2 The Basics of Provable Security

Until very recently, cryptography seemed doomed to be a cat-and-mouse game. Someone would come up with an encryption method, someone else would find a way to break it, and this process would repeat again and again. Crypto-enthusiast Edgar Allen Poe wrote in 1840,

> *"Human ingenuity cannot concoct a cypher*
> *which human ingenuity cannot resolve."*

With the benefit of 21st-century knowledge, I would argue that Poe's sentiment is not true. The code-makers *can* win against the code-breakers. Modern cryptography is full of schemes that we can **prove** are secure in a very specific sense.

In order to *prove* things about security, we must be very precise about what exactly we mean by "security." Our study revolves around formal *security definitions.* In this chapter, we will learn how to write, understand, and interpret the meaning of a security definition; how to prove security using the technique of *hybrids*; and how to demonstrate insecurity by showing an attack violating a security definition.

2.1 Library-Based Security Reasoning

Motivation

Suppose you write a program $\mathcal{P}$ that calls a SORT subroutine, which is implemented using BubbleSort. Changing the implementation of SORT to use MergeSort might make $\mathcal{P}$ run faster but it will not change anything about what $\mathcal{P}$ actually computes. This is because on every input both BubbleSort and MergeSort will give the same output — they have identical input-output behavior. You could say that the calling program $\mathcal{P}$ can't tell whether it's calling SORT implemented by BubbleSort and QuickSort.[1]

What if the output behavior of a subroutine is *randomized?* In this case, we say that two subroutines have identical input-output behavior if for every input, both generate the *same output distribution.* Consider the following two subroutines:

```
ROLL-D8():
  d ← ℤ_8
  return d
```

```
ROLL-D8():
  a ← {0,1}
  b ← {0,1}
  c ← {0,1}
  return 4a + 2b + c
```

[1] Here we are making an assumption that the SORT subroutine can influence the program's behavior *only* through how it computes its output. One example of how to violate this assumption would be if the program measures how much time the subroutine takes. We discuss this assumption later on.

Both produce a return value that is distributed uniformly in $\mathbb{Z}_8$ (think of the first one as tossing an 8-sided die, and the other one as tossing 3 coins). Hence, the subroutines have identical input-output behavior. As above, replacing one by the other will not change the overall behavior of a program. The calling program can't tell which of the two subroutines is being used.

In both cases, the calling program can call a subroutine but can't tell which of two possible implementations is actually used. **This information is hidden** from the calling program! The most fundamental goal in cryptography is to hide information, which suggests to use the terminology of subroutines, input-output behavior, calling programs, etc. to reason about security.

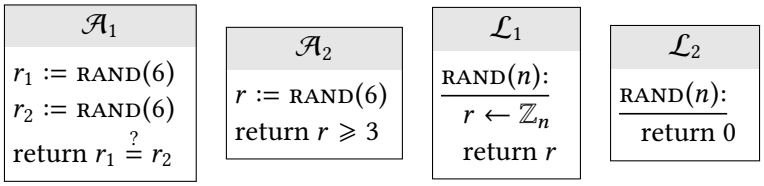
Libraries & Interfaces

The main theme of this chapter is that if two subroutines have identical input-output behavior, then no calling program can tell which one is being used. The choice of subroutine is hidden from the calling program.

We now introduce some terminology and notation for these concepts.

Definition 2.1 (Libraries) *A **library** $\mathcal{L}$ is a collection of subroutines and private/static variables. A library's **interface** consists of the names, argument types, and output type of all of its subroutines. If a program $\mathcal{A}$ includes calls to subroutines in the interface of $\mathcal{L}$, then we write $\mathcal{A} \diamond \mathcal{L}$ to denote the result of **linking** $\mathcal{A}$ to $\mathcal{L}$ in the natural way (answering those subroutine calls using the implementation specified in $\mathcal{L}$). We write $\mathcal{A} \diamond \mathcal{L} \Rightarrow z$ to denote the event that program $\mathcal{A} \diamond \mathcal{L}$ outputs the value z.*

Example *Below are two calling programs $\mathcal{A}_1, \mathcal{A}_2$ and two libraries $\mathcal{L}_1, \mathcal{L}_2$ with a common interface:*



We can consider all 4 ways of linking a calling program to a library:

