Opuscula Math. 45, no. 2 (2025), 145–177
https://doi.org/10.7494/OpMath.2025.45.2.145 Opuscula Mathematica

REPRESENTATION OF SOLUTIONS OF LINEAR DISCRETE SYSTEMS WITH CONSTANT COEFFICIENTS AND WITH DELAYS

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Communicated by P.A. Cojuhari

Abstract. The paper surveys the results achieved in representing solutions of linear non-homogeneous discrete systems with constant coefficients and with delays and their fractional variants by using special matrices called discrete delayed-type matrices. These are used to express solutions of initial problems in a closed and often simple form. Then, results are briefly discussed achieved by such representations of solutions in stability, controllability and other fields. In addition, a similar topic is dealt with concerning linear non-homogeneous differential equations with delays and their variants. Moreover, some comments are given to this parallel direction pointing some important moments in the developing this theory. An outline of future perspectives in this direction is discussed as well.

Keywords: discrete linear system, constants coefficients, delay, discrete matrix functions, representation of solutions, commutative matrices, non-commutative matrices, controllability.

Mathematics Subject Classification: 39A06, 39A12, 39A27, 39A30.

1. THE PROBLEM AND INTRODUCTION

The paper's primary aim is to survey the results achieved in representing solutions of linear non-homogeneous discrete systems with constant coefficients and with delays and their fractional variants, by special matrices called discrete delayed-type matrices. These are used to express the solutions of initial problems in a closed and often simple form. The results mentioned in the paper have been motivated by four papers introducing discrete special matrix functions. While two of such papers, highly cited, written by Diblík and Khusainov [17, 18] at Oberwolfach, Germany (Mathematisches Forschungsinstitut Oberwolfach) during the RiP (research in pair) stay at April 2005, consider linear discrete systems with the first-order difference of the unknown variable and with a single delay, the papers by Diblík and Mencáková [11, 14], published recently, consider such systems but with the second-order difference of the unknown variable. These motivating papers have found numerous followers and the present paper tries to map their achievements.

Then, we briefly refer to results in stability, controllability and in other fields achieved by such representations of solutions. In addition to this survey, a similar topic is dealt with concerning linear non-homogeneous differential equations with delays and their fractional variants. Moreover, we give some comments to this parallel direction pointing out some important moments in developing this theory. An outline of future perspectives in this direction is discussed as well.

Let us recall some customary notations used throughout the paper. For integers s and q such that $s \le q$, we define a set $\mathbb{Z}_s^q := \{s, s+1, \dots, q\}$, where the case of $s = -\infty$ or $q = \infty$ is admitted, too. Using the notation $\mathbb{Z}_s^q$ we assume $s \le q$. The values of an empty sum and empty product are defined by formulas

$$\sum_{i=k+s}^{k} \dots := 0, \qquad \prod_{i=k+s}^{k} \dots := 1,$$

whenever s is a positive integer and dots "..." are replaced by the expression considered in a given computation. If operations with matrices are considered, we denote the zero matrix by Θ and the unit matrix by I rather than 0 and 1 in the right-hand sides of formulas. The binomial coefficients used are defined as

$$\binom{p}{q} := \begin{cases} \dfrac{p!}{q! \cdot (p-q)!} & \text{if } p \ge q \ge 0, \\ 0 & \text{otherwise,} \end{cases}$$

where p, q are integers and $0! := 1$. If a vector-function ϕ, where

$$\phi = (\phi_1, \dots, \phi_n)^T : \mathbb{Z}_\ell^\infty \to \mathbb{R}^n$$

and ℓ is an integer, is given, then its first-order forward difference $\Delta\phi(k)$, $k \in \mathbb{Z}_\ell^\infty$, and its forward differences of order greater than the first one $\Delta^i\phi(k)$, $i \ge 2$, are defined as

$$\Delta\phi(k) = \phi(k+1) - \phi(k), \quad \Delta^i\phi(k) = \Delta(\Delta^{i-1}\phi(k)).$$

For $t \in \mathbb{R}$, the function $\lfloor t \rfloor$ denotes the floor integer function while the notation $\lceil t \rceil$ is used for the ceiling integer function. For other definitions and notations we will refer when these are used.

In the paper, we consider equations with first-order or second-order differences as well as their variants appearing in the literature. Below, the description of basic equations is given with some of their modifications being described when particular cases are dealt with.

1.1. SYSTEMS WITH FIRST-ORDER DIFFERENCES

Let B_i, $i = 1, \dots, p$, be $n \times n$ real matrices, let m_i, $i = 1, \dots, p$, be integers satisfying $0 \le m_1 < m_2 < \dots < m_p$, and let $x = (x_1, \dots, x_n)^T$ be a vector. In the paper, we consider a system of equations (or its particular cases or variants) with the first-order difference

$$\Delta x(k) = \sum_{i=1}^{p} B_i x(k - m_i) + f(k), \tag{1.1}$$

where k is the independent variable varying in the domain $\mathbb{Z}_0^\infty$ and

$$f = (f_1, \ldots, f_n)^T : \mathbb{Z}_0^\infty \to \mathbb{R}^n$$

is a given discrete function (the superscript ^{T} denotes the transpose of a given vector or a matrix). The variable $x\colon \mathbb{Z}_0^\infty \to \mathbb{R}^n$ is dependent (unknown function). The constants m_i, $i = 1, \ldots, p$, are often called delays (except for m_1 if it equals zero). If a function $x = \psi$, where

$$\psi = (\psi_1, \ldots, \psi_n)^T : \mathbb{Z}_{-m_p}^\infty \to \mathbb{R}^n$$

satisfies equation (1.1) for all $k \in \mathbb{Z}_0^\infty$, we say that it is a solution to (1.1) on $\mathbb{Z}_0^\infty$. It is easy to prove that the initial problem (1.1), (1.2), where

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m_p}^0, \tag{1.2}$$

and $\varphi\colon \mathbb{Z}_{-m_p}^0 \to \mathbb{R}^n$, defines a unique solution to system (1.1). Since a similar statement is valid for various restrictions or extensions of this system, we will not mention it further.

1.2. SYSTEMS WITH SECOND-ORDER DIFFERENCES

We will also focus our attention to problems of representing solutions of systems (or its particular cases or variants) with the second-order forward difference

$$\Delta^2 x(k) + \sum_{i=1}^{p} B_i x(k - m_i) = f(k), \quad k \in \mathbb{Z}_0^\infty, \tag{1.3}$$

where B_i, m_i, p and f are described in part 1.1. In much the same way is defined a solution to (1.3) and the existence of a unique solution of initial problem.

1.3. THE STRUCTURE OF THE PAPER

The survey deals, as explained in parts 1.1, 1.2 with systems (1.1), (1.3) and with their variants. The formulas expressing solutions of initial problems use matrix functions called discrete "delayed" matrices. The structure of the paper is the following. Part 2 surveys the relevant results concerning the systems (1.1) and their variants while Part 3 deals with the progress achieved for systems (1.3) and their modifications. The formulas derived are expressed, in accordance with the theory of linear equations, as the sum of the solution of the adjoint homogeneous problem and a particular solution of a non-homogeneous system. Formulation of the problems, etc. will be repeated whenever it is reasonable as the authors often use different notations. In Part 4 some applications of the results are mentioned and in Part 5 some results concerning parallel investigation of continuous variants of the problem of representations of solutions are referred to. The paper closes with Part 6 listing the advantages of the representations derived and formulating some problems still unsolved.

2. SYSTEMS WITH FIRST-ORDER DIFFERENCES

2.1. TWO STARTING PAPERS – THE SIMPLEST CASES – SINGLE DELAY

In 2006, two papers by Diblík and Khusainov [17, 18] were published, presenting results that became the starting point for various generalizations and modifications. In these papers, a so-called discrete delayed matrix exponential is used. Below, we provide its definition.

The discrete matrix delayed exponential of an $n \times n$ constant matrix B is a discrete matrix function $\exp_m(Bk)$ such that

$$\mathrm{e}_m^{Bk} := \begin{cases} \Theta & \text{if} \quad k \in \mathbb{Z}_{-\infty}^{-m-1}, \\ I & \text{if} \quad k \in \mathbb{Z}_{-m}^{0}, \\ I + B \cdot \binom{k}{1} & \text{if} \quad k \in \mathbb{Z}_{1}^{m+1}, \\ I + B \cdot \binom{k}{1} + B^2 \cdot \binom{k-m}{2} & \text{if} \quad k \in \mathbb{Z}_{(m+1)+1}^{2(m+1)}, \\ \dots & \\ I + B \cdot \binom{k}{1} + B^2 \cdot \binom{k-m}{2} + \dots + B^{\ell} \cdot \binom{k-(\ell-1)m}{\ell} & \\ \text{if} \quad k \in \mathbb{Z}_{(\ell-1)(m+1)+1}^{\ell(m+1)},\ \ell = 0, 1, 2, \dots & \end{cases} \tag{2.1}$$

The above definition can be shortened to

$$\mathrm{e}_m^{Bk} := I + \sum_{j=1}^{\ell} B^j \cdot \binom{k-(j-1)m}{j} \tag{2.2}$$

for $k = (\ell-1)(m+1)+1, \dots, \ell(m+1)$ and $\ell = 0, 1, \dots$

The basic property of the discrete matrix exponential $\exp_m(Bk)$ can be formulated as follows (we refer to Theorem 2.1 in [18]):

Theorem 2.1. *For $k \in \mathbb{Z}_{-m}^{\infty}$, we have*

$$\Delta\ \mathrm{e}_m^{Bk} = B\, \mathrm{e}_m^{B(k-m)}\,. \tag{2.3}$$

The proof in [18] uses computations with binomial numbers being based, in principle, on repeated application of the well-known method of steps. The formula (2.3) states that the matrix $X(k) = \exp_m(Bk)$, $k \in \mathbb{Z}_{-m}^{\infty}$, is the unique matrix solution of the initial problem to the matrix equation:

$$\begin{aligned} \Delta X(k) &= BX(k-m), && k \in \mathbb{Z}_0^{\infty}, \\ X(k) &= I, && k \in \mathbb{Z}_{-m}^{0}. \end{aligned}$$

The delayed matrix exponential is used in the formula describing the solution of the initial problem:

$$\Delta x(k) = Bx(k-m) + f(k), \quad k \in \mathbb{Z}_0^\infty, \tag{2.4}$$

$$x(k) = \varphi(k), \qquad k \in \mathbb{Z}_{-m}^0, \tag{2.5}$$

where $f\colon \mathbb{Z}_0^\infty \to \mathbb{R}^n$ is a given function and $\varphi\colon \mathbb{Z}_{-m}^0 \to \mathbb{R}^n$ is the initial function. The solution of the problem (2.4), (2.5) is expressed by a formula in the following theorem (Theorem 3.6 in [18]).

Theorem 2.2. *On $\mathbb{Z}_{-m}^\infty$, the solution $x = x(k)$ of the initial Cauchy problem* (2.4), (2.5) *can be represented in the form*

$$x(k) = \mathrm{e}_m^{Bk}\ \varphi(-m) + \sum_{j=-m+1}^{0} \mathrm{e}_m^{B(k-m-j)}\ \Delta\varphi(j-1) + \sum_{j=1}^{k} \mathrm{e}_m^{B(k-m-j)}\ f(j-1). \tag{2.6}$$

The paper [17] considers a system

$$x(k+1) = Ax(k) + Bx(k-m) + f(k), \tag{2.7}$$

more general than the system (2.4). Assume that $m \geq 1$ is a fixed integer, $k \in \mathbb{Z}_0^\infty$, A and B are constant $n \times n$ matrices admitting commutative property

$$AB = BA, \tag{2.8}$$

and $\det A \neq 0$. The choice $A = I$ changes (2.7) to (2.4). The following transformation of (2.7) transforms this system into a system of the type (2.4) as well. Substituting

$$x(k) = A^k y(k)$$

with $k \in \mathbb{Z}_{-m}^\infty$ into (2.7), we get

$$y(k+1) = y(k) + B_1 y(k-m) + A^{-k-1} f(k) \tag{2.9}$$

with the variable matrix $B_1 = A^{-k-1} B A^{k-m}$. Due to property (2.8), we obtain

$$B_1 = A^{-1} B A^{-m}$$

and the matrix B_1 becomes constant. Using the difference operator, an equivalent form to (2.7) is

$$\Delta y(k) = B_1 y(k-m) + A^{-k-1} f(k), \quad k \in \mathbb{Z}_0^\infty.$$

Together with equation (2.7), consider the initial problem (2.5). Then, the corresponding equivalent initial data for system (2.9) are

$$y(k) = A^{-k}\varphi(k), \quad k \in \mathbb{Z}_{-m}^0.$$

Before giving a formula for a representation of solution to problem (2.7), (2.5), consider an initial problem for the homogeneous linear matrix equation:

$$X(k+1) = AX(k) + BX(k-m), \quad k \in \mathbb{Z}_0^\infty, \tag{2.10}$$

$$X(k) = A^k, \qquad k \in \mathbb{Z}_{-m}^0. \tag{2.11}$$

Here $X\colon \mathbb{Z}_{-m}^\infty \to \mathbb{R}^{n\times n}$ is an unknown matrix.

