and $S = A \cup B$ form a partition $\{I, S\}$ of T^* into an ID-set I and TD-set S. Thus, T^* is a TI-graph, and so

$$\mathrm{ti}(T) \leq |A| = n - |B| - |C| \leq n - 2|B| = n - 2s. \tag{4.1}$$

If $\mathrm{ti}(T) \leq n - 2s - 1$, then we obtain the result of part (c). Hence, we may assume that $\mathrm{ti}(T) = n - 2s$, for otherwise the desired result follows. With this assumption, we must have equality throughout Inequality Chain (4.1), implying that $|B| = |C| = s$ and $\mathrm{ti}(T) = |A| = n - 2s$.

Recall that $|A| \geq 2$. Let v be an arbitrary vertex in A. We now consider the tree T' obtained from T by adding an edge from each vertex in $A \setminus \{v\}$ to a vertex in C, and we consider the sets $I' = C \cup \{v\}$ and $S' = (A \cup B) \setminus \{v\}$. By construction, the set I' is an independent set in T'. If S' is a TD-set of T', then $\{I', S'\}$ is a partition of T' into an ID-set I' and TD-set S', implying that $\mathrm{ti}(T) \leq |A| - 1$, a contradiction. Hence, S' is a dominating set of T' but is not a TD-set of T', implying that there exists a vertex $v_b \in S'$ that is isolated in $T[S']$. Every vertex in A has degree at least 2 in T. Thus, every vertex in $A \setminus \{v\}$ has at least one neighbor in S' and is therefore not isolated in $T[S']$. We therefore infer that $v_b \in B$. Further, the vertex v_b has degree 2 in T and is adjacent to the vertex v and to a vertex, say v_c, in C. Hence, vv_bv_c is a path in T where v_c is a leaf in T (that belongs to C) and v_b is a support vertex of degree 2 in T (that belongs to B). We note that the vertex v may possibly have many such neighbors v_b of degree 2 in T that are support vertices.

Let u and v be two distinct vertices in A. By our earlier properties, the vertex u has a neighbor, say u_b, of degree 2 that belongs to the set B, and the vertex v has a neighbor, say v_b, of degree 2 that belongs to the set B. Thus, each of u_b and v_b are support vertices in T. Let u_c be the neighbor of u_b that belongs to C, and let v_c be the neighbor of v_b that belongs to C. An example of such a tree T is illustrated in Figure 1.

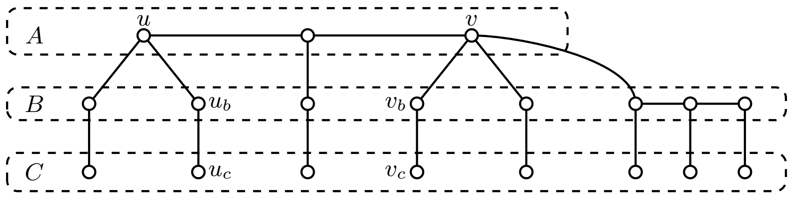

Fig. 1. A possible tree T in the proof of Theorem 4.5

We now consider the tree T'' obtained from T by adding the edge u_cv_v and adding an edge from each vertex in $A \setminus \{u, v\}$ to a vertex in C, and we consider the sets $I'' = (C \setminus \{u_c, v_c\}) \cup \{u_b, v_b\}$ and $S'' = A \cup (B \setminus \{u_b, v_b\}) \cup \{u_c, v_c\}$. The sets I'' and S'' form a partition $\{I'', S''\}$ of T'' into an ID-set and TD-set, respectively. Therefore, T'' is a TI-graph, implying that $\mathrm{ti}(T) \leq (|A| - 2) + 1 = |A| - 1$, a contradiction. If T is the tree illustrated in Figure 1, then an example of a tree T'' is illustrated in Figure 2, where the vertices in I'' are indicated by the shaded vertices and the vertices in S'' by the white vertices.

