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ON EXPANSIVE THREE-ISOMETRIES

Laurian Suciu

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Abstract. The sub-Brownian 3-isometries in Hilbert spaces are the natural counterparts of the 2-isometries, because all of them have Brownian-type extensions in the sense of J. Agler and M. Stankus. We show that all powers T^n for $n \geq 2$ of every expansive 3-isometry T are sub-Brownian, even if T does not have such a property. This fact induces some useful relations between the corresponding covariance operators of T. We analyze two matrix representations of T in order to get some conditions under which T is sub-Brownian, or T admits the Wold-type decomposition in the sense of S. Shimorin. We show that the restriction of T to its range is sub-Brownian of McCullough's type, and that under some conditions on $\mathcal{N}(T^*)$, T itself is sub-Brownian, and it admits the Wold-type decomposition.

Keywords: Wold decomposition, 3-isometry, sub-Brownian 3-isometry.

Mathematics Subject Classification: 47A05, 47A15, 47A20, 47A63.

1. INTRODUCTION AND PRELIMINARIES

Throughout the paper $\mathcal{H}$ stands for a complex Hilbert space with the inner product $(\cdot,\cdot)$, while $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators on $\mathcal{H}$, with the identity operator I ($= I_{\mathcal{H}}$ when it is the case). As usual, $\mathcal{R}(T)$, $\mathcal{N}(T)$ and T^* stand for the range, the kernel and the adjoint of an operator $T \in \mathcal{B}(\mathcal{H})$. For such T we define $\Delta_T = T^*T - I$, $\Delta_T^{(2)} = T^*\Delta_T T - \Delta_T$ and $\Delta_T^{(3)} = T^*\Delta_T^{(2)}T - \Delta_T^{(2)}$. They are the *covariance operators* of order 1, 2 or 3, respectively.

An operator $A \in \mathcal{B}(\mathcal{H})$ is positive (which we note as $A \geq 0$) if $(Ah, h) \geq 0$ for each $h \in \mathcal{H}$. If $A \geq 0$, then $A^{1/2}$ denotes the square root of A.

The operator $T \in \mathcal{B}(\mathcal{H})$ is called expansive (contractive) if $\Delta_T \geq 0$ ($-\Delta_T \geq 0$). Also, T is an isometry (resp. unitary) when $\Delta_T = 0$ (resp. $\Delta_T = 0$ and $\Delta_{T^*} = 0$). More generally, T is a 2-*isometry* if $\Delta_T^{(2)} = 0$, and in this case T is expansive.

In this paper we deal in principle with 3-*isometries*, namely, the operators T with the property that $\Delta_T^{(3)} = 0$, i.e. those which satisfy the relation

$$T^{*2}\Delta_T T^2 - 2T^*\Delta_T T + \Delta_T = 0. \tag{1.1}$$

If T is a 3-isometry, then T is convex, which means that $\Delta_T^{(2)} \geq 0$. However, it is not expansive, in general. We restrict our analysis to the expansive 3-isometries, in particular to those Δ_T-bounded, which means that there exists a constant $c \geq 1$ such that

$$T^*\Delta_T T \leq c\Delta_T. \tag{1.2}$$

A 3-isometry T which satisfies the condition (1.2) is called to be *sub-Brownian*, according to the definition from [6]. The case $c = 1$ in (1.2) imposes that T is 2-isometric, while when $c > 1$ one has $0 \leq \Delta_T^{(2)} \leq (c-1)\Delta_T$. Hence, the condition (1.2) ensures that T is expansive.

The sub-Brownian operators in the context of 3-isometries firstly appeared in [9] (the case $c = 2$), where the problem of their extensions to Brownian type operators was formulated. Such extensions for 2-isometries were studied by Agler and Stankus in [3, 4]. In [6] it was proved that every sub-Brownian 3-isometry has a 3-Brownian unitary extension, but here we do not refer to these extensions.

According to the notation and the terminology from [10], for $T \in \mathcal{B}(\mathcal{H})$ such that T^*T is invertible, $T' = T(T^*T)^{-1}$ is called the *Cauchy dual operator* of T. For such T we denote

$$\mathcal{H}_\infty(T) = \bigcap_{n\geq 1} T^n\mathcal{H}, \quad \mathcal{H}_\infty(T') = \bigcap_{n\geq 1} T'^n\mathcal{H}.$$

It follows from [10] that if T is an expansive 3-isometry, then $\mathcal{H}_\infty(T)$ is reducing for T and $T|_{\mathcal{H}_\infty(T)}$ is unitary. Moreover, $\mathcal{H}_\infty(T) \subset \mathcal{H}_\infty(T')$ and

$$\mathcal{H} = \mathcal{H}_\infty(T) \oplus (\bigvee_{n\geq 0} T'^n\mathcal{N}(T^*)) = \mathcal{H}_\infty(T') \oplus (\bigvee_{n\geq 0} T^n\mathcal{N}(T^*)).$$

For a subset $\mathcal{M}$ of $\mathcal{H}$, by $\bigvee \mathcal{M}$ we understand the closed linear span of $\mathcal{M}$. If $\mathcal{M}$ is a subspace of $\mathcal{H}$, then $\mathcal{M}^\perp = \mathcal{H} \ominus \mathcal{M}$ is the orthogonal complement of $\mathcal{M}$ in $\mathcal{H}$, and $P_\mathcal{M} \in \mathcal{B}(\mathcal{H})$ stands for the orthogonal projection onto $\mathcal{M}$.

We say that an operator T is *analytic* if $\mathcal{H}_\infty(T) = \{0\}$, has the *wandering subspace property* if $\mathcal{H}_\infty(T') = \{0\}$, and admits the *Wold-type decomposition* if $\mathcal{H}_\infty(T) = \mathcal{H}_\infty(T')$.

An important open problem is to describe the expansive 3-isometries which admit Wold-type decompositions. This problem is unsolved and even more interesting for those sub-Brownian. Below we obtain some results in the context of sub-Brownian 3-isometries, continuing the study started in [6].

The organization of the paper is the following. In Section 2 we show that all powers T^n for $n \geq 2$ of an expansive 3-isometry T are sub-Brownian. An interesting fact is that the constant of Δ_{T^n}-boundedness is independent of n, and different for even powers and odd powers. We also derive some useful relations between the kernels of Δ_{T^n} and $\Delta_T^{(2)}$, as well as the fact that $\mathcal{N}(\Delta_{T^2}) = \mathcal{N}(\Delta_T) \cap \mathcal{N}(\Delta_T^{(2)})$ is the maximum invariant subspace for T contained in $\mathcal{N}(\Delta_T)$.

In Section 3 we firstly refer to the matrix structure of an expansive 3-isometry T. We describe its block matrix under the decomposition of $\mathcal{H}$ induced by $\mathcal{N}(\Delta_{T^2})$. Some cases when T is sub-Brownian or even a 2-isometry are analyzed with respect to the entries of the matrix. Next, we describe the sub-Brownian 3-isometries in the terms of the Cauchy dual operator, which plays an essential role in Shimorin's theory [10]. An important result obtained is that the restriction of T to its range is sub-Brownian of covariance 2. Also, we show that under the kernel condition investigated in [7], T and its restriction to $\mathcal{R}(T)$ admit Wold-type decompositions.

2. POWERS OF EXPANSIVE 3-ISOMETRIES

It is known (see [6]) that an expansive 3-isometry T is not necessarily sub-Brownian. Nevertheless, the powers T^n have this property, as we shall see in the following proposition.

Proposition 2.1. *If $T \in \mathcal{B}(\mathcal{H})$ is an expansive 3-isometry, then T^n is a sub-Brownian 3-isometry for each integer $n \geq 2$. More precisely, we have*

$$T^{*2n}\Delta_{T^{2n}}T^{2n} \leq 5\Delta_{T^{2n}}, \quad T^{*(2n+1)}\Delta_{T^{2n+1}}T^{2n+1} \leq 17\Delta_{T^{2n+1}} \tag{2.1}$$

for all $n \geq 1$.

