CS 321: Exam 2 Study Guide

1. True or false? If true, give brief justification. If false, give counterexample:

   (a) If A is context-free and B is regular then $A \setminus B$ is **always** context-free

   (b) If A and B are context-free then $A \cap B$ is **always** context-free

   (c) If A is context-free then its complement is **always** context-free

2. Below are some context-free languages. Use them to practice the following activities:

   - directly designing a PDA
   - directly designing a CFG
   - converting between PDA and CFG
   - converting CFG to Chomsky normal form
   - arguing that something is CFL using other methods like closure properties (probably only relevant to some of these)

   (a) $\{a^m b^n a^n b^m \mid m, n \in \mathbb{N}\}$

   (b) $\{a^m b^m a^n b^n \mid m, n \in \mathbb{N}\}$

   (c) $\{a^m b^n a^k b^m \mid m, n, k \in \mathbb{N},\ \text{and } n \neq k\}$

   (d) $\{a^m b^n a^k b^j \mid m < j \text{ and } n > k\}$

   (e) $\{w \in \{a, b\}^* \mid w \text{ is not a palindrome}\}$

   (f) $\{xy \in \{a, b\}^* \mid x \text{ is a palindrome and } y \text{ has equal number of } a\text{'s and } b\text{'s}\}$

   (g) $\{w \in \{0, 1\}^* \mid w \text{ is a multiple of 3 in binary}\}$

3. For each of the grammars below, do the following:

   - Using set notation (or a regular expression), say what language it generates.
   - Find a string that has two different *parse trees* (and draw them):

   (a) $S \to aaaX \mid aaaaaY$
   $X \to aaX \mid \varepsilon$
   $Y \to aaaY \mid \varepsilon$

   (b) $S \to aSb \mid aS \mid \varepsilon$

   (c) $S \to XC \mid AY$
   $X \to aXb \mid \varepsilon$
   $Y \to bYc \mid \varepsilon$
   $A \to aA \mid \varepsilon$
   $C \to cC \mid \varepsilon$

> For reference, here is the *context-free pumping lemma game* (for language A):
>
> 1. Adversary picks a number $p \geq 0$.
> 2. You pick a string $s \in A$, such that $|s| \geq p$.
> 3. Adversary breaks s into $s = uvwxy$, such that $|vwx| \leq p$ and $|vx| > 0$.
> 4. You pick a number $i \geq 0$. If $uv^i wx^i y \notin A$, then you win.
>
> If you can describe a strategy in which you always win, then A is not context-free.

4. Show that the following are not CFL (using pumping lemma or otherwise):

   (a) $\{xyx \mid y = \text{rev}(x)\}$

   (b) the set of strings of properly nested parentheses with two types of parentheses: `[ ] ( )`, where the number of "`[`" characters is equal to the number of "`(`" characters.

   * (any of the languages from the next problem that are non-context-free)

5. State whether the language is [regular], [context-free but not regular], [not context-free]. If it's regular or context-free, give a short justification (can describe a DFA/NFA/regex/PDA/CFG or argue in terms of closure properties). If it's not context free, then either describe a TM (at a high level) or describe the string that you would choose in the pumping lemma (you don't have to do the entire pumping lemma proof, just say what string you will choose).

   (a) $\{a^n a^m b^m b^k \mid n, m, k \in \mathbb{N}\}$

   (b) $\{w \mid w$ is not a palindrome$\}$

   (c) $\{w \in \{a, b, c\}^* \mid$ the middle character of w is $c\}$

   (d) $\{xy \mid x, y \in \{a, b\}^*$ and $|x| = |y|$ and x is a palindrome$\}$

   (e) $\{xcy \mid x, y \in \{a, b\}^*$ and $|x| = 2|y|\}$

   (f) $\{w \in \{0, 1\}^* \mid w$ is a palindrome and $\text{bin}(w)$ is a multiple of 3$\}$

   (g) $\{w \in \{a, b, c\}^* \mid \text{num}(a, x) = \text{num}(b, x)$ and $\text{num}(a, x) < \text{num}(c, x)\}$

   (h) $\{a^n b^m \mid a + 2m$ is a multiple of 3$\}$

   (i) $\{xcy \mid x, y \in \{a, b\}^*$ and x is a substring of $y\}$

   (j) $\{xy \mid x, y \in \{a, b\}^*$ and $|x| = |y|\}$

6. TM design / idioms. Assume the input to the TM is a string from $\{a, b\}^*$ (except the last problem). Describe how the TM works, in terms of its tape head movements and tape markings. Don't bother with a complete transition table. Don't include any steps that are more high-level than "scan L/R until you see this character", "if you see this character, do this", "add a mark to a character", "remove a mark from a character", etc.

   (a) A TM that places a special mark (e.g., replace a with $\widehat{a}$; replace b with $\widehat{b}$) on the *middle* character of the input, or rejects if the input has even length.

   (b) A TM that reverses its input string.

   (c) A TM that "doubles" its input string. On input w, it should halt with ww on its tape.

   (d) A TM that starts with a binary integer on its tape, and increments the integer by one.

7. Undecidability. Suppose you had a subroutine that always gave a correct answer to problem A. Show that such a subroutine could be used to decide problem B (hence, if B is undecidable, then so is A).

   (a) A: given a TM M and string w, output 1 if M accepts w, and output 0 otherwise (including if M runs forever on w).

   B: [halting problem] Given M and w, output 1 if M halts on w and output 0 otherwise.

(b) A: [halting problem]

B: given a TM M and string w, output 1 if M accepts w, and output 0 otherwise (including if M runs forever on w).

(c) A: Given M, output 1 if M halts on input ε, and output 0 otherwise.

B: [halting problem]