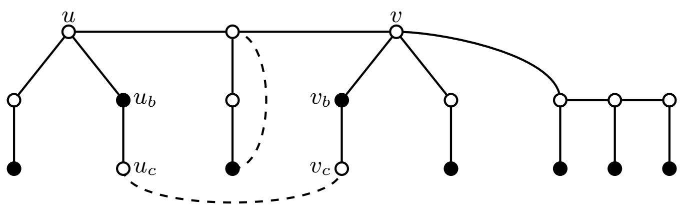

Fig. 2. A possible tree T'' in the proof of Theorem 4.5

This completes the proof of Theorem 4.5. □

As a consequence of Theorems 4.1 and 4.5, we have the following upper bound on the TI-augmentation number of a tree in terms of its order.

Theorem 4.6. *If T is a tree of order $n \geq 5$, then* $\mathrm{ti}(T) \leq \frac{1}{5}n$.

Proof. Let T be a tree of order $n \geq 5$ with s support vertices. We note that $n \geq 2s$. If $s = 1$, then T is the star $K_{1,n-1}$ and by Proposition 3.3 we have $\mathrm{ti}(T) = 1 \leq \frac{1}{5}n$. If $n = 2s$, then by Theorem 4.5(a) we have $\mathrm{ti}(T) = 0 < \frac{1}{5}n$. If $n = 2s + 1$, then by Theorem 4.5(b) we have $\mathrm{ti}(T) \leq 1 \leq \frac{1}{5}n$. Hence, we may assume that $s \geq 2$ and $n \geq 2s + 2$, for otherwise the desired result follows. With these assumptions, if $s \leq \frac{2}{5}n$, then by Theorem 4.1 we have $\mathrm{ti}(T) \leq \frac{1}{2}s \leq \frac{1}{5}n$, as desired. Moreover, if $s > \frac{2}{5}n$, then by Theorem 4.5(c) we have $\mathrm{ti}(T) \leq n - 2s - 1 < n - \frac{4}{5}n - 1 < \frac{1}{5}(n - 1)$. □

We remark that the upper bound in Theorem 4.6 is the best possible. The simplest example of a tree T of order n satisfying $\mathrm{ti}(T) = \frac{1}{5}n$ is a path $T = P_5$. However, when n is sufficiently large, it is not known if the bound is asymptotically the best possible, or whether the upper bound can be improved in this case.

5. BOUNDS IN TERMS OF THE INDEPENDENCE NUMBER

In this section, we establish an upper bound on the TI-augmentation number $\mathrm{ti}(G)$ of a graph G in terms of its order n and its vertex independence number $\alpha(G)$.

Theorem 5.1. *If G is an isolate-free graph of order n, then* $\mathrm{ti}(G) \leq \left\lceil \frac{n-\alpha(G)}{2} \right\rceil$.

Proof. Let G be an isolate-free graph of order n with vertex set $V = V(G)$, and let I be an α-set of G. Thus, I is an ID-set of G and since G is isolate-free, every vertex in I has a neighbor in $V \setminus I$. Thus, $V \setminus I$ is a dominating set. Let X be the set of isolated vertices in $G[V \setminus I]$. If $X = \emptyset$, then $V \setminus I$ is a TD-set of G, and so G is a TI-graph and $\mathrm{ti}(G) = 0$. Hence, we may assume that $X \neq \emptyset$. If $|X|$ is even, then we pair the vertices of X and add edges of a perfect matching in $G[X]$; and if $|X|$ is odd, then we add a matching of edges on $|X| - 1$ vertices of X along with one more edge to the remaining vertex in X. Thus, in the resulting graph $V \setminus I$ is a TD-set. Therefore, $\mathrm{ti}(G) \leq \left\lceil \frac{|X|}{2} \right\rceil \leq \left\lceil \frac{n-\alpha(G)}{2} \right\rceil$. □

