Simple 2-out-of-2 secret-sharing scheme:

$$\Sigma:$$

$$\mathcal{M} = \{0,1\}^{\ell} \qquad \frac{\text{Share}(m):}{s_1 \leftarrow \{0,1\}^{\ell}}$$

$$t = 2$$

$$n = 2$$

$$Share(m):$$

$$s_1 \leftarrow \{0,1\}^{\ell}$$

$$s_2 := s_1 \oplus m$$

$$return (s_1, s_2)$$

$$Reconstruct(s_1, s_2):$$

$$return s_1 \oplus s_2$$

Claim:

 Σ is a secure 2-out-of-2 secret-sharing scheme. That is,

$$\mathcal{L}^{\Sigma}_{\mathsf{tsss-L}} \equiv \mathcal{L}^{\Sigma}_{\mathsf{tsss-R}}.$$

We will **use** the fact that one-time pad has one-time security $(\mathcal{L}_{ots-L}^{OTP} \equiv \mathcal{L}_{ots-R}^{OTP})$.

Overview:

Want to show:

$$\begin{array}{|c|c|} \hline \mathcal{L}_{\mathsf{tsss-L}}^{\Sigma} \\ \hline \underline{\mathsf{QUERY}(m_L, m_R, U):} \\ \hline \mathsf{if} \ |U| \geq 2: \ \mathsf{return} \ \mathsf{err} \\ \pmb{s} \leftarrow \Sigma. \mathsf{Share}(m_L) \\ \\ \mathsf{return} \ (s_i)_{i \in U} \end{array} \end{array} \equiv \begin{array}{|c|c|} \hline \mathcal{L}_{\mathsf{tsss-R}}^{\Sigma} \\ \hline \underline{\mathsf{QUERY}(m_L, m_R, U):} \\ \hline \mathsf{if} \ |U| \geq 2: \ \mathsf{return} \ \mathsf{err} \\ \pmb{s} \leftarrow \Sigma. \mathsf{Share}(m_R) \\ \\ \mathsf{return} \ (s_i)_{i \in U} \end{array}$$

Standard hybrid technique:

- Starting with $\mathcal{L}^{\Sigma}_{\mathsf{tsss-L}}$, make a sequence of small modifications
- Each modification has no effect on calling program / adversary
- lacksquare Sequence of modifications ends with $\mathcal{L}_{ ext{tsss-R}}^{\Sigma}$



 $\mathcal{L}_{\mathsf{tsss-L}}^{\Sigma}$ $\underline{\mathsf{QUERY}(m_L, m_R, U)}:$ $\mathsf{if} \ |U| \ge 2: \mathsf{return} \ \mathsf{err}$ $\boldsymbol{s} \leftarrow \Sigma.\mathsf{Share}(m_L)$ $\mathsf{return} \ (s_i)_{i \in U}$

Starting point is $\mathcal{L}_{tsss-L}^{\Sigma}$.

```
\mathcal{L}_{\mathsf{tsss-L}}^{\Sigma}
\underline{\mathsf{QUERY}(m_L, m_R, U):}
\overline{\mathsf{if} \ |U| \ge 2: \ \mathsf{return} \ \mathsf{err}}
s \leftarrow \Sigma.\mathsf{Share}(m_L)
\mathrm{return} \ (s_i)_{i \in U}
```

Starting point is $\mathcal{L}_{tsss-L}^{\Sigma}$. Fill in details of Σ



 $\mathcal{L}^{\Sigma}_{tsss\text{-L}}$ QUERY (m_L, m_R, U) : if $|U| \ge 2$: return err $s_1 \leftarrow \{0,1\}^{\ell}$ $s_2 := s_1 \oplus m_L$ return $(s_i)_{i \in U}$

Details of Σ filled in.