Proof. Let T be as before, that is, satisfying the conditions

$$T^{*2}\Delta_T T^2 - 2T^*\Delta_T T + \Delta_T = 0, \quad \Delta_T = T^*T - I \geq 0. \tag{2.2}$$

It is clear that $\Delta_{T^2} = T^{*2}T^2 - T^*T + \Delta_T = T^*\Delta_T T + \Delta_T$. Since T^n is also an expansive 3-isometry for $n \geq 1$, we have from (2.2) that

$$\Delta_{T^{2n}} = T^{*n}\Delta_{T^n}T^n + \Delta_{T^n}. \tag{2.3}$$

Now using this relation together with (2.2) for T^n in the form

$$T^{*2n}\Delta_{T^n}T^{2n} = 2T^{*n}\Delta_{T^n}T^n - \Delta_{T^n}$$

we obtain

$$\begin{aligned} T^{*2n}\Delta_{T^{2n}}T^{2n} &= T^{*3n}\Delta_{T^n}T^{3n} + T^{*2n}\Delta_{T^n}T^{2n} \\ &= 2T^{*2n}\Delta_{T^n}T^{2n} + T^{*n}\Delta_{T^n}T^n - \Delta_{T^n} \\ &= 5T^{*n}\Delta_{T^n}T^n - 3\Delta_{T^n} = 5\Delta_{T^{2n}} - 8\Delta_{T^n}. \end{aligned}$$

Taking into account that $\Delta_{T^n} \geq 0$, we infer from this relation the first inequality in (2.1), which justifies that T^{2n} is sub-Brownian for $n \geq 1$.

For the odd powers we have

$$\Delta_{T^{2n+1}} = T^*\Delta_{T^{2n}}T + \Delta_T \quad (n \geq 1). \tag{2.4}$$

Using (2.4) and the previous relations concerning $\Delta_{T^{2n}}$ we obtain

$$\begin{aligned}
&T^{*(2n+1)}\Delta_{T^{2n+1}}T^{2n+1} = T^{*2(n+1)}\Delta_{T^{2n}}T^{2(n+1)} + T^{*(2n+1)}\Delta_T T^{2n+1}\\
&= T^{*2}(5\Delta_{T^{2n}} - 8\Delta_{T^n})T^2 + T^{*2}\Delta_{T^{2n}}T^2 + T^{*2}T^2 - T^{*(2n+1)}T^{2n+1}\\
&= 6T^{*2}\Delta_{T^{2n}}T^2 - 8T^{*2}\Delta_{T^n}T^2 - \Delta_{T^{2n+1}} + \Delta_{T^2}\\
&= 6(T^*\Delta_{T^{2n+1}}T - T^*\Delta_T T) - \Delta_{T^{2n+1}} - 8T^{*2}\Delta_{T^n}T^2 + T^*\Delta_T T + \Delta_T\\
&= 6(T^{*(2n-1)}T^{*3}T^3T^{2n-1} - T^{*2}T^2) - \Delta_{T^{2n+1}} - 8T^{*2}\Delta_{T^n}T^2 - 5T^{*2}T^2 - I\\
&= 6T^{*(2n-1)}(3T^{*2}T^2 - 3T^*T + I)T^{2n-1} - \Delta_{T^{2n+1}} - 8T^{*2}\Delta_{T^n}T^2 - 5T^{*2}T^2 - I\\
&= 18T^{*(2n+1)}T^{2n+1} - 18T^{*2n}T^{2n} + 6T^{*(2n-1)}T^{2n-1} - \Delta_{T^{2n+1}}\\
&\quad - 8T^{*2}\Delta_{T^n}T^2 - 5T^{*2}T^2 - I\\
&= 17\Delta_{T^{2n+1}} - 18\Delta_{T^{2n}} + 6\Delta_{T^{2n-1}} - 8T^{*2}\Delta_{T^n}T^2 - 5\Delta_{T^2}\\
&= 17\Delta_{T^{2n+1}} - 6(3\Delta_{T^{2n}} - \Delta_{T^{2n-1}}) - 8T^{*2}\Delta_{T^n}T^2 - 5\Delta_{T^2}.
\end{aligned}$$

Since T is expansive, we have $\Delta_{T^j} \geq 0$ for $j \in \{2, n, 2n-1, 2n\}$ and $\Delta_{T^{2n}} - \Delta_{T^{2n-1}} \geq 0$. So from the above relation we infer the second inequality in (2.1). We conclude that

$$T^{*n}\Delta_{T^n}T^n \leq 17\Delta_{T^n}$$

for every integer $n \geq 2$, hence T^n is a sub-Brownian 3-isometry. □

Remark 2.2. The boundedness constant in the inequalities (2.1) does not depend on the exponent n from the power of T, while for the even powers we get a smaller constant. However, these constants (5 and 17) are not optimal, in general. We illustrate this fact in the following remark.

Remark 2.3. If $T \in \mathcal{B}(\mathcal{H})$ is an expansive 3-isometry, then

$$T^{*n}\Delta_{T^n}T^n \leq 4\Delta_{T^n}, \quad n \in \{3, 4\}.$$

Indeed, in the case of $n = 3$ we get from (2.2) that

$$\Delta_{T^3} = 2T^*\Delta_T T + T^{*2}T^2 - T^*T = 3T^*\Delta_T T.$$

Also, by (2.2) this implies

$$T^{*3}\Delta_{T^3}T^3 = 3T^{*4}\Delta_T T^4 = 3T^{*2}(2T^*\Delta_T T - \Delta_T)T^2 = 4\Delta_{T^3} - 9\Delta_T,$$

which gives the inequality $T^{*3}\Delta_{T^3}T^3 \leq 4\Delta_{T^3}$.

When $n = 4$ we obtain the same boundedness constant. Indeed, we have

$$\begin{aligned}
\Delta_{T^4} &= T^{*3}\Delta_T T^3 + \Delta_{T^3}\\
&= T^*(2T^*\Delta_T T - \Delta_T)T + 3T^*\Delta_T T\\
&= 6T^*\Delta_T T - 2\Delta_T = 2(\Delta_{T^3} - \Delta_T).
\end{aligned}$$

This together with the previous relation concerning Δ_{T^3} give the relations

$$\begin{aligned} T^{*4}\Delta_{T^4}T^4 &= 2T^{*4}(\Delta_{T^3} - \Delta_T)T^4 \\ &= 2T^*[T^{*3}\Delta_{T^3}T^3 - T^*(2T^*\Delta_T T - \Delta_T)T]T \\ &= 4\Delta_{T^4} - 2\Delta_{T^3} + 4T^*\Delta_T T - 10\Delta_T = 4\Delta_{T^4} - 2(T^*\Delta_T T + 5\Delta_T). \end{aligned}$$

So one obtains the inequality $T^{*4}\Delta_{T^4}T^4 \leq 4\Delta_{T^4}$.

If we compare this with the relations (2.1), we see that the boundedness constant in the cases $n = 3$ and $n = 4$ is less than those obtained in the general case. But those constants are relevant only for the covariance of sub-Brownian 3-isometries T^n $(n \geq 2)$, a concept used in [6], but which is not discussed here. Next we refer to the kernel of Δ_{T^n}.