Theorem 2.3 ([17, Theorem 2.2])**.** *The matrix*

$$X = X_0(k) := A^k \, \mathrm{e}_m^{B_1 k}, \quad k \in \mathbb{Z}_{-m}^\infty,$$

solves the problem (2.10)*,* (2.11)*.*

Now it is possible to give a formula solving the problem (2.7), (2.5) (we refer to [17, Theorem 3.5]).

Theorem 2.4. *On $\mathbb{Z}_{-m}^\infty$, the solution $x = x(k)$ of the initial problem* (2.7)*,* (2.5) *can be represented as*

$$x(k) = X_0(k)A^{-m}\varphi(-m) + A^m \sum_{j=-m+1}^{0} X_0(k-m-j)\,[\varphi(j) - A\varphi(j-1)] + A^m \sum_{j=1}^{k} X_0(k-m-j) f(j-1).$$

Results of [17] are used in [53], where the problem

$$x(k+1) = Ax(k) + Bx(k-m) + f(k), \quad k \in \mathbb{Z}_0^\infty, \tag{2.12}$$

$$x(k) = \varphi(k), \qquad k \in \mathbb{Z}_{-m}^0 \tag{2.13}$$

with impulses

$$x(k+1) = Cx(k+1-0) + J_{k+1}, \quad k \in \mathbb{Z}_0^\infty, \tag{2.14}$$

is studied. In (2.12)–(2.14), A, B and C are regular $n\times n$ matrices, $f(k)$, $k \in \mathbb{Z}_0^\infty$, $\varphi(k)$, $k \in \mathbb{Z}_{-m}^0$ and J_{k+1} are n-dimensional column vectors, $m > 0$ is an integer and x is an unknown n-dimensional column vector. By $x(k+1-0)$ is denoted the value of x at the point $k+1-0$ (i.e. the limit on the left) which is immediately switched to the value $x(k+1)$ by (2.14). The following formula for the solutions of the problem (2.12)–(2.14) is derived ([53, Theorem 1]).

Theorem 2.5. *Let A, B, C be constant regular $n\times n$ matrices satisfying $ACB = BCA$, let m be a fixed positive integer, and let $J_i \in \mathbb{R}^n$, $i \in \mathbb{Z}_1^\infty$. Then, the solution of the problem* (2.12)–(2.14) *can be expressed as*

$$x(k) = X_0(k)(CA)^m\varphi(-m) + (CA)^m \sum_{j=-m+1}^{0} X_0(k-m-j)\,[\varphi(j) - CA\varphi(j-1)] + (CA)^m \sum_{i=1}^{k} X_0(k-m-i)\,[Cf(i-1) + J_i]\,,$$

where $k \in \mathbb{Z}_{-m}^\infty$, $X_0(k) = (CA)^k \mathrm{e}_m^{B_1 k}$, $B_1 = (CA)^{-1}CB(CA)^{-m}$.

2.2. TWO DELAYS AND COMMUTATIVE MATRICES

The paper by Diblík and Morávková [15] provides a definition of the discrete matrix function $\exp_m(Bk)$, given by formulas (2.1), (2.2), extended to two matrices and two delays. Two variants of this generalization are presented.

Define a discrete $n \times n$ matrix function e_{mr}^{BCk}, $k \in \mathbb{Z}_{-\infty}^{+\infty}$, called the discrete matrix delayed exponential for two delays $m, r \in \mathbb{N}$, $m \neq r$ and for two $n \times n$ commuting constant matrices B, C as follows. Set

$$p_{(k)} := \left\lfloor \frac{k+m}{m+1} \right\rfloor \quad \text{and} \quad q_{(k)} := \left\lfloor \frac{k+r}{r+1} \right\rfloor .$$

Definition 2.6 ([15, Definition 2])**.** Let B, C be constant $n \times n$ matrices with $BC = CB$ and let $m, r \in \mathbb{N}$, $m < r$ be fixed integers. We define a discrete $n \times n$ matrix function e_{mr}^{BCk} called the discrete matrix delayed exponential for two delays m, r and for two $n \times n$ constant matrices B, C by the formula

$$\mathrm{e}_{mr}^{BCk} := \begin{cases} \Theta & \text{if} \quad k \in \mathbb{Z}_{-\infty}^{-\max\{m,r\}-1}, \\ I & \text{if} \quad k \in \mathbb{Z}_{-\max\{m,r\}}^{0}, \\ I + (B+C) \sum\limits_{i=0,j=0}^{p_{(k)}-1, q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-mi-rj}{i+j+1} & \text{if} \quad k \in \mathbb{Z}_1^{\infty}. \end{cases} \tag{2.15}$$

The main property of e_{mr}^{BCk} is given by the following theorem.

Theorem 2.7 ([15, Theorem 2])**.** *Let B, C be constant $n \times n$ matrices with $BC = CB$ and let $m, r \in \mathbb{N}$, where $m \neq r$, be fixed integers. Then,*

$$\Delta\, \mathrm{e}_{mr}^{BCk} = B\, \mathrm{e}_{mr}^{BC(k-m)} + C\, \mathrm{e}_{mr}^{BC(k-r)} \tag{2.16}$$

holds for $k \in \mathbb{Z}_0^{\infty}$.

Another form of a discrete matrix delayed exponential $\tilde{\mathrm{e}}_{mr}^{BCk}$ can be found in the paper by Diblík and Morávková [16].

Definition 2.8 ([16, Definition 6])**.** Let B, C be constant $n \times n$ matrices with $BC = CB$ and let $m, r \in \mathbb{N}$, where $m < r$, be fixed integers. We define a discrete $n \times n$ matrix function $\tilde{\mathrm{e}}_{mr}^{BCk}$ called the discrete matrix delayed exponential for two delays m, r and for two $n \times n$ constant matrices B, C as follows:

$$\tilde{\mathrm{e}}_{mr}^{BCk} := \begin{cases} \Theta & \text{if} \quad k \in \mathbb{Z}_{-\infty}^{-1}, \\ I & \text{if} \quad k \in \mathbb{Z}_0^{m}, \\ I + B \sum_{i=0,j=0}^{p_{(k)}-1, q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-m-mi-rj}{i+j+1} & \\ \quad + C \sum_{i=0,j=0}^{p_{(k)}-1, q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-r-mi-rj}{i+j+1} & \text{if} \quad k \in \mathbb{Z}_{m+1}^{\infty}. \end{cases} \tag{2.17}$$

Remark 2.9. For $k \in \mathbb{Z}_0^r$, it is easy to deduce that $\tilde{\mathrm{e}}_{mr}^{BCk} = \mathrm{e}_m^{B(k-m)}$.

The main property of $\tilde{\mathrm{e}}_{mn}^{BCk}$ copies Theorem 2.7 and is given by following theorem.

Theorem 2.10 ([16, Theorem 9]). *Let B, C be constant $n\times n$ matrices with $BC = CB$ and let $m, r \in \mathbb{N}$, where $m < r$, be fixed integers. Then*

$$\Delta \tilde{\mathrm{e}}_{mr}^{BCk} = B\tilde{\mathrm{e}}_{mr}^{BC(k-m)} + C\tilde{\mathrm{e}}_{mr}^{BC(k-r)} \tag{2.18}$$

for $k \in \mathbb{Z}_0^\infty$.

Both Theorem 2.7 and Theorem 2.10 were proved by direct analogy with the proof of Theorem 2.1 in [18], using properties of binomial numbers and the method of steps.

Consider the discrete system

$$\Delta x(k) = Bx(k-m) + Cx(k-r) + f(k), \tag{2.19}$$

where $m, r \in \mathbb{N}$, $m \neq r$, are fixed, $k \in \mathbb{Z}_0^\infty$, B and C are constant $n \times n$ matrices, $f\colon \mathbb{Z}_0^\infty \to \mathbb{R}^n$ is a given $n \times 1$ vector, and $x\colon \mathbb{Z}_0^\infty \to \mathbb{R}^n$ is an $n \times 1$ unknown vector. Together with equation (2.19), we consider the initial problem

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-r}^0. \tag{2.20}$$

With the aid of both discrete matrix delayed exponentials, formulas for the solution to the initial problem (2.19), (2.20) are given.

Theorem 2.11 ([16, Theorem 14]). *Let B, C be constant $n \times n$ matrices with $BC = CB$, $\det(B + C) \neq 0$, and let $m, r \in \mathbb{N}$, where $m < r$, be fixed integers. Then, the solution of the initial problem (2.19), (2.20) can be expressed in the form*

$$x(k) = \sum_{j=0}^{r} \mathrm{e}_{mr}^{BC(k+j)}\, v_j + \sum_{\ell=1}^{k} \tilde{\mathrm{e}}_{mr}^{BC(k-\ell)} f(\ell-1),$$

where $k \in \mathbb{Z}_{-r}^\infty$ and

$$v_0 = \varphi(-r) - \sum_{s=1}^{r} v_s,$$

$$v_\ell = (B+C)^{-1}\left[\Delta\varphi(-\ell) - \sum_{t=1}^{r-\ell} \Delta\, \mathrm{e}_{mr}^{BCt}\, v_{t+\ell}\right], \quad \ell \in \mathbb{Z}_1^r.$$

Theorem 2.12 ([16, Theorem 15]). *Let B, C be constant $n \times n$ matrices with $BC = CB$ and let $m, r \in \mathbb{N}$, where $m < r$, be fixed integers. Then, the solution of the initial problem (2.19), (2.20) can be expressed in the form*

$$x(k) = \sum_{j=0}^{r} \tilde{\mathrm{e}}_{mr}^{BC(k+j)} w_j + \sum_{\ell=1}^{k} \tilde{\mathrm{e}}_{mr}^{BC(k-\ell)} f(\ell-1),$$

where $k \in \mathbb{Z}_{-r}^\infty$ and

$$\begin{aligned}
w_\ell &= \Delta\varphi(-\ell-1) - \Delta\tilde{\mathrm{e}}_{mr}^{BC(-\ell+r-1)}\varphi(-r)\\
&\quad - \sum_{s=-r}^{-\ell-m-2} \Delta\tilde{\mathrm{e}}_{mr}^{BC(-\ell-s-2)}\Delta\varphi(s), \qquad \ell\in\mathbb{Z}_0^{r-m-1},\\
w_\ell &= \Delta\varphi(-\ell-1), \qquad\qquad\qquad\qquad\qquad \ell\in\mathbb{Z}_{r-m}^{r-1},\\
w_r &= \varphi(-r).
\end{aligned}$$

In the paper [49] by Medved'and Pospíšil, the homogeneous problem is considered. However, their approach employs a different construction of the delayed matrix exponential. We will illustrate this using the case of two matrices. Consider the matrix equation

$$\Delta Y(k) = B_1 Y(k-m_1) + B_2 Y(k-m_2), \quad k\in\mathbb{Z}_0^\infty \tag{2.21}$$

with the initial problem

$$Y(k) = \begin{cases} \Theta, & k\in\mathbb{Z}_{-\infty}^{-1},\\ I, & k=0. \end{cases} \tag{2.22}$$

Set

$$X(k) = \mathrm{e}_{m_1}^{B(k-m_1)}. \tag{2.23}$$

This matrix function solves (by Theorem 2.1) the matrix equation

$$\Delta X(k) = B_1 X(k-m_1)$$

and satisfies the same initial data (2.22), that is,

$$X(k) = \begin{cases} \Theta, & k\in\mathbb{Z}_{-\infty}^{-1},\\ I, & k=0. \end{cases}$$

Define a matrix exponential for two matrices (which differs from definitions given by (2.15) and (2.17))

$$\hat{\mathrm{e}}_{m_1,m_2}^{B_1B_2k} = \begin{cases} \Theta \quad \text{if} \quad k\in\mathbb{Z}_{-\infty}^{-m_2-1},\\ X(k+m_2) + B_2\sum_{j_1=1}^{k} X(k-j_1)X(j_1-1)+\dots\\ +B_2^l\sum_{j_1=(l-1)(m_2+1)+1}^{k}\sum_{j_2=(l-1)(m_2+1)+1}^{j_1}\dots\sum_{j_l=(l-1)(m_2+1)+1}^{j_{l-1}} X(k-j_1)\\ \times\prod_{i=1}^{l-1} X(j_i-j_{i+1})X(j_l-(l-1)(m_2+1)-1)\\ \text{if} \quad k\in\mathbb{Z}_{(l-1)(m_2+1)+1}^{l(m_2+1)},\ l\in\mathbb{Z}_0^\infty. \end{cases}$$

Then, (we refer to Theorem 2.2 in [49]) the below theorem can be proved (the proof is based on the method of induction).