 $\mathcal{L}^{\Sigma}_{tsss\text{-L}}$

QUERY (m_L, m_R, U) :

if $|U| \ge 2$: return err

 $s_1 \leftarrow \{0,1\}^{\ell}$

 $s_2 := s_1 \oplus m_L$

return $(s_i)_{i \in U}$

Details of Σ filled in.

```
QUERY(m_L, m_R, U):
  if |U| \ge 2: return err
  if U = \{1\}:
    s_1 \leftarrow \{0, 1\}^{\ell}
      s_2 := s_1 \oplus m_L
      return s<sub>1</sub>
  elsif U = \{2\}:
     s_1 \leftarrow \{0,1\}^{\ell}
      s_2 := s_1 \oplus m_I
      return s2
  else return null
```

Duplicate body for the 3 possible unauthorized sets: $\{1\}, \{2\}, \emptyset$.

```
QUERY(m_L, m_R, U):
  if |U| \ge 2: return err
  if U = \{1\}:
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      s_2 := s_1 \oplus m_L
      return s2
  else return null
```

s₂ not used in this branch.

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  if |U| \ge 2: return err
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 s_2 not used in this branch, so we can change how it is assigned.

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  if U = \{1\}:
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      s_2 := s_1 \oplus m_R
      return s<sub>1</sub>
  elsif U = \{2\}:
     s_1 \leftarrow \{0,1\}^{\ell}
     s_2 := s_1 \oplus m_L
      return s2
  else return null
```

Recognize s_2 as OTP encryption of m_L .



```
QUERY(m_L, m_R, U):
  if |U| \ge 2: return err
  if U = \{1\}:
     s_1 \leftarrow \{0,1\}^{\ell}
     s_2 := s_1 \oplus m_R
      return s<sub>1</sub>
  elsif U = \{2\}:
     s_2 \leftarrow \text{QUERY}'(m_L, m_R)
      return s2
  else return null
```

```
\frac{\text{QUERY}'(m_L, m_R):}{k \leftarrow \{0, 1\}^{\ell}}
    c := k \oplus m_l
    return c
```

Write it in terms of the "left" OTP security library.



```
QUERY(m_L, m_R, U):
  if |U| \ge 2: return err
  if U = \{1\}:
     s_1 \leftarrow \{0,1\}^{\ell}
      s_2 := s_1 \oplus m_R
      return s<sub>1</sub>
  elsif U = \{2\}:
      s_2 \leftarrow \text{QUERY}'(m_L, m_R)
      return s<sub>2</sub>
  else return null
```

$$\begin{cases}
\mathcal{L}_{\text{ots-L}}^{\text{OTP}} \\
\frac{\text{QUERY}'(m_L, m_R):}{k \leftarrow \{0, 1\}^{\ell}} \\
c := k \oplus m_L \\
\text{return } c
\end{cases}$$

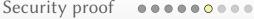
Write it in terms of the "left" OTP security library.



```
QUERY(m_L, m_R, U):
  if |U| \ge 2: return err
  if U = \{1\}:
     s_1 \leftarrow \{0,1\}^{\ell}
      s_2 := s_1 \oplus m_R
      return s<sub>1</sub>
  elsif U = \{2\}:
      s_2 \leftarrow \text{QUERY}'(m_L, m_R)
      return s<sub>2</sub>
  else return null
```

```
QUERY'(m_L, m_R):
  k \leftarrow \{0,1\}^{\ell}
  c := k \oplus m_R
  return c
```

OTP security says we can replace \mathcal{L}_{ots-L} with \mathcal{L}_{ots-R} .



```
QUERY(m_L, m_R, U):
  if |U| \ge 2: return err
  if U = \{1\}:
     s_1 \leftarrow \{0,1\}^{\ell}
      s_2 := s_1 \oplus m_R
      return s<sub>1</sub>
  elsif U = \{2\}:
      s_2 \leftarrow \text{QUERY}'(m_L, m_R)
      return s<sub>2</sub>
  else return null
```

$$\Rightarrow \begin{array}{c}
\mathcal{L}_{\text{ots-R}}^{\text{OTP}} \\
\Rightarrow \\
\downarrow \\
\frac{\text{QUERY}'(m_L, m_R):}{k \leftarrow \{0, 1\}^{\ell}} \\
c := k \oplus m_R \\
\text{return } c
\end{array}$$

OTP security says we can replace \mathcal{L}_{ots-L} with \mathcal{L}_{ots-R} .