$\mathcal{A}_1 \diamond \mathcal{L}_1$: *rolls two fair 6-sided dice and checks whether they match. This combined program outputs* `true` *with probability 1/6.*

$\mathcal{A}_1 \diamond \mathcal{L}_2$: *sets r_1 and r_2 to both be 0, so outputs* `true` *with probability 1.*

$\mathcal{A}_2 \diamond \mathcal{L}_1$: *rolls a 6-sided die (with faces 0 through 5) and checks whether the result is at least 3. This combined program outputs* `true` *with probability 1/2.*

$\mathcal{A}_2 \diamond \mathcal{L}_2$: *sets r to 0, so outputs* `true` *with probability 0.*

Example *A library can contain several subroutines and variables that are kept static between subroutine calls. For example, the following library allows the caller to specify a set S and then choose*

randomly from this set without replacement. *Every time a value is chosen from S, it is removed from S:*

$\mathcal{L}$

$S := \emptyset$

RESET(S'):
  $S := S'$

SAMP():
  if $S = \emptyset$ then return err
  $r \leftarrow S$
  $S := S \setminus \{r\}$
  return r

Discussion

- If $\mathcal{A} \diamond \mathcal{L}$ is a program that makes random choices, then its output is also a random variable / probability distribution. It is often useful to consider expressions like $\Pr[\mathcal{A} \diamond \mathcal{L} \Rightarrow z]$, as we did in the previous examples.
- We can consider compound programs like $\mathcal{A} \diamond \mathcal{L}_1 \diamond \mathcal{L}_2$. Our convention is that subroutine calls only happen from left to right across the $\diamond$ symbol, so in this example, $\mathcal{L}_2$ doesn't call subroutines of $\mathcal{A}$. We can then think of $\mathcal{A} \diamond \mathcal{L}_1 \diamond \mathcal{L}_2$ as $(\mathcal{A} \diamond \mathcal{L}_1) \diamond \mathcal{L}_2$ (a compound program linked to $\mathcal{L}_2$) or as $\mathcal{A} \diamond (\mathcal{L}_1 \diamond \mathcal{L}_2)$ ($\mathcal{A}$ linked to a compound library), whichever is convenient.

Semantics & Scope

We will use a pseudocode to specify libraries, and most aspects of that pseudocode will (hopefully) be straight-forward and self-explanitory. But we will make one important assumption about the meaning of these programs & libraries:

> The **only** thing a calling program can do with a library is to call its subroutines (on any arguments of its choice) and receive the output of subroutines.

One important consequence of this is that we assume all variables in a library to be *privately scoped* to the library (*e.g.*, the S variable in the previous example). Calling programs cannot directly access internal variables in a library. If we want a calling program to have access to some internal variables, we must explicitly add a subroutine/accessor to the library.

This is where the analogy to a "real-world software library" breaks down somewhat. In real-world software, when a program is linked to a library there are sneaky ways for the calling program to get information stored in the library beyond just the advertised interface. For example, a calling program might be able to peek into a library's internal memory, or measure the response time of a subroutine call, or see whether some memory access triggers a cache miss / page fault, etc.[2]

[2] In my experience, students who are interested in cryptography are also the most likely to be interested in these kinds of side channels.

In this course, we use the libraries to precisely model what an attacker can do in some situation, and then reason about the consequences. It simply works out best if all the adversary's capabilities are *explicit* in the library. So it's best to think of the libraries more as *mathematical abstractions* than realistic software.

We can still use these libraries to reason about attacks where an adversary has side-channels of information on our cryptographic implementations. The only catch is that if you want to prove something about what an adversary can do in the presence of such a side channel, then that side channel has to be explicit *in the library you're reasoning about*, even if its purpose is to model a channel that is implicit in the real world.

Interchangeability

We have already seen several examples of different libraries that have the same input-output behavior. A library that provides a SORT algorithm might implement it using quicksort or mergesort. A library that provides a ROLL-D8 algorithm might implement it using an 8-sided die or 3 coin flips.

We have also argued that if two libraries have identical input-output behavior, then no calling program will behave differently when linked to one or the other. This turns out to be a convenient way to *define* what it means to have identical input-output behavior, and it works even if the libraries use randomness.