Theorem 2.13. *Let $m_1, m_2 \geq 1$ and B_1, B_2 be $n\times n$ matrices such that $B_1B_2 = B_2B_1$. Then, the matrix solution of equation* (2.21) *satisfying* (2.22) *has the form*

$$Y(k) = \hat{\mathrm{e}}_{m_1,m_2}^{B_1B_2(k-m_2)}.$$

Consider an equation

$$\Delta x(k) = B_1 x(k-m_1) + B_2 x(k-m_2), \quad k \in \mathbb{Z}_0^{\infty} \tag{2.24}$$

with the initial problem

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}^{0}, \tag{2.25}$$

where m is fixed below. The solution formula of the problem (2.24), (2.25) is given below (we refer to Theorem 2.6 in [49]).

Theorem 2.14. *Let $1 \le m_1 \le m := m_2$, $\varphi\colon \mathbb{Z}_{-m}^{0} \to \mathbb{R}^n$ be a given function and B_1, B_2 be $n \times n$ permutable matrices. Then, the solution of the problem* (2.24), (2.25) *has the form*

$$\begin{aligned} x(k) = {} & Y(k+m)\varphi(-m) + \sum_{j=-m+1}^{0} Y(k-j)\Delta\varphi(j-1) \\ & - B_1 \sum_{j=-m}^{-1-m_1} Y(k-1-m_1-j)\varphi(j), \quad \forall\, k \in \mathbb{Z}_{-m}^{\infty}. \end{aligned}$$

2.3. MULTIPLE DELAYS AND COMMUTATIVE MATRICES

Theorems 2.13, 2.14 are generalized in the same paper [49] to the case of an arbitrary number of delays. The authors consider the matrix equation

$$\Delta Y(k) = B_1 Y(k-m_1) + \ldots + B_p Y(k-m_p), \quad k \in \mathbb{Z}_0^{\infty} \tag{2.26}$$

and initial problem (2.22). It is assumed that $p > 1$, $m_i \ge 1$, $i = 1, \ldots, p$, with $B_1, \ldots, B_p$ being pairwise permutable matrices, i.e., $B_i B_j = B_j B_i$ for each $i, j \in \{1, \ldots, p\}$. In accordance with Definition 3.1 in [49], for each $j = 2, \ldots, p$, define a discrete multi-delayed matrix exponential corresponding to delays $m_i \ge 1$, $i = 1, \ldots, j$ and to matrices $B_1, \ldots, B_j$ as follows:

$$\hat{\mathrm{e}}_{m_1,\ldots,m_j}^{B_1,\ldots,B_j k} = \begin{cases} \Theta \quad \text{if} \quad k \in \mathbb{Z}_{-\infty}^{-m_j-1}, \\ X_{j-1}(k+m_j) + B_j \sum_{i_1=1}^{k} X_{j-1}(k-i_1) X_{j-1}(i_1-1) + \ldots \\ + B_j^l \sum_{i_1=(l-1)(m_j+1)+1}^{k} \sum_{i_2=(l-1)(m_j+1)+1}^{i_1} \\ \quad \ldots \sum_{i_l=(l-1)(m_j+1)+1}^{i_{l-1}} X_{j-1}(k-i_1) \\ \times \prod_{s=1}^{l-1} X_{j-1}(i_s - i_{s+1}) X_{j-1}(i_l - (l-1)(m_j+1) - 1) \\ \text{if} \quad k \in \mathbb{Z}_{(l-1)(m_j+1)+1}^{l(m_j+1)},\ l \in \mathbb{Z}_0^{\infty}, \end{cases}$$

where

$$X_{j-1}(k) = \hat{\mathrm{e}}_{m_1,\ldots,m_{j-1}}^{B_1,\ldots,B_{j-1}(k-m_{j-1})},$$

and, if $j = 2$, by formula (2.23),

$$X_1(k) = \hat{\mathrm{e}}_{m_1}^{B_1(k-m_1)} := X(k) = \mathrm{e}_{m_1}^{B(k-m_1)}.$$

The following theorem is Theorem 3.2 in [49].

Theorem 2.15. *Let $p \geq 1$, $m_i \geq 1$, $i = 1, \dots, p$, and $B_1, \dots, B_p$ be pairwise permutable matrices. Then, the matrix solution of equation* (2.26) *satisfying condition* (2.22) *has the form*

$$Y(k) = \begin{cases} \hat{\mathrm{e}}_{m_1}^{B_1(k-m_1)} & \text{if } \ p = 1, \\ \hat{\mathrm{e}}_{m_1,\dots,m_p}^{B_1,\dots,B_p(k-m_p)} & \text{if } \ p > 1. \end{cases}$$

Consider a problem

$$\Delta x(k) = \sum_{i=1}^{p} B_i x(k - m_i) + f(k), \quad k \in \mathbb{Z}_0^{\infty}, \tag{2.27}$$

$$x(k) = \psi(k), \qquad k \in \mathbb{Z}_{-m}^{0}, \tag{2.28}$$

where $m = \max\{m_1, \dots, m_p\}$. By Theorem 2.15, the following formula is derived for the solution of nonhomogeneous linear difference initial problem with multiple delays (2.27), (2.28) ([49, Theorem 3.6]).

Theorem 2.16. *Let $p > 1$, $m_1, \dots, m_p \geq 1$, $B_1, \dots, B_p$ be $n \times n$ pairwise permutable matrices, $f\colon \mathbb{Z}_0^{\infty} \to \mathbb{R}^n$, and $\psi\colon \mathbb{Z}_{-m}^{0} \to \mathbb{R}^n$. Then, the solution $x = x(k)$ of the initial value problem* (2.27), (2.28) *is on $\mathbb{Z}_{-m}^{\infty}$ represented by the formula*

$$\begin{aligned} x(k) &= Y(k+m)\psi(-m) + \sum_{j=-m+1}^{0} Y(k-j)\Delta\psi(j-1) \\ &\quad - \sum_{i=1}^{p} B_i \sum_{j=-m}^{-1-m_i} Y(k-1-m_i-j)\psi(j) + \sum_{j=1}^{k} Y(k-j)f(j-1), \end{aligned}$$

where

$$Y(k) = \hat{\mathrm{e}}_{m_1,\dots,m_p}^{B_1,\dots,B_p(k-m_p)}.$$

If in (2.27), a non-delayed term appears, i.e., if the problem (2.27), (2.28) is interchanged with a problem (2.29), (2.28), where

$$x(k+1) = Ax(k) + \sum_{i=1}^{p} B_i x(k - m_i) + f(k), \quad k \in \mathbb{Z}_0^{\infty}, \tag{2.29}$$

then, in accordance with the Corollary 3.7 in [49], we obtain the following result.

Theorem 2.17. *Let $p > 1$, $m_1, \dots, m_p \geq 1$, $A, B_1, \dots, B_p$ be $n \times n$ pairwise permutable matrices, $\det A \neq 0$, $f\colon \mathbb{Z}_0^{\infty} \to \mathbb{R}^n$, and $\psi\colon \mathbb{Z}_{-m}^{0}, \to \mathbb{R}^n$. Then, the solution $x = x(k)$ of the initial value problem* (2.29), (2.28) *is on $\mathbb{Z}_{-m}^{\infty}$ represented by the formula*

$$\begin{aligned} x(k) &= Y^*(k+m)\psi(-m) + \sum_{j=-m+1}^{0} Y^*(k-j)[\psi(j) - A\psi(j-1)] \\ &\quad - \sum_{i=1}^{p} B_i \sum_{j=-m}^{-1-m_i} Y^*(k-1-m_i-j)\psi(j) + \sum_{j=1}^{k} Y^*(k-j)f(j-1), \end{aligned}$$

where

$$Y^*(k) = A^k \hat{\mathrm{e}}_{m_1,\dots,m_p}^{B_1^*,\dots,B_p^*(k-m_p)}, \quad \textit{and} \quad B_i^* = B^i A^{-1-m_i}, \quad i = 1,\dots,p.$$

Although the latter results formulated in Theorems 2.15, 2.16, and 2.17 were successfully applied in the same paper to prove results on exponential stability, nevertheless, for practical computations, they are not very suitable. The paper by Pospíšil [56] provides a representation of a solution of a nonhomogeneous linear difference equation with multiple delays in a closed form, and directly generalizes the results given in [16, 18]. The main results are proved by using the $\mathcal{Z}$-transform provided that the nonhomogeneous vector is exponentially bounded. Then, this assumption may be omitted because the formulas derived hold without such a restriction.

In [56] the following problem is treated. Let us consider the system (1.1), i.e.,

$$\Delta x(k) = \sum_{i=1}^{p} B_i x(k-m_i) + f(k), \tag{2.30}$$

where matrices $B_1, B_2, \dots, B_p$ are pairwise permutable, $f\colon \mathbb{Z}_0^\infty \to \mathbb{R}^n$, and the initial problem (1.2), i.e.,

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}^0, \tag{2.31}$$

where $\varphi\colon \mathbb{Z}_{-m}^0 \to \mathbb{R}^n$ and $m = \max\{m_1,\dots,m_p\}$. The proof of the results on a representation of a solution to the problem (2.30), (2.31) is performed via $\mathcal{Z}$ transform.

The paper [56] derives an alternative formula to formula (2.6) solving the initial problem (2.4), (2.5). Let us use the notation $\mathcal{A}(k) = \mathrm{e}_m^{Bk}$ for delayed matrix exponential e_m^{Bk}, where formula

$$\mathcal{A}(k) = \sum_{i=0}^{\left\lfloor \frac{k+m}{m+1} \right\rfloor} B^i \binom{k-m(i-1)}{i}$$

holds. Theorem 3.1 and Corollary 3.3 in [56] imply the following theorem.

Theorem 2.18. *On $\mathbb{Z}_1^\infty$, the solution $x = x(k)$ of the initial value problem* (2.4), (2.5) *can be represented in the form*

$$\begin{aligned} x(k) = {} & \mathcal{A}(k-m)\varphi(0) + B \sum_{j=-m}^{-1} \mathcal{A}(k-1-2m-j)\varphi(j) \\ & + \sum_{j=1}^{k} \mathcal{A}(k-m-j) f(j-1). \end{aligned} \tag{2.32}$$

To generalize Theorem 2.18 to the problem with multiple delays, we use a multinomial coefficient

$$\binom{a}{b_1\ \dots\ b_m} = \frac{a!}{b_1!\ \dots\ b_m!},$$

for an integer $m > 0$ and $a, b_1, \dots, b_m \in \mathbb{Z}_0^\infty$, and the matrix

$$\mathcal{B}(k) = \sum_{\substack{i_1,\dots,i_p \geq 0, \\ \sum_j^p (m_j+1)i_j \leq k}} \binom{k - \sum_{j=1}^p m_j i_j}{i_1, \dots, i_p, k - \sum_{j=1}^p (m_j+1) i_j} \prod_{j=1}^p B_j^{i_j}.$$

Theorem 3.4 and Corollary 3.6 in [56] give the following representation of solution.