We present next some consequences of Theorem 5.1. For any bipartite graph G of order n, we have $\alpha(G) \geq \frac{1}{2}n$. The 4-Color Theorem implies that every planar graph G of order n satisfies $\alpha(G) \geq \frac{1}{4}n$. For $d \geq 1$, a graph G is *d-degenerate* if every subgraph of G has at least one vertex of degree at most d. The *degeneracy* of G is the smallest integer d such that G is d-degenerate. The *coloring number*, $\mathrm{col}(G)$, of G is the smallest positive integer d for which there exists an ordering $v_1, v_2, \ldots, v_n$ of the vertices of G such that every v_i has fewer than d neighbors $v_j \in N_G(v_i)$ with $j < i$. The coloring number and degeneracy are related as $\mathrm{col}(G)$ is equal to one more than the degeneracy of G. If G is an isolate-free graph with degeneracy d, then we infer that $\alpha(G) \geq n/\mathrm{col}(G) = n/(d + 1)$. The above observations yield the following consequences of Theorem 5.1.

Corollary 5.2. *If G is an isolate-free of order n with $\delta(G) \geq 3$, then the following hold.*

(a) *If G is a bipartite graph, then* $\mathrm{ti}(G) \leq \lceil \frac{1}{4}n \rceil$.

(b) *If G is a planar graph, then* $\mathrm{ti}(G) \leq \lceil \frac{3}{8}n \rceil$.

(c) *If G is a graph with degeneracy d, then* $\mathrm{ti}(G) \leq \left\lceil \frac{d}{2(d-1)}n \right\rceil$.

We can improve the bound of Theorem 5.1 slightly as follows. Let $\mathcal{F}$ denote the family of graphs defined as follows. A graph G is in $\mathcal{F}$ if G is a bipartite graph such that for any α-set I of G, every vertex in I has degree 1 or 2 and $|V \setminus I| = n - \alpha(G)$ is odd.

Theorem 5.3. *If $G \notin \mathcal{F}$ is an isolate-free graph of order n, then* $\mathrm{ti}(G) \leq \frac{1}{2}(n - \alpha(G))$.

Proof. Let G be an isolate-free graph such that $G \notin \mathcal{F}$, and let I be an α-set of G. From the proof of Theorem 5.1, we deduce that $\mathrm{ti}(G) \leq \frac{1}{2}(n - \alpha(G))$ except possibly for the case where $V \setminus I$ is independent and $|V \setminus I|$ is odd. Assume that this holds. Since $G \notin \mathcal{F}$, there exists a vertex $v \in I$ such that v has degree at least 3. In this case, we show that the result holds by adding edges to G to form G^* and giving a black-white TI-coloring of G^* such that the black vertices form a ID-set and the white vertices form an ID-set in G^*.

Let $X = V \setminus (I \cup N(v))$ and add edges such that every vertex in $G^*[X]$ has a neighbor in X. Thus, at most $\frac{1}{2}(|X| + 1)$ edges must be added to the vertices of X. Furthermore, we add the edge vy, where y is a vertex in $I \setminus \{v\}$. Color the vertices of $I \setminus \{v\}$ black, the vertices of $(V \setminus I) \cup \{v\}$ white. Then the black vertices form a ID-set and the white vertices form an TD-set of G^*. Therefore,

$$\begin{aligned}
\mathrm{ti}(G) &\leq \frac{1}{2}(|X| + 1) + 1 \\
&= \frac{1}{2}(n - |I| - \deg_G(v) + 1) + 1 \\
&\leq \frac{1}{2}(n - \alpha(G) - 3 + 1) + 1 \\
&= \frac{1}{2}(n - \alpha(G)),
\end{aligned}$$

yielding the desired upper bound. □

As an immediate consequence of Theorem 5.3, we have the following result.

Corollary 5.4. *If G is a graph of order n with $\delta(G) \geq 3$, then* $\mathrm{ti}(G) \leq \frac{1}{2}(n - \alpha(G))$.

Recall that for any bipartite graph G of order n, we have $\alpha(G) \geq \frac{1}{2}n$. Moreover, every planar graph G of order n satisfies $\alpha(G) \geq \frac{1}{4}n$, and every graph with degeneracy d satisfies $\alpha(G) \geq n/(d+1)$. Hence, we have the following consequences of Corollary 5.4.