Proposition 2.4. *If $T \in \mathcal{B}(\mathcal{H})$ is an expansive 3-isometry, then*

$$\mathcal{N}(\Delta_{T^n}) = \mathcal{N}(T^*\Delta_T T) \subset \mathcal{N}(\Delta_T) \tag{2.5}$$

for all integers $n \geq 2$. Moreover, the kernel $\mathcal{N}(\Delta_{T^n})$ is invariant for T, and the inclusion in (2.5) *becomes equality if and only if $\mathcal{N}(\Delta_T)$ is invariant for T.*

Proof. For T as above and each integer $n \geq 1$, by the relations (2.1), (2.3) and (2.4), we have

$$\begin{aligned} \Delta_{T^{2n+1}} &= T^{*(n+1)}\Delta_{T^n}T^{n+1} + T^*\Delta_{T^n}T + \Delta_T \\ &\leq 18T^*\Delta_{T^n}T + \Delta_T \leq 18T^*\Delta_{T^{2n-1}}T + \Delta_T \\ &= 18\Delta_{T^{2n}} - 17\Delta_T \leq 18\Delta_{T^{2n}}. \end{aligned}$$

Here we used that $T^*T \geq I$, which ensures that $\Delta_{T^n} \leq \Delta_{T^{2n-1}}$, as well as that $\Delta_{T^{2n}} \leq \Delta_{T^{2n+1}}$. As a consequence, we get $\mathcal{N}(\Delta_{T^{2n+1}}) = \mathcal{N}(\Delta_{T^{2n}})$.

On the other hand, by (2.1) and (2.3), we have

$$\Delta_{T^{2n}} = T^{*n}\Delta_{T^n}T^n + \Delta_{T^n} \leq 18\Delta_{T^n} \leq 18\Delta_{T^{2n-1}}.$$

These inequalities together with the above conclusion imply

$$\mathcal{N}(\Delta_{T^{2n-1}}) = \mathcal{N}(\Delta_{T^{2n}}) = \mathcal{N}(\Delta_{T^{2n+1}}),$$

hence all kernels $\mathcal{N}(\Delta_{T^n})$ coincide, for $n \geq 2$. But $\Delta_{T^2} = T^*\Delta_T T + \Delta_T$, so

$$\mathcal{N}(\Delta_{T^2}) = \mathcal{N}(T^*\Delta_T T) \cap \mathcal{N}(\Delta_T) = \mathcal{N}(T^*\Delta_T T) \subset \mathcal{N}(\Delta_T),$$

because $0 \leq \Delta_T \leq T^*\Delta_T T$ and T is a 3-isometry. Thus, (2.5) holds.

Now we show that $T\mathcal{N}(\Delta_{T^2}) \subset \mathcal{N}(\Delta_{T^2})$. Let $h \in \mathcal{N}(\Delta_{T^2})$. Then $\Delta_{T^2}h = \Delta_{T^4}h = \Delta_T h = 0$ (by (2.5)), and so

$$(\Delta_{T^3}Th, Th) = ((T^{*4}T^4 - T^*T)h, h) = ((\Delta_{T^4} - \Delta_T)h, h) = 0.$$

Since $\Delta_{T^3} \geq 0$, we get $\Delta_{T^3}Th = 0$, that is, $Th \in \mathcal{N}(\Delta_{T^3}) = \mathcal{N}(\Delta_{T^2})$.

Finally, if $T\mathcal{N}(\Delta_T) \subset \mathcal{N}(\Delta_T)$, then $\mathcal{N}(\Delta_T) \subset \mathcal{N}(T^*\Delta_T T)$ which means (by (2.5)) that $\mathcal{N}(\Delta_T) = \mathcal{N}(T^*\Delta_T T) = \mathcal{N}(\Delta_{T^n})$, for $n \geq 2$. This proves an implication from the last assertion of the proposition, while the converse implication is trivial. □

A more precise description of $\mathcal{N}(\Delta_{T^2})$ is obtained in the following result.

Proposition 2.5. *If $T \in \mathcal{B}(\mathcal{H})$ is an expansive 3-isometry, then $\mathcal{N}(\Delta_{T^2})$ is the maximum invariant subspace for T contained in $\mathcal{N}(\Delta_T)$. In addition, $\mathcal{N}(\Delta_{T^2}) = \mathcal{N}(\Delta_T) \cap \mathcal{N}(\Delta_T^{(2)})$, where $\Delta_T^{(2)} = T^*\Delta_T T - \Delta_T$.*

Proof. It is known from [8, Lemma 2.1] that the maximum subspace of $\mathcal{N}(\Delta_T)$ which is invariant for T is $\mathcal{H}_0 = \mathcal{N}(I - S_C)$, where S_C is the asymptotic limit of the contraction $C = P_{\mathcal{N}(\Delta_T)} T|_{\mathcal{N}(\Delta_T)}$, that is,

$$C = \lim_{n\to\infty} C^{*n}C^n \quad \text{strongly in } \mathcal{B}(\mathcal{N}(\Delta_T)).$$

Since $\mathcal{N}(\Delta_{T^2}) \subset \mathcal{N}(\Delta_T)$ and $\mathcal{N}(\Delta_{T^2})$ is invariant for T (by Proposition 2.4), it follows that $\mathcal{N}(\Delta_{T^2}) \subset \mathcal{H}_0$.

Conversely, let $h \in \mathcal{H}_0$. Then $T^*Th = h$. But $Th \in \mathcal{H}_0$, therefore $T^*T^2h = Th$, hence $T^{*2}T^2h = T^*Th = h$. Thus, $h \in \mathcal{N}(\Delta_{T^2})$ and this yields $\mathcal{H}_0 \subset \mathcal{N}(\Delta_{T^2})$. Consequently, $\mathcal{N}(\Delta_{T^2}) = \mathcal{H}_0$.

Furthermore, by (2.5), we have $\mathcal{N}(\Delta_{T^2}) = \mathcal{N}(T^*\Delta_T T) \subset \mathcal{N}(\Delta_T)$, which implies $\mathcal{N}(\Delta_{T^2}) \subset \mathcal{N}(\Delta_T) \cap \mathcal{N}(\Delta_T^{(2)})$. Since the converse inclusion is also true (by (2.5)), the proof is finished. □

We see that in Proposition 2.5 (by (2.5)), for each $n \geq 2$, $\mathcal{N}(\Delta_{T^n})$ is expressed in terms of $\mathcal{N}(\Delta_T)$ and $\mathcal{N}(\Delta_T^{(2)})$, the last subspace being invariant for T, on which T is 2-isometric. On the other hand, $\mathcal{N}(\Delta_T^{(2)})$ itself can also be expressed in the terms of Δ_{T^n} for $n \geq 1$. This fact is based on a relation between Δ_{T^n}, Δ_T and $T^*\Delta_T T$ given in the following result.

Proposition 2.6. *Let $T \in \mathcal{B}(\mathcal{H})$ be an expansive 3-isometry. Then for each integer $n \geq 2$ the following relations hold:*

$$\Delta_{T^n} = \frac{(n-1)n}{2} T^*\Delta_T T - \frac{(n-3)n}{2}\Delta_T, \quad n\Delta_T^{(2)} = T^*\Delta_{T^n}T - \Delta_{T^n}, \tag{2.6}$$

and

$$\mathcal{N}(\Delta_T^{(2)}) = \mathcal{N}(\Delta_{T^n} - n\Delta_T) = \mathcal{N}(T^*\Delta_{T^n}T - \Delta_{T^n}). \tag{2.7}$$

Proof. Recall that $\Delta_{T^2} = T^*\Delta_T T + \Delta_T$, $\Delta_{T^3} = 3T^*\Delta_T T$ and $\Delta_{T^4} = 6T^*\Delta_T T - 2\Delta_T$, so Δ_{T^n} has the form (2.6) for $n = 2, 3, 4$ (see Remark 2.3). Using the induction hypothesis on Δ_{T^n} in (2.6) for some $n > 4$, we obtain (by (2.2))

$$\begin{aligned}
\Delta_{T^{n+1}} &= T^*\Delta_{T^n}T + \Delta_T \\
&= \frac{(n-1)n}{2} T^{*2}\Delta_T T^2 - \frac{(n-3)n}{2} T^*\Delta_T T + \Delta_T \\
&= \frac{(n-1)n}{2}(2T^*\Delta_T T - \Delta_T) - \frac{(n-3)n}{2} T^*\Delta_T T + \Delta_T \\
&= \frac{n(n+1)}{2} T^*\Delta_T T - \frac{(n-2)(n+1)}{2}\Delta_T.
\end{aligned}$$

Hence, Δ_{T^n} satisfies the first relation in (2.6) for every integer $n \geq 1$.