Theorem 2.19. *Let $p > 1$, $m_i \geq 1$, $i = 1, \dots, p$, $B_1, \dots, B_p$ be pairwise permutable $n \times n$ matrices, $f \colon \mathbb{Z}_0^\infty \to \mathbb{R}^n$, and $\varphi \colon \mathbb{Z}_{-m}^0, \to \mathbb{R}^n$. On $\mathbb{Z}_0^\infty$, the solution $x = x(k)$ of the initial value problem* (2.30), (2.31) *can be represented in the form*

$$\begin{aligned} x(k) = \mathcal{B}(k)\varphi(0) &+ \sum_{j=1}^p B_j \sum_{i=-m_j}^{-1} \mathcal{B}(k - i - m_j - 1)\varphi(i) \\ &+ \sum_{j=1}^k \mathcal{B}(k-j) f(j-1). \end{aligned} \tag{2.33}$$

Let us remark that there is a formal analogy between formulas (2.32) and (2.33). If $p = 1$, the matrix function $\mathcal{B}$ becomes the matrix function $\mathcal{A}(\cdot - m)$, where $B = B_1$, $m = m_1$. Moreover, the matrix function $\mathcal{B}$ solves the matrix equation

$$\Delta X(k) = \sum_{i=1}^{m_p} B_i X(k - m_i), \quad k \in \mathbb{Z}_0^\infty$$

with initial problem

$$X(k) = \begin{cases} \Theta, & k \in \mathbb{Z}_{-\infty}^{-1}, \\ I, & k = 0. \end{cases}$$

Then one can see, by analogy with (2.2) and Definitions 2.6, 2.8, that

$$X(k) = \mathcal{B}(k) = \mathrm{e}_{m_1 \dots m_p}^{B_1 \dots B_p (k - m_p)}, \quad k \in \mathbb{Z}_{-\infty}^\infty$$

and, for $p = 2$ (we refer to formula (2.17)),

$$\mathcal{B}(k) = \tilde{\mathrm{e}}_{m_1 m_2}^{B_1 B_2 k}, \quad k \in \mathbb{Z}_{-\infty}^\infty.$$

2.3.1. Formula solving the problem with a nondelayed term

Consider a system with nondelayed linear term

$$x(k+1) = Ax(k) + \sum_{i=1}^p B_i x(k - m_i) + f(k), \quad k \in \mathbb{Z}_0^\infty, \tag{2.34}$$

where $\det A \neq 0$, matrices $A, B_1, B_2, \dots, B_p$ are pairwise permutable, with the initial problem being

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m_p}^0. \tag{2.35}$$

Set

$$\tilde{\mathcal{B}}(k) = A^k \sum_{\substack{i_1,\dots,i_p \geq 0, \\ \sum_j^p (m_j+1) i_j \leq k}} \binom{k - \sum_{j=1}^p m_j i_j}{i_1, \dots, i_p, k - \sum_{j=1}^p (m_j+1) i_j} \prod_{j=1}^p \tilde{B}_j^{i_j}$$

and

$$\tilde{B}_i = B_i A^{-1-m_i}, \quad i = 1, \dots, p.$$

The following result reproduces Theorem 3.7 in [56].

Theorem 2.20. *Let $p > 1$, $m_i \geq 1$, $i = 1, \dots, p$, $A, B_1, \dots, B_p$ be pairwise permutable $n \times n$ matrices, $\det A \neq 0$, $f\colon \mathbb{Z}_0^\infty \to \mathbb{R}^n$, and $\varphi\colon \mathbb{Z}_{-m}^0, \to \mathbb{R}^n$. On $\mathbb{Z}_1^\infty$, the solution $x = x(k)$ of the initial value problem* (2.34), (2.35) *can be represented in the form*

$$x(k) = \tilde{\mathcal{B}}(k)\varphi(0) + \sum_{j=1}^p B_j \sum_{i=-m_j}^{-1} \tilde{\mathcal{B}}(k-i-m_j-1)\varphi(i) + \sum_{j=1}^k \tilde{\mathcal{B}}(k-j) f(j-1).$$

2.4. NONCOMMUTATIVE MATRICES

Obviously, the assumption of pairwise commutativity of the matrices of linear terms is very restrictive. Therefore, further efforts were made to dispense with this assumption and to derive formulas representing solutions in such a case. The non-commutativity of matrices does not make it possible to change the order when multiplying matrices so that the problem to get formulas for solutions becomes much more difficult. The paper by Mahmudov [42] is pioneering in this direction overcoming the restriction of commutativity. Another novelty is that the matrices of the delayed parts are non-constant. Below we list the principal achievements of this paper. Consider a discrete system

$$x(k+1) = Ax(k) + B_k x(k-m) + f(k), \quad k \in \mathbb{Z}_0^\infty, \tag{2.36}$$

where $m \geq 1$ is a fixed integer, A and B_k are $n \times n$ matrices (their commutativity is not assumed), $\det A \neq 0$, $f\colon \mathbb{Z}_0^\infty \to \mathbb{R}^n$, and the initial problem

$$x(k) = \psi(k), \quad k \in \mathbb{Z}_{-m}^0, \tag{2.37}$$

$\psi\colon \mathbb{Z}_{-m}^0 \to \mathbb{R}^n$. Substituting in (2.36), (2.37)

$$z(k) := A^{-k} x(k), \quad D_k := A^{-k-1} B_k A^{k-m}, \quad k \in \mathbb{Z}_0^\infty,$$

we get an equivalent linear system of the form

$$\begin{aligned} z(k+1) &= z(k) + D_k z(k-m) + A^{-k-1} f(k), && k \in \mathbb{Z}_0^\infty, \\ z(k) &= A^{-k}\psi(k), && k \in \mathbb{Z}_{-m}^0. \end{aligned}$$

Below a discrete delayed matrix exponential is introduced depending on the sequence of matrices $\mathcal{D} = \{D_1, D_2, \dots\}$. The following matrix is introduced:

$$P^{\mathcal{D}}(k,d) = \begin{cases} I \quad \text{if} \quad l = d = 0, \\ \sum\limits_{j_1=(d-1)(m+1)}^{k-1} D_{j_1} \sum\limits_{j_2=(d-1)(m+1)}^{j_1} D_{j_2-m-1} \\ \quad \dots \sum\limits_{j_d=(d-1)(m+1)}^{j_d-1} D_{j_d-(d-1)(m+1)}, \\ \text{if} \quad k \in \mathbb{Z}_{(d-1)(m+1)+1}^{l(m+1)} \quad \text{and} \quad l \in \mathbb{Z}_1^{\infty}, \quad 1 \le d \le l. \end{cases}$$

Applying $P^{\mathcal{D}}(k,d)$, the delayed exponential depending on the sequence of matrices is defined as

$$e_m^{\mathcal{D}}(k) = \begin{cases} \Theta & \text{if } k \in \mathbb{Z}_{-\infty}^{-(m+1)}, \\ I & \text{if } k \in \mathbb{Z}_{-m}^{0}, \\ I + \sum\limits_{d=1}^{l} P^{\mathcal{D}}(k,d) & \text{if } k \in \mathbb{Z}_{(l-1)(m+1)+1}^{l(m+1)} \quad \text{and} \quad l \in \mathbb{Z}_1^{\infty}. \end{cases}$$

By such a matrix delayed exponential, it is possible to solve the following linear matrix equation:

$$\Phi(k+1) = A\Phi(k) + D_k\Phi(k-m), \quad k \in \mathbb{Z}_0^{\infty}, \tag{2.38}$$
$$\Phi(k) = A^k, \qquad k \in \mathbb{Z}_{-m}^{0}. \tag{2.39}$$

Theorem 2.21 ([42, Theorem 3])**.** *The matrix*

$$\Phi(k) = \Psi(k) := A^k e_m^{\mathcal{D}}(k), \quad k \in \mathbb{Z}_{-m}^{\infty},$$

solves the problem (2.38)*,* (2.39)*.*

Using $\Psi(k)$, the representation of the solution to the nonhomogeneous delay problem (2.36), (2.37) (we refer to Corollary 6 in [42]) is the following.

Theorem 2.22. *On $\mathbb{Z}_{-m}^{\infty}$, the solution $x = x(k)$ of the problem* (2.36)*,* (2.37) *can be represented in the form*

$$\begin{aligned} x(k) = \Psi(k)A^{-m}\psi(-m) + A^m \sum_{j=-m+1}^{0} A^j \Psi(k-m-j)A^{-j}\left[\psi(j) - A\psi(j-1)\right] \\ + A^m \sum_{j=1}^{k} A^j \Psi(k-m-j)A^{-j} f(j-1). \end{aligned}$$

For the case of two matrices, the paper by Mahmudov [45] gives a partial answer to an open problem from the paper by Diblík and Mencáková [14] related to the applicability of $\mathcal{Z}$ transform to systems with non-permutable matrices. The problem considered is not as general as (2.36), (2.37) and concerns the following system with constant matrices

$$x(k+1) = Ax(k) + Bx(k-m) + f(k), \quad k \in \mathbb{Z}_0^{\infty}, \tag{2.40}$$

where $m \geq 1$ is a fixed integer, A and B are constant non-permutable $n \times n$ matrices and, unlike of the system (2.36), the regularity of A is not required, $f\colon \mathbb{Z}_0^{\infty} \to \mathbb{R}^n$, and the initial value problem is

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}^{0}, \tag{2.41}$$

$\varphi\colon \mathbb{Z}_{-m}^{0} \to \mathbb{R}^n$. The substantial novelty here is a modified approach – for the representation of solutions, a delayed perturbation of discrete matrix exponential is defined and used. For the given matrices A, B and delay $m \geq 1$, a delayed perturbation of discrete matrix exponential $X_m^{A,B}(k)\colon \mathbb{Z}_0^{\infty} \to \mathbb{R}^{n\times n}$ is defined by the formula

$$X_m^{A,B}(k) = \sum_{j=0}^{\left\lfloor \frac{k+m}{m+1} \right\rfloor} Q(k+m-mj; j),$$

where

$$Q(k;j) = \begin{cases} \Theta & \text{if} \quad j \in \mathbb{Z}_{-\infty}^{-1}, \\ A^k \sigma(k) & \text{if} \quad j = 0, \\ \sum\limits_{l=j}^{k} A^{k-l} B Q(l-1; j-1)\sigma(k-j) & \text{if} \quad j \in \mathbb{Z}_1^{\infty} \end{cases}$$

and σ is the Heaviside step function defined as

$$\sigma(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases} \tag{2.42}$$

The following theorem is proved (Theorem 3.6 in [45]).

Theorem 2.23. *Let $m \geq 1$, A, B be constant $n \times n$ matrices, $\varphi\colon \mathbb{Z}_{-m}^{0} \to \mathbb{R}^n$. The solution $x = x(k)$ of the problem* (2.40), (2.41) *has the following form*

$$x(k) = X_m^{A,B}(k-m)\varphi(0) + \sum_{i=-m}^{-1} X_m^{A,B}(k-1-2m-i)B\varphi(i) + \sum_{i=1}^{k} X_m^{A,B}(k-m-i) f(i-1)$$

for $k \in \mathbb{Z}_0^{\infty}$.

A variant of this theorem is proved in [45], Theorem 3.1, assuming that $f\colon \mathbb{Z}_0^{\infty} \to \mathbb{R}^n$ is exponentially bounded, by applying the $\mathcal{Z}$-transform. Then, the result is verified without this restriction.

