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Dedicated to Prof. Wai Chee Shiu
on the occasion of his 66th birthday

COMPLETE CHARACTERIZATION OF GRAPHS WITH LOCAL TOTAL ANTIMAGIC CHROMATIC NUMBER 3

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Abstract. A total labeling of a graph $G = (V, E)$ is said to be local total antimagic if it is a bijection $f : V \cup E \to \{1, \ldots, |V|+|E|\}$ such that adjacent vertices, adjacent edges, and pairs of an incident vertex and edge have distinct induced weights where the induced weight of a vertex v is $w_f(v) = \sum f(e)$ with e ranging over all the edges incident to v, and the induced weight of an edge uv is $w_f(uv) = f(u)+f(v)$. The local total antimagic chromatic number of G, denoted by $\chi_{lt}(G)$, is the minimum number of distinct induced vertex and edge weights over all local total antimagic labelings of G. In this paper, we first obtain general lower and upper bounds for $\chi_{lt}(G)$ and sufficient conditions to construct a graph H with k pendant edges and $\chi_{lt}(H) \in \{\Delta(H) + 1, k + 1\}$. We then completely characterize graphs G with $\chi_{lt}(G) = 3$. Many families of (disconnected) graphs H with k pendant edges and $\chi_{lt}(H) \in \{\Delta(H) + 1, k + 1\}$ are also obtained.

Keywords: local total antimagic, local total antimagic chromatic number.

Mathematics Subject Classification: 05C78, 05C69.

1. INTRODUCTION

Let $G = (V, E)$ be a simple and loopless graph of order p and size q. For integers $a < b$, let $[a, b] = \{n \in \mathbb{Z} \mid a \le n \le b\}$. If no adjacent vertices of G are assigned the same color, then G has a proper (vertex) coloring. The smallest required number of colors is the chromatic number of G, denoted $\chi(G)$. The most famous problem on the chromatic number is the 4-Color Conjecture which states that every planar graph G has $\chi(G) = 4$. Interested readers may refer to [2] for the proof by Appel and Haken. If no two adjacent edges of G are assigned the same color, then G has a proper edge coloring. The smallest required number of colors is the chromatic index of G, denoted $\chi'(G)$. Interested readers may refer to [16] for an interesting brief history. A total coloring of G is a coloring of the vertices and edges of G such that x and y are assigned distinct colors whenever x and y are adjacent vertices or edges, or incident vertex and edge. The smallest number of colors required for a total coloring is the total chromatic number of G, denoted $\chi_t(G)$. The conjecture that

$$\Delta(G)+1 \leq \chi_t(G) \leq \Delta(G)+2$$

remained unsolved (see [4, 6]) where $\Delta(G)$ is the maximum degree of G.

A bijection $f : E \to [1, q]$ is a local antimagic labeling if $f^+(u) \neq f^+(v)$ for every edge uv of G where $f^+(u)$ is the sum of all the incident edge label(s) of u under f. Let $f^+(u)$ be the color of u. Clearly, f^+ is a proper vertex coloring of G induced by f. The smallest number of vertex colors taken over all local antimagic labeling f is called the local antimagic chromatic number of G, denoted $\chi_{la}(G)$ (see [3, 7–9, 11, 12] for the many results available).

A bijection $g : V \to [1, p]$ is a local edge antimagic labeling if $g_{le}(e_1) \neq g_{le}(e_2)$ for every two adjacent edges e_1 and e_2 of G where $g_{le}(uv) = g(u) + g(v)$. Let $g_{le}(e)$ be the induced edge color of e under g. Clearly, g_{le} is a proper edge coloring of G induced by g. The smallest number of edge color(s) over all local edge antimagic labeling g is called the local edge antimagic chromatic number of G, denoted $\chi_{lea}(G)$ [1].

A bijection $f : V \cup E \to [1, p+q]$ is a local antimagic total labeling if $w^+(u) \neq w^+(v)$ for every two adjacent vertices u and v where $w^+(u) = f(u) + \sum f(e)$ over all edge(s) e incident to u. Let $w^+(u)$ be the induced vertex color of u under f. Clearly, w^+ is a proper vertex coloring of G induced by f. The smallest number of vertex colors over all local antimagic total labeling f is called the local antimagic total chromatic number of G, denoted $\chi_{lat}(G)$. In [10, 13], the authors proved that every graph is local antimagic total and obtained the exact $\chi_{lat}(G)$ for many families of graphs G.

Motivated by the concepts of local (edge) antimagic labeling and total labeling of G, Sandhiya and Nalliah [14] defined the concept of local total antimagic labeling. A bijection $f : V \cup E \to [1, p+q]$ such that the weight of a vertex u is $w_f(u) = \sum f(e)$ over all edge(s) e incident to u, and the weight of an edge $e = uv$ is $w_f(e) = f(u)+f(v)$, is called a *local total antimagic labeling* if (i) every two adjacent vertices and adjacent edges have distinct weights, and (ii) every vertex and its incident edge(s) have distinct weights. The mapping w_f (or w if no ambiguity) is called a *local total antimagic labeling of G induced by f*, and the weights assigned to vertices and edges are called *induced total colors* under f.

The *local total color number* of a local total antimagic labeling f is the number of distinct induced total colors under f, denoted by $w(f)$. Moreover, f is called a *local total antimagic $w(f)$-coloring* and G is *local total antimagic $w(f)$-colorable.* The *local total antimagic chromatic number* $\chi_{lt}(G)$ is defined to be the minimum number of colors taken over all local total antimagic colorings of G induced by local total antimagic labelings of G. Let $G + H$ and mG denote the disjoint union of graphs G and H, and m copies of G, respectively. By definition, any graph with an isolated vertex or a K_2 component does not admit a local total antimagic labeling. In this paper, we only consider graph G of order $p \geq 3$ and size $q \geq 2$. Thus, G contains neither isolated vertex nor K_2 component.

In Section 2, we obtained various sufficient conditions on bounds of $\chi_{lt}(G)$. In Section 3, we give complete characterization of graphs G with $\chi_{lt}(G) = 3$. In Section 4, we first obtained various families of disconnected graph with k pendant edges and

$\chi_{lt}(G) = k + 1$. Using the results in Section 3, we then obtained various new families of graphs H, constructed from G, having s pendant edges and $\chi_{lt}(G) = s + 1$. Open problems and conjectures are given in Section 5.

2. SUFFICIENT CONDITIONS

In [14], the authors proved that a graph G with $k \geq 1$ pendant edge(s) has $\chi_{lt}(G) \geq k + 1$. We now give a necessary condition and a proof for equality to hold.

Lemma 2.1. *Suppose G has $k \geq 1$ pendant edges, then $\chi_{lt}(G) \geq k+1$. If $\chi_{lt}(G) = k+1$, then $p + q$ is assigned to a pendant vertex or a pendant edge.*

Proof. For completeness, we only need to prove the second statement. Assume that $\chi_{lt}(G) = k+1$. Suppose $p+q$ is assigned to a non-pendant vertex, say x. Now, all the k pendant vertices have distinct weights at most $p+q-1$. Moreover, x is incident to two edges, say e and e'. Clearly, $w(e), w(e')$ are distinct weights at least $p+q+1$. Suppose $p + q$ is assigned to a non-pendant edge, say xy. Now, all the k pendant vertices have distinct weights at most $p+q-1$ and $w(x), w(y)$ are distinct weights at least $p+q+1$. In both cases, there are at least $k + 2$ distinct weight so that $\chi_{lt}(G) \geq k + 2$. This contradicts $\chi_{lt}(G) = k + 1$. □

Corollary 2.2. *Suppose G is a graph with $k \geq 1$ pendant edges, then*

$$\chi_{lt}(G) \geq \max\{\Delta(G) + 1, k + 1\} \geq 3.$$

Proof. By definition and Lemma 2.1, $\chi_{lt}(G) \geq \max\{\Delta(G) + 1, k + 1\}$. Thus, if $\chi_{lt}(G) = 2$, then $\Delta(G) = 1$ or $k = 1$. If the former holds, G is a 1-regular graph that does not admit a local total antimagic labeling. If the latter holds, $\Delta(G) \geq 3$ so that $\chi_{lt}(G) \geq \chi_t(G) \geq 4$, a contradiction. Consequently, $\chi_{lt}(G) \geq 3$ if G admits a local total antimagic labeling. □

In [14], the authors also proved that if G is an r-regular graph, $r \geq 2$, that admits a $\chi_{la}(G)$-labeling and a $\chi_{lea}(G)$-labeling, then $\chi_{lt}(G) \leq \chi_{la}(G) + \chi_{lea}(G)$. We now give a more general result.

Theorem 2.3. *Suppose G admits a $\chi_{lea}(G)$-labeling. If G has* (i) *$\delta(G) \geq 2$; or else* (ii) *$\delta(G) = 1$ so that G has $k \geq 1$ pendant vertices, v_p non-pendant vertices and e_p non-pendant edges with $e_p > v_p + k - 2$ that admits a $\chi_{la}(G)$-labeling that assigns the non-pendant edges by integers in $[1, e_p]$, then G admits a local total antimagic labeling. Moreover, $\chi_{lt}(G) \leq \chi_{la}(G) + \chi_{lea}(G)$.*

Proof. Suppose G has order p and size q. By definition, G has no K_2 components. Let $f : E \to [1, q]$ (and $h : V \to [1, p]$) be a $\chi_{la}(G)$- (and $\chi_{lea}(G)$-) labeling of G. Define a bijective total labeling $g : V \cup E \to [1, p + q]$ such that $g(u) = h(u)$ for each vertex u of G, and $g(e) = f(e) + p$ for each edge e of G. Therefore, $\{g(u)\} = [1, p]$ and $\{g(e)\} = [p + 1, p + q]$. Clearly, $w_g(u) = w_g(v)$ if and only if $f^+(u) = f^+(v)$, and $w_g(e_1) = w_g(e_2)$ if and only if $h_{le}(e_1) = h_{le}(e_2)$ for all $u, v \in V(G)$ and $e_1, e_2 \in E(G)$.

(i) $\delta \geq 2$. In this case, G has no pendant edges. Thus, every edge weight under g is at most $2p-1$ and every vertex weight under g is at least $2p+3$. Therefore, every edge weight is less than every vertex weight. Consequently, g is a local total antimagic labeling that induces $\chi_{la}(G)+\chi_{lea}(G)$ distinct weights.

(ii) $\delta = 1$. Thus, G has order $p = v_p + k$ and size $q = e_p + k$. By the given assumption, the $\chi_{la}(G)$-labeling f of G assigns each non-pendant edge with an integer in $[1, e_p]$. In this case, under g, every vertex is assigned an integer in $[1, v_p + k]$, every non-pendant edge is assigned an integer in $[v_p+k+1, v_p+k+e_p]$ while every pendant edge is assigned an integer in $[v_p+k+e_p+1, v_p+2k+e_p]$. Now, every edge weight is at most $2(v_p+k)-1$. Every pendant vertex weight is at least $v_p+k+e_p+1 > 2(v_p+k)-1$ and every non pendant vertex weight is at least $2(v_p+k)+3$. Therefore, every edge weight is less than every vertex weight. Consequently, g is a local total antimagic labeling that induces $\chi_{la}(G)+\chi_{lea}(G)$ distinct weights. □

Suppose G is a graph with $\chi_{lt}(G) = \Delta(G)+1$. We now give sufficient condition to construct new graph H with $k \geq 2$ pendants edges from G to have $\chi_{lt}(H) = \Delta(H)+1$.