Definition 2.2 (Interchangeable)

*Let $\mathcal{L}_{\text{left}}$ and $\mathcal{L}_{\text{right}}$ be two libraries with a common interface. We say that $\mathcal{L}_{\text{left}}$ and $\mathcal{L}_{\text{right}}$ are **interchangeable**, and write $\mathcal{L}_{\text{left}} \equiv \mathcal{L}_{\text{right}}$, if for all programs $\mathcal{A}$ that output a single bit,* $\Pr[A \diamond \mathcal{L}_{\text{left}} \Rightarrow 1] = \Pr[A \diamond \mathcal{L}_{\text{right}} \Rightarrow 1]$.

Discussion

- We consider calling programs that produce only a single bit of output, which might seem unnecessarily restrictive. However, the definition says that the two libraries have the same effect on *all* calling programs. In particular, the libraries must have the same effect on a calling program $\mathcal{A}$ whose only goal is to **distinguish** between these particular libraries. A single output bit is necessary for this distinguishing task — just interpret the output bit as a "guess" for which library $\mathcal{A}$ thinks it is linked to. For this reason, we will often refer to the calling program $\mathcal{A}$ as a **distinguisher**.

- There is nothing special about defining interchangeability in terms of the calling program giving output 1. Since the only possible outputs are 0 and 1, we have:

$$\begin{aligned} & \Pr[A \diamond \mathcal{L}_{\text{left}} \Rightarrow 1] = \Pr[A \diamond \mathcal{L}_{\text{right}} \Rightarrow 1] \\ \Leftrightarrow \quad & 1 - \Pr[A \diamond \mathcal{L}_{\text{left}} \Rightarrow 1] = 1 - \Pr[A \diamond \mathcal{L}_{\text{right}} \Rightarrow 1] \\ \Leftrightarrow \quad & \Pr[A \diamond \mathcal{L}_{\text{left}} \Rightarrow 0] = \Pr[A \diamond \mathcal{L}_{\text{right}} \Rightarrow 0]. \end{aligned}$$

- It is a common pitfall to imagine the program $\mathcal{A}$ being *simultaneously* linked to both libraries. But in the definition, calling program $\mathcal{A}$ is only ever linked to one of the libraries at a time.

- Taking the previous observation even further, the definition applies against calling programs $\mathcal{A}$ that "know everything" about (more formally, whose code is allowed to depend arbitrarily on) the two libraries. This is a reflection of **Kerckhoffs' principle**, which roughly says "assume that the attacker has full knowledge of the system."[3]

  There is, however, a subtlety that deserves some careful attention, though. Our definitions will typically involve libraries that use internal randomness. Kerckhoffs' principle allows the calling program to know *which libraries* are used, which in this case corresponds to *how* a library will choose randomness (i.e., from which distribution). It doesn't mean that the adversary will know *the result* of the libararies' choice of randomness (i.e., the values of all internal variables in the library). It's the difference between knowing that you will choose a random card from a deck (i.e., the uniform distribution on a set of 52 items) versus reading your mind to know exactly what card you chose.

  This subtlety is reflected in Definition 2.2 in the following way. First, we specify two libaries, *then* we consider a particular distinguisher, and *only then* do we link and execute the distinguisher with a library. The distinguisher cannot depend on the random choices made by the library, since the choice of randomness "happens after" the distinguisher is fixed.

  > **Kerckhoff's Principle, adapted to our terminology:**
  >
  > *Assume that the distinguisher knows every fact in the universe, except for:*
  >
  > 1. *which of the two libraries it is linked to, and*
  > 2. *the outcomes of random choices made by the library (often assigned to privately-scoped variables within the library).*

- The definitions here are general enough that they can apply to many future situations. Our first examples of libraries will be very simple, consisting of just a single subroutine. Later, our discussions of security will require libraries with multiple subroutines and persistent state between different subroutine calls (via static variables).

  Defining interchangeability in terms of distinguishers may not seem entirely natural, but this definition allows us to ease into future concepts as well. Instead of requiring two libraries to have identical input-output behavior, we will eventually consider libraries that are "similar enough." Distinguishers provide a conceptually simple way to measure the similarity between two libraries.