Corollary 5.5. *If G is a graph of order n with $\delta(G) \geq 3$, then the following hold.*

(a) *If G is a bipartite graph, then* $\mathrm{ti}(G) \leq \frac{1}{4}n$.

(b) *If G is a planar graph, then* $\mathrm{ti}(G) \leq \frac{3}{8}n$.

(c) *If G has degeneracy d, then* $\mathrm{ti}(G) \leq \left(\frac{d}{2(d-1)}\right) n$.

We establish next an upper bound on the TI-augmentation number of a cubic graph.

Corollary 5.6. *If G is a cubic graph of order n, then* $\mathrm{ti}(G) \le \frac{1}{3}n$.

Proof. Let G be a cubic graph of order n. If C is a K_4-component of G, then C is a TI-graph noting that if v is an arbitrary vertex in the component C, then the sets $I = \{v\}$ and $V(C) \setminus \{v\}$ form a partition $\{I, S\}$ of C into an ID-set I and TD-set S. Hence, we may assume that no component of G is a K_4-component. By Brooks' Coloring Theorem, the cubic graph G therefore has a vertex coloring with at most three colors. The largest color class in such a 3-coloring in G has at least $n/3$ vertices. Therefore, G has an independent set of at least $n/3$ vertices, and so $\alpha(G) \ge \frac{1}{3}n$. Hence, by Corollary 5.4, $\mathrm{ti}(G) \le \frac{1}{2}(n - \alpha(G)) \le \frac{1}{2}(n - \frac{1}{3}n) = \frac{1}{3}n$. □

In 1995 Fraughnaugh and Locke [7] described a family $\mathcal{F}$ of six graphs (each of order at most 22) and conjectured that every triangle-free graph G of order n with maximum degree at most 3 having no subgraph isomorphic to a member of $\mathcal{F}$ satisfies $\alpha(G) \ge \frac{3}{8}n$. This conjecture was proven in 2020 by Cames van Batenburg, Goedgebeur, and Joret [3]. We remark that as a consequence of their result and the bound given in Theorem 5.1, we can infer the following result (we omit the proof details) which is an improvement of the bound given in Corollary 5.6.

Corollary 5.7. *If G is a triangle-free cubic graph of order n, then* $\mathrm{ti}(G) \le \frac{5}{16}n$.

6. GRAPHS WITH MINIMUM DEGREE THREE

As shown in Corollary 5.6, if G is a cubic graph of order n, then $\mathrm{ti}(G) \le \frac{1}{3}n$. However, it is unlikely that this bound is achievable. We pose the following conjecture.

Conjecture 6.1. If G is a graph of order n with $\delta(G) \ge 3$, then $\mathrm{ti}(G) \le \frac{1}{6}n$.

We remark that if Conjecture 6.1 is true, then the bound is the best possible as may be seen by considering, for example, the complete bipartite graph $G = K_{3,3}$ of order $n = 6$ that satisfies $\mathrm{ti}(G) = 1 = \frac{1}{6}n$. We show next that if Conjecture 6.1 is true, then the bound is asymptotically sharp in the sense that there exist connected graphs G of sufficiently large order n with $\delta(G) \ge 3$ such that the ratio $\frac{\mathrm{ti}(G)}{n}$ can be made arbitrarily close to $\frac{1}{6}$.

Proposition 6.2. *Given any value of $\varepsilon > 0$, there exist connected graphs G of sufficiently large order n with $\delta(G) = 3$ satisfying*

$$\frac{\mathrm{ti}(G)}{n} > \frac{1}{6} - \varepsilon.$$

Proof. Let $\varepsilon > 0$ be given. For $k \ge 2$, let $G_{i,j}$ be the complete bipartite graph $K_{3,3}$ with partite sets $X_{i,j} = \{x_{i,j,1}, x_{i,j,2}, x_{i,j,3}\}$ and $Y_{i,j} = \{y_{i,j,1}, y_{i,j,2}, y_{i,j,3}\}$ where $i \in [k]$ and $j \in [k]$. Let G_i be the graph obtained from the disjoint union of the k graphs $G_{i,1}, G_{i,2}, \ldots, G_{i,k}$ by adding a new vertex v_i and adding the edges $v_i x_{i,j,1}$ and $v_i y_{i,j,1}$ for all $j \in [k]$. The graph G_i has order $6k + 1$ for all $i \in [k]$. See Figure 3.