Theorem 2.4. *For $k, s \geq 1, ks \geq 2$, let G be a graph of order p and size q with $d \geq 0$ pendant edges such that*

(a) *$\chi_{lt}(G) = t = \Delta(G)+1$ and the corresponding local total antimagic labeling f assigns k to a maximum degree vertex v of G;*
(b) *for $k \geq 1, s \geq 1$,*

$$w_f(v) + \sum_{j=1}^{s}\sum_{i=1}^{k}(p+q+2jk+1-i) \neq w_f(x),$$

for each vertex x adjacent to v;
(c) *v is the only element with weight $w_f(v)$, or else there is a non-pendant vertex element in $V(G)\cup E(G)$ with weight in $\{p+q+(2j-1)k+i \,|\, 1 \leq j \leq s, 1 \leq i \leq k\}$.*

Suppose $G_v(k,s)$ is obtained from G by attaching $ks \geq 2$ pendant edges to v. If Conditions (a), (b) *are satisfied, then*

$$\Delta(G_v(k,s)) + 1 \leq \chi_{lt}(G_v(k,s)) \leq \Delta(G_v(k,s)) + 2.$$

The lower bound holds if Condition (c) *is also satisfied.*

Proof. Let G be a graph satisfying Condition (a). Clearly, v is not a pendant vertex. Lemma 2.1 implies that $\Delta(G) \geq d$ if $d > 0$. Note that $G_v(k,s)$ has $ks+d$ pendant edges. So,

$$\Delta(G_v(k,s)) = \Delta(G) + ks \geq ks + d.$$

By definition,

$$\chi_{lt}(G_v(k,s)) \geq \Delta(G_v(k,s)) + 1 = \Delta(G) + ks + 1.$$

Let the $ks \geq 2$ pendant edges added to G to get H be $e_{j,i} = vx_{j,i}$ for $1 \leq j \leq s$, $1 \leq i \leq k$. For simplicity, if $k = 1$, let $e_{j,1} = e_j$ and $x_{j,1} = x_j$. Define a total labeling $g : V(G_v(k,s)) \cup E(G_v(k,s)) \to [1, p+q+2ks]$ such that $g(x) = f(x)$ for each $x \in V(G) \cup E(G)$. For the remaining vertices and edges, we do as follows.

(i) For $k = 1$, let $g(x_j) = p + q + 2j - 1$ and $g(e_j) = p + q + 2(s - j + 1)$. If $s \geq 3$ is odd, then swap the edge labels of $e_{(s+1)/2}$ and $e_{(s-1)/2}$.
(ii) For $k \geq 2$, let $g(x_{j,i}) = p + q + 2(j-1)k + i$, and $g(e_{j,i}) = p + q + 2jk + 1 - i$ for $1 \leq j \leq s, 1 \leq i \leq k$. If $k \geq 3$ is odd, then swap the edge labels of $e_{j,(k+1)/2}$ and $e_{j,(k-1)/2}$ for each $1 \leq j \leq s$.

Clearly, for distinct $x, y \in (V(G) \cup E(G)) \setminus \{v\}$, $w_g(x) = w_g(y)$ if and only if $w_f(x) = w_f(y)$. Observe that

$$\begin{aligned}
\{g(e_{j,i}) \mid 1 \leq j \leq s, 1 \leq i \leq k\} &= \bigcup_{j=1}^{s} [p + q + (2j-1)k + 1, p + q + 2jk] \\
&= \{w_g(e_{j,i}) \mid 1 \leq j \leq s, 1 \leq i \leq k\} \\
&= \{w_g(x_{j,i}) \mid 1 \leq j \leq s, 1 \leq i \leq k\},
\end{aligned}$$

denoted W. Clearly, all the elements in W are distinct. Moreover, every edge e of G that is incident to v has weight

$$\begin{aligned}
w_g(e) = w_f(e) &\leq 2p + 2q - 1 \\
&< w_f(v) + \sum_{j=1}^{s} \sum_{i=1}^{k} (p + q + (2j-1)k + i) = w_g(v).
\end{aligned}$$

So, Condition (b) implies that g is a local total antimagic labeling of H with weights set $W_g = W \cup (W_f \setminus \{w_f(v)\}) \cup \{w_g(v)\}$ that has size at most

$$ks + t + 1 = \Delta(G) + ks + 2 = \Delta(G_v(k,s)) + 2.$$

Thus,

$$\Delta(G_v(k,s)) + 1 \leq \chi_{lt}(G_v(k,s)) \leq \Delta(G_v(k,s)) + 2.$$

Consider Condition (c). Suppose v is the only element with weight $w_f(v)$. Now g induces $t - 1$ distinct weights among the elements in $(V(G) \cup E(G)) \setminus \{v\}$. Therefore, W_g has size

$$ks + (t-1) + 1 = ks + t = \Delta(G_v(k,s)) + 1.$$

Otherwise, since there is a non-pendant vertex element in $(V(G) \cup E(G)) \setminus \{v\}$ with weight in $\{p + q + (2j-1)k + i \,|\, 1 \leq j \leq s, 1 \leq i \leq k\}$, we also have W_g has size $\Delta(G_v(k,s)) + 1$. This completes the proof. $\square$

Suppose G has $d_G \geq 1$ pendant edges and $\chi_{lt}(G) = d_G + 1 \geq \Delta(G) + 1$. Let H be obtained from G by attaching pendant edges to a vertex of G such that H has d_H pendant edges. We now give sufficient condition for $\chi_{lt}(H) = d_H + 1$. Let $\deg_G(v)$ be the degree of vertex v in G.

Theorem 2.5. *For $k, s \geq 1$, $ks \geq 2$, let G be a graph of order p and size q with d_G pendant edges such that*

(i) $\chi_{lt}(G) = d_G + 1 \geq \Delta(G) + 1 \geq 3$ *and the corresponding local total antimagic labeling f assigns k to a vertex v of G,*

(ii) *if x is a vertex adjacent to v, then*

$$w_f(x) \neq w_f(v) + \sum_{j=1}^{s}\sum_{i=1}^{k}(p + q + (2j-1)k + i),$$

(iii) *if v is not the only element with weight $w_f(v)$, then w or else $w_f(v) \in \{g(e_{j,i}) \mid 1 \leq j \leq s, 1 \leq i \leq k\}$, where w is the only weight under f which is not a pendant edge label.*

Suppose $G_v(k,s)$ is obtained from G by attaching ks pendant edges to v.

(a) *If v is a pendant vertex, then*

$$d_H + 1 = ks + d_G \leq \chi_{lt}(G_v(k,s)) \leq ks + d_G + 1 = d_H + 2.$$

The lower bound holds if (i) *there is a weight of G under f equal to $p+q+(2j-1)k+i$ for $1 \leq j \leq s, 1 \leq i \leq k$; or else* (ii) *$v$ is the only element with weight $w_f(v)$ under f.*

(b) *If v is not a pendant vertex, then $\chi_{lt}(G_v(k,s)) = ks + d_G + 1$.*

Proof. Let the ks vertices added to G to get $H = G_v(k,s)$ be $x_{j,i}$ for $1 \leq j \leq s$, $1 \leq i \leq k$ and let $vx_{j,i}$ be $e_{j,i}$. Since H has order $p + ks$ and size $q + ks$, we define a total labeling $g : V(H) \cup E(H) \to [1, p + q + 2ks]$ such that g is as defined in the proof of Theorem 2.4. Obviously, every edge incident to v has weight less than $w_g(e_{j,i})$. Moreover, all but one of the weights of G under f, say w, must be an edge label of G under f. Moreover,

$$w_g(v) = w_f(v) + \sum_{j=1}^{s}\sum_{i=1}^{k}(p + q + (2j-1)k + i).$$

By definition and Lemma 2.1, we know $d_G \geq \Delta(G) \geq 2$.

(a) If v is a pendant vertex, then H has $d_H = ks + d_G - 1 \geq ks + 1$ pendant edges with $\deg_H(v) = ks + 1$. Since $ks \geq 2$, we also have $ks + d_G - 1 \geq d_G + 1 > \Delta(G)$ so that $ks + d_G - 1 \geq \max\{ks + 1, \Delta(G)\} = \Delta(H)$. By Lemma 2.1, $\chi_{lt}(H) \geq ks + d_G = d_H + 1$. Note that $w_g(x) = w_g(y)$ if and only if $w_f(x) = w_f(y)$ for each $x, y \in (V(G) \cup E(G)) \setminus \{v\}$. Clearly, $w_f(v)$ is unique. If w does not equal to $p + q + (2j-1)k + i$ for $1 \leq j \leq s, 1 \leq i \leq k$, then Condition (ii) implies that g is a local total antimagic labeling that induces $d_G + ks + 1$ distinct weights so that $d_G + ks \leq \chi_{lt}(H) \leq d_G + ks + 1$. Otherwise, g induces $d_G + ks$ distinct weights and the lower bound holds.

(b) If v is not a pendant vertex, then $\Delta(G) \geq \deg_G(v) = r \geq 2$ and H has $ks + d_G$ pendant edges with $\deg_H(v) = ks + r$ so that $ks + d_G \geq \max\{ks + r, \Delta(G)\} = \Delta(H)$.

By Lemma 2.1, $\chi_{lt}(H) \geq ks + d_G + 1$. Note that $w_g(x) = w_g(y)$ if and only if $w_f(x) = w_f(y)$ for each $x, y \in (V(G) \cup E(G)) \setminus \{v\}$. If v is the only element with weight $w_f(v)$, then Condition (ii) implies that g is a local total antimagic labeling that induces $ks + d_G + 1$ distinct weights. Otherwise, Condition (iii) implies that g is a local total antimagic chromatic labeling that induces $ks + d_G + 1$ distinct weights. Thus, $\chi_{lt}(H) = ks + d_G + 1$.

This completes the proof. □

Theorem 2.6. *Suppose G is a graph of order p and size q with exactly one vertex of maximum degree $\Delta \geq 3$ which is not adjacent to any pendant vertex and all other vertices of G has degree at most $m < \Delta$. Moreover, G has $k \geq \Delta \geq 2$ pendant edges such that*

$$\Delta(\Delta + 1) > \max\{m[2(p + q) - m + 1], 4(p + q) - 2\}.$$

If G admits a local total antimagic labeling, then $\chi_{lt}(G) \geq k + 2$.

Proof. Let f be a local total antimagic labeling of G. If $p+q$ is assigned to a non-pendant vertex or edge, by Lemma 2.1, $\chi_{lt}(G) \geq k+2$. Suppose $p+q$ is assigned to a pendant edge that has a non-pendant end-vertex x. Now, all the k induced pendant vertex labels are distinct and at most $p + q$. Suppose u is the vertex of maximum degree Δ, then $w(u) \geq \Delta(\Delta + 1)/2$ and $p + q + 1 \leq w(x) \leq m[2(p + q) - m + 1]/2$. By the given hypothesis, $w(u) > w(x) > w(y)$ for every pendant vertex y. Thus, f induces at least $k + 2$ distinct vertex weights.