Suppose two libraries $\mathcal{L}_{\text{left}}$ and $\mathcal{L}_{\text{right}}$ are interchangeable: two libraries that are internally different but appear the same to all calling programs. Think of the internal *differences*

[3] *"Il faut qu'il n'exige pas le secret, et qu'il puisse sans inconvénient tomber entre les mains de l'ennemi."* Auguste Kerckhoffs, 1883. Translation: [The method] must not be required to be secret, and it can fall into the enemy's hands without causing inconvenience.

between the two libraries as **information that is perfectly hidden** to the calling program. If the information weren't perfectly hidden, then the calling program could get a whiff of whether it was linked to $\mathcal{L}_{\text{left}}$ or $\mathcal{L}_{\text{right}}$, and use it to act differently in those two cases. But such a calling program would contradict the fact that $\mathcal{L}_{\text{left}} \equiv \mathcal{L}_{\text{right}}$.

This line of reasoning leads to:

> **The Prime Directive of Security Definitions:**
>
> *If two libraries are interchangeable, then their common interface leaks no information about their internal differences.*

We can use this principle to define security, as we will see shortly (and throughout the entire course). It is typical in cryptography to want to hide some sensitive information. To argue that the information is really hidden, we define two libraries with a common interface, which formally specifies what an adversary is allowed to do & learn. The two libraries are typically identical except in the choice of the sensitive information. The Prime Directive tells us that when the resulting two libraries are interchangeable, the sensitive information is indeed hidden when an adversary is allowed to do the things permitted by the libraries' interface.

An example: One-time pad's uniformity property

In Claim 1.3 we proved that a particular subroutine VIEW generates uniformly distributed output, no matter what input is given to it. We can restate that claim in the new terminology of libraries:

Claim 2.3 (OTP rule) *The following two libraries are interchangeable (*i.e., $\mathcal{L}_{\text{otp-real}} \equiv \mathcal{L}_{\text{otp-rand}}$*):*

$\mathcal{L}_{\text{otp-real}}$
QUERY$(m \in \{0,1\}^\lambda)$:
$k \leftarrow \{0,1\}^\lambda$
return $k \oplus m$

$\mathcal{L}_{\text{otp-rand}}$
QUERY$(m \in \{0,1\}^\lambda)$:
$c \leftarrow \{0,1\}^\lambda$
return c

You can tell just by looking that the two libraries $\mathcal{L}_{\text{otp-}\star}$ have the same *interface.* Claim 2.3 says something more specific: the two libraries in fact have the same input-output *behavior.*

2.2 A General-Purpose Security Definition for Encryption

It's important and useful to be able to talk about security definitions that can apply to *any* encryption scheme. With such a general-purpose security definition, we can design a system in a *modular* way, saying "my system is secure as long as the encryption scheme being used has such-and-such property." If concerns arise about a particular choice of encryption scheme, then we can easily swap it out for a different one, thanks to the clear abstraction boundary.

Claim 2.3 is a good security property, but it is rather specific to one-time pad. In this section, we develop a general-purpose security definition for encryption. That means it's time to face the question, *what are we really asking for when we want a "secure encryption method?"*

Syntax of Encryption

Before tackling what it means to be secure, it is helpful to define what it means to be an encryption method in the first place.

Definition 2.4 (Encryption syntax)

*A **symmetric-key encryption (SKE) scheme** consists of the following algorithms:*

- KeyGen*: a randomized algorithm that outputs a **key** $k \in \mathcal{K}$.*
- Enc*: a (possibly randomized) algorithm that takes a key $k \in \mathcal{K}$ and **plaintext** $m \in \mathcal{M}$ as input, and outputs a **ciphertext** $c \in \mathcal{C}$.*
- Dec*: a deterministic algorithm that takes a key $k \in \mathcal{K}$ and ciphertext $c \in \mathcal{C}$ as input, and outputs a plaintext $m \in \mathcal{M}$.*

*We call $\mathcal{K}$ the **key space**, $\mathcal{M}$ the **message space**, and $\mathcal{C}$ the **ciphertext space** of the scheme. When we use a single variable — say, Σ — to refer to the scheme as a whole and distinguish one scheme from another, we write Σ.KeyGen, Σ.Enc, Σ.Dec, $\Sigma.\mathcal{K}$, $\Sigma.\mathcal{M}$, and $\Sigma.\mathcal{C}$ to refer to its components.*

Because the same key is used for encryption and decryption, we refer to this style of encryption scheme as **symmetric-key**. It's also sometimes refered to as *secret-key* or *private-key* encryption; these terms are somewhat confusing because even other styles of encryption involve things that are called secret/private keys.