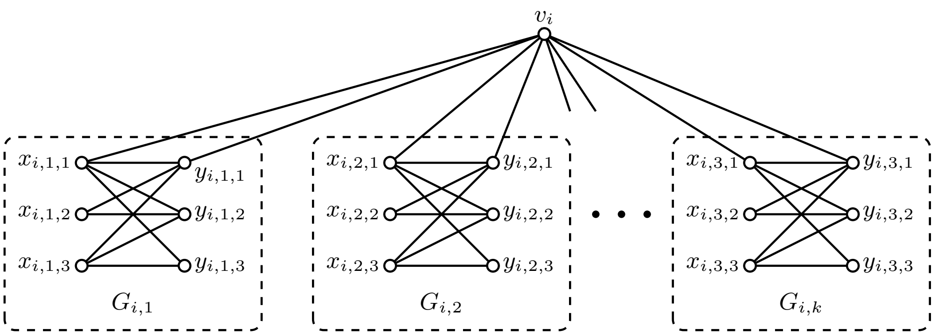

Fig. 3. The graph G_i in the proof of Proposition 6.2

Let H_k be the graph obtained from the disjoint union of the k graphs $G_1, G_2, \ldots, G_k$ by forming a clique on the set $\{v_1, v_2, \ldots, v_k\}$ of k vertices. The graph H_k has order $n = k(6k+1)$ and minimum degree $\delta(H_k) = 3$. For example, when $k = 3$ the graph H_k is illustrated in Figure 4, albeit without the vertices labels.

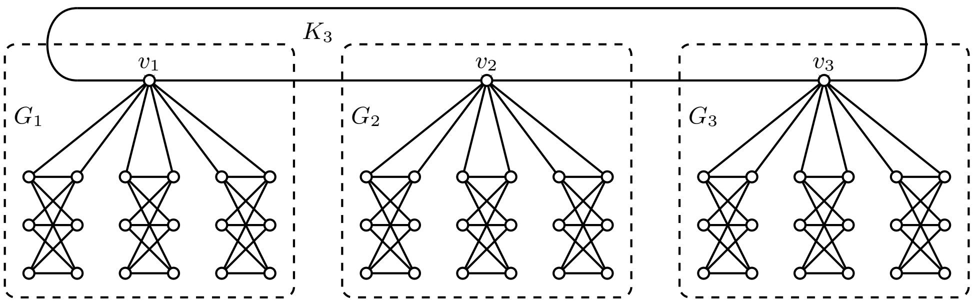

Fig. 4. The graph H_3 in the proof of Proposition 6.2

We show that $\mathrm{ti}(H_k) = k(k-1)$. We augment H_k to form the graph H_k^* by adding the $k-1$ edges $e_i^j = y_{i,j,2}y_{i,j,3}$ for all i and j where $i \in [k-1]$ and $j \in [k]$. Let

$$I^* = \{v_k\} \cup \left(\bigcup_{j=1}^{k} \{x_{k,j,2}, x_{k,j,3}\} \right) \cup \left(\bigcup_{i=1}^{k-1} \bigcup_{j=1}^{k} X_{i,j} \right)$$

and let $S^* = V(H_k) \setminus I^*$. The set I^* and S^* form a partition $\{I^*, S^*\}$ of H_k^* into an ID-set I^* and TD-set S^*. Thus, H_k^* is a TI-graph, and so $\mathrm{ti}(H_k) \le k(k-1)$. For example, the graph H_3^* and the associated sets I^* and S^* are illustrated in Figure 5, albeit without the vertices labels, where I^* is the set of shaded vertices and S^* is the set of white vertices.