A representation of solutions of a discrete problem with impulses is considered by Jin, Wang and Shen in [29]. Consider a system

$$x(t+1) = Ax(t) + A_1 x(t-m) + f(t), \quad t \in \mathbb{Z}_0^{\infty}, \tag{2.43}$$

together with the initial problem

$$x(t) = \varphi(t), \quad t \in \mathbb{Z}_{-m}^{0} \tag{2.44}$$

and an impulse requirement

$$x(t+1) = A_2 x(t+1-0) + J_{t+1}, \quad t \in \mathbb{Z}_0^{\infty}, \tag{2.45}$$

where A, A_1, A_2 are $n \times n$ matrices (their commutativity is not assumed and the invertibility of the matrix A is dropped), the delay $m \geq 1$ is a given integer, $x \colon \mathbb{Z}_{-m}^{\infty} \to \mathbb{R}^n$, $f \colon \mathbb{Z}_0^{\infty} \to \mathbb{R}^n$, $\varphi \colon \mathbb{Z}_{-m}^{0} \to \mathbb{R}^n$, and $J_{t+1} \in \mathbb{R}^n$ are impulses. Among others, the paper brings an explicit form of the solutions to (2.43)–(2.45) derived with the aid of a modified discrete delay matrix exponential (with its definition inspired by [45])

$$X_m^{A_2 A, A_2 A_1}(t) := \begin{cases} \Theta & \text{if} \quad t \in \mathbb{Z}_{-\infty}^{-(m+1)}, \\ (A_2 A)^{t+m} & \text{if} \quad t \in \mathbb{Z}_{-m}^{0}, \\ \sum\limits_{j=0}^{l} Q(t+m-mj; j) & \text{if} \quad t \in \mathbb{Z}_{(l-1)(m+1)+1}^{l(m+1)} \quad \text{and} \quad l \in \mathbb{Z}_1^{\infty}, \end{cases}$$

where $l = \left\lfloor \frac{t+m}{m+1} \right\rfloor$,

$$Q(t; j) := \begin{cases} \Theta & \text{if} \quad j \in \mathbb{Z}_{-\infty}^{-1}, \\ (A_2 A)^t \sigma(t) & \text{if} \quad j = 0, \\ \sum\limits_{i=j}^{t} (A_2 A)^{t-i} A_2 A_1 Q(i-1; j-1) \sigma(t-j) & \text{if} \quad j \in \mathbb{Z}_1^{\infty}, \end{cases}$$

and σ is the Heaviside step function defined by (2.42). Note that $X_m^{A_2 A, A_2 A_1}(t)$ satisfies for $t \in \mathbb{Z}_{-\infty}^{\infty}$, $t \neq -m-1$, the matrix equation (we refer to [29, Lemma 3.2])

$$X_m^{A_2 A, A_2 A_1}(t+1) = A_2 A X_m^{A_2 A, A_2 A_1}(t) + A_2 A_1 X_m^{A_2 A, A_2 A_1}(t-m).$$

The main result, generalizing some results in [52] (we also refer to [53]) is shown below (we refer to Theorem 3.1 and Theorem 3.3 in [29]).

Theorem 2.24. *The solution to problem* (2.43)–(2.45) *is given by the formula*

$$x(t) := \begin{cases} \varphi(t), & t \in \mathbb{Z}_{-m}^{0}, \\ X_m^{A_2 A, A_2 A_1}(t-m)\varphi(0) + \sum\limits_{i=-m}^{-1} X_m^{A_2 A, A_2 A_1}(t-1-2m-i) A_2 A_1 \varphi(0) & \\ \quad + \sum\limits_{i=1}^{t} X_m^{A_2 A, A_2 A_1}(t-m-i)[A_2 f(i-1) + J_i], & t \in \mathbb{Z}_0^{\infty}. \end{cases}$$

In the proof, the $\mathcal{Z}$ transform is used. At first, it is assumed that the function f is exponentially bounded, then this assumption is dropped.

The next step was recently made by Mahmudov in the paper [48], where the non-reduced problem (1.1), (1.2) is considered with non-commutative matrices. Let us show the problem in its original notation used in [48]. Consider a system

$$x(k+1) = Ax(k) + \sum_{j=1}^{d} A_j x(k-\lambda_j) + f(k), \quad k \in \mathbb{Z}_0^{\infty} \tag{2.46}$$

and initial problem

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-\lambda_{\max}}^{0}, \tag{2.47}$$

where $\lambda_j \in \mathbb{Z}_1^{\infty}$, $\lambda_{\max} = \max\{\lambda_1, \dots, \lambda_d\}$, $x\colon \mathbb{Z}_{-\lambda_{\max}}^{\infty} \to \mathbb{R}^n$, $f\colon \mathbb{Z}_0^{\infty} \to \mathbb{R}^n$, and $\varphi\colon \mathbb{Z}_{-\lambda_{\max}}^{0} \to \mathbb{R}^n$. The regularity of A is not assumed. Results are described in terms of so-called multi-delayed discrete matrix exponential and this notion is explained below.

Let $\mathbf{A} = (A, A_1, \dots, A_d)$. Define d-dimensional vectors $\mathbf{i} = (i_1, \dots, i_d)$, $\mathbf{1} = (1, \dots, 1)$ and $\mathbf{\Lambda} = (\lambda_1, \dots, \lambda_d)$. Moreover, let $\mathbf{e}$ be a canonical basis of $\mathbb{R}^d$ defined by unit vectors $\mathbf{e}_i$, $i = 1, \dots, d$. We begin by establishing a recursive definition for the determining matrix function $Q^{\mathbf{A}}(k; \mathbf{i})$ playing a crucial role in the definition of multi-delayed discrete matrix exponential.

The determination of the matrix function $Q^{\mathbf{A}}(k; \mathbf{i}) \in \mathbb{M}_{n\times n}$, where $\mathbb{M}_{n\times n}$ is the set of $n \times n$ matrices with coefficients in $\mathbb{R}$, is given inductively:

$$\begin{aligned} &Q^{\mathbf{A}}(k+1; \mathbf{i}) = AQ^{\mathbf{A}}(k; \mathbf{i}) + \sum_{j=1}^{d} A_j Q^{\mathbf{A}}(k; \mathbf{i} - \mathbf{e}_j) \quad \text{if} \quad \mathbf{i} \in (\mathbb{Z}_0^{\infty})^d, \quad k \in \mathbb{Z}_1^{\infty}, \\ &Q^{\mathbf{A}}(0; \mathbf{i}) = \Theta, \quad Q^{\mathbf{A}}(1; \mathbf{0}) = I, \end{aligned}$$

and

$$Q^{\mathbf{A}}(k; -1, i_2, \dots, i_d) = Q^{\mathbf{A}}(k; i_1, -1, \dots, i_d) = \ldots = Q^{\mathbf{A}}(k; i_1, i_2, \dots, -1) = \Theta,$$

where $k \in \mathbb{Z}_1^{\infty}$. The multi-delayed discrete matrix exponential $X_{\mathbf{\Lambda}}^{\mathbf{A}}\colon \mathbb{Z}_0^{\infty} \to \mathbb{M}_{n\times n}$ is defined as follows:

$$X_{\mathbf{\Lambda}}^{\mathbf{A}}(k) = \sum_{\substack{\mathbf{i}\cdot(\mathbf{\Lambda}+\mathbf{1}) \le k, \\ i_1, \dots, i_d \ge 0}} Q^{\mathbf{A}}(k + 1 - \mathbf{i}\cdot\mathbf{\Lambda}; \mathbf{i}),$$

where $\mathbf{i}\cdot\mathbf{\Lambda}$ and $\mathbf{i}\cdot(\mathbf{\Lambda}+\mathbf{1})$ are scalar products. The main result of the paper [48], i.e., Theorem 4.2 follows.

Theorem 2.25. *Let $\mathbf{A} = (A, A_1, \dots, A_d)$ and let φ be as above. The solution $x = x(k)$ of the problem* (2.46), (2.47) *has the following explicit form*

$$x(k) = X_{\mathbf{\Lambda}}^{\mathbf{A}}(k)\varphi(0) + \sum_{j=1}^{d} \sum_{r=-\lambda_j}^{-1} X_{\mathbf{\Lambda}}^{\mathbf{A}}(k - 1 - \lambda_j - r) A_j \varphi(r) + \sum_{l=1}^{k} X_{\mathbf{\Lambda}}^{\mathbf{A}}(k - l) f(l - 1)$$

for $k \in \mathbb{Z}_0^{\infty}$.

2.5. FRACTIONAL DISCRETE EQUATIONS

For the rudiments of discrete fractional calculus used in this part, we refer to [26]. The initial idea used in the papers [17, 18] of constructing the delayed matrix exponential has been transfered to some classes of fractional difference equations as well. In the paper by Du and Lu [23] (we refer as well to the paper [22] by Du and Jia) this idea is applied to the Riemann–Liouville fractional-order delay difference system

$$\nabla^{\gamma}_{-\xi}x(t) = Ax(t-\xi) + f(t), \quad t \in \mathbb{Z}_1^{\infty}, \tag{2.48}$$

$$x(t) = \varphi(t), \qquad t \in \mathbb{Z}_{1-\xi}^{0}, \tag{2.49}$$

where $\xi \in \mathbb{Z}_2^{\infty}$ is a fixed delay, $\nabla^{\gamma}_{-\xi}$ is the Riemann–Liouville fractional-order difference of order $\gamma \in (0,1)$, $x\colon \mathbb{Z}_1^{\infty} \to \mathbb{R}^n$, $f\colon \mathbb{Z}_1^{\infty} \to \mathbb{R}^n$, A is a real $n \times n$ constant matrix, and $\varphi\colon \mathbb{Z}_{1-\xi}^{0} \to \mathbb{R}^n$ is an initial function. From the point of view of solution representation, the main result of the paper is following. A discrete delayed Mittag–Leffler matrix function is introduced and the general solution of non-homogeneous Riemann–Liouville fractional-order delay difference problem (2.48), (2.49) is derived.

Define the discrete delayed Mittag–Leffler-type matrix function $\mathbb{F}_{\xi}^{At^{\overline{\gamma}}}$ as follows:

$$\mathbb{F}_{\xi}^{At^{\overline{\gamma}}} = \begin{cases} \Theta & \text{if} \quad t \in \mathbb{Z}_{-\infty}^{-\xi}, \\ I\,\frac{(t+\xi)^{\overline{\gamma-1}}}{\Gamma(\gamma)} & \text{if} \quad t \in \mathbb{Z}_{1-\xi}^{0}, \\ I\,\frac{(t+\xi)^{\overline{\gamma-1}}}{\Gamma(\gamma)} + \sum_{j=0}^{m} A^j\,\frac{(t-(j-1)\xi)^{\overline{(j+1)\gamma-1}}}{\Gamma((j+1)\gamma)} & \text{if} \quad t \in \mathbb{Z}_{(m-1)\xi+1}^{m\xi}, \end{cases}$$

where $m \in \mathbb{Z}_1^{\infty}$ and $y^{\overline{w}} := \Gamma(y+w)/\Gamma(y)$. Define also, for $\gamma \notin \mathbb{Z}_{-\infty}^{-1}$, the fractional-order Taylor monomial $h_{\gamma}(t,a)$ with fractional-order γ as follows:

$$h_{\gamma}(t,a) = \frac{(t-a)^{\overline{\gamma}}}{\Gamma(\gamma+1)}.$$

The below theorem reproduces Theorem 1 of [23].

Theorem 2.26. *Assume that $0 < \gamma < 1$. Then, the discrete Mittag–Leffler-type matrix $\mathbb{F}_{\xi}^{At^{\overline{\gamma}}}$ is a solution of the matrix initial value problem*

$$\nabla^{\gamma}_{-\xi}X(t) = AX(t-\xi), \qquad t \in \mathbb{Z}_1^{\infty},$$

$$X(t) = Ih_{\gamma-1}(t,-\xi), \quad t \in \mathbb{Z}_{1-\xi}^{0}.$$

The main result on the representation of the solution of problem (2.48), (2.49) is given in [23, Theorem 4] and here we reproduce its content.