Suppose $p + q$ is assigned to a pendant vertex, then the adjacent pendant edge, say e, has $p + q + 1 \leq w(e) \leq 2(p + q) - 1$. Moreover, all the k induced pendant vertex labels are distinct and at most $p + q - 1$. By the given hypothesis, $w(u) \geq \Delta(\Delta + 1)/2 > 2(p + q) - 1 \geq w(e)$. Thus, f induces at least $k + 2$ distinct vertex weights. □

Let f_n be the friendship graph obtained from $n \geq 2$ copies of K_3 with a common vertex c. Let $f_n(k), k \leq 2n - 3$, be obtained from f_n by attaching exactly k pendant edges to every degree 2 vertex of f_n. Now, $f_n(k)$ has order $p = n(2k + 2) + 1$ and size $q = n(2k + 3)$ with exactly one vertex c with maximum degree $\Delta = 2n$ with is not adjacent to any pendant vertex and all other vertices has degree at most $m = k + 2 \leq 2n - 1 < \Delta$. Thus,

$$\begin{aligned} \max\{m[(2(p + q) - m + 1], 4(p + q) - 2\} &= m[2(p + q) - m + 1] \\ &= 2n(k + 2)(4k + 5) - (k + 2)(k - 1). \end{aligned}$$

Thus,

$$\Delta(\Delta + 1) > m[2(p + q) - m + 1]$$

implies that

$$2n(2n + 1) - 2n(k + 2)(4k + 5) + (k + 2)(k - 1) > 0.$$

By Lemma 2.1 and Theorem 2.6, we have the following.

Corollary 2.7. *For* $2n \geq k+3 \geq 4$, *and* $2n(2n+1)-2n(k+2)(4k+5)+(k+2)(k-1) > 0$, $\chi_{lt}(f_n(k)) \geq 2nk+2$.

Note that the above corollary is always attainable. For example, take $k = 1$, we have $n > 13$.

3. GRAPHS WITH $\chi_{lt} = 3$

By Corollary 2.2, $\chi_{lt}(G) = 3$ only if $\Delta(G) = 2$ possibly with exactly one path component of order at least 3. Let $P_n = u_1u_2\ldots u_n$ be the path of order $n \geq 3$ with $e_i = u_iu_{i+1}$ for $1 \leq i \leq n-1$. The authors in [14] also obtained $\chi_{lt}(P_n) = 3$ for $n = 3, 6$ and $\chi_{lt}(P_n) = 4$ for $n = 4$ and odd $n \geq 5$. Moreover, $3 \leq \chi_{lt}(P_n) \leq 5$ for even $n \geq 8$. We first improve the lower bound of the last statement in the following lemma.

Lemma 3.1. *For* $n = 3, 6$, $\chi_{lt}(P_n) = 3$. *If* $n = 4$ *or* $n \geq 5$ *is odd,* $\chi_{lt}(P_n) = 4$. *Otherwise,* $4 \leq \chi_{lt}(P_n) \leq 5$ *for even* $n \geq 8$.

Proof. From the proofs in [14, Theorems 2.3 and 2.5], we only need to show that $\chi_{lt}(P_n) \geq 4$ for $n \geq 7$. Suppose $\chi_{lt}(P_n) = 3$ and f is a corresponding local total antimagic 3-coloring. Without loss of generality, we may assume the 3 distinct weights are a, b, c such that

$$\begin{aligned} a &= w(u_1) = w(e_2) = w(u_4) = w(e_5), \\ b &= w(e_1) = w(u_3) = w(e_4) = w(u_6) \\ c &= w(u_2) = w(e_3) = w(u_5) = w(e_6). \end{aligned}$$

Thus, we have

$$\begin{aligned} a &= f(e_1) = f(e_3) + f(e_4) \\ b &= f(e_2) + f(e_3) = f(e_5) + f(e_6) \\ c &= f(e_1) + f(e_2) = f(e_4) + f(e_5). \end{aligned}$$

Thus, from $a + b - c$, we get $f(e_3) = f(e_3) + f(e_6)$, a contradiction. Therefore, $\chi_{lt}(P_n) \geq 4$. The lemma holds. $\square$

By Corollary 2.2, the proofs in [14, Theorems 2.3 and 2.5] and the argument of Lemma 3.1, we have the following.

Corollary 3.2. *If* $\chi_{lt}(G) = 3$, *then* G *is a 2-regular graph or a path* $P_n, n = 3, 6$, *or* $H + P_3$ *or* $H + P_6$, *where* H *is a 2-regular graph.*

In what follows, if a 2-regular graph has i-th component of order $n \geq 3$, then the consecutive vertices and edges are $u_{i,1}, e_{i,1}, u_{i,2}, e_{i,2}, \ldots, u_{i,n}, e_{i,n}$. In particular, a cycle of order n has consecutive vertices and edges $u_1, e_1, u_2, e_2, \ldots, u_n, e_n$. We first give a family of 2-regular graphs G with $\chi_{lt}(G) = 3$.

Theorem 3.3. *For $m \geq 1$, $\chi_{lt}(mC_6) = 3$.*

Proof. Since mC_6 has $6m$ vertices and $6m$ edges, we define a bijection $f : V(mC_6) \cup E(mC_6) \to [1, 12m]$ such that for $1 \leq i \leq m$,

(i) $f(u_{i,1}) = 3i - 2,\ f(u_{i,3}) = 3i - 1,\ f(u_{i,5}) = 3i,$
(ii) $f(u_{i,2}) = 12m + 3 - 3i,\ f(u_{i,4}) = 12m + 1 - 3i,\ f(u_{i,6}) = 12m + 2 - 3i,$
(iii) $f(e_{i,1}) = 6m - 1 + 3i,\ f(e_{i,3}) = 6m + 3i,\ f(e_{i,5}) = 6m - 2 + 3i,$
(iv) $f(e_{i,2}) = 6m + 1 - 3i,\ f(e_{i,4}) = 6m + 2 - 3i,\ f(e_{i,6}) = 6m + 3 - 3i.$

Clearly, the weights of $u_{i,1}$, $e_{i,1}$, $u_{i,2}$, $e_{i,2}$, $u_{i,3}$, $e_{i,3}$, $u_{i,4}$, $e_{i,4}$, $u_{i,5}$, $e_{i,5}$, $u_{i,6}$, $e_{i,6}$ are $12m + 2$, $12m + 1$, $12m$ repeatedly for $1 \leq i \leq m$. Thus, $\chi_{lt}(mC_6) \leq 3$. Since $\chi_{lt}(mC_6) \geq \chi_t(mC_6) = 3$, the theorem holds. □

Example 3.4. Figure 1 gives the local total antimagic 3-coloring of $2C_6$ as defined above with induced weights $24, 25, 26$.

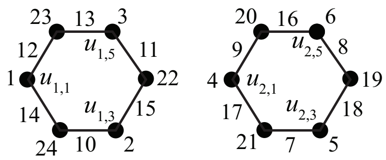

Fig. 1. $2C_6$ with local total antimagic 3-coloring

Theorem 3.5. *Suppose G is a 2-regular graph or else the disjoint union of a 2-regular graph and a path of order at least 3. If G has a C_n component, $n \neq 6$, then $\chi_{lt}(G) \geq 4$. Moreover, for $m \geq 1$, $\chi_{lt}(C_3) = \chi_{lt}(mC_4) = \chi_{lt}(C_5) = 4$, and $\chi_{lt}(C_8) \leq 5$.*

Proof. By definition, $\chi_{lt}(G) \geq 3$. If $n \equiv 1, 2 \pmod 3$, then $\chi_{lt}(G) \geq \chi_t(G) \geq 4$. Suppose $n \equiv 0 \pmod 3$ and that $\chi_{lt}(G) = \chi_t(G) = 3$. Let f be a required local total antimagic 3-coloring of G. Suppose $n = 3$. Without loss of generality, we may assume the 3 distinct weights are a, b, c such that $a = w(u_1) = w(e_2)$, $b = w(e_1) = w(u_3)$ and $c = w(u_2) = w(e_3)$. Thus, we have

$$\begin{aligned} a &= f(e_1) + f(e_3) = f(u_2) + f(u_3) \\ b &= f(e_2) + f(e_3) = f(u_1) + f(u_2) \\ c &= f(e_1) + f(e_2) = f(u_1) + f(u_3). \end{aligned}$$

Now, $a + b - c$ gives $f(e_3) = f(u_2)$, a contradiction.

Suppose $n \geq 9$. Similarly, we may assume $a = w(u_j), j \equiv 1 \pmod 3$, $b = w(u_j)$, $j \equiv 0 \pmod 3$, $c = w(u_j)$, $j \equiv 2 \pmod 3$. Thus, we have

$$\begin{aligned} a &= f(e_3) + f(e_4) = f(e_6) + f(e_7), \\ b &= f(e_2) + f(e_3) = f(e_5) + f(e_6), \\ c &= f(e_1) + f(e_2) = f(e_4) + f(e_5). \end{aligned}$$

From $a-b+c$, we get $f(e_4)+f(e_1) = f(e_7)+f(e_4)$ so that $f(e_1) = f(e_7)$, a contradiction. Thus, $\chi_{lt}(G) \geq 4$ if G has a component of order $n \neq 6$.

Consider C_3, label the vertices and edges $u_{1,1}$, $e_{1,1}$, $u_{1,2}$, $e_{1,2}$, $u_{1,3}$, $e_{1,3}$ by $1, 3, 5, 4, 6, 2$ bijectively so that the corresponding weights are $5, 6, 7, 11, 6, 7$. Thus, $\chi_{lt}(C_3) = 4$.

Consider mC_4. Since mC_4 has $4m$ vertices and $4m$ edges, we define a bijection $f : V(mC_4) \cup E(mC_4) \to [1, 8m]$ such that for $1 \leq i \leq m$,

(i) $f(u_{i,1}) = i$, $f(u_{i,2}) = 7m + 1 - i$, $f(u_{i,3}) = 3m + i$, $f(u_{i,4}) = 6m + 1 - i$,
(ii) $f(e_{i,1}) = m + i$, $f(e_{i,2}) = 5m + 1 - i$, $f(e_{i,3}) = 2m + i$, $f(e_{i,4}) = 8m + 1 - i$.

Clearly, the weights of $u_{i,1}$, $e_{i,1}$, $u_{i,2}$, $e_{i,2}$, $u_{i,3}$, $e_{i,3}$, $u_{i,4}$, $e_{i,4}$ are $9m + 1$, $7m + 1$, $6m + 1$, $10m + 1$, $7m + 1$, $9m + 1$, $10m + 1$, $6m + 1$ respectively. Thus, $\chi_{lt}(mC_4) = 4$.