On the other hand, we remark that the coefficients $p_n = \frac{(n-1)n}{2}$ and $q_n = \frac{n(3-n)}{2}$ in (2.6) verify the relation $p_n + q_n = n$. Using this fact we infer that $\Delta_T^{(2)}$ can be expressed by Δ_{T^n} as

$$\Delta_{T^n} - n\Delta_T = p_n \Delta_T^{(2)} \quad (n \geq 2),$$

whence we have the first equality in (2.7). Also, by this relation we get

$$T^*\Delta_{T^n}T - \Delta_{T^n} - n(T^*\Delta_T T - \Delta_T) = p_n(T^*\Delta_T^{(2)}T - \Delta_T^{(2)}) = p_n\Delta_T^{(3)} = 0.$$

Hence, $n\Delta_T^{(2)} = T^*\Delta_{T^n}T - \Delta_{T^n}$, so the second relation in (2.6) is true, which gives the last equality in (2.7). $\square$

3. SUB-BROWNIAN OPERATORS AND WOLD DECOMPOSITIONS

In this section we refer to sub-Brownian 3-isometries T on $\mathcal{H}$. Such an operator is expansive, and from Proposition 2.1 we infer that T has the form $T = T'^*S$, where T' is the Cauchy dual operator of T, while $S = T^2$ is sub-Brownian. Therefore, every expansive 3-isometry is a multiplicative and contractive perturbation of a sub-Brownian 3-isometry. In addition, if T itself is sub-Brownian, then $\mathcal{N}(\Delta_T)$ is invariant for T and $\mathcal{N}(\Delta_T) = \mathcal{N}(\Delta_{T^n})$ for $n \geq 2$ (in (2.5)). Also, for the expansive operators we have a matrix description suggested by Proposition 2.5.

Theorem 3.1. *An operator $T \in \mathcal{B}(\mathcal{H})$ is an expansive 3-isometry if and only if T has a matrix representation of the form*

$$T = \begin{pmatrix} V & X \\ 0 & Y \end{pmatrix} \tag{3.1}$$

under an orthogonal decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, such that V is an isometry with $V^*X = 0$, $Z = \begin{pmatrix} X & Y \end{pmatrix}^{\mathrm{tr}} : \mathcal{H}_1 \to \mathcal{H}$ is expansive, and Y, Z satisfy the relation

$$Y^{*2}\Delta_Z Y^2 - 2Y^*\Delta_Z Y + \Delta_Z = 0. \tag{3.2}$$

In this case $\mathcal{H}_0 \subset \mathcal{N}(\Delta_{T^2})$, and the following statements hold.

(i) *$\mathcal{H}_0 = \mathcal{N}(\Delta_{T^2})$ if and only if $\Delta_Z Y$ is injective. Furthermore, $\mathcal{H}_0 = \mathcal{N}(\Delta_T)$ if and only if Δ_Z is injective.*
(ii) *T of the form (3.1) is sub-Brownian if and only if Y is Δ_Z-bounded, and in this case $\mathcal{H}_0 = \mathcal{N}(\Delta_T)$. In addition, if X in (3.1) is expansive, then T is sub-Brownian.*

Proof. If T is a 3-isometry with $\Delta_T \geq 0$, then $\mathcal{H}_0 := \mathcal{N}(\Delta_{T^2})$ is invariant for T, so T has a block matrix (3.1) on $\mathcal{H} = \mathcal{H}_0 \oplus (\mathcal{H} \ominus \mathcal{H}_0)$, where $V = T|_{\mathcal{H}_0}$ is an isometry. Since $\Delta_T \geq 0$, we have $V^*X = 0$. Consequently,

$$\Delta_T = 0 \oplus (X^*X + Y^*Y - I) = 0 \oplus \Delta_Z \quad \text{on} \quad \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1, \tag{3.3}$$

where Z is the operator from Theorem 3.1 with $\Delta_Z \geq 0$.

This gives
$$\Delta_T^{(2)} = T^*\Delta_T T - \Delta_T = 0 \oplus (Y^*\Delta_Z Y - \Delta_Z), \tag{3.4}$$
and also
$$T^*\Delta_T^{(2)} T = 0 \oplus Y^*(Y^*\Delta_Z Y - \Delta_Z)Y.$$
Since $T^*\Delta_T^{(2)}T = \Delta_T^{(2)}$, T being 3-isometric, it follows that Y and Δ_Z satisfy the relation (3.2).

To prove the converse implication let us assume that T has the form (3.1) under some decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, with entries V, X, Y and Z as in the statement of the theorem. Then, as before, $\Delta_T = 0 \oplus \Delta_Z \geq 0$ because $\Delta_Z \geq 0$, so T is expansive. Moreover, the relations (3.2), (3.3) and (3.4) show that $T^*\Delta_T^{(2)}T = \Delta_T^{(2)}$, which means that T is 3-isometric.

Now taking into account the fact that $V = T|_{\mathcal{H}_0}$ is an isometry and that $T^*T\mathcal{H}_0 \subset \mathcal{H}_0$ (as $V^*X = 0$), we infer that the subspace $\mathcal{H}_0$ is contained in $\mathcal{N}(\Delta_T)$. But, by Proposition 2.5 this implies that $\mathcal{H}_0 \subset \mathcal{N}(\Delta_{T^2})$. The first assertion (of the equivalence) in the theorem is proved.

If $\mathcal{H}_0 = \mathcal{N}(\Delta_{T^2})$, then $\mathcal{H}_0 = \mathcal{N}(T^*\Delta_T T)$ by (2.5), so $P_{\mathcal{H}_1}\mathcal{N}(T^*\Delta_T T) = \{0\}$. But this means (by (3.1)) that $\mathcal{N}(Y^*\Delta_Z Y) = \{0\}$ or that $\Delta_Z Y$ is injective (as $\Delta_Z \geq 0$). Conversely, if $\mathcal{N}(\Delta_Z Y) = \{0\}$, then $\mathcal{N}(\Delta_{T^2}) = \mathcal{N}(T^*\Delta_T T) = \mathcal{H}_0$ (by (3.1)).

It is clear from (3.3) that $\mathcal{H}_0 = \mathcal{N}(\Delta_T)$ if and only if Δ_Z is injective, or equivalently (by Proposition 2.4), when $\mathcal{N}(\Delta_T)$ is invariant for T.

Next, from (3.3) and the fact that $T^*\Delta_T T = 0 \oplus Y^*\Delta_Z Y$ on $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ we deduce that the 3-isometry T is sub-Brownian if and only if the operator Y is Δ_Z-bounded. In this case, $\mathcal{N}(\Delta_T)$ is invariant for T, so $\mathcal{H}_0 = \mathcal{N}(\Delta_T)$. The assertions of (i) are proved.

Finally, let us assume that the operator $X = P_{\mathcal{H}_0}T|_{\mathcal{H}_1}$ in (3.1) is expansive, i.e. $X^*X \geq I$. We claim (by the previous conclusion) that Y is Δ_Z-bounded. Let $c = \|X^*X + Y^*Y\| > 1$. Then
$$\begin{aligned} Y^*\Delta_Z Y &= Y^*(X^*X + Y^*Y)Y - Y^*Y \leq (c-1)Y^*Y \\ &= (c-1)(Y^*Y + \Delta_X) - (c-1)\Delta_X \leq (c-1)\Delta_Z. \end{aligned}$$
Hence, Y is Δ_Z-bounded, which implies that T is Δ_T-bounded, that is, T is sub-Brownian. This proves the assertions of (ii) and completes the proof. □

In what follows, we analyze three special cases of the representation (3.1).