Theorem 2.27. *The solution $x = x(t)$ of the problem* (2.48), (2.49) *can be written as*

$$\begin{aligned} x(t) = \mathbb{F}_{\xi}^{At^{\overline{\gamma}}}\varphi(1-\xi) &+ \int\limits_{1-\xi}^{0} \mathbb{F}_{\xi}^{A(t-\xi-\rho(s))^{\overline{\gamma}}}\nabla^{\gamma}_{-\xi}\varphi(s)\nabla s \\ &+ \int\limits_{0}^{t} \mathbb{F}_{\xi}^{A(t-\xi-\rho(s))^{\overline{\gamma}}} f(s)\nabla s, \quad t \in \mathbb{Z}_{1-\xi}^{\infty}, \end{aligned} \tag{2.50}$$

where $\rho(s) = s - 1$.

Let us remark that, in the paper by Yuting Chen [7], Theorem 5.2 uses an alternative formula differing from the formula (2.50). A particular solution is expressed in terms of a two-parameter discrete delayed Mittag–Leffler matrix function (formula (22) in [7]).

Now describe the results achieved by Awadalla, Mahmudov and Alahmadi presented in [2]. In this paper a new discrete delayed Mittag–Leffler matrix function generated by two noncommutative matrices is introduced. An explicit formula for the solution of homogeneous (and nonhomogeneous) nabla fractional delay difference systems is derived using a discrete delayed Mittag–Leffler matrix function.

Consider the following linear retarded Riemann–Liouville fractional difference system,

$$\nabla^{\gamma}_{-1} z(\xi) = \mathfrak{A} z(\xi) + \mathfrak{B} z(\xi - r) + f(\xi), \quad \xi \in \mathbb{Z}_1^{\infty} \tag{2.51}$$

and initial problem

$$z(\xi) = \psi(\xi), \quad \xi \in \mathbb{Z}^0_{1-r}, \tag{2.52}$$

where $r \in \mathbb{Z}_2^{\infty}$ is a fixed delay, ∇^{γ}_{-1} is the Riemann–Liouville fractional difference of order γ, $0 < \gamma < 1$, $z\colon \mathbb{Z}_1^{\infty} \to \mathbb{R}^n$, $f\colon \mathbb{Z}_1^{\infty} \to \mathbb{R}^n$, $\mathfrak{A}, \mathfrak{B}$ are real constant $n \times n$ matrices (their commutativity is not assumed), and $\psi\colon \mathbb{Z}^0_{1-r} \to \mathbb{R}^n$ is an initial function.

Let us introduce some auxiliary notions. For some of them, we refer to [26, 44]. For a matrix $A = \{a_{ij}\}_{i,j=1}^n$ define $\|A\| = \max_{1\le j \le n} \sum_{i=1}^n |a_{ij}|$. Assume $\|\mathfrak{A}\| \le 1$, $\gamma > 0$, $\mu \in \mathbb{R}$ and define the delayed nabla Mittag–Leffler matrix function $\mathfrak{D}^{\mathfrak{A},\mathfrak{B}}_{\gamma,\mu,r}$ generated by $\mathfrak{A}$, $\mathfrak{B}$, as follows:

$$\mathfrak{D}^{\mathfrak{A},\mathfrak{B}}_{\gamma,\mu,r}(\xi) := \begin{cases} \Theta & \text{if} \quad \xi \in \mathbb{Z}^{-1}_{-\infty}, \\ \sum\limits_{i=0}^{\infty} \mathfrak{A}^i \frac{(\xi)_+^{\overline{i\gamma+\mu-1}}}{\Gamma(i\gamma+\mu)} + \sum\limits_{i=1}^{\infty} Q(i+1,1) \frac{(\xi-r)_+^{\overline{i\gamma+\mu-1}}}{\Gamma(i\gamma+\mu)} & \\ \quad + \ldots + \sum\limits_{i=p}^{\infty} Q(i+1,p) \frac{(\xi-pr)_+^{\overline{i\gamma+\mu-1}}}{\Gamma(i\gamma+\mu)} & \text{if} \quad \xi \in \mathbb{Z}_{pr}^{(p+1)r} \text{ and } p \in \mathbb{Z}_0^{\infty}, \end{cases}$$

where, for $q, s \in \mathbb{Z}_0^{\infty}$, $Q(q, s)$ are matrices defined by the recursive form

$$Q(q+1, s) = \mathfrak{A} Q(q, s) + \mathfrak{B} Q(q, s-1), \quad Q(q, -1) = \Theta, \quad Q(1, 0) = I.$$

The below Theorem 32 in [2] gives, in a sense, an analogy of the properties (2.3), (2.16), and (2.18).

Theorem 2.28. *The nabla Mittag–Leffler matrix function $\mathfrak{D}^{\mathfrak{A},\mathfrak{B}}_{\gamma,\gamma,r}$ satisfies homogeneous matrix system*

$$\begin{aligned} \nabla^{\gamma}_{-1} Z(\xi) &= \mathfrak{A} Z(\xi) + \mathfrak{B} Z(\xi - r), && \xi \in \mathbb{Z}_1^{\infty}, \\ Z(\xi) &= \Theta, && \xi \in \mathbb{Z}^0_{1-r}, \end{aligned}$$

that is,

$$\nabla^{\gamma}_{-1} \mathfrak{D}^{\mathfrak{A},\mathfrak{B}}_{\gamma,\gamma,r}(\xi) = \mathfrak{A} \mathfrak{D}^{\mathfrak{A},\mathfrak{B}}_{\gamma,\gamma,r}(\xi) + \mathfrak{B} \mathfrak{D}^{\mathfrak{A},\mathfrak{B}}_{\gamma,\gamma,r}(\xi - r), \quad \xi \in \mathbb{Z}_1^{\infty}.$$

The following theorem on representation of solution of problem (2.51), (2.52) is given in [2, Theorem 34]. Its proof uses Theorem 2.28.

Theorem 2.29. *Assume $f\colon \mathbb{Z}_1^\infty \to \mathbb{R}^n$, $\|\mathfrak{U}\| \le 1$ and $0<\gamma<1$. Then the unique solution of the fractional problem* (2.51), (2.52) *is given by*

$$\begin{aligned} z(\xi) &= \mathfrak{D}^{\mathfrak{A},\mathfrak{B}}_{\gamma,\gamma,r}(\xi+1)\,[(I-\mathfrak{A})\phi(0)-f(0)] - \mathfrak{D}^{\mathfrak{A},\mathfrak{B}}_{\gamma,\gamma,r}(\xi+1-r)\mathfrak{B}\phi(0) \\ &\quad + \int_{-r}^{0} \mathfrak{D}^{\mathfrak{A},\mathfrak{B}}_{\gamma,\gamma,r}(\xi-\rho(m)-r)\mathfrak{B}\phi(m)\nabla m + \int_{-1}^{\xi} \mathfrak{D}^{\mathfrak{A},\mathfrak{B}}_{\gamma,\gamma,r}(\xi-\rho(s))f(s)\nabla s, \quad \xi\in\mathbb{Z}_1^\infty, \end{aligned}$$

where $\rho(m)=m-1$.

3. SYSTEMS WITH SECOND-ORDER DIFFERENCES

The investigation of a representation of solutions of equations with second-order difference of the form given below was initiated in papers by Diblík and Mencáková (let us refer to [11–14]). First, equations with the second-power of the matrix of linear terms

$$\Delta^2 x(k) + A^2 x(k-m) = f(k), \quad k\in\mathbb{Z}_0^\infty \tag{3.1}$$

were considered. In the literature, such an equation (or its analogies) is often called an oscillating equation referring to an ordinary differential equation of second-order describing a harmonic oscillator. If $\varphi\colon \mathbb{Z}_{-m}^1\to\mathbb{R}^n$ is a given function, let

$$x(k)=\varphi(k)=(\varphi_1(k),\dots,\varphi_n(k))^T, \quad k\in\mathbb{Z}_{-m}^1. \tag{3.2}$$

Assuming that A is a regular matrix, two auxiliary matrix functions

$$\mathrm{Cos}_m\, Ak\colon \mathbb{Z}\to\mathbb{R}^{n\times n}, \quad \mathrm{Sin}_m\, Ak\colon \mathbb{Z}\to\mathbb{R}^{n\times n}$$

called delayed discrete cosine and delayed discrete sine were defined as follows:

$$\mathrm{Cos}_m\, Ak := \begin{cases} \Theta & \text{if } k\in\mathbb{Z}_{-\infty}^{-m-1}, \\ I & \text{if } k\in\mathbb{Z}_{-m}^{1}, \\ \dots & \\ I - A^2\binom{k}{2} + A^4\binom{k-m}{4} & \\ \quad + \dots + (-1)^\ell A^{2\ell}\binom{k-(\ell-1)m}{2\ell} & \text{if } k\in\mathbb{Z}_{(\ell-1)(m+2)+2}^{\ell(m+2)+1}, \quad \ell=0,1,2,\dots, \end{cases}$$

and

$$\mathrm{Sin}_m\, Ak := \begin{cases} \Theta & \text{if } k\in\mathbb{Z}_{-\infty}^{-m}, \\ A\binom{k+m}{1} & \text{if } k\in\mathbb{Z}_{-m+1}^{2}, \\ \dots & \\ A\binom{k+m}{1} - A^3\binom{k}{3} + A^5\binom{k-m}{5} & \\ \quad + \dots + (-1)^\ell A^{2\ell+1}\binom{k-(\ell-1)m}{2\ell+1} & \text{if } k\in\mathbb{Z}_{(\ell-1)(m+2)+3}^{\ell(m+2)+2}, \quad \ell=0,1,2,\dots \end{cases}$$

They have the following short forms (for $k \in \mathbb{Z}$):

$$\operatorname{Cos}_m Ak := \sum_{j=0}^{\lceil \frac{k-1}{m+2} \rceil} (-1)^j A^{2j} \binom{k-(j-1)m}{2j},$$

$$\operatorname{Sin}_m Ak := \sum_{j=0}^{\lceil \frac{k-2}{m+2} \rceil} (-1)^j A^{2j+1} \binom{k-(j-1)m}{2j+1}.$$

Let us explain why the square A^2 is used in (1.3) rather than A. The definition of $\operatorname{Sin}_m Ak$ uses odd powers of A. If in (1.3), matrix A is considered (rather than A^2), then the definition of $\operatorname{Sin}_m Ak$ must be corrected. The square root $\sqrt{A}$ should be used instead of A. But the square root of a matrix may not exist or may be not unique. Functions $\operatorname{Cos}_m Ak$, $\operatorname{Sin}_m Ak$ satisfy

$$\Delta^2 \operatorname{Cos}_m Ak + A^2 \operatorname{Cos}_m A(k-m) = \Theta, \quad k \in \mathbb{Z} \setminus \{-m-2, -m-1\}, \tag{3.3}$$

and

$$\Delta^2 \operatorname{Sin}_m Ak + A^2 \operatorname{Sin}_m A(k-m) = \Theta, \quad k \in \mathbb{Z} \setminus \{-m-1\}, \tag{3.4}$$

i.e., they solve the matrix equation

$$\Delta^2 X(k) + A^2 X(k-m) = \Theta,$$

for the domains indicated in (3.3) and (3.4). The main result on a representation of the solution of the problem (3.1), published in [11] and repeated in [13], is shown below.