For C_5, label the vertices and edges $u_{1,1}$, $e_{1,1}$, $u_{1,2}$, $e_{1,2}, \ldots, u_{1,5}$, $e_{1,5}$ by 1, 2, 7, 8, 5, 6, 3, 4, 9, 10 bijectively so that the corresponding weights are 12, 8, 10, 12, 14, 8, 10, 12, 14, 10. Thus, $\chi_{lt}(C_n) = 4$ for $n = 4, 5$. For C_8, label the vertices and edges $u_{1,1}$, $e_{1,1}$, $u_{1,2}$, $e_{1,2}, \ldots, u_{1,8}$, $e_{1,8}$ by 1, 10, 16, 5, 2, 11, 13, 6, 3, 12, 14, 7, 4, 9, 15, 8 bijectively so that the corresponding weights are 18, 17, 15, 18, 16, 15, 19, 16, 18, 17, 19, 18, 16, 19, 17, 16. Thus, $\chi_{lt}(C_8) \leq 5$. □

Theorem 3.6. *For $m \geq 1$, $\chi_{lt}(G) = 3$ if and only if $G = P_n, n = 3, 6$ or $G = mC_6$ or $G = mC_6 + P_6$, $(m \geq 0)$.*

Proof. Necessity: Suppose $\chi_{lt}(G) = 3$. By Corollary 3.2, Theorem 3.3 and the proof of Theorem 3.5, we know $G = P_n, n = 3, 6$ or $G = mC_6$, $G = mC_6 + P_3$ or $G = mC_6 + P_6$ for $m \geq 1$. We shall prove that $\chi_{lt}(mC_6 + P_3) \neq 3$. Suppose equality holds and the P_3 component has vertices and edges v_1, h_1, v_2, h_2, v_3 consecutively. Let f be a local total antimagic 3-coloring of $mC_6 + P_3$ with induced weights a, b, c. Without loss of generality, we may assume the weights of $u_{i,1}, e_{i,1}, \ldots, u_{i,6}$, $e_{i,6}$ of the i-th C_6 are a, b, c repeatedly, and the weights of v_1, h_1, v_2, h_2, v_3 are a, b, c, a, b respectively. Thus, we have

$$\begin{aligned}
a &= f(e_{i,6}) + f(e_{i,1}) = f(h_1),\\
b &= f(e_{i,2}) + f(e_{i,3}) = f(h_2),\\
c &= f(e_{i,1}) + f(e_{i,2}) = f(h_1) + f(h_2).
\end{aligned}$$

The right hand side above gives $a+b-c = 0$ so that $f(e_{i,6})+f(e_{i,3}) = 0$, a contradiction. Thus, $\chi_{lt}(mC_6 + P_3) \geq 4$. Therefore, $G = P_n, n = 3, 6$ or $G = mC_6$ or $G = mC_6 + P_6$ for $m \geq 1$.

Sufficiency: By Lemma 3.1 and Theorem 3.3, we only need to show that $\chi_{lt}(mC_6 + P_6) = 3$ for $m \geq 1$. Suppose the P_6 component has vertices and edges $v_1, h_1, \ldots, v_5, h_5, v_6$ consecutively. Since G has $6m + 6$ vertices and $6m + 5$ edges, we define a bijection $f : V(G) \cup E(G) \to [1, 12m + 11]$ such that for $1 \leq i \leq m$,

(i) $f(v_1) = 6m+7$, $f(v_2) = 6m+3$, $f(v_3) = 6m+8$, $f(v_4) = 6m+4$, $f(v_5) = 6m+6$, $f(v_6) = 6m + 5$,
(ii) $f(h_1) = 12m + 11, f(h_2) = 1, f(h_3) = 12m + 9, f(h_4) = 2, f(h_5) = 12m + 10$,

(iii) $f(u_{i,1}) = 3i$, $f(u_{i,3}) = 3i + 1$, $f(u_{i,5}) = 3i + 2$,
(iv) $f(u_{i,2}) = 12m + 11 - 3i$, $f(u_{i,4}) = 12m + 9 - 3i$, $f(u_{i,6}) = 12m + 10 - 3i$,
(v) $f(e_{i,1}) = 6m + 7 + 3i$, $f(e_{i,3}) = 6m + 8 + 3i$, $f(e_{i,5}) = 6m + 6 + 3i$,
(vi) $f(e_{i,2}) = 6m + 3 - 3i$, $f(e_{i,4}) = 6m + 4 - 3i$, $f(e_{i,6}) = 6m + 5 - 3i$.

Clearly, the weights of $u_{i,1}, e_{i,1}, \ldots, u_{i,6}, e_{i,6}$ are $12m + 12$, $12m + 11$, $12m + 10$ repeatedly for $1 \le i \le m$ whereas the weights of $v_1, h_1, \ldots, v_5, h_5, v_6$ are $12m + 11, 12m + 10, 12m + 12, 12m + 11, 12m + 10, 12m + 12, 12m + 11, 12m + 10$, respectively. Thus, f is a local total antimagic labeling and $\chi_{lt}(G) = 3$. □

Example 3.7. Figure 2 gives the local total antimagic 3-coloring of $2C_6 + P_6$ as defined above with induced weights $34, 35, 36$.

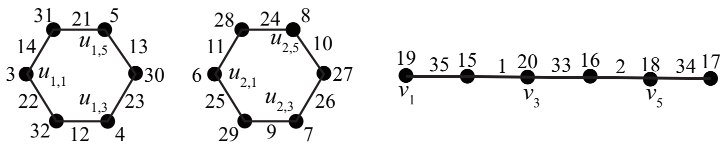

Fig. 2. $2C_6 + P_6$ with local total antimagic 3-coloring

Corollary 3.8. *Suppose $m \ge 1$ is odd and $s \ge \frac{m+1}{2}$. If $G_v(s)$ is obtained from mC_4 by attaching $s \ge 2$ pendant edges to a single vertex of mC_4, then*

$$\Delta(G_v(s)) + 1 = s + 3 \le \chi_{lt}(G_v(s)) \le s + 4 = \Delta(G_v(s)) + 2.$$

Proof. Let f be the local total antimagic labeling of mC_4 as defined in the proof of Theorem 3.5. Without loss of generality, assume the s edges are attached to $u_{1,1}$. Define a labeling g of $G_v(s)$ such that $g(x) = f(x)$ if $x \in V(G) \cup E(G)$, and label the vertices and edges of the s added pendant edges as in the proof of Theorem 2.4 for $k = 1$. Now, g is a local total antimagic labeling that induces weights $9m + 1$, $7m + 1$, $6m + 1$, $10m + 1$, $8m + 2i$, $1 \le i \le s$, $\sum_{i=1}^{s}(8m + 2i) = 8ms + s(s + 1)$. Since $s \ge \frac{m+1}{2}$, $9m + 1 \in \{8m + 2i \,|\, 1 \le i \le s\}$, there are exactly $s + 4$ distinct weight so that $\chi_{lt}(G_v(s)) \le s + 4$. Since $\Delta(G_v(s)) = s + 2$ and $G_v(s)$ has s pendant edges, by Corollary 2.2, $\chi_{lt}(G_v(s)) \ge s + 3$. This completes the proof. □

4. χ_{lt} = NUMBER OF PENDANT EDGES + 1

We first note that applying Theorem 2.4 to the vertex of $mC_6, m \ge 1$ with label 1 as in Theorem 3.3, we get the following corollary.

Corollary 4.1. *Suppose $m \ge 1$. If $G_v(s)$ is obtained from mC_6 by attaching $s \ge 2$ pendant edges to a single vertex of mC_6, then $\chi_{lt}(G_v(s)) = \Delta(G_v(s)) + 1 = s + 3$.*

In [15], the authors proved that $\chi_{lt}(nP_3) = 2n + 1$ for $n \ge 1$. We can now extend the obtained labeling to the following theorem.

Theorem 4.2. *For $m \geq 1, n \geq 2$, $\chi_{lt}(mC_6 + P_3) = 4$ and $\chi_{lt}(mC_6 + nP_3) = 2n + 1$.*

Proof. Consider $G = mC_6 + P_3$. From the proof of Theorem 3.6, we know that $\chi_{lt}(G) \geq 4$. Since G has $6m + 3$ vertices and $6m + 2$ edges, we define a bijection $f : V(G) \cup E(G) \to [1, 12m + 5]$ such that for $1 \leq i \leq m$,

(i) $f(v_1) = 1$, $f(h_1) = 12m + 5$, $f(v_2) = 12m + 3$, $f(h_2) = 12m + 4$, $f(v_3) = 2$,
(ii) $f(u_{i,1}) = 3i$, $f(u_{i,3}) = 3i + 1$, $f(u_{i,5}) = 3i + 2$,
(iii) $f(u_{i,2}) = 12m + 5 - 3i, f(u_{i,4}) = 12m + 3 - 3i, f(u_{i,6}) = 12m + 4 - 3i$,
(iv) $f(e_{i,1}) = 6m + 1 + 3i, f(e_{i,3}) = 6m + 2 + 3i, f(e_{i,5}) = 6m + 3i$,
(v) $f(e_{i,2}) = 6m + 3 - 3i, f(e_{i,4}) = 6m + 4 - 3i, f(e_{i,6}) = 6m + 5 - 3i$.

Clearly, the weights of $u_{i,1}$, $e_{i,1}$, $u_{i,2}$, $e_{i,2}$, $u_{i,3}$, $e_{i,3}$, $u_{i,4}$, $e_{i,4}$, $u_{i,5}$, $e_{i,5}$, $u_{i,6}$, $e_{i,6}$ are $12m + 6$, $12m + 5$, $12m + 4$ repeatedly for $1 \leq i \leq m$ whereas the weights of v_1, h_1, v_2, h_2, v_3 are $12m + 5, 12m + 4, 24m + 9, 12m + 5, 12m + 4$ respectively. Thus, $\chi_{lt}(G) \leq 4$. Consequently, $\chi_{lt}(G) = 4$.

Consider $n \geq 2$. Now, $G = mC_6 + nP_3$ has $2n$ pendant vertices and maximum degree 3. By Corollary 2.2, we have $\chi_{lt}(G) \geq 2n + 1$. Suppose the nP_3 has vertex set $\{v_{j,1}, v_{j,2}, v_{j,3}\}$ and edge set $\{v_{j,1}v_{j,2}, v_{j,2}v_{j,3}\}$ for $1 \leq j \leq n$. Define a total labeling $f : V(G) \cup E(G) \to [1, 12m + 5n]$ such that for $1 \leq i \leq m, 1 \leq j \leq n$,

(a) $f(v_{j,1}) = j$, $f(v_{j,1}v_{j,2}) = 5n + 12m + 1 - j$, $f(v_{j,2}) = 3n + 12m + 1 - j$, $f(v_{j,2}v_{j,3}) = 3n + 12m + j, f(v_{j,3}) = n + j$,
(b) $f(u_{i,1}) = 2n + 3i - 2$, $f(u_{i,3}) = 2n + 3i - 1$, $f(u_{i,5}) = 2n + 3i$,
(c) $f(u_{i,2}) = 2n+12m+3-3i$, $f(u_{i,4}) = 2n+12m+1-3i$, $f(u_{i,6}) = 2n+12m+2-3i$,
(d) $f(e_{i,1}) = 2n + 6m - 1 + 3i$, $f(e_{i,3}) = 2n + 6m + 3i$, $f(e_{i,5}) = 2n + 6m - 2 + 3i$,
(e) $f(e_{i,2}) = 2n + 6m + 1 - 3i$, $f(e_{i,4}) = 2n + 6m + 2 - 3i$, $f(e_{i,6}) = 2n + 6m + 3 - 3i$.

We now have $w(v_{j,1}) = 5n + 12m + 1 - j$, $w(v_{j,3}) = 3n + 12m + j$ for $1 \leq j \leq n$, $w(v_{j,2}) = 8n + 24m + 1$, $w(v_{j,1}v_{j,2}) = 3n + 12m + 1, w(v_{j,2}v_{j,3}) = 4n + 12m + 1$ so that the nP_3 components have $2n + 1$ distinct weights. Moreover, the weights of $u_{i,1}$, $e_{i,1}$, $u_{i,2}$, $e_{i,2}$, $u_{i,3}$, $e_{i,3}$, $u_{i,4}$, $e_{i,4}$, $u_{i,5}$, $e_{i,5}$, $u_{i,6}$, $e_{i,6}$ are $4n+12m+2, 4n+12m+1, 4n+12m$ (that are the $w(v_{n-1,1}), w(v_{n,1}), w(v_{n,3})$) repeatedly for $1 \leq i \leq m$. Thus, f is a local total antimagic labeling that induces $2n + 1$ distinct weights so that $\chi_{lt}(G) \leq 2n + 1$. Consequently, $\chi_{lt}(G) = 2n + 1$. □

Example 4.3. Figure 3 gives the local total antimagic 5-coloring of $2C_6 + 2P_3$ with induced weights $31, 32, 33, 34, 65$ as defined above.