Theorem 3.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be an expansive 3-isometry with a block matrix of the form (3.1) on $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, having the entries V, X, Y as in Theorem 3.1. The following statements hold.*

(i) *$Y = P_{\mathcal{H}_1}T|_{\mathcal{H}_1}$ is a 3-isometry if and only if Y and $X = P_{\mathcal{H}_0}T|_{\mathcal{H}_1}$ satisfy the relation*
$$Y^{*2}X^*XY^2 - 2Y^*X^*XY + X^*X = 0. \tag{3.5}$$

(ii) *If X is an isometry, then Y is a 2-isometry on $\overline{\mathcal{R}(Y)}$ and Y is injective on $\mathcal{H}_1 = \overline{\mathcal{R}(\Delta_T)}$. In this case T is a 2-isometry on $\mathcal{H}$.*

(iii) *If $X^*X = Y^*Y$ and Y is 2-isometric on $\mathcal{R}(Y)$, then Y is 2-isometric on $\mathcal{H}_1 = \overline{\mathcal{R}(\Delta_T)}$ and T is sub-Brownian. In this case, T is 2-isometric if and only if Y (and so X) is an isometry.*

Proof. Preserving the notation $\Delta_Z = X^*X + Y^*Y - I$, by (3.2), we have

$$Y^{*2}X^*XY^2 - 2Y^*X^*XY + X^*X = -(Y^{*3}Y^3 - 3Y^{*2}Y^2 + 3Y^*Y - I).$$

Thus, the relation (3.5) is equivalent to the fact that Y is 3-isometric, which means the assertion (i).

To prove (ii) we assume that $X^*X = I$. Then the relation (3.2) becomes

$$Y^{*3}Y^3 - 2Y^{*2}Y^2 + Y^*Y = 0, \tag{3.6}$$

which means that Y is 2-isometric on $\mathcal{R}(Y)$. As Y is Δ_Z-bounded (by Theorem 3.1), it follows that $\mathcal{H}_0 = \mathcal{N}(\Delta_T)$, and later that Y is injective on $\mathcal{H}_1 = \overline{\mathcal{R}(\Delta_T)}$.

Now the relation (3.4) with $\Delta_Z = \Delta_Y$ together with (3.6) give

$$\Delta_T^{(2)} = T^*\Delta_T^{(2)}T = 0 \oplus Y^*(Y^*\Delta_Y Y - \Delta_Y)Y = 0 \oplus Y^*\Delta_Y^{(2)}Y = 0,$$

taking also into account that T is 3-isometric. Hence, T is even a 2-isometry on $\mathcal{H}$, what proves the assertion (ii).

Finally, assume that $X^*X = Y^*Y$ and $Y^*\Delta_Y^{(2)}Y = 0$. Then Y satisfies the relation (3.5) with $X^*X = Y^*Y$, and therefore Y is 3-isometric (from the assertion (i)). Thus, we have

$$\Delta_Y^{(2)} = Y^*\Delta_Y^{(2)}Y = 0,$$

which means that Y is 2-isometric. So Y is expansive which ensures that the operator $\Delta_Z = Y^*Y + \Delta_Y$ is invertible. Hence,$\mathcal{N}(\Delta_T) = \mathcal{H}_0 \oplus \mathcal{N}(\Delta_Z) = \mathcal{H}_0$, $\mathcal{H}_1 = \overline{\mathcal{R}(\Delta_T)}$ and T is sub-Brownian (by Theorem 3.1). Also, since Y is 2-isometric, from the relation (3.4) with $\Delta_Z = 2Y^*Y - I$ we get

$$\Delta_T^{(2)} = 0 \oplus (2Y^{*2}Y^2 - 3Y^*Y + I) = 0 \oplus (Y^{*2}Y^2 - Y^*Y) = 0 \oplus (Y^*Y - I).$$

This shows that T is 2-isometric on $\mathcal{H}$ if and only if Y (and so X) is an isometry. The assertion (iii) is proved. □

Remark 3.3. In the case $\mathcal{H}_0 = \mathcal{N}(\Delta_{T^2})$ the assertions (ii) and (iii) on Y in Theorem 3.2 can be completed by Theorems 2.1, 3.1 or 3.5 in [8] to get the quasi-Brownian isometric reducing part $\mathcal{H}_q$ of T in $\mathcal{H}$. This means that $\mathcal{H}_q$ is the maximum reducing subspace for T, and $T_q = T|_{\mathcal{H}_q}$ is a 2-isometry satisfying the condition

$$\Delta_{T_q}T_q = \Delta_{T_q}^{1/2}T_q\Delta_{T_q}^{1/2}.$$

The case previously mentioned will also lead to another matrix structure of T, which can be useful in applications. We omit here the details.

Theorem 3.2 shows some properties of T and Y on their ranges. Other information on T can be obtained by analyzing its block matrix with respect to the orthogonal decomposition $\mathcal{H} = \mathcal{R}(T) \oplus \mathcal{N}(T^*)$.

Theorem 3.4. *Let $T \in \mathcal{B}(\mathcal{H})$ be an expansive 3-isometry, $T_0 = T|_{\mathcal{R}(T)}$ and $T_1 = P_{\mathcal{R}(T)}T|_{\mathcal{N}(T^*)}$. Then T_0 is sub-Brownian as a 3-isometry. In fact, we have $T_0^*\Delta_{T_0}T_0 \leq 2\Delta_{T_0}$, and T_1 is expansive. Furthermore, the following statements hold.*

(i) *T is sub-Brownian if and only if the Cauchy dual operator $T' = T(T^*T)^{-1}$ of T is $D^2_{T'}$-bounded, where $D^2_{T'} = I - T'^*T'$.*

(ii) *T satisfies the kernel condition $T^*T\mathcal{N}(T^*) \subset \mathcal{N}(T^*)$ if and only if $\mathcal{R}(T_1) = \mathcal{N}(T_0^*)$. If this is the case, then T and T_0 simultaneously admit Wold-type decompositions.*

Proof. Since T is a 3-isometry with $\Delta_T = T^*T - I \geq 0$, we have

$$T^{*2}\Delta_T T^2 = 2T^*\Delta_T T - \Delta_T \leq 2T^*\Delta_T T.$$

As $T_0^* = P_{\mathcal{R}(T)}T^*|_{\mathcal{R}(T)}$, $\mathcal{R}(T)$ being invariant for T, we get for $h \in \mathcal{H}$ that

$$(T_0^*\Delta_{T_0}T_0Th, Th) = (T^{*2}\Delta_T T^2h, h) \leq 2(T^*\Delta_T Th, h) = 2(\Delta_{T_0}Th, Th).$$

Hence, $T_0^*\Delta_{T_0}T_0 \leq 2\Delta_{T_0}$ in $\mathcal{B}(\mathcal{R}(T))$, and consequently the 3-isometry $T_0 = T|_{\mathcal{H}_0}$ is sub-Brownian.

On the other hand, it is obvious that

$$T_1^*T_1 = P_{\mathcal{N}(T^*)}T^*P_{\mathcal{R}(T)}T|_{\mathcal{N}(T^*)} = P_{\mathcal{N}(T^*)}T^*T|_{\mathcal{N}(T^*)} \geq I,$$

therefore T_1 is expansive. The first assertion of the theorem is proved.