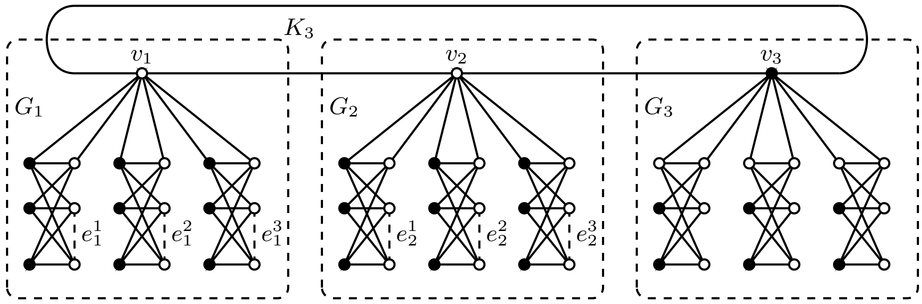

Fig. 5. The graph H_3^* in the proof of Proposition 6.2

We show next that $\mathrm{ti}(H_k) \geq k(k-1)$. Let H'_k be the graph obtained from H_k by adding $\mathrm{ti}(H_k)$ edges in such a way as to produce a TI-graph, and let $\{I', S'\}$ be a partition of H'_k into an ID-set I' and TD-set S'. Since the vertices v_1, $v_2, \ldots v_k$ induce a clique in H_k, we note that at most one of these vertices belongs to the set I'. Hence, the vertex v_i belongs to S' for at least $k-1$ values of $i \in [k]$. We consider such subgraphs G_i where $v_i \in S'$, and show that we can uniquely associate at least $k-1$ augmented edges with each such subgraph G_i.

Suppose that $\{x_{i,j,1}, y_{i,j,1}\} \subseteq S'$ for some $j \in [k]$. We note that $X_{i,j} \subseteq S'$ or $Y_{i,j} \subseteq S'$ (or both $X_{i,j} \subseteq S'$ and $Y_{i,j} \subseteq S'$). By symmetry, we may assume that $Y_{i,j} \subseteq S'$, implying that the vertex $x_{i,j,1}$ is incident with an augmented edge that joins it to a vertex in I'. We can therefore uniquely associate an augmented edge with the subgraph $G_{i,j}$.

Suppose that $\{x_{i,j,1}, y_{i,j,1}\} \nsubseteq S'$ for some $j \in [k]$. By symmetry, we may assume that $x_{i,j,1} \in I'$, implying that $Y_{i,j} \subseteq S'$. If $\{x_{i,j,2}, x_{i,j,3}\} \subseteq I'$, then each of $y_{i,j,2}$ and $y_{i,j,3}$ is incident with an augmented edge that joins it to some other vertex in S', and so at least two vertices of $G_{i,j}$ are incident with augmented edges. If $x_{i,j,2} \in S'$ and $x_{i,j,3} \in I'$, then the vertex $x_{i,j,2}$ is incident with an augmented edge that joins it to a vertex in I', implying that we can uniquely associate an augmented edge with $G_{i,j}$. Analogously, if $x_{i,j,2} \in I'$ and $x_{i,j,3} \in S'$, then we can uniquely associate an augmented edge with $G_{i,j}$. If $\{x_{i,j,2}, x_{i,j,3}\} \subseteq S'$, then each of $x_{i,j,2}$ and $x_{i,j,3}$ is incident with an augmented edge that joins it to a vertex in I', implying that we can uniquely associate two augmented edges with $G_{i,j}$.

In all the above cases, we can uniquely associate at least one augmented edge with $G_{i,j}$. This is true for at least $k-1$ values of $i \in [k]$. Therefore, noting that $j \in [k]$, we infer that at least $k(k-1)$ augmented edges were added to H_k when constructing H'_k, implying that $\mathrm{ti}(H_k) \geq k(k-1)$. As observed earlier, $\mathrm{ti}(H_k) \leq k(k-1)$. Consequently, $\mathrm{ti}(H_k) = k(k-1)$. Recall that the graph H_k has order $n = k(6k+1)$ and minimum degree $\delta(H_k) = 3$.