This definition only says what kind of object an "encryption scheme" is. This information is the **syntax** of encryption.

Correctness

The syntax definition states what algorithms comprise an encryption scheme, but it does not say what behaviors the scheme should have. In particular, the syntax definition allows for KeyGen, Enc, Dec to always output all zeroes, but this would not be a very useful encryption scheme. An encryption scheme is only useful for communication if the receiver can learn the sender's intended plaintext:

Definition 2.5

*An encryption scheme Σ satisfies **correctness** if for all $k \in \Sigma.\mathcal{K}$ and all $m \in \Sigma.\mathcal{M}$,*

$$\Pr[\Sigma.\mathsf{Dec}(k, \Sigma.\mathsf{Enc}(k, m)) = m] = 1.$$

The definition is expressed in terms of a probability, because Enc is allowed to be a randomized algorithm.

Note that correctness has little to do with security. An encryption scheme where $\mathsf{Enc}(k, m) = m$ has no security at all but still satisfies the correctness property. One way to see that the definition has nothing to do with security is to recognize the absence of an adversary in the definition. A security property must refer to something that happens in the presence of an adversary.

One-Time Secrecy

We now develop a general-purpose security definition that makes sense for any encryption scheme. We will be considering an arbitrary, unspecified encryption scheme Σ. Let's first consider a very simplistic scenario in which an eavesdropper sees the encryption of some plaintext. It should make sense that we are considering an eavesdropper who sees encrypted messages — if you are confident that no attacker will ever see your ciphertexts, then what is the point of encrypting? We will start with the following informal idea:

seeing a ciphertext should leak no information about the choice of plaintext.

Our goal is to formalize this property as a statement about interchangeable libraries.

We can work backwards from the Prime Directive. We will show two libraries whose common interface allows the calling program to see a ciphertext, and whose only internal difference was in the choice of plaintext that was encrypted. Saying that these two libraries are interchangeable is equivalent to saying that their common interface (seeing a ciphertext) leaks no information about the internal differences (the choice of plaintext).

The two libraries should look something like:

```
QUERY(??):
  k ← Σ.KeyGen
  c ← Σ.Enc(k, m_L)
  return c
```

and

```
QUERY(??):
  k ← Σ.KeyGen
  c ← Σ.Enc(k, m_R)
  return c
```

.

Indeed, the common interface of these libraries allows the calling program to learn a ciphertext, and the libraries differ only in the choice of plaintext m_L vs. m_R (highlighted). We are getting very close! However, the libraries are still underspecified. Variables m_L and m_R are undefined — where do they come from? Should they be fixed, hard-coded into the libraries? Should the libraries choose them randomly?

A good approach is actually to let the *calling program itself* choose m_L and m_R. Think of this as giving the calling program control over precisely what the difference is between the two libraries. If the libraries are still interchangeable, then seeing a ciphertext leaks no information about the choice of plaintext, *even if you already knew some partial information* about the choice of plaintext, even if you knew that it was one of only two options, even if you got to *choose* those two options!

Putting these ideas together, we obtain the following definition:

Definition 2.6 (One-time secrecy) *Let Σ be an encryption scheme. We say that Σ is **(perfectly) one-time secret** if $\mathcal{L}^{\Sigma}_{\text{ots-L}} \equiv \mathcal{L}^{\Sigma}_{\text{ots-R}}$, where:*

$\mathcal{L}^{\Sigma}_{\text{ots-L}}$

```
QUERY(m_L, m_R ∈ Σ.M):
  k ← Σ.KeyGen
  c ← Σ.Enc(k, m_L)
  return c
```

$\mathcal{L}^{\Sigma}_{\text{ots-R}}$

```
QUERY(m_L, m_R ∈ Σ.M):
  k ← Σ.KeyGen
  c ← Σ.Enc(k, m_R)
  return c
```

This security notion is often called *perfect secrecy* in other sources.[4]

Interpreting/Critiquing a Security Definition

Just because some author writes something down and calls it a security definition, there's no guarantee that it's a *good* definition! In fact, some security definitions are weak, some are strong, some are trivially weak, some are impossibly strong (no scheme can satisfy them), some are a good model of real-world applications of encryption, some are unrealistic. *All* security definitions have limitations.