For the assertion (ii) we use the representations of T and T^*T on $\mathcal{H} = \mathcal{R}(T) \oplus \mathcal{N}(T^*)$ in the form

$$T = \begin{pmatrix} T_0 & T_1 \\ 0 & 0 \end{pmatrix}, \quad T^*T = \begin{pmatrix} T_0^*T_0 & T_0^*T_1 \\ T_1^*T_0 & T_1^*T_1 \end{pmatrix}. \tag{3.7}$$

They show that $T^*T\mathcal{N}(T^*) \subset \mathcal{N}(T^*)$ if and only if $\mathcal{R}(T_1) \subset \mathcal{N}(T_0^*)$. In this case

$$\mathcal{N}(T_0^*) \ominus \mathcal{R}(T_1) \subset \mathcal{N}(T_0^*) \cap \mathcal{N}(T_1^*) \subset \mathcal{R}(T) \cap \mathcal{N}(T^*) = \{0\}$$

(by (3.7)), so $\mathcal{R}(T_1) = \mathcal{N}(T_0^*)$. The first assertion of (ii) is proved.

Next, assuming this last equality, we denote

$$\mathcal{H}_\infty(T) = \bigcap_{n\geq 1} T^n\mathcal{H} \quad \text{and} \quad \mathcal{H}_\infty(T_0) = \bigcap_{n\geq 1} T_0^n\mathcal{R}(T).$$

Then

$$\mathcal{H}_\infty(T) = \bigcap_{n\geq 1} T_0^{n-1}\mathcal{R}(T) = \mathcal{H}_\infty(T_0).$$

On the other hand, with similar notations $\mathcal{H}_\infty(T')$ and $\mathcal{H}_\infty(T_0')$ for the Cauchy dual operators T' of T and T_0' of T_0, respectively, we can get that $\mathcal{H} \ominus \mathcal{H}_\infty(T') = \mathcal{H} \ominus \mathcal{H}_\infty(T_0')$. Indeed, by (3.7), we have

$$T\mathcal{N}(T^*) = T_1\mathcal{N}(T^*) \oplus \{0\} = \mathcal{N}(T_0^*) \oplus \{0\},$$

which leads to

$$T^2\mathcal{N}(T^*) = T(\mathcal{N}(T_0^*) \oplus \{0\}) = T_0\mathcal{N}(T_0^*) \oplus \{0\}.$$

To use the induction, we assume that

$$T^{n-1}\mathcal{N}(T^*) = T_0^{n-2}\mathcal{N}(T_0^*) \oplus \{0\}.$$

Then (by (3.7))

$$T^n\mathcal{N}(T^*) = T(T_0^{n-2}\mathcal{N}(T_0^*) \oplus \{0\}) = T_0^{n-1}\mathcal{N}(T_0^*) \oplus \{0\}.$$

Thus, we infer that

$$\begin{aligned}\mathcal{H} \ominus \mathcal{H}_\infty(T') &= \bigvee_{n\geq 0} T^n\mathcal{N}(T^*) = \mathcal{N}(T^*) \oplus \bigvee_{n\geq 1} T^n\mathcal{N}(T^*)\\ &= \mathcal{N}(T^*) \oplus \bigvee_{n\geq 0} T_0^n\mathcal{N}(T_0^*) = \mathcal{N}(T^*) \oplus (\mathcal{R}(T) \ominus \mathcal{H}_\infty(T_0'))\\ &= \mathcal{H} \ominus \mathcal{H}_\infty(T_0').\end{aligned}$$

We conclude that the relations

$$\mathcal{H} = \mathcal{H}_\infty(T) \oplus (\mathcal{H} \ominus \mathcal{H}_\infty(T')) \quad \text{and} \quad \mathcal{H} = \mathcal{H}_\infty(T_0) \oplus (\mathcal{H} \ominus \mathcal{H}_\infty(T_0'))$$

are simultaneously satisfied. In other words, T admits the Wold-type decomposition if and only if T_0 admits the Wold-type decomposition. The assertions of (ii) are proved.

For the statement (i), we assume firstly that T is sub-Brownian, that is,

$$T^*\Delta_T T \leq c\Delta_T$$

for some constant $c \geq 1$. Then, as $T^*T = (T'^*T')^{-1} \geq 1$, we get

$$\begin{aligned}T'^*D_{T'}^2T' &\leq T'^*D_{T'}T^*TD_{T'}T' = T'^*(T^*T - I)T' = T'^*T'T^*\Delta_T TT'^*T'\\ &\leq c(T'^*T')^2\Delta_T = cT'^*T'(I - T'^*T') \leq cD_{T'}^2.\end{aligned}$$

Hence, T' is $D_{T'}^2$-bounded.

Conversely, let us assume that $T'^*D_{T'}^2T' \leq cD_{T'}$ with $c > 0$. Then there exists an operator W on $\overline{\mathcal{R}(D_{T'})} = \overline{\mathcal{R}(\Delta_T)}$ such that $WD_{T'} = D_{T'}T'$. This means

$$W\Delta_T^{1/2}(T'^*T')^{1/2} = (T'^*T')^{1/2}\Delta_T^{1/2}TT'^*T',$$

whence

$$(T^*T)^{1/2}W\Delta_T^{1/2} = \Delta_T^{1/2}T(T'^*T')^{1/2}.$$

Next, we infer that

$$(T^*T)^{-1/2}T^*\Delta_T T(T^*T)^{-1/2} = \Delta_T^{1/2}W^*(T^*T)W\Delta_T^{1/2} \leq \|T\|^2\|W\|^2\Delta_T,$$

whence we also get

$$T^*\Delta_T T \leq \Delta_T^{1/2}(T^*T)^{1/2}W^*(T^*T)W(T^*T)^{1/2}\Delta_T^{1/2} \leq \|T\|^4\|W\|^2\Delta_T.$$

This finally implies $T^*\Delta_T T \leq c\Delta_T$, where $c = \|T\|^4\|W\|^2 > 0$. Here $\|T\| \geq 1$ and $W \neq 0$. In fact, $W = 0$ implies that $\Delta_T T = D_{T'}T' = 0$, whence $T^*\Delta_T T = 0$, which yields $\Delta_T = 2T^*\Delta_T T - T^{*2}\Delta_T T^2 = 0$, that is, T is an isometry (a trivial case). From the above inequality we conclude that the 3-isometry T is sub-Brownian. This proves the assertion (i). □

An additional information on sub-Brownian 3-isometries T can be obtained as a consequence of the assertion (i).

Corollary 3.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be an expansive 3-isometry with $T^*T\mathcal{N}(T^*) \subset \mathcal{N}(T^*)$ and T' be its Cauchy dual operator. Then T is sub-Brownian if and only if there exist an operator W and a 2-isometry S in $\mathcal{B}(\mathcal{H})$ such that $WD_{T'} = D_{T'}T'$ and $W(T^*T)^{1/2} = (T^*T)^{-1/2}S$.*

Proof. Assume that T is sub-Brownian. So T' is $D^2_{T'}$-bounded (by Theorem 3.4) and there is $W \in \mathcal{B}(\mathcal{H})$ with $WD_{T'} = D_{T'}T'$. This means that

$$W(T^*T)^{-1/2}\Delta_T^{1/2} = (T^*T)^{-1/2}\Delta_T T(T^*T)^{-1},$$

or equivalently

$$\Delta_T^{1/2}T = (T^*T)^{1/2}W(T^*T)^{-1/2}\Delta_T^{1/2}T^*T = (T^*T)^{1/2}W(T^*T)^{1/2}\Delta_T^{1/2}.$$

Define the operator

$$S = (T^*T)^{1/2}W(T^*T)^{1/2}.$$

Then $\Delta_T^{1/2}T = S\Delta_T^{1/2}$, so $\Delta_T^{1/2}S^*S\Delta_T^{1/2} = T^*\Delta_T T$. As T is 3-isometric, it follows that S is 2-isometric such that $(T^*T)^{-1/2}S = W(T^*T)^{1/2}$.