Choosing k sufficiently large so that

$$k > \frac{1}{6}\left(\frac{7}{6\varepsilon} - 1\right),$$

we have

$$\begin{aligned}\left(\frac{1}{6}-\varepsilon\right)n &= \left(\frac{1}{6}-\varepsilon\right)\cdot k(6k+1)\\ &< \left(\frac{1}{6}-\frac{7}{6(6k+1)}\right)\cdot k(6k+1)\\ &= k(k-1),\end{aligned}$$

and so

$$\mathrm{ti}(H_k) = k(k-1) > \left(\frac{1}{6}-\varepsilon\right)n. \qquad \square$$

7. CONCLUDING REMARKS AND OPEN PROBLEMS

In this paper, we considered the problem of augmenting a graph with at least three vertices to have a disjoint total dominating set and an independent set, that is, we considered the question of how many edges must be added to the graph G to ensure a partition of $V(G)$ into a TD-set and an ID-set in the resulting graph. We denoted this minimum number as the TI-augmentation number of G, denoted by $\mathrm{ti}(G)$.

The class $\mathcal{G}_\delta$. For $\delta \ge 1$, let $\mathcal{G}_\delta$ be the class of all graphs G of order at least 3 with minimum degree $\delta(G) \ge \delta$.

We pose the following problem.

Problem 7.1. Determine or estimate the best possible constants $c_{\mathrm{ti},\delta}$ (which depend only on δ) for each $\delta \ge 1$, such that $\mathrm{ti}(G) \le c_{\mathrm{ti},\delta} \times n(G)$ for all $G \in \mathcal{G}_\delta$. These constants are given by

$$c_{\mathrm{ti},\delta} = \sup_{G\in\mathcal{G}_\delta} \frac{\mathrm{ti}(G)}{n(G)}.$$

The following result presents lower bounds on the constants $c_{\mathrm{ti},\delta}$ for all $\delta \ge 1$.

Proposition 7.2. *The following lower bounds hold for the constants $c_{\mathrm{ti},\delta}$.*

(a) $c_{\mathrm{ti},1} \ge \frac{1}{3}$.
(b) $c_{\mathrm{ti},\delta} \ge \frac{1}{2\delta}$ *for all $\delta \ge 2$.*

Proof. If G is a path P_3, then we observe that G has order $n = 3$ and $\mathrm{ti}(G) = 1$, and so $c_{\mathrm{ti},\delta} \ge \frac{1}{3}$. For $\delta \ge 2$, let G be a complete bipartite graph $K_{\delta,\delta}$ with partite sets X and Y. The sets X and Y are the only two ID-sets of G. However, the partition $\{X, Y\}$ is not a partition of G into an ID-set and a TD-set noting that neither set X or Y is a TD-set of G. Hence, $\mathrm{ti}(G) \ge 1$. However, letting x and x' be two arbitrary (distinct) vertices that belong to the set X, the set $I' = X \setminus \{x\}$ and $S' = Y \cup \{x\}$ form a partition $\{I', S'\}$ of the graph $G' = G + xx'$ into an ID-set I' and TD-set S', and so the augmented graph G' is a TI-graph, implying that $\mathrm{ti}(G) \le 1$. Consequently, $\mathrm{ti}(G) = 1$, and so $c_{\mathrm{ti},\delta} \ge \frac{\mathrm{ti}(G)}{n(G)} = \frac{1}{2\delta}$. $\square$

It would be interesting to determine the exact values of the constants $c_{\mathrm{ti},\delta}$ for any given value of $\delta \geq 1$. By Proposition 7.2, we have $c_{\mathrm{ti},2} \geq \frac{1}{4}$ and $c_{\mathrm{ti},3} \geq \frac{1}{6}$. We pose the following conjectures.

Conjecture 7.3. $c_{\mathrm{ti},2} = \frac{1}{4}$.

Conjecture 7.4. $c_{\mathrm{ti},3} = \frac{1}{6}$.

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