Theorem 3.1. *The solution $x = x_\varphi(k)$, $k \in \mathbb{Z}_{-m}^{\infty}$ of initial problem* (3.1)*,* (3.2) *is*

$$\begin{aligned} x_\varphi(k) &= (\operatorname{Cos}_m Ak)\,\varphi(-m) \\ &\quad + A^{-1}\Big[(\operatorname{Sin}_m Ak)\,\Delta\varphi(-m) + \sum_{j=-m+1}^{0} \operatorname{Sin}_m A(k-m-j)\cdot \Delta^2\varphi(j-1)\Big] \\ &\quad + A^{-1}\cdot \sum_{j=0}^{k-2} \operatorname{Sin}_m A(k-1-m-j)\cdot f(j). \end{aligned} \tag{3.5}$$

Unfortunately, Theorem 3.1 covers only a narrow class of systems because the system (3.1) uses squared matrix A^2 such that $\det A \neq 0$ (the matrix A^{-1} is used in the formula (3.5)). In the paper [14] an improvement is given without this requirement and with an equation

$$\Delta^2 x(k) + A\, x(k-m) = f(k), \quad k \in \mathbb{Z}_0^{\infty} \tag{3.6}$$

considered in which, unlike in system (3.1), the matrix A is arbitrary. The progress has been achieved by employing two special discrete delayed matrices defined in a way different from the above matrix functions $\mathrm{Cos}_m\, Ak$ and $\mathrm{Sin}_m\, Ak$. For a given matrix A and delay m, define two delayed discrete matrix functions $\mathcal{M}_c(k, A, m)$ and $\mathcal{M}_s(k, A, m)$ by the following formulas. The delayed discrete matrix $\mathcal{M}_c(k, A, m)$ is defined as:

$$\mathcal{M}_c(k, A, m) := \begin{cases} \Theta & \text{if } k \in \mathbb{Z}_{-\infty}^{-m-1}, \\ I & \text{if } k \in \mathbb{Z}_{-m}^{1}, \\ \dots & \\ I - A\binom{k}{2} + A^2\binom{k-m}{4} & \\ \quad + \dots + (-1)^\ell A^\ell \binom{k-(\ell-1)m}{2\ell} & \text{if } k \in \mathbb{Z}_{(\ell-1)(m+2)+2}^{\ell(m+2)+1},\ \ell = 0, 1, 2, \dots \end{cases}$$

The delayed discrete matrix $\mathcal{M}_s(k, A, m)$ is defined as:

$$\mathcal{M}_s(k, A, m) := \begin{cases} \Theta & \text{if } k \in \mathbb{Z}_{-\infty}^{-m}, \\ I\binom{k+m}{1} & \text{if } k \in \mathbb{Z}_{-m+1}^{2}, \\ \dots & \\ I\binom{k+m}{1} - A\binom{k}{3} & \\ \quad + \dots + (-1)^\ell A^\ell \binom{k-(\ell-1)m}{2\ell+1} & \text{if } k \in \mathbb{Z}_{(\ell-1)(m+2)+3}^{\ell(m+2)+2}, \\ & \quad \ell = 0, 1, 2, \dots \end{cases}$$

We will reduce the notation of $\mathcal{M}_c(k, B, m)$, $\mathcal{M}_s(k, B, m)$ to a mere $\mathcal{M}_c(k)$ and $\mathcal{M}_s(k)$ since there is no danger of ambiguity in the sequel. The main property of $\mathcal{M}_c(k)$ and $\mathcal{M}_s(k)$ is described below (we refer to Corollary 1 in [14]).

Theorem 3.2. *Functions $\mathcal{M}_c(k)$ and $\mathcal{M}_s(k)$ satisfy*

$$\Delta^2 \mathcal{M}_c(k) + A\mathcal{M}_c(k - m) = \Theta, \quad k \in \mathbb{Z} \setminus \{-m - 2, -m - 1\},$$

and

$$\Delta^2 \mathcal{M}_s(k) + A\mathcal{M}_s(k - m) = \Theta, \quad k \in \mathbb{Z} \setminus \{-m - 1\}.$$

Using these properties, the main result of the paper [14, Theorem 2] giving a formula for a solution to problem (3.6), (3.2) is proved. Here is how it is formulated.

Theorem 3.3. *The solution of initial problem* (3.6)*,* (3.2) *can be expressed in the form*

$$\begin{aligned} x(k) = {} & \mathcal{M}_c(k)\varphi(-m) + \mathcal{M}_s(k)\Delta\varphi(-m) \\ & + \sum_{j=-m+1}^{0} \mathcal{M}_s(k - m - j)\Delta^2\varphi(j - 1) + \sum_{j=0}^{k-2} \mathcal{M}_s(k - 1 - m - j) f(j), \end{aligned}$$

where $k \in \mathbb{Z}_{-m}^{\infty}$.

3.1. OSCILLATING EQUATION – THE CASE OF MULTIPLE DELAYS

In [14], Diblík and Mencáková formulated an open problem to generalize the results of this paper to the case of linear discrete systems with multiple delays and permutable or nonpermutable matrices. The paper by Elshenhab and Wang [25] solves this problem positively. Motivated by [14], an explicit representation of solutions of the problem

$$\Delta^2 x(k) = -A_1 x(k-m_1) - \ldots - A_p x(k-m_p) + f(k), \quad k \in \mathbb{Z}_0^{\infty}, \tag{3.7}$$

$$x(k) = \psi(k) = (\psi_1(k), \ldots, \psi_n(k))^T, \qquad k \in \mathbb{Z}_{-m}^{1} \tag{3.8}$$

is given. Let us describe this approach. In (3.7), (3.8), $m_q \in \mathbb{Z}_1^{\infty}$, $q = 1, \ldots, p$, are delays, $m = \max\{m_1, \ldots, m_p\}$, $x = (x_1, \ldots, x_n)^T : \mathbb{Z}_{-m}^{\infty} \to \mathbb{R}^n$ is a solution satisfying (3.7), (3.8) for every $k \in \mathbb{Z}_0^{\infty}$, $A_1, \ldots, A_p$ are arbitrary $n \times n$ constant real nonzero matrices (their permutability is not assumed), $f\colon \mathbb{Z}_0^{\infty} \to \mathbb{R}^n$, and $\psi\colon \mathbb{Z}_{-m}^{1} \to \mathbb{R}^n$.

Define, for $k \in \mathbb{Z}_0^{\infty}$, functions $\mathcal{H}, \mathcal{M}\colon \mathbb{Z}_0^{\infty} \to \mathbb{R}^{n\times n}$ as follows:

$$\mathcal{H}(k) := \sum_{\substack{\beta_1,\ldots,\beta_p \geq 0, \\ \sum_{q=1}^{p}(m_q+2)\beta_q \leq k}} (-1)^{|\beta|} \binom{k - \sum_{q=1}^{p} m_q\beta_q}{2|\beta|} \mathcal{P}_{\beta}^{A}, \tag{3.9}$$

$$\mathcal{M}(k) := \sum_{\substack{\beta_1,\ldots,\beta_p \geq 0, \\ \sum_{q=1}^{p}(m_q+2)\beta_q \leq k-1}} (-1)^{|\beta|} \binom{k - \sum_{q=1}^{p} m_q\beta_q}{1+2|\beta|} \mathcal{P}_{\beta}^{A}, \tag{3.10}$$

where $\beta = (\beta_1, \ldots, \beta_p)$, $|\beta| = \sum_{i=1}^{p} \beta_i$, $A = (A_1, \ldots, A_p)$,

$$\mathcal{P}_{\beta}^{A} := \sum_{\sigma \in S_{|\beta|}^{\mathcal{R}^{\beta}}} \prod_{i=1}^{|\beta|} A_{\mathcal{R}^{\beta}_{\sigma(i)}} \quad \text{if} \quad |\beta| > 0 \quad \text{and} \quad \mathcal{P}_{(0,\ldots,0)}^{A} := I,$$

the symbol $S_{|\beta|}^{\mathcal{R}^{\beta}}$ denotes all permutations of the set $\{1, \ldots, |\beta|\}$,

$$\mathcal{R}^{\beta} := (\underbrace{1, \ldots, 1}_{\beta_1}, \underbrace{2, \ldots, 2}_{\beta_2}, \ldots, \underbrace{p, \ldots, p}_{\beta_p}) \in (\mathbb{Z}_1^{\infty})^{|\beta|},$$

and $\mathcal{R}_i^{\beta}$ is the i-th coordinate of the vector $\mathcal{R}^{\beta}$. The main result, formulated below, is given in Theorem 1 and Corollary 1 in [25]. It is first proved (in Theorem 1) by using the $\mathcal{Z}$-transform assuming that f is exponentially bounded. This assumption is then omitted (in Corollary 1).

Theorem 3.4. *Let $p \geq 1$, $m_1, \ldots, m_p \geq 1$. The solution $x(k)$, $k \in \mathbb{Z}_{-m}^{\infty}$ of initial problem* (3.7), (3.8) *is given by the formula*

$$x(k) = \begin{cases} \psi(k) \quad \text{if} \quad k \in \mathbb{Z}_{-m}^{1}, \\ \mathcal{H}(k)\psi(0) + \mathcal{M}(k)\Delta\psi(0) - \sum_{q=1}^{p} A_q \sum_{i=-m_q}^{-1} \mathcal{M}(k-m_q-i-1)\psi(i) \\ \quad + \sum_{q=0}^{k-2} \mathcal{M}(k-q-1) f(q) \quad \text{if} \quad k \in \mathbb{Z}_0^{\infty}. \end{cases} \tag{3.11}$$

If matrices $A_1, \dots, A_p$ are permutable, i.e., if $A_i A_j = A_j A_i$ for each $i, j = i, \dots, p$, then the representation (3.11) can be reduced, we refer to Corollary 2 in [25]. Below we formulate this result. Define, for $k \in \mathbb{Z}_0^\infty$, the functions $\mathcal{X}, \mathcal{Y} \colon \mathbb{Z}_0^\infty \to \mathbb{R}^{n \times n}$ as follows:

$$\mathcal{X}(k) := \sum_{\substack{\beta_1, \dots, \beta_p \geq 0, \\ \sum_{q=1}^p (m_q+2)\beta_q \leq k}} (-1)^{|\beta|} \binom{|\beta|}{\beta_1 \dots, \beta_p} \binom{k - \sum_{q=1}^p m_q \beta_q}{2|\beta|} \prod_{q=1}^p A_q^{\beta_q}, \tag{3.12}$$

$$\mathcal{Y}(k) := \sum_{\substack{\beta_1, \dots, \beta_p \geq 0, \\ \sum_{q=1}^p (m_q+2)\beta_q \leq k-1}} (-1)^{|\beta|} \binom{|\beta|}{\beta_1 \dots, \beta_p} \binom{k - \sum_{q=1}^p m_q \beta_q}{1 + 2|\beta|} \prod_{q=1}^p A_q^{\beta_q}. \tag{3.13}$$

Theorem 3.5. *Let $p \geq 1$, $m_1, \dots, m_p \geq 1$. Let matrices $A_1, \dots, A_p$ be permutable. Then, the solution $x(k)$, $k \in \mathbb{Z}_{-m}^\infty$ of initial problem* (3.7), (3.8) *is given by the formula*

$$x(k) = \begin{cases} \psi(k) \quad \textit{if} \quad k \in \mathbb{Z}_{-m}^1, \\ \mathcal{X}(k)\psi(0) + \mathcal{Y}(k)\Delta\psi(0) - \sum_{q=1}^p A_q \sum_{i=-m_q}^{-1} \mathcal{Y}(k - m_q - i - 1)\psi(i) \\ \quad + \sum_{q=0}^{k-2} \mathcal{Y}(k - q - 1) f(q) \quad \textit{if} \quad k \in \mathbb{Z}_0^\infty. \end{cases}$$