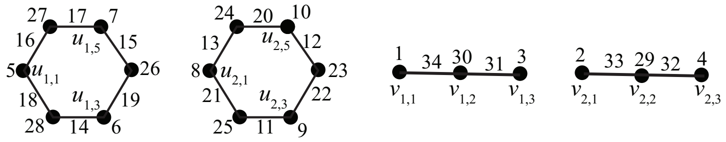

Fig. 3. $2C_6 + 2P_3$ with local total antimagic 5-coloring

In [14], the authors also proved that

$$\chi_{lt}(mP_6) = 2m + 1$$

for $m \geq 2$, and that

$$\chi_{lt}(mP_6 + nP_3) = 2m + 2n + 1$$

for $m \geq 1, n \geq 2$. We can now prove the following theorems.

Theorem 4.4. *For $m, n \geq 1$, $\chi_{lt}(mC_6 + nP_6) = 2n + 1$.*

Proof. Let $G = mC_6 + nP_6$. By Theorem 3.6, we only need to consider $m \geq 1, n \geq 2$. Since G has $2n$ pendant edges and $\Delta(G) = 2$, by Corollary 2.2, $\chi_{lt}(G) \geq 2n + 1$. We shall show that $\chi_{lt}(G) \leq 2n+1$. Keep the notations for the mC_6, $m \geq 1$. For $1 \leq t \leq n$, let the vertex and edge sets of the t-th component of the nP_6 be $\{y_{t,a} \mid 1 \leq a \leq 6\}$ and $\{z_{t,a} = y_{t,a}y_{t,a+1} \mid 1 \leq a \leq 5\}$ respectively. Since G has order $6m + 6n$ and size $6m + 5n$, we define a bijection $f : V(G) \cup E(G) \to [1, 12m + 11n]$ such that for $1 \leq i \leq m$, $1 \leq t \leq n$,

(a) $f(u_{i,1}) = 2n + 3i - 2$, $f(u_{i,3}) = 2n + 3i - 1$, $f(u_{i,5}) = 2n + 3i$,
(b) $f(u_{i,2}) = 8n+12m-3i+3$, $f(u_{i,4}) = 8n+12m-3i+1$, $f(u_{i,6}) = 8n+12m-3i+2$,
(c) $f(e_{i,1}) = 8n + 6m + 3i - 1$, $f(e_{i,3}) = 8n + 6m + 3i$, $f(e_{i,5}) = 8n + 6m + 3i + 1$,
(d) $f(e_{i,2}) = 2n + 6m - 3i + 1$, $f(e_{i,4}) = 2n + 6m - 3i + 2$, $f(e_{i,6}) = 2n + 6m - 3i + 3$,
(e) $f(y_{t,1}) = 8n + 6m - 3t + 2$, $f(y_{t,3}) = 8n + 6m - 3t + 3$, $f(y_{t,5}) = 8n + 6m - 3t + 1$,
(f) $f(y_{t,2}) = 2n + 6m + 3t - 2$, $f(y_{t,4}) = 2n + 6m + 3t - 1$, $f(y_{t,6}) = 2n + 6m + 3t$,
(g) $f(z_{t,1}) = 11n+12m+1-t$, $f(z_{t,3}) = 9n+12m+1-t$, $f(z_{t,5}) = 10n+12m+1-t$,
(h) $f(z_{t,2}) = t$, $f(z_{t,4}) = n + t$.

One can check that the induced weights of each component of the mC_6, starting from $w(u_{i,1})$ and $w(e_{i,1})$, are $10n + 12m + 2, 10n + 12m + 1, 10n + 12m$ repeatedly and the induced non-pendant vertex weights of each component of the nP_6, starting from $w(z_{t,1})$, are $10n + 12m$, $11n + 12m + 1$, $10n + 12m + 1$, $9n + 12m + 1$, $10n + 12m + 2$, $10n + 12m + 1$, $10n + 12m$, $11n + 12m + 1$, $10n + 12m + 1$ consecutively, while $w(y_{t,1}) = 11n + 12m + 1 - t$ and $w(y_{t,6}) = 10n + 12m + 1 - t$ for $1 \leq t \leq n$. Thus, f is a local total antimagic labeling that induces $2n + 1$ distinct weights so that $\chi_{lt}(G) \leq 2n + 1$. This completes the proof. □

Example 4.5. Figure 4 gives the local total antimagic 5-coloring of $2C_6 + 2P_6$ as defined above with induced weights $43, 44, 45, 46, 47$.

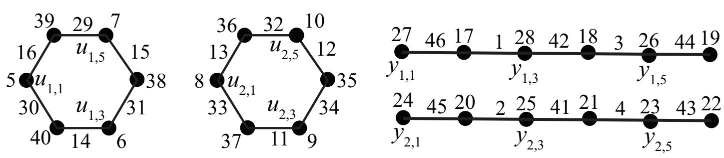

Fig. 4. $2C_6 + 2P_6$ with local total antimagic 5-coloring

Theorem 4.6. *For $m, n \geq 1, a \geq 2$, if $n \geq 2a+2$ or else $a \geq 2n$, then*

$$\chi_{lt}(mC_6 + nP_6 + aP_3) = 2n + 2a + 1.$$

Otherwise,

(i) *if $n = 2a+1$ or else $s \in [n+1, 2n-1]$, then*

$$2n + 2a + 1 \leq \chi_{lt}(mC_6 + nP_6 + aP_3) \leq 2n + 2a + 2.$$

(ii) *if $n = 2a$ or else $a = n$, then*

$$2n + 2a + 1 \leq \chi_{lt}(mC_6 + nP_6 + aP_3) \leq 2n + 2a + 3.$$

(iii) *if $n \in [a+1, 2a-1]$ or else $a = n-1$, then*

$$2n + 2a + 1 \leq \chi_{lt}(mC_6 + nP_6 + aP_3) \leq 2n + 2a + 4.$$

(iv) *if $n \leq a$ or else $a \leq n-2$, then*

$$2n + 2a + 1 \leq \chi_{lt}(mC_6 + nP_6 + aP_3) \leq 2n + 2a + 5.$$

Proof. Let $G = mC_6 + nP_6 + aP_3$ for $m, n \geq 1, a \geq 2$. Since G has $2n + 2a$ pendant edges and $\Delta(G) = 2$, by Corollary 2.2, $\chi_{lt}(G) \geq 2n + 2a + 1$. We shall show that $\chi_{lt}(G)$ has the given upper bound under the given conditions. Keep the notations of $mC_6 + nP_6$ as defined in the proof of Theorem 4.4. Let f be the total labeling as defined in the proof of Theorem 4.4. For $1 \leq r \leq a$, let the vertex and edge sets of the r-th component of the aP_3 be $\{v_{r,b} \mid 1 \leq b \leq 3\}$ and $\{h_{r,b} = v_{r,b}v_{r,b+1} \mid 1 \leq b \leq 2\}$ respectively.

Since G has order $6m + 6n + 3a$ and size $6m + 5n + 2a$, we define a bijection $g : V(G) \cup E(G) \to [1, 12m + 11n + 5a]$ such that for $1 \leq i \leq m, 1 \leq t \leq n, 1 \leq r \leq a$, $g(x) = f(x) + 2a$ for $x \in V(mC_6 + nP_6) \cup E(mC_6 + nP_6)$. Moreover,

$$\begin{aligned}
g(v_{r,1}) &= r, \\
g(h_{r,1}) &= 12m + 11n + 5a + 1 - r, \\
g(v_{r,2}) &= 12m + 11n + 3a + 1 - r, \\
g(h_{r,2}) &= 12m + 11n + 3a + r, \\
g(v_{r,3}) &= a + r.
\end{aligned}$$

Similar to the proof of Theorem 4.4, one can check that the induced weights of each component of the mC_6 are $10n + 12m + 4a + 2$, $10n + 12m + 4a + 1$, $10n + 12m + 4a$ repeatedly, the induced non-pendant vertex weights of each component of the nP_6 are $10n + 12m + 4a$, $11n + 12m + 4a + 1$, $10n + 12m + 4a + 1$, $9n + 12m + 4a + 1$,

$10n + 12m + 4a + 2$, $10n + 12m + 4a + 1$, $10n + 12m + 4a$, $11n + 12m + 4a + 1$, $10n + 12m + 4a + 1$, while

$$\begin{aligned}
w_g(y_{t,1}) &= 11n + 12m + 2a + 1 - t,\\
w_g(y_{t,5}) &= 10n + 12m + 2a + 1 - t,\\
w_g(v_{r,1}) &= 11n + 12m + 5a + 1 - r,\\
w_g(h_{r,1}) &= 11n + 12m + 3a + 1,\\
w_g(v_{r,2}) &= 22n + 24m + 8a + 1,\\
w_g(h_{r,2}) &= 11n + 12m + 4a + 1,\\
w_g(v_{r,3}) &= 11n + 12m + 3a + r
\end{aligned}$$

for $1 \le t \le n, 1 \le r \le a$. Clearly, $w_g(v_{r,2}) = 22n + 24m + 8a + 1$ is not in the pendant vertex weights set

$$W = [9n + 12m + 2a + 1, 11n + 12m + 2a] \cup [11n + 12m + 3a + 1, 11n + 12m + 5a].$$

Note that this labeling is local total antimagic. Moreover, there are exactly $2n + 2a + 1$ distinct weights if and only if all the induced weights set of the non-pendant vertices of $mC_6 + nP_6$ components, namely $\{9n + 12m + 4a + 1, 10n + 12m + 4a, 10n + 12m + 4a + 1, 10n + 12m + 4a + 2, 11n + 12m + 4a + 1\}$ is a subset of W. This implies that

(a) $2n \ge 2a + 1$, or else $a \ge 2n$ for $9n + 12m + 4a + 1 \in W$,
(b) $n \ge 2a$, or else $a \ge n + 1$ for $10n + 12m + 4a \in W$,
(c) $n \ge 2a + 1$, or else $a \ge n$ for $10n + 12m + 4a + 1 \in W$,
(d) $n \ge 2a + 2$, or else $a \ge n - 1$ for $10n + 12m + 4a + 2 \in W$.

Therefore, if $n \ge 2a + 2$ or else $a \ge 2n$, then

$$\chi_{lt}(mC_6 + nP_6 + aP_3) \le 2n + 2a + 1.$$

Otherwise,

(i) if $n = 2a+1$ or else $a \in [n+1, 2n-1]$, then there are $2n+2a+2$ distinct weights so that
$$2n + 2a + 1 \le \chi_{lt}(mC_6 + nP_6 + aP_3) \le 2n + 2a + 2.$$

(ii) if $n = 2a$ or else $a = n$, then there are $2n + 2a + 3$ distinct weights so that
$$2n + 2a + 1 \le \chi_{lt}(mC_6 + nP_6 + sP_3) \le 2n + 2a + 3.$$

(iii) if $n \in [a + 1, 2a - 1]$ or else $a = n - 1$, then there are $2n + 2a + 4$ distinct weights so that
$$2n + 2a + 1 \le \chi_{lt}(mC_6 + nP_6 + aP_3) \le 2n + 2s + 4.$$

(iv) if $n \le a$ or else $a \le n - 2$, then there are $2n + 2a + 5$ distinct weights so that
$$2n + 2a + 1 \le \chi_{lt}(mC_6 + nP_6 + aP_3) \le 2n + 2a + 5.$$

This completes the proof. □

Example 4.7. Figures 5 and 6 give the local total antimagic 11-coloring of $C_6+4P_6+P_3$ with $n=4$, $a=1$ and induced weights $51, 52, \ldots, 58, 60, 61, 121$, as well as local total antimagic 9-coloring of $C_6 + P_6 + 3P_3$ with $n = 1$, $a = 3$ and induced weights $28, 29, 33, 34, \ldots, 38, 71$ as defined above.