In short, **security definitions should always be viewed critically.** A security definition is a contract that says, "a cryptographic scheme provides a particular guarantee, when it is used in a very specific context." You might want to use the scheme in a different context than the one in the security definition, in which you might not get *any* security guarantee. Even in the right context, you might want a different guarantee than the one promised by the definition.

Never simply say, *this scheme satisfies a "security definition," so I can safely use it for whatever I want.*

To make things concrete, let's talk about the limitations of one-time secrecy as a security definition. Here are a few things that are *not* covered by the definition:

- The definition assumes that a key is used to encrypt only a single plaintext: each time QUERY is called, a fresh key is chosen as k. Of course, it is *possible* for different calls to QUERY to use the same value k by chance. But there is no way for the calling program to *ensure* that two plaintexts are encrypted under the same key. This means that one-time secrecy gives no guarantee about what happens when a key is (purposefully, consistently — not just by chance) used to encrypt several plaintexts.

- The definition doesn't involve the Dec algorithm of the scheme at all! This fact has many consequences. One consequence is that one-time secrecy does not prevent an adversary from generating valid-looking ciphertexts. That is, an adversary can generate a ciphertext that a receiver would happily decrypt (and presumably act upon). If you want a scheme where the receiver will *detect/reject* ciphertexts that were not generated by the sender (who knows k), then you have to find a security definition that captures that guarantee. It is not a guarantee provided by the one-time security definition.

- The definition assumes that the adversary (calling program) has *no information* about the key (other than its length, which is a publicly-known parameter): the variable k is private and falls out of scope at the end of the call to QUERY. One-time secrecy gives no guarantee when the adversary has some partial information about the key (*e.g.*, the adversary knows that the key has an odd number of 1s).

- The definition does not guarantee that the ciphertext hides *the fact that a plaintext comes from the set* $\mathcal{M}$. In both libraries, the plaintext (m_L or m_R) is an element of $\mathcal{M}$,

[4] Personally, I think that using the term "perfect" leads to an impression that one-time pad should *always* be favored over any other kind of encryption scheme (presumably with only "imperfect" security). But if you want encryption, then you should almost never favor plain old one-time pad.

whereas only the *differences* between the libraries are hidden. More concretely, for one-time pad we have $\mathcal{M} = \{0,1\}^\lambda$. The one-time secrecy definition does not hide the fact that the plaintext is λ bits long, and indeed, you can quite clearly deduce the length of the plaintext from the length of the one-time pad ciphertext. If you want to hide the length of a plaintext, then you need to find a security definition that captures that guarantee. It is not a guarantee provided by the one-time secrecy definition.

- Here is a very subtle issue. The definition assumes that the choice of plaintext does not depend on the key: the calling program's choice of m_L, m_R happens before the library chooses k. There is no way for the calling program to force the library to use the key itself as a plaintext (again, unless it happens by chance). Therefore one-time secrecy gives no guarantee about what happens when users (intentionally) encrypt their own keys. Indeed, encryptions of the key can "look different" than encryptions of other things (see Exercise 1.3), even though the scheme satisfies one-time secrecy.

- The definition doesn't specify *how* the sender and receiver come to know a common key k in the real-world. That problem is considered out of scope for encryption (it is known as *key distribution*). You can think of the problem of (symmetric-key) encryption as how two users can securely communicate once they have established a shared key.

The point of all this is not to pick on one-time secrecy, even though it is a relatively weak security definition. The point is merely to show that every security definition has limitations, and to get you in the habit of interpreting/understanding security definitions in this way. In fact, only the first two limitations in this list are specific to one-time secrecy, while the other limitations are shared by even the strongest security definitions in this book.

2.3 How to Prove Security with the Hybrid Technique

We now have a general-purpose security definition (one-time secrecy) and we know of one encryption scheme (one-time pad). The natural next step is to show that one-time pad satisfies one-time secrecy.

A Useful Chaining Lemma

Before proving the security of one-time pad, we first show a helpful lemma that will be used in essentially every security proof that deals with libraries.