```
QUERY(m_L, m_R, U):
  if |U| \ge 2: return err
  if U = \{1\}:
      s_1 \leftarrow \{0,1\}^{\ell}
      s_2 := s_1 \oplus m_R
      return s<sub>1</sub>
  elsif U = \{2\}:
      s_2 \leftarrow \text{QUERY}'(m_L, m_R)
      return s<sub>2</sub>
  else return null
```

$$\Rightarrow \frac{\mathcal{L}_{\text{ots-R}}^{\text{OTP}}}{k \leftarrow \{0,1\}^{\ell}}$$

$$c := k \oplus m_R$$

$$return c$$

Inline the subroutine call.



```
QUERY(m_L, m_R, U):
  if |U| \ge 2: return err
  if U = \{1\}:
     s_1 \leftarrow \{0,1\}^{\ell}
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      return s<sub>1</sub>
  elsif U = \{2\}:
     s_1 \leftarrow \{0,1\}^{\ell}
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      return s2
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Inline the subroutine call.



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Inline the subroutine call.



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  else return null
```

Three branches of if-statement can be unified.



```
QUERY(m_L, m_R, U):
  if |U| \ge 2: return err
  s_1 \leftarrow \{0,1\}^{\ell}
  s_2 := s_1 \oplus m_R
  return (s_i)_{i \in U}
```

Three branches of if-statement can be unified..



QUERY (m_L, m_R, U) :

if $|U| \ge 2$: return err $s_1 \leftarrow \{0,1\}^{\ell}$ $s_2 := s_1 \oplus m_R$ return $(s_i)_{i \in U}$

Three branches of if-statement can be unified..



QUERY (m_L, m_R, U) :

if $|U| \ge 2$: return err $s_1 \leftarrow \{0,1\}^{\ell}$ $s_2 := s_1 \oplus m_R$ return $(s_i)_{i \in U}$

This happens to be $\mathcal{L}_{tsss-R}^{\Sigma}$.



```
\mathcal{L}_{\mathsf{tsss-R}}^{\Sigma}
QUERY(m_L, m_R, U):
   if |U| \ge 2: return err
   s \leftarrow \Sigma.Share(m_R)
   return (s_i)_{i \in U}
```

This happens to be $\mathcal{L}_{tsss-R}^{\Sigma}$.



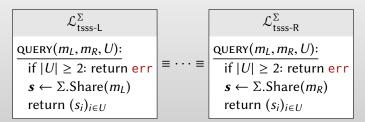
 $\mathcal{L}_{\mathsf{tsss-R}}^{\Sigma}$ QUERY (m_L, m_R, U) : if $|U| \ge 2$: return err

 $s \leftarrow \Sigma$.Share (m_R) return $(s_i)_{i \in U}$

This happens to be $\mathcal{L}_{tsss-R}^{\Sigma}$.

Summary

We showed:



So $\boldsymbol{\Sigma}$ is a secure 2-out-of-2 secret-sharing scheme.

Generalization:

If $\mathcal E$ is any encryption scheme with one-time secrecy, then the following is a secure 2-out-of-2 threshold secret sharing scheme:

```
\mathcal{M} = \underbrace{\mathcal{E}.\mathcal{M}}_{t = 2}
t = 2
n = 2
Share(m):
s_1 \leftarrow \underbrace{\mathcal{E}.KeyGen}_{s_2 := \underbrace{\mathcal{E}.Enc(s_1, m)}_{return}}
return(s_1, s_2)
Reconstruct(s_1, s_2):
return(s_1, s_2)
```