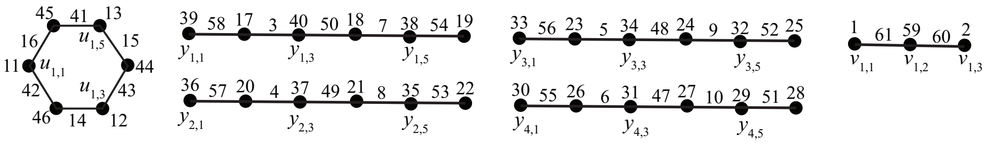

Fig. 5. $C_6 + 4P_6 + P_3$ with local total antimagic 13-coloring

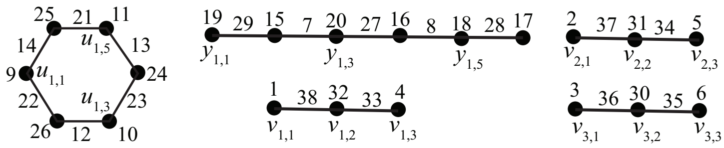

Fig. 6. $C_6 + P_6 + 3P_3$ with local total antimagic 9-coloring

Theorem 4.8. *For $m \geq 1, n \geq 2$, let $G = mC_6 + nP_3$ with a local total antimagic labeling as defined in the proof of Theorem 4.2. Suppose $s \geq 1$, $1 \leq j' \leq s$, $1 \leq i' \leq k$, $1 \leq i \leq m$, $1 \leq j \leq n$, $ks \geq 2$ and the following ten conditions:*

(a) $v = v_{1,1}$, $k = 1$, *n is odd and* $s \geq (3n + 24m + 1)/2$,
(b) $v = v_{j,1}$ *for* $j \in [2, n]$, $k = j$ *and* $2(j' - 1)j + 1 \leq 3n + 12m + 1 \leq 2j'j$,
(c) $v = v_{j,3}$ *for* $j \in [1, n]$, $k = n + j$ *and*

$$(2j' - 1)(n + j) + 1 \leq 3n + 12m + 1 \leq 2j'(n + j),$$

(d) $v = v_{j,2}$, $k = 3n + 12m + 1 - j, s \geq 1$,
(e) $v = u_{i,1}$, $k = 2n + 3i - 2$, *and*

$$8n+24m+1 \in \bigcup_{j'=1}^{s} [5n+12m+(2j'-1)(2n+3i-2)+1, 5n+12m+2j'(2n+3i-2)],$$

(f) $v = u_{i,3}$, $k = 2n + 3i - 1$, *and*

$$8n+24m+1 \in \bigcup_{j'=1}^{s} [5n+12m+(2j'-1)(2n+3i-1)+1, 5n+12m+2j'(2n+3i-1)],$$

(g) $v = u_{i,5}$, $k = 2n + 3i$, *and*

$$8n + 24m + 1 \in \bigcup_{j'=1}^{s} [5n + 12m + (2j' - 1)(2n + 3i) + 1, 5n + 12m + 2j'(2n + 3i)],$$

(h) $v = u_{i,2}$, $k = 2n + 12m + 3 - 3i, s \geq 1$,
(i) $v = u_{i,4}$, $k = 2n + 12m + 1 - 3i, s \geq 1$,
(j) $v = u_{i,6}$, $k = 2n + 12m + 2 - 3i, s \geq 1$.

Each of Conditions (a) *to* (c) *implies that* $\chi_{lt}(G_v(k,s)) = ks + 2n$, *whereas each of Conditions* (d) *to* (j) *implies that* $\chi_{lt}(G_v(k,s)) = ks + 2n + 1$.

Proof. Let f be the local total antimagic labeling of G as defined in the proof of Theorem 4.2. Clearly, if v is a pendant vertex, then $v \in \{v_{j,1}, v_{j,3} \mid 1 \leq j \leq n\}$ and $f(v) = k = j$ for $v = v_{j,1}$ and $k = n + j$ for $v = v_{j,3}$. Moreover, $\Delta(G_v(k,s)) = ks + 1$ and $G_v(k,s)$ has $ks + 2n - 1$ pendant edges. By Corollary 2.2, $\chi_{lt}(G_v(k,s)) \geq ks + 2n$. If v is not a pendant vertex, then

$$v \in \{u_{i,1}, \ldots, u_{i,6}, v_{j,2} \,|\, 1 \leq i \leq m, 1 \leq j \leq n\}$$

and

$$\begin{aligned} f(v) = k \in \{&2n + 3i - 2, 2n + 3i - 1, 2n + 3i, 2n + 12m + 3 - 3i, 2n + 12m + 2 - 3i,\\ &2n + 12m + 1 - 3i, 3n + 12m + 1 - j \,|\, 1 \leq i \leq m, 1 \leq j \leq n\}. \end{aligned}$$

Moreover, $\Delta(G_v(k,s)) = ks + 2$ and $G_v(k,s)$ has $ks + 2n$ pendant edges. Suppose the added ks pendant edges incident to v are $e_{i',j'}$ and the corresponding pendant vertices are $x_{i',j'}$ for $1 \leq j' \leq s, 1 \leq i' \leq k$. By Corollary 2.2, $\chi_{lt}(G_v(k,s)) \geq ks + 2n + 1$.

Since $G_v(k,s)$ has $3n + 6m + ks$ vertices and $2n + 6m + ks$ edges, we define a total labeling $g : V(G_v(k,s)) \cup E(G_v(k,s)) \to [1, 5n + 12m + 2ks]$ such that $g(z) = f(z)$ for $z \in V(G) \cup E(G)$. Otherwise, $g(z)$ is as defined in the proof of Theorems 2.4 or 2.5. One can check that

$$w_g(v) = w_f(v) + \sum_{j'=1}^{s} \sum_{i'=1}^{k} (12m + 5n + (2j' - 1)k + i'),$$

and

$$\begin{aligned} w(z) \in\ &[3n + 12m + 1, 5n + 12m] \cup \{8n + 24m + 1\}\\ &\cup \{5n + 12m + (2j' - 1)k + i' \,|\, 1 \leq j' \leq s, 1 \leq i' \leq k\} \end{aligned}$$

for $z \in (V(G_v(k,s)) \cup E(G_v(k,s))) \setminus \{v\}$. Moreover, g is a local total antimagic labeling. Thus, $\chi_{lt}(G_v(k,s)) \leq ks + 2n + 2$ if all the weights are distinct. We shall need to check the conditions under Theorem 2.5 in the following 10 cases.

(a) Suppose $v = v_{1,1}$ so that $k = 1$. Thus,

$$\{w_g(e_{j',1})\} = \{5n + 12m + 2j' \mid 1 \leq j' \leq s, s \geq 2\}.$$

Therefore, $8n + 24m + 1 \in \{w_g(e_{j',1})\}$ so that n is odd and

$$s \geq (3n + 24m + 1)/2.$$

(b) Suppose $v = v_{j,1}, 2 \le j \le n$ so that $k = j \ge 2$. Therefore,

$$\begin{aligned}8n + 24m + 1 &\in \{w_g(e_{j',i'})\}\\ &= \bigcup_{j'=1}^{s} [5n + 12m + (2j' - 1)j + 1, 5n + 12m + 2j'j],\end{aligned}$$

$2 \le j \le n$, so that

$$2(j' - 1)j + 1 \le 3n + 12m + 1 \le 2j'j.$$

As an example, take $n = 3, m = 1$, we can choose $j = 3, s \ge j' = 4$ to get

$$8n + 24m + 1 = 49 \in [49, 51] = [5n + 12m + (2j' - 1)j + 1, 5n + 12m + 2j'j]$$

as required.

(c) Suppose $v = v_{j,3}, 1 \le j \le n$ so that $k = n + j$. Therefore,

$$\begin{aligned}8n + 24m + 1 &\in \{w_g(e_{j',i'})\}\\ &= \bigcup_{j'=1}^{s} [5n + 12m + (2j' - 1)(n + j) + 1, 5n + 12m + 2j'(n + j)],\end{aligned}$$

$1 \le j \le n$, so that

$$(2j' - 1)(n + j) + 1 \le 3n + 12m + 1 \le 2j'(n + j).$$

As an example, take $n = m = 3$, we can choose $j = 1, s \ge j' = 6$ to get

$$\begin{aligned}8n + 24m + 1 &= 97 \in [96, 99]\\ &= [5n + 12m + (2j' - 1)(n + j) + 1, 5n + 12m + 2j'(n + j)]\end{aligned}$$

as required.

(d) Suppose $v = v_{j,2}$ so that $k = 3n + 12m + 1 - j$. Thus,

$$\begin{aligned}&\{w_g(e_{j',i'})\}\\ &= \{5n + 12m + (2j' - 1)(3n + 12m + 1 - j) + i' \mid 1 \le i' \le k, 1 \le j' \le s\}\\ &= \bigcup_{j'=1}^{s} [5n + 12m + (2j' - 1)(3n + 12m + 1 - j) + 1,\\ &\qquad 5n + 12m + 2j'(3n + 12m + 1 - j)].\end{aligned}$$

If $j' = 1$, then

$$8n + 24m + 1 \in [8n + 24m + 2 - j, 11n + 36m + 2 - 2j]$$

since $j \le n$. Thus, g induces $ks + 2n + 1$ distinct weights. Therefore, if $G = mC_6 + nP_3$, then

$$\chi_{lt}(G_{v_{j,2}}(3n + 12m + 1 - j, s)) = (3n + 12m + 1 - j)s + 2n + 1$$

for $s \ge 1$.

(e) Suppose $v = u_{i,1}$ so that $k = 2n + 3i - 2$. Thus,

$$\begin{aligned}\{w_g(e_{j',i'})\} &= \{5n + 12m + (2j' - 1)k + i' \mid 1 \le i' \le k, 1 \le j' \le s\}\\ &= \bigcup_{j'=1}^{s} [5n + 12m + (2j' - 1)(2n + 3i - 2) + 1, 5n + 12m + 2j'(2n + 3i - 2)],\end{aligned}$$

denoted U, for $1 \le i \le m, 1 \le j' \le s$ Thus, if $8n + 24m + 1 \in U$, then g induces $ks + 2n + 1$ distinct weights. As an example, take $m = 1, n = 3, i = 1, k = 7, j' = 2$, we get $8n + 24m + 1 = 49$ and

$$[5n + 12m + (2j' - 1)(2n + 3i - 2) + 1, 5n + 12m + 2j'(2n + 3i - 2)] = [49, 55].$$

Thus, for $G = C_6 + 3P_3$,

$$\chi_{lt}(G_{u_{1,1}}(7, s)) = 7s + 7$$

for $s \ge 2$.