Conversely, if there are two operators S and W satisfying the quoted relations, then

$$WD_{T'}T^*T = D_{T'}T = (T^*T)^{-1/2}\Delta_T^{1/2}T.$$

This implies that

$$(T^*T)^{-1/2}\Delta_T^{1/2}T = WT^*T(T^*T)^{-1/2}\Delta_T^{1/2} = W(T^*T)^{1/2}\Delta_T^{1/2} = (T^*T)^{-1/2}S\Delta_T^{1/2}.$$

Consequently, $\Delta_T^{1/2}T = S\Delta_T^{1/2}$. Finally, we get

$$T^*\Delta_T T = \Delta_T^{1/2}S^*S\Delta_T^{1/2} \leq \|S\|^2\Delta_T,$$

hence T is sub-Brownian. □

Next we consider the representation (3.7) of T with respect to the orthogonal decomposition $\mathcal{H} = \mathcal{R}(T) \oplus \mathcal{N}(T^*)$ in a useful form for applications.

Remark 3.6. Recall (from Theorem 3.4 (ii)) that for an expansive 3-isometry T as in (3.7) we have $T^*T\mathcal{N}(T^*) \subset \mathcal{N}(T^*)$ if and only if $\mathcal{R}(T_1) = \mathcal{N}(T_0^*)$. On the other hand, we have $T^{*2}T^2\mathcal{N}(T^*) \subset \mathcal{N}(T^*)$ if and only if $T_0^*T_0\mathcal{R}(T_1) \subset \mathcal{N}(T_0^*)$. Indeed, by (3.7), we obtain

$$T^2 = \begin{pmatrix} T_0^2 & T_0T_1 \\ 0 & 0 \end{pmatrix}, \quad T^{*2}T^2 = \begin{pmatrix} T_0^{*2}T_0^2 & T_0^{*2}T_0T_1 \\ T_1^*T_0^*T_0^2 & T_1^*T_0^*T_0T_1 \end{pmatrix}.$$

Therefore, $T^{*2}T^2\mathcal{N}(T^*) \subset \mathcal{N}(T^*)$ if and only if $T_0^{*2}T_0T_1 = 0$, that is, $T_0^*T_0\mathcal{R}(T_1) \subset \mathcal{N}(T_0^*)$. This inclusion becomes $T_0^*T_0\mathcal{N}(T_0^*) \subset \mathcal{N}(T_0^*)$ when $T^*T\mathcal{N}(T^*) \subset \mathcal{N}(T^*)$.

Using the terminology from [7] we say that T satisfies the *2-kernel condition* if $\mathcal{N}(T^*)$ is invariant for T^*T and $T^{*2}T^2$. In these terms, we present below some cases when T is sub-Brownian.

Proposition 3.7. *Let $T \in \mathcal{B}(\mathcal{H})$ be an expansive 3-isometry which satisfies the 2-kernel condition and let $T_0 = T|_{\mathcal{R}(T)}$, $T_1 = P_{\mathcal{R}(T)}T|_{\mathcal{N}(T^*)}$. Then the following assertions hold.*

(i) *T is sub-Brownian if and only if $\mathcal{R}(\Delta_{T_1}^{1/2}) = \mathcal{R}(T_1^*\Delta_{T_0}^{1/2}T_1)$.*
(ii) *$T\mathcal{N}(\Delta_T) \subset \mathcal{N}(\Delta_T)$ if and only if $T_1\mathcal{N}(\Delta_{T_1}) \subset \mathcal{N}(\Delta_{T_0})$. In this case, if $\mathcal{R}(\Delta_{T_1})$ is closed then T is sub-Brownian.*

Proof. Assume T as above, so T having the representation (3.7) with $T_0^*T_1 = 0$ and $T_0^{*2}T_0T_1 = 0$. Then on $\mathcal{H} = \mathcal{R}(T) \oplus \mathcal{N}(T^*)$ we get

$$\Delta_T = \Delta_{T_0} \oplus \Delta_{T_1}, \quad T^*\Delta_T T = T_0^*\Delta_{T_0}T_0 \oplus T_1^*\Delta_{T_0}T_1. \tag{3.8}$$

To prove (i) we use that T_0 is sub-Brownian (by Theorem (3.4)). Therefore, T is sub-Brownian if and only if $T_1^*\Delta_{T_0}T_1 \leq c\Delta_{T_1}$, for a constant $c > 0$. But by the Douglas criterion on ranges, this inequality is equivalent with $\mathcal{R}(T_1^*\Delta_{T_0}^{1/2}) \subset \mathcal{R}(\Delta_{T_1}^{1/2})$. In fact, this inclusion becomes equality because $\Delta_T \leq T^*\Delta_T T$ (T being 3-isometric and so convex), which gives by (3.8) that $\Delta_{T_1} \leq T_1^*\Delta_{T_0}T_1$, that is, $\mathcal{R}(\Delta_{T_1}^{1/2}) \subset \mathcal{R}(T_1^*\Delta_{T_0}^{1/2})$. But $T_0^*T_0\mathcal{N}(T_1^*) \subset \mathcal{N}(T_1^*)$ by the 2-kernel condition for T, therefore

$$T_1^*\Delta_{T_0}^{1/2}\mathcal{R}(T) = T_1^*\Delta_{T_0}^{1/2}(\mathcal{R}(T_1) \oplus \mathcal{N}(T_1^*)) = T_1^*\Delta_{T_0}^{1/2}\mathcal{R}(T_1).$$

Hence, $\mathcal{R}(\Delta_{T_1}^{1/2}) = \mathcal{R}(T_1^*\Delta_{T_0}^{1/2}) = \mathcal{R}(T_1^*\Delta_{T_0}^{1/2}T_1)$, when T is sub-Brownian, while the reverse implication is obvious by 2-kernel condition of T and Douglas's criterion. This gives the assertion (i).

For (ii) we use the representations (3.7) for T and (3.8) for Δ_T, which give

$$\Delta_T T = \begin{pmatrix} \Delta_{T_0}T_0 & \Delta_{T_0}T_1 \\ 0 & 0 \end{pmatrix} \quad \text{on } \mathcal{H} = \mathcal{R}(T) \oplus \mathcal{N}(T^*).$$

As $T_0\mathcal{N}(\Delta_{T_0}) \subset \mathcal{N}(\Delta_{T_0})$, T_0 being sub-Brownian, it follows that $T\mathcal{N}(\Delta_T) \subset \mathcal{N}(\Delta_T)$ if and only if $T_1\mathcal{N}(\Delta_{T_1}) \subset \mathcal{N}(\Delta_{T_0})$. But this last relation implies $T_1^*\mathcal{R}(\Delta_{T_0}^{1/2}) \subset \mathcal{R}(\Delta_{T_1}^{1/2})$, when $\mathcal{R}(\Delta_{T_1})$ is closed. If this happens, then using the 2-kernel condition of T and the relation $\Delta_{T_1} \leq T_1^* \Delta_{T_0} T_1$, it follows that

$$\mathcal{R}(T_1^* \Delta_{T_0}^{1/2} T_1) = \mathcal{R}(T_1^* \Delta_{T_0}^{1/2}) = \mathcal{R}(\Delta_{T_1}^{1/2}).$$

But by the assertion (i) before, this means that T is sub-Brownian. □

Because the condition $T\mathcal{N}(\Delta_T) \subset \mathcal{N}(\Delta_T)$ is necessary for T to be sub-Brownian, the important assumption in the last assertion of the proposition is that $\mathcal{R}(\Delta_{T_1})$ is closed. A special case when this happens, appearing in some applications, is when Δ_{T_1} is a scalar multiple of an orthogonal projection.

We now refer to another more particular case when T has the properties discussed above.