3.2. FRACTIONAL DELAY OSCILLATION DIFFERENCE EQUATIONS

As in part 2.5, for the rudiments of discrete fractional calculus used in this part, we refer to [26]. Among other things, the paper by Yuting Chen [7] gives an explicit solution of the homogeneous fractional delay oscillation difference equation of order $1 < \imath < 2$. The progress is achieved by constructing discrete sine-type and cosine-type delayed Mittag–Leffler functions. Then, the discrete Laplace transform technique is used to solve the equation with a nonhomogeneous term. Consider the problem

$${}^C\nabla_0^\imath w(x) = -Qw(x - \sigma) + z(x), \quad x \in \mathbb{Z}_1^\infty, \quad 1 < \imath < 2, \tag{3.14}$$

$$w(x) = \psi(x), \quad \nabla w(x) = \nabla \psi(x), \quad x \in \mathbb{Z}_{1-\sigma}^0, \tag{3.15}$$

where $w \colon \mathbb{Z}_{1-\sigma}^\infty \to \mathbb{R}^n$, $\sigma \in \mathbb{Z}_2^\infty$ is a fixed delay time, $Q \in \mathbb{R}$ is an $n \times n$ constant matrix, $\psi \colon \mathbb{Z}_{1-\sigma}^0 \to \mathbb{R}^n$ is the initial function, and $z \colon \mathbb{Z}_1^\infty \to \mathbb{R}^m$ is a given function. Define discrete cosine-type delayed Mittag–Leffler function $C_{\bar\imath}(x)$ as

$$C_{\bar\imath}(x) = \begin{cases} \Theta & \text{if} \quad x \in \mathbb{Z}_{-\infty}^{-\sigma}, \\ I & \text{if} \quad x \in \mathbb{Z}_{1-\sigma}^0, \\ I - Q\frac{x^{\bar\imath}}{\Gamma(\imath+1)} + Q^2 \frac{(x-\sigma)^{\overline{2\imath}}}{\Gamma(2\imath+1)} & \\ \quad + \dots + (-1)^i Q^i \frac{(x-(i-1)\sigma)^{\overline{i\imath}}}{\Gamma(i\imath+1)} & \text{if} \quad x \in \mathbb{Z}_{(i-1)\sigma+1}^{i\sigma} \text{ and } i \in \mathbb{Z}_1^\infty, \end{cases}$$

and discrete sine-type delayed Mittag–Leffler function $S_{\bar\imath}(x)$ as

$$S_{\bar\imath}(x)=\begin{cases}\Theta & \text{if} \quad x\in\mathbb{Z}_{-\infty}^{-\sigma},\\ (x+\sigma)I & \text{if} \quad x\in\mathbb{Z}_{1-\sigma}^{0},\\ (x+\sigma)I-Q\frac{x^{\overline{\imath+1}}}{\Gamma(\imath+2)}+Q^2\frac{(x-\sigma)^{\overline{2\imath+1}}}{\Gamma(2\imath+2)} & \\ \quad +\ldots+(-1)^iQ^i\frac{(x-(i-1)\sigma)^{\overline{i\imath+1}}}{\Gamma(i\imath+2)} & \text{if} \quad x\in\mathbb{Z}_{(i-1)\sigma+1}^{i\sigma} \text{ and } i\in\mathbb{Z}_1^{\infty},\end{cases}$$

where $x^{\bar\imath}:=\Gamma(x+\imath)/\Gamma(x)$ and $0^{\bar\imath}:=0$. We list the properties of $C_{\bar\imath}(x)$ and $S_{\bar\imath}(x)$ (these are given in Theorem 3.1 in [7]).

Theorem 3.6. *For $C_{\bar\imath}(x)$ and $S_{\bar\imath}(x)$, we have:*

(i) $\Delta S_{\bar\imath}(x)=C_{\bar\imath}(x)$, *for all* $x\in\mathbb{Z}_{-\infty}^{\infty}$,
(ii) $C_{\bar\imath}(x)$ *solves the matrix problem*

$$\begin{aligned}&{}^C\nabla_0^{\imath}W(x)=-QW(x-\sigma), \quad x\in\mathbb{Z}_1^{\infty}, \quad 1<\imath<2,\\ &W(x)=I, \quad \nabla W(x)=\Theta, \quad t\in\mathbb{Z}_{1-\sigma}^{0},\end{aligned}$$

(iii) $S_{\bar\imath}(x)$ *solves the matrix problem*

$$\begin{aligned}&{}^C\nabla_0^{\imath}W(x)=-QW(x-\sigma), \quad x\in\mathbb{Z}_1^{\infty}, \quad 1<\imath<2,\\ &W(x)=(x+\sigma)I, \quad \nabla W(x)=I, \quad t\in\mathbb{Z}_{1-\sigma}^{0}.\end{aligned}$$

The main result of the paper in [7] (Theorem 3.5) related to the formula solving the problem (3.14), (3.15) is the following.

Theorem 3.7. *An exact solution of the initial problem* (3.14), (3.15) *can be expressed in the form*

$$\begin{aligned}w(x)&=C_{\bar\imath}(x)[\psi(1-\sigma)-\nabla\psi(1-\sigma)]+S_{\bar\imath}(x)\nabla\psi(1-\sigma)\\ &\quad+\int_{1-\sigma}^{0}S_{\bar\imath}(x-\sigma-\rho(\vartheta))\nabla^2\psi(\vartheta)\nabla\vartheta+{}^C\nabla_0^{2-\imath}S_{\bar\imath}(x-\sigma)*z(x), \quad x\in\mathbb{Z}_{1-\sigma}^{\infty},\end{aligned}$$

where $\rho(\vartheta)=\vartheta-1$.

4. APPLICATIONS

In the papers [17, 18] we ventured a prognosis that the formulas representing the solutions of initial problems can be used in investigating the asymptotic properties of solutions. This has been confirmed and, in addition, the formulas and their numerous generalizations have found many applications in numerous fields, such as in control theory, iterative learning control, and stability analysis. Let us look at some of them.

Problems of relative controllability are solved in papers by Diblík, Khusainov, and Růžičková [19], by Diblík in [9], by Diblík and Mencáková in [13, 51], by Pospíšil, Diblík and Fečkan in [57, 58], by Diblík, Fečkan and Pospíšil in [21], by Pospíšil in [55]. The papers by Jin and Wang [27], by Liang, Wang, and Shen [39], by Jin, Wang, and Shen [28], by Liang, Wang, and Fečkan [38], by Yang, Fečkan, and Wang [65], and by Luo, Wang and Shen [41], apply special matrix functions to problems of iterative learning control and learning ability analysis. Paper by Jin, Fečkan, and Wang [30] studies the relative controllability of impulsive linear discrete delay systems with constant coefficients and a pure delay. Relative controllability for delayed linear discrete system with second-order differences is studied by Yang, Fečkan, and Wang in [64].

Exponential stability of systems is studied by Medved' and Pospíšil in [49], by Medved' and Škripková in [50], and by Jin, Wang and Shen in [29]. Yang, Fečkan, and Wang study in [66] Ulam's type stability of delayed discrete system with second-order differences. Ulam–Hyers stability of nabla fractional delay difference systems is established in [2] by Awadalla, Mahmudov, and Alahmadi. Finite time stability is studied by Du and Lu in [23], and in the paper [22] by Du and Jia.

5. REMARK ON THE CONTINUOUS CASE

The initial papers mentioned in the Introduction are discrete analogues of continuous variants published in two papers [31, 32]. These papers served as starting points for similar studies in the field of delayed differential equations, fractional differential equations, and differential equations with quaternions. Let us mention some papers related to this branch of development.

The paper by Li and Wang [35] considers conformable fractional oscillating delay systems. The relative controllability of semilinear delay differential systems with linear parts defined by permutable matrices is studied by Wang, Luo, and Fečkan in [61].

Liang, Wang, and O'Regan in [40] and Mahmudov in [43, 46] introduce fractional delayed matrices cosine and sine. These are fractional analogues of the delayed matrix sine and cosine functions defined in the paper by Khusainov, Diblík, Růžičková, and Lukáčová [32]. Mahmudov also considers multi-delayed perturbation of Mittag–Leffler type matrix functions in [47].

Diblík, Khusainov, Lukáčová, and Růžičková [20] use delayed matrix sine and delayed matrix cosine to develop the conditions of relative controllability for oscillating second-order systems of differential equations with a single delay. Cao and Wang [6] study the finite-time stability of a class of oscillating systems with two delays, using delayed matrix polynomials.

Svoboda [60] studies the asymptotic properties of delayed matrix sine and delayed matrix cosine. The controllability of nonlinear delay oscillating systems is examined in the paper [37] by Liang, Wang, and O'Regan. Zhou in [69] investigates exponential stability and relative controllability of nonsingular conformable delay systems using the delayed matrix exponential. Hyers–Ulam stability of fractional equations with double delays is studied by Liang, Shi, and Fan in [36].

The convergence of iterative learning control for linear delay systems with deterministic and random impulses, through the representation of solutions involving the concept of the delayed exponential matrix, is studied by Wang, Luo, and Shen in [62]. Applications of delayed matrix exponential to Fredholm's boundary-value problems are given by Boichuk, Diblík, Khusainov, and Růžičková in [5]. M. Pospíšil develops, among others, representations of systems of functional differential equations with multiple variable delays in [54]. Sathiyaraj and Wang construct representations of solutions suitable for non-instantaneous impulsive stochastic multiple delays system in [59].

The present paper only mentions several papers related to the continuous case. A more comprehensive survey of the progress made from the initial papers [31, 32] to the present is suggested as a topic for a future survey. In this context, we would like to mention the recent book [63] by Wang, Fečkan, and Li, which contains results related to stability and control for delay (and also discrete) systems. Finally, it is worth noting that the book by Khusainov, Diblík, and Růžičková [33] lists, among others, some results related with the continuous case.

6. CONCLUDING REMARKS

This survey paper is concerned with the development of a simple initial idea of utilizing a step method in constructing representations of solutions of linear delayed discrete equations with the first-order and second-order forward differences. The progress has been achieved by involving some special discrete delayed matrix functions, which enable the representations to be written in a simple form. In the literature, general theoretical formulas can be found (e.g., for discrete case of initial systems being moved to non-delayed systems in [1, 24, 34], for functional differential equations in [4] as well as for equations with aftereffect in [3]). The formulas described in the paper are, among others, suitable for computational purposes. In particular, these can be used to design algorithms suitable for numerical computations. We refer, e.g., to paper by Pospíšil [56], where the sum in $\mathcal{B}(k)$, used in formula (2.33), is rewritten in the form of iterated sums, i.e.,

$$\sum_{\substack{i_1,\dots,i_p\geq 0,\\ \sum_{j=1}^{p}(m_j+1)i_j\leq k}} \dots = \sum_{i_1=0}^{\left\lfloor \frac{k}{m_1+1}\right\rfloor} \sum_{i_2=0}^{\left\lfloor \frac{k-(m_1+1)i_1}{m_2+1}\right\rfloor} \dots \sum_{i_p=0}^{\left\lfloor \frac{k-\sum_{j=1}^{p}(m_j+1)i_j}{m_p+1}\right\rfloor} \dots,$$

or to the paper by Elsenhab and Wang [25], where, in formulas (3.9), (3.10), (3.12) and (3.13) iterated sums can be used as well, i.e.,

$$\sum_{\substack{\beta_1,\dots,\beta_p\geq 0,\\ \sum_{q=1}^{p}(m_q+2)\beta_q\leq k-r}} \dots = \sum_{\beta_1=0}^{\left\lfloor \frac{k-r}{m_1+2}\right\rfloor} \sum_{\beta_2=0}^{\left\lfloor \frac{k-r-(m_1+2)\beta_1}{m_2+2}\right\rfloor} \dots \sum_{\beta_p=0}^{\left\lfloor \frac{k-r-\sum_{q=1}^{p-1}(m_q+2)\beta_q}{m_p+2}\right\rfloor} \dots,$$

and $r = 0, 1$. The method of representation of solutions summarized in the paper can also be used in examining some boundary value problems for linear discrete systems with constant coefficients on finite intervals. Moreover, the results obtained are useful in investigating such asymptotic problems as describing the asymptotic behavior of solutions concerning boundedness, convergence or stability of solutions.

Based on the survey presented in the paper, several problems awaiting solutions can be listed (the author is aware that the list reflects the situation current at the period of preparing the paper).

(1) Generalize the results of the paper by Yuting Chen [7] described in part 3.2 to the case of multiple delays with mutually commuting matrices and non-commuting matrices.
(2) Generalize the results of the paper by Jin, Wang and Shen [29] described in part 2.4 to the case of discrete equations with second-order differences and multiple delays with mutually commuting or non-commuting matrices.
(3) Find the classes of Volterra discrete systems (we refer to [10] and to the references therein) such that simple formulas for their solutions can be derived.
(4) Construct formulas for discrete equations with third-order forward differences and multiple delays with mutually commuting or non-commuting matrices.
(5) An important challenge is the application of methods of spectral analysis to find the representation of solutions in various spaces (e.g.in spaces $l_p(\mathbb{Z}, \mathbb{C}^d)$, $1 \le p < \infty$, $l_\infty(\mathbb{Z}, \mathbb{C}^d)$). This approach should give, in addition to various formulas representing solutions, an insight into the properties of solutions such as their stability or convergence to zero. For some aspects of this approach we refer, e.g., to [8, 67, 68].

Acknowledgements

The author has been supported by the Czech Science Foundation under the project 23-06476S.

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Received: November 21, 2024.
Accepted: December 23, 2024.