(f) Suppose $v = u_{i,3}$ so that $k = 2n + 3i - 1$. By a similar argument, we get g induces $ks + 2n + 1$ distinct weights if

$$\begin{aligned}&8n + 24m + 1 \in\\ &\bigcup_{j'=1}^{s} [5n + 12m + (2j' - 1)(2n + 3i - 1) + 1, 5n + 12m + 2j'(2n + 3i - 1)]\end{aligned}$$

for $1 \le i \le m, 1 \le j' \le s$. As an example: take $m = 2, n = 3, i = 2, k = 11, j' = 2$, we get $8n + 24m + 1 = 73$ and

$$[5n + 12m + (2j' - 1)(2n + 3i - 1) + 1, 5n + 12m + 2j'(2n + 3i - 1)] = [73, 83].$$

Thus, for $G = 2C_6 + 3P_3$,

$$\chi_{lt}(G_{u_{2,3}}(11, s)) = 11s + 7$$

for $s \ge 2$.

(g) Suppose $v = u_{i,5}$ so that $k = 2n + 3i$. By a similar argument, we get g induces $ks + 2n + 1$ distinct weights if

$$8n + 24m + 1 \in \bigcup_{j'=1}^{s} [5n + 12m + (2j' - 1)(2n + 3i) + 1, 5n + 12m + 2j'(2n + 3i)].$$

As an example: take $m = n = i = 3$, $k = 15$, $j' = 2$, we get $8n + 24m + 1 = 97$ and

$$[5n + 12m + (2j' - 1)(2n + 3i) + 1, 5n + 12m + 2j'(2n + 3i)] = [97, 111].$$

Thus, for $G = 3C_6 + 3P_3$,

$$\chi_{lt}(G_{u_{3,5}}(15, s)) = 15s + 7$$

for $s \ge 2$.

(h) Suppose $v = u_{i,2}$ so that $k = 2n + 12m + 3 - 3i$. We get g induces $ks + 2n + 1$ distinct weights if

$$8n + 24m + 1 \in \bigcup_{j'=1}^{s} [5n + 12m + (2j' - 1)(2n + 12m + 3 - 3i) + 1, \\ 5n + 12m + 2j'(2n + 12m + 3 - 3i)].$$

For $j' = 1$, we get

$$8n + 24m + 1 \in [7n + 12m + 4 - 3i, 9n + 36m + 6 - 6i]$$

since $i \leq m$. Thus, for $G = mC_6 + nP_3$,

$$\chi_{lt}(G_{u_{i,2}}(2n + 12m + 3 - 3i, s)) = (2n + 12m + 3 - 3i)s + 2n + 1$$

for $s \geq 1$.

(i) Suppose $v = u_{i,4}$ so that $k = 2n + 12m + 1 - 3i$. We get g induces $ks + 2n + 1$ distinct weights if

$$8n + 24m + 1 \in \bigcup_{j'=1}^{s} [5n + 12m + (2j' - 1)(2n + 12m + 1 - 3i) + 1, \\ 5n + 12m + 2j'(2n + 12m + 1 - 3i)].$$

For $j' = 1$, we get

$$8n + 24m + 1 \in [7n + 24m + 2 - 3i, 9n + 36m + 3 - 6i]$$

since $i \leq m$. Thus, for $G = mC_6 + nP_3$,

$$\chi_{lt}(G_{u_{i,4}}(2n + 12m + 1 - 3i, s)) = (2n + 12m + 1 - 3i)s + 2n + 1$$

for $s \geq 1$.

(j) Suppose $v = u_{i,6}$ so that $k = 2n + 12m + 2 - 3i$. We get g induces $ks + 2n + 1$ distinct weights if

$$8n + 24m + 1 \in \bigcup_{j'=1}^{s} [5n + 12m + (2j' - 1)(2n + 12m + 2 - 3i) + 1, \\ 5n + 12m + 2j'(2n + 12m + 2 - 3i)].$$

For $j' = 1$, we get

$$8n + 24m + 1 \in [7n + 24m + 3 - 3i, 9n + 36m + 4 - 6i]$$

since $i \leq m$. Thus, for $G = mC_6 + nP_3$,

$$\chi_{lt}(G_{u_{i,6}}(2n + 12m + 2 - 3i, s)) = (2n + 12m + 2 - 3i)s + 2n + 1$$

for $s \geq 1$.

For (a) to (c), Theorem 2.5(a)(i) implies that $\chi_{lt}(G_v(k,s)) = ks + 2n$. For (d) to (j), Theorem 2.5(b) implies that $\chi_{lt}(G_v(k,s)) = ks+2n+1$. This completes the proof. $\square$

Example 4.9. Figure 7 gives the local total antimagic 65-coloring of $G_v(k,s)$ for $G = 2C_6+2P_3, v = v_{1,2}, k = 30, s = 2$ with induced weights $31, 32, 33, 34, 65, 66, \ldots, 94$, $125, 126, \ldots$, 154 and 6635 being the weight of $v_{1,2}$ in $G_v(k,s)$ as defined above.

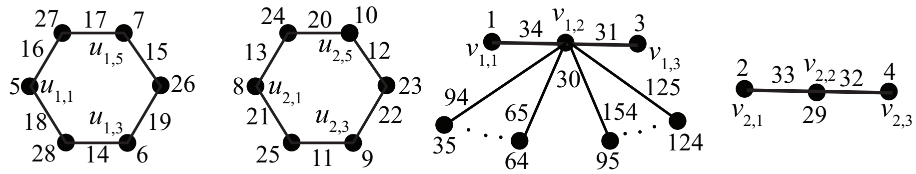

Fig. 7. $G_v(k,s)$, $v = v_{1,2}$, $k = 30, s = 2$ with local total antimagic 65-coloring

Note that Theorem 2.5 is not applicable to $mC_6 + nP_6$ for $m,n \geq 1$. Let $G = mC_6+nP_6+aP_3$ with vertex and edge sets as defined in the proof of Theorem 4.6. We now have the following theorem.

Theorem 4.10. *For $m,n \geq 1$, $a \geq 2$ and $G = mC_6 + nP_6 + aP_3$ such that $n \geq 2a+2$ or else $a \geq 2n$ with the following 15 conditions:*

(1) $v = y_{t,1}$, $k = 8n + 6m - 3t + 2a + 2$, $t \leq (5n + 3)/6$,
(2) $v = y_{t,6}$, $k = 2n + 6m + 3t + 2a$ *and*
  (a) $s \geq 1, t \in [1,n], a \geq 2n$ *only, or else*
  (b) $s \geq 2, (3n - 12m - 5a + 1)/12 \leq t \leq (5n - 6m - 3a + 1)/9$, *or else*
  (c) $s \geq 3, t \leq (n - 18m - 7a + 1)/15$.
(3) $v = v_{r,1}, r \in [1,a], k = r, (2j' - 1)r \leq 11n + 12m + 3a + 1 \leq 2j'r$,
(4) $v = v_{r,3}, r \in [1,a], k = a + r$,

$$(2j' - 1)(a + r) \leq 11n + 12m + 3a + 1 \leq 2j'(a + r),$$

(5) $v = u_{i,1}, i \in [1,m], k = 2n + 2a + 3i - 2$,

$$(2j' - 1)(2n + 2a + 3i - 2) \leq 11n + 12m + 3a + 1 \leq 2j'(2n + 2a + 3i - 2),$$

(6) $v = u_{i,3}, i \in [1,m], k = 2n + 2a + 3i - 1$,

$$(2j' - 1)(2n + 2a + 3i - 1) \leq 11n + 12m + 3a + 1 \leq 2j'(2n + 2a + 3i - 1),$$

(7) $v = u_{i,5}, i \in [1,m], k = 2n + 2a + 3i$,

$$(2j' - 1)(2n + 2a + 3i) \leq 11n + 12m + 3a + 1 \leq 2j'(2n + 2a + 3i),$$

(8) $v = u_{i,2}, i \in [1,m], k = 8n + 12m + 2a - 3i + 3, s \geq 1$,
(9) $v = u_{i,4}, i \in [1,m], k = 8n + 12m + 2a - 3i + 1, s \geq 1$,
(10) $v = u_{i,6}, i \in [1,m], k = 8n + 12m + 2a - 3i + 2, s \geq 1$,

(11) $v = y_{t,3}, k = 8n+6m+2a-3t+3, t \le (5n+a+5)/6$, *$n \ge a+5$ for $a = 1,2$ and $n \ge 2a+2$ otherwise,*
(12) $v = y_{t,5}, k = 8n+6m+2a-3t+1, t \le (5n+a+1)/6$ *and $n \ge 2a+2$ only,*
(13) $v = y_{t,2}, k = 2n+6m+2a+3t-2$ *and*
   (a) $s \ge 1, t \in [1,n], a \ge n+5$ *for $a = 1,2,3,4$ and $a \ge 2n$ otherwise, or else*
   (b) $s \ge 2, (3n-12m-5a+9)/12 \le t \le (5n-6m-a+7)/3$, *or else*
   (c) $s \ge 3, t \le (n-18m-7a+13)/15$,
(14) $v = y_{t,4}$, $k = 2n+6m+2a+3t-1$, *and*
   (a) $s \ge 1, t \in [1,n], a \ge n+3$ *for $a = 1,2$ and $a \ge 2n$ otherwise, or else*
   (b) $s \ge 2, (3n-12m-5+5)/12 \le t \le (5n-6m-3a+4)/9$, *or else*
   (c) $s \ge 3, t \le (n-18m-7a+6)/15$.
(15) $v = v_{r,2}, r \in [1,a]$, $k = 12m+11n+3a+1-r, s \ge 1$.

Each of Conditions (1) to (4) implies that $\chi_{lt}(G_v(k,s)) = 2n+2a+ks$, whereas each of Conditions (5) to (15) implies that $\chi_{lt}(G_v(k,s)) = 2n+2a+ks+1$.

Proof. Let g be the local total antimagic labeling of G as defined in the proof of Theorem 4.6. By an argument similar to the proof of Theorem 4.8 and Theorem 2.5(iii), we shall show that the only weight $w_g(v_{r,2}) = 22n+24m+8a+1$ which is not a pendant edge label under g must be in

$$\{w_g(e_{j',i'})\} = \bigcup_{j'=1}^{s} [11n+12m+5a+(2j'-1)k+1, 11n+12m+5a+2j'k].$$

We also check the conditions under Theorem 2.5. We have the following first four cases for v being a pendant vertex and next 11 cases for v not a pendant vertex.

(1) Suppose $v = y_{t,1}, t \in [1,n]$ with $k = 8n+6m-3t+2a+2$. Therefore, $22n+24m+8a+1 \in \{w_g(e_{j',i'})\}$ means

$$(2j'-1)(8n+6m-3t+2a+2)+1 \le 11n+12m+3a+1 \le 2j'(8n+6m-3t+2a+2).$$

Thus, $j' = 1$ so that

$$8n+6m-3t+2a+3 \le 11n+12m+3a+1 \le 16n+12m-6t+4a+4.$$

Consequently, $t \le (5n+3)/6$.