Theorem 3.8. *Let T be an expansive 3-isometry with $T^*T\mathcal{N}(T^*) \subset \mathcal{N}(T^*)$ and T' be its Cauchy dual operator. The following statements hold.*

(i) *$T^{*2}T^2\mathcal{N}(T^*) \subset \mathcal{N}(T^*)$ if and only if $TT'\mathcal{N}(T^*) = T'T\mathcal{N}(T^*)$. In this case T admits Wold-type decomposition.*
(ii) *$\mathcal{N}(T^*) \subset \mathcal{N}(TT' - T'T)$ if and only if $\Delta_{T_0} T_1 = T_1 \Delta_{T_1}$, where $T_0 = T|_{\mathcal{R}(T)}$, $T_1 = P_{\mathcal{R}(T)} T|_{\mathcal{N}(T^*)}$. In this case T is sub-Brownian.*

Proof. The condition $T^*T\mathcal{N}(T^*) \subset \mathcal{N}(T^*)$ ensures by (3.7) that $T^*T = T_0^*T_0 \oplus T_1^*T_1$ on $\mathcal{H} = \mathcal{R}(T) \oplus \mathcal{N}(T^*)$. So like T in (3.7), we get for T' the representation

$$T' = T(T^*T)^{-1} = \begin{pmatrix} T_0 & T_1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (T_0^*T_0)^{-1} & 0 \\ 0 & (T_1^*T_1)^{-1} \end{pmatrix} = \begin{pmatrix} T_0' & T_1' \\ 0 & 0 \end{pmatrix},$$

where T_0' and T_1' are Cauchy dual operators for T_0 and T_1, respectively. Then

$$TT' = \begin{pmatrix} T_0T_0' & T_0T_1' \\ 0 & 0 \end{pmatrix}, \quad T'T = \begin{pmatrix} T_0'T_0 & T_0'T_1 \\ 0 & 0 \end{pmatrix}.$$

(i) Assume that $TT'\mathcal{N}(T^*) = T'T\mathcal{N}(T^*)$. This means that

$$T_0T_1'\mathcal{N}(T^*) = T_0'T_1\mathcal{N}(T^*),$$

or

$$T_0T_1\mathcal{N}(T^*) = T_0(T_0^*T_0)^{-1}T_1\mathcal{N}(T^*),$$

taking into account that

$$T_1^*T_1\mathcal{N}(T^*) = \mathcal{N}(T^*) = (T_1^*T_1)^{-1}\mathcal{N}(T^*).$$

As $T_0'^*T_0 = I$, we get

$$T_1\mathcal{N}(T^*) = (T_0^*T_0)^{-1}T_1\mathcal{N}(T^*),$$

that is,

$$T_0^*T_0T_1\mathcal{N}(T^*) = T_1\mathcal{N}(T^*) = \mathcal{R}(T_1).$$

But this means that $T_0^*T_0\mathcal{N}(T_0^*) = \mathcal{N}(T_0^*)$, which is equivalent with the inclusion $T^{*2}T^2\mathcal{N}(T^*) \subset \mathcal{N}(T^*)$, by Remark 3.6.

For the converse implication of the former assertion in (i), we show that $TT'\mathcal{N}(T^*) = T'T\mathcal{N}(T^*)$, when T satisfies the 2-kernel condition. Because T_0 also satisfies the 2-kernel condition we have $T_0^*T_0\mathcal{N}(T_0^*) = \mathcal{N}(T_0^*)$, or in another form $(T_0^*T_0)^{-1}T_1\mathcal{N}(T^*) = T_1\mathcal{N}(T^*)$. This later gives

$$T_0'T_1\mathcal{N}(T^*) = T_0(T_0^*T_0)^{-1}T_1\mathcal{N}(T^*) = T_0T_1\mathcal{N}(T^*) = T_0T_1'\mathcal{N}(T^*),$$

hence $T'T\mathcal{N}(T^*) = TT'\mathcal{N}(T^*)$. The equivalence from (i) is now proved.

To prove the second assertion in (i), we assume that T satisfies the 2-kernel condition. Then $T'\mathcal{N}(T^*) = T\mathcal{N}(T^*)$, and as above we have $\mathcal{N}(T_0^*) = \mathcal{R}(T_1) = T_0^*T_0\mathcal{R}(T_1)$. So we get

$$\begin{aligned} T'^2\mathcal{N}(T^*) &= T(T^*T)^{-1}T'\mathcal{N}(T^*) = T(T^*T)^{-1}T\mathcal{N}(T^*) \\ &= T(T^*T)^{-1}(T_1\mathcal{N}(T^*) \oplus \{0\}) \\ &= T((T_0^*T_0)^{-1}T_1\mathcal{N}(T^*) \oplus \{0\}) \\ &= T(T_1\mathcal{N}(T^*) \oplus \{0\}) = T^2\mathcal{N}(T^*). \end{aligned}$$

Notice that the 2-kernel condition for T ensures that $T^{*n}T^n\mathcal{N}(T^*) \subset \mathcal{N}(T^*)$ for every integer $n \geq 1$ (T being a 3-isometry), while this implies later that T_0 also satisfies the 2-kernel condition (as a 3-isometry on $\mathcal{R}(T)$). By a similar argument as above, we have $T_0'^2\mathcal{N}(T_0^*) = T_0^2\mathcal{N}(T_0^*)$. Thus, we get

$$\begin{aligned} T'^3\mathcal{N}(T^*) &= T'^2T\mathcal{N}(T^*) = T'^2(T_1\mathcal{N}(T^*) \oplus \{0\}) \\ &= T_0'^2\mathcal{N}(T_0^*) \oplus \{0\} = T_0^2\mathcal{N}(T_0^*) \oplus \{0\} = T^3\mathcal{N}(T^*). \end{aligned}$$

Using the induction argument it will follow that $T'^n\mathcal{N}(T^*) = T^n\mathcal{N}(T^*)$ for $n \geq 1$. With the notation from the proof of Theorem 3.4 we obtain

$$\mathcal{H} \ominus \mathcal{H}_\infty = \bigvee_{n\geq 0} T'^n\mathcal{N}(T^*) = \bigvee_{n\geq 0} T^n\mathcal{N}(T^*) = \mathcal{H} \ominus \mathcal{H}_\infty(T'),$$

which means that T admits the Wold-type decomposition. The assertion (i) is proved.

(ii) Assume firstly that $TT' = T'T$ on $\mathcal{N}(T^*)$. So $T_0T_1' = T_0'T_1$ on $\mathcal{N}(T^*)$ which yields $T_0^*T_0T_1 = T_1T_1^*T_1$, that is, $\Delta_{T_0}T_1 = T_1\Delta_{T_1}$. Next, from this relation it follows that

$$T_1^*\Delta_{T_0}T_1 = T_1^*T_1\Delta_{T_1} = \Delta_{T_1}^{1/2}T_1^*T_1\Delta_{T_1}^{1/2} \leq \|T_1\|^2\Delta_{T_1}.$$

Having in view that $T_0^*\Delta_{T_0}T_0 \leq 2\Delta_{T_0}$ (by Theorem 3.4) as well as the relations (3.8) we get

$$T^*\Delta_T T = T_0^*\Delta_{T_0}T_0 \oplus T_1^*\Delta_{T_0}T_1 \leq \max\{2, \|T_1\|^2\}\Delta_T.$$

Hence, T is sub-Brownian as a 3-isometry.

Remark finally that the relation $\Delta_{T_0}T_1 = T_1\Delta_{T_1}$ implies $TT'h = T'Th$ for $h \in \mathcal{N}(T^*)$, by the reverse argument of above. Thus, the assertion (ii) is proved, which ends the proof. □

Notice that the assertion from (i) concerning the Wold decomposition also results from [7, Theorem 4.2]. But in this context our proof is slightly different from that given in [7], and it is included here for the sake of completion.

An open question in this context is if an expansive 3-isometry T satisfying $T^*T\mathcal{N}(T^*) \subset \mathcal{N}(T^*)$ admits the Wold-type decomposition. This is unknown even in the case when T is sub-Brownian.

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Laurian Suciu
laurian.suciu@ulbsibiu.ro
https://orcid.org/0000-0002-5675-0519

Universitatea Lucian Blaga din Sibiu
Departamentul de Matematica si Informatica
Sibiu, Romania

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