(2) Suppose $v = y_{t,6}, t \in [1,n]$ with $k = 2n+6m+3t+2a$. Therefore, we must have

$$(2j'-1)(2n+6m+3t+2a) \le 11n+12m+3a+1 \le 2j'(2n+6m+3t+2a).$$

Thus, $j' \le 3$. If $j' = 1$, then

$$2n+6m+3t+2a \le 11n+12m+3a+1 \le 4n+12m+6t+4a$$

so that

$$(7n-a+1)/6 \le t \le (9n+6m+a+1)/3.$$

Since $t \le n$, we then have $a \ge n+1$. The conditions $n \ge 2a+2$ or else $a \ge 2n$ further implies that we must have $a \ge 2n$ for this case. If $s \ge j' = 2$, then

$$6n + 18m + 9t + 6a \le 11n + 12m + 3a + 1 \le 8n + 24m + 12t + 8a$$

so that

$$(3n - 12m - 5a + 1)/12 \le t \le (5n - 6m - 3a + 1)/9.$$

If $s \ge j' = 3$, then

$$10n + 30m + 15t + 10a \le 11n + 12m + 3a + 1 \le 12n + 36m + 18t + 12a$$

so that

$$t \le (n - 18m - 7a + 1)/15.$$

(3) Suppose $v = v_{r,1}, r \in [1, a]$ with $k = r$. Therefore, we must have

$$(2j' - 1)r \le 11n + 12m + 3a + 1 \le 2j'r.$$

Note that $s \ge 2$ if $r = 1$, and $s \ge 1$ otherwise.

(4) Suppose $v = v_{r,3}$ with $k = a + r$. Therefore, we must have

$$(2j' - 1)(a + r) \le 11n + 12m + 3a + 1 \le 2j'(a + r).$$

(5) Suppose $v = u_{i,1}$, $i \in [1, m]$ with $k = 2n + 2a + 3i - 2$. Therefore, we must have

$$(2j' - 1)(2n + 2a + 3i - 2) \le 11n + 12m + 3a + 1 \le 2j'(2n + 3a + 3i - 2).$$

(6) Suppose $v = u_{i,3}$ with $k = 2n + 2a + 3i - 1$. Therefore, we must have

$$(2j' - 1)(2n + 2a + 3i - 1) \le 11n + 12m + 3a + 1 \le 2j'(2n + 3a + 3i - 1).$$

(7) Suppose $v = u_{i,5}$ with $k = 2n + 2a + 3i$. Therefore, we must have

$$(2j' - 1)(2n + 2a + 3i) \le 11n + 12m + 3a + 1 \le 2j'(2n + 3a + 3i).$$

(8) Suppose $v = u_{i,2}$ with $k = 8n + 12m + 2a - 3i + 3$. Therefore, we must have

$$(2j'-1)(8n+12m+2a-3i+3) \le 11n+12m+3a+1 \le 2j'(8n+12m+2a-3i+3).$$

Since $i \le m$, we must have $j' = 1$ and the equality always holds.

(9) Suppose $v = u_{i,4}$ with $k = 8n + 12m + 2a - 3i + 1$. Therefore, we must have

$$(2j'-1)(8n+12m+2a-3i+1) \le 11n+12m+3a+1 \le 2j'(8n+12m+2a-3i+1).$$

Similar to (8), the equality always hold.

(10) Suppose $v = u_{i,6}$ with $k = 8n + 12m + 2a - 3i + 2$. Therefore, we must have

$$(2j'-1)(8n+12m+2a-3i+2) \le 11n+12m+3a+1 \le 2j'(8n+12m+2a-3i+2).$$

Similar to (8), the equality always hold.

(11) Suppose $v = y_{t,3}, t \in [1,n]$ with $k = 8n + 6m + 2a - 3t + 3$. Therefore, we must have

$$(2j'-1)(8n+6m+2a-3t+3) \leq 11n+12m+3a+1 \leq 2j'(8n+6m+2a-3t+3).$$

Thus, $j' = 1$ so that $t \leq (5n + a + 5)/6$. Since $t \leq n$, we have $a + 5 \leq n$. The conditions $n \geq 2a + 2$ or else $a \geq 2n$ further implies that we must have $n \geq a + 5$ for $a = 1, 2$, and $n \geq 2a + 2$ otherwise.

(12) Suppose $v = y_{t,5}$ with $k = 8n + 6m + 2a - 3t + 1$. Therefore, we must have

$$(2j'-1)(8n+6m+2a-3t+1) \leq 11n+12m+3a+1 \leq 2j'(8n+6m+2a-3t+1).$$

Thus, $j' = 1$ so that $t \leq (5n + a + 1)/6$. Since $t \leq n$, we must have $a + 1 \leq n$. Similar to (11), we must have $n \geq 2a + 2$ for this case.

(13) Suppose $v = y_{t,2}$ with $k = 2n + 6m + 2a + 3t - 2$. Therefore, we must have

$$(2j'-1)(2n+6m+2a+3t-2) \leq 11n+12m+3a+1 \leq 2j'(2n+6m+2a+3t-2).$$

Thus, $j' \leq 3$. If $j' = 1$, then

$$2n + 6m + 2a + 3t - 2 \leq 11n + 12m + 3a + 1 \leq 4n + 12m + 4a + 6t - 4$$

so that

$$(7n - a + 5)/6 \leq t \leq (9n + 6m + a + 3)/3.$$

Since $t \leq n$, we then have $a \geq n + 5$. The conditions $n \geq 2a + 2$ or else $a \geq 2n$ further implies that we must have $a \geq n+5$ for $a = 1, 2, 3, 4$ and $a \geq 2n$ otherwise. If $s \geq j' = 2$, then

$$6n + 18m + 6a + 9t - 6 \leq 11n + 12m + 3a + 1 \leq 8n + 24m + 8a + 12t - 8$$

so that

$$(3n - 12m - 5a + 9)/12 \leq t \leq (5n - 6m - 3a + 7)/9.$$

If $s \geq j' = 3$, then

$$10n + 30m + 10a + 15t - 10 \leq 11n + 12m + 3a + 1 \leq 12n + 36m + 12a + 18t - 12$$

so that

$$t \leq (n - 18m - 7a + 13)/15.$$

(14) Suppose $v = y_{t,4}$ with $k = 2n + 6m + 2a + 3t - 1$. Therefore, we must have

$$(2j'-1)(2n+6m+2a+3t-1) \leq 11n+12m+3a+1 \leq 2j'(2n+6m+2a+3t-1).$$

Thus, $j' \leq 3$. If $j' = 1$, then

$$2n + 6m + 2a + 3t - 1 \leq 11n + 12m + 3a + 1 \leq 4n + 12m + 4a + 6t - 2$$

so that

$$(7n - a + 3)/6 \leq t \leq (9n + 6m + a + 2)/3.$$

Since $t \le n$, we then have $a \ge n+3$. The conditions $n \ge 2a+2$ or else $a \ge 2n$ further implies that we must have $a \ge n+3$ for $a = 1,2$ and $a \ge 2n$ otherwise. If $s \ge j' = 2$, then

$$6n + 18m + 6a + 9t - 3 \le 11n + 12m + 3a + 1 \le 8n + 24m + 8a + 12t - 4$$

so that

$$(3n - 12m - 5a + 5)/12 \le t \le (5n - 6m - 3a + 4)/9.$$

If $s \ge j' = 3$, then

$$10n + 30m + 10a + 15t - 5 \le 11n + 12m + 3a + 1 \le 12n + 36m + 12a + 18t - 6$$

so that

$$t \le (n - 18m - 7a + 6)/15.$$

(15) Suppose $v = v_{r,2}, r \in [1,a]$ with $k = 12m + 11n + 3a + 1 - r$. Therefore, we must have

$$(2j'-1)(12m+11n+3a+1-r) \le 11n+12m+3a+1 \le 2j'(12m+11n+3a+1-r).$$

Since $r \le a$, we must have $j' = 1$ and the equality always holds.

For (1) to (4), Theorem 2.5(a)(i) implies that $\chi_{lt}(G_v(k,s)) = 2n + 2a + ks$. For (5) to (15), By Theorem 2.5(b), we have $\chi_{lt}(G_v(k,s)) = 2n + 2a + ks + 1$. This completes the proof. □

Example 4.11. Figure 8 gives the local total antimagic 39-coloring of $G_v(k,s)$ for $G = C_6 + P_6 + 3P_3$, $v = v_{3,2}$, $k = 30$, $s = 1$ with induced weights 28, 29, 33, 34, 35, 36, 37, 38, $69, 70, \ldots, 98$ and 2576 being the weight of $v_{3,2}$ in $G_v(k,s)$ as defined above.

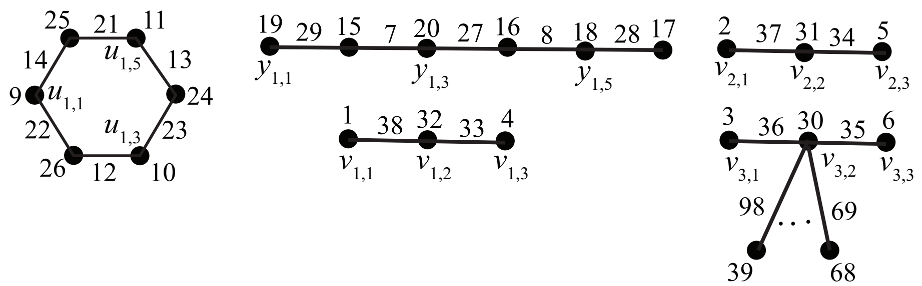

Fig. 8. $G_v(k,s)$, $v = v_{3,2}$, $k = 30, s = 1$ with local total antimagic 39-coloring

5. CONCLUSIONS

In this paper, we successfully characterize graphs with $\chi_{lt}(G) = 3$. Moreover, we obtain a sufficient condition for a graph G with $k \ge 2$ pendant edges such that $\chi_{lt}(G) \ge k+2$.

A family of graphs G with $\chi_{lt}(G) \geq k+2$ is given. We then obtain several families of graphs with (i) $\Delta(G) = k+2$ and $\chi_{lt}(G) = k+3$, or (ii) $k \geq \Delta(G)$ and $\chi_{lt}(G) = k+1$. The following problems arise naturally.

Problem 5.1. For $2n \geq k+3 \geq 4$ and $2n(2n+1)-2n(k+2)(5k+5)+(k+2)(k-1) > 0$, show that $\chi_{lt}(f_n(k)) = 2nk+2$. Otherwise, $\chi_{lt}(f_n(k)) = 2nk+1$.

Problem 5.2. Determine $\chi_{lt}(G)$ for $G \cong P_n, C_n$ for $n \geq 3$ completely.

Problem 5.3. Determine $\chi_{lt}(G_v(k,s))$ for $G = mC_6 + nP_6$, $m \geq 1, n \geq 1$.

Problem 5.4. Determine $\chi_{lt}(mC_6 + nP_6 + aP_3)$ for $m, a \geq 1, (a+1)/2 \leq n \leq 2a+1$.

We note that $P_n, n \geq 3$ is the only graph with $\Delta(P_n) = 2 =$ the number of pendant edges. Thus, we also pose the following problem.

Problem 5.5. Determine $\chi_{lt}(G)$ for G with $k \geq 3$ pendant edges and $\Delta(G) = k$.

In all the known results on graphs G with $k \geq \Delta(G)$ pendant edges, we have $k+1 \leq \chi_{lt}(G) \leq k+2$. Similar to the conjecture of $\Delta(G)+1 \leq \chi_t(G) \leq \Delta(G)+2$, we end the paper with the following conjectures.

Conjecture 5.6. If G has $k \leq \Delta(G)$ pendant edges, then

$$\Delta(G) + 1 \leq \chi_{lt}(G) \leq \Delta(G) + 2.$$

Conjecture 5.7. If G has $k \geq \Delta(G)$ pendant edges, then

$$k + 1 \leq \chi_{lt}(G) \leq k + 2.$$

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