Lemma 2.7 (Chaining) *If $\mathcal{L}_{\text{left}} \equiv \mathcal{L}_{\text{right}}$ then $\mathcal{L}^* \diamond \mathcal{L}_{\text{left}} \equiv \mathcal{L}^* \diamond \mathcal{L}_{\text{right}}$ for any library $\mathcal{L}^*$.*

Proof Take an arbitrary calling program $\mathcal{A}$. We would like to show that $\mathcal{A} \diamond (\mathcal{L}^* \diamond \mathcal{L}_{\text{left}})$ and $\mathcal{A} \diamond (\mathcal{L}^* \diamond \mathcal{L}_{\text{right}})$ have identical output distribution. We can interpret $\mathcal{A} \diamond \mathcal{L}^* \diamond \mathcal{L}_{\text{left}}$ as a calling program $\mathcal{A}$ linked to the library $\mathcal{L}^* \diamond \mathcal{L}_{\text{left}}$, but also as a calling program $\mathcal{A} \diamond \mathcal{L}^*$ linked to the library $\mathcal{L}_{\text{left}}$. After all, $\mathcal{A} \diamond \mathcal{L}^*$ is some program that makes calls to the

interface of $\mathcal{L}_{\text{left}}$. Since $\mathcal{L}_{\text{left}} \equiv \mathcal{L}_{\text{right}}$, swapping $\mathcal{L}_{\text{left}}$ for $\mathcal{L}_{\text{right}}$ has no effect on the output of any calling program. In particular, it has no effect when the calling program happens to be $\mathcal{A} \diamond \mathcal{L}^*$. Hence we have:

$$\begin{aligned}
\Pr[\mathcal{A} \diamond (\mathcal{L}^* \diamond \mathcal{L}_{\text{left}}) \Rightarrow 1] &= \Pr[(\mathcal{A} \diamond \mathcal{L}^*) \diamond \mathcal{L}_{\text{left}} \Rightarrow 1] && \text{(change of perspective)} \\
&= \Pr[(\mathcal{A} \diamond \mathcal{L}^*) \diamond \mathcal{L}_{\text{right}} \Rightarrow 1] && \text{(since } \mathcal{L}_{\text{left}} \equiv \mathcal{L}_{\text{right}}\text{)} \\
&= \Pr[\mathcal{A} \diamond (\mathcal{L}^* \diamond \mathcal{L}_{\text{right}}) \Rightarrow 1]. && \text{(change of perspective)}
\end{aligned}$$

Since $\mathcal{A}$ was arbitrary, we have proved the lemma. ■

One-Time Secrecy of One-Time Pad

Remember that to prove a security property we must show that two libraries are interchangeable.

Theorem 2.8 *Let* OTP *denote the one-time pad encryption scheme (Construction 1.1). Then* OTP *has one-time secrecy. That is,* $\mathcal{L}^{\text{OTP}}_{\text{ots-L}} \equiv \mathcal{L}^{\text{OTP}}_{\text{ots-R}}$.

Given what we already know about one-time pad, it's not out of the question that we could simply "eyeball" the claim $\mathcal{L}^{\text{OTP}}_{\text{ots-L}} \equiv \mathcal{L}^{\text{OTP}}_{\text{ots-R}}$. Indeed, we have already shown that in both libraries the QUERY subroutine simply returns a uniformly random string. A direct proof along these lines is certainly possible.

Instead of directly relating the behavior of the two libraries, however, we will instead show that:

$$\mathcal{L}^{\text{OTP}}_{\text{ots-L}} \equiv \mathcal{L}_{\text{hyb-1}} \equiv \mathcal{L}_{\text{hyb-2}} \equiv \mathcal{L}_{\text{hyb-3}} \equiv \mathcal{L}_{\text{hyb-4}} \equiv \mathcal{L}^{\text{OTP}}_{\text{ots-R}},$$

where $\mathcal{L}_{\text{hyb-1}}, \ldots, \mathcal{L}_{\text{hyb-4}}$ are a sequence of what we call **hybrid** libraries. (It is not hard to see that the "$\equiv$" relation is transitive, so this proves that $\mathcal{L}^{\text{OTP}}_{\text{ots-L}} \equiv \mathcal{L}^{\text{OTP}}_{\text{ots-R}}$.) This proof technique is called the **hybrid technique.**

Again, the hybrid technique is likely overkill for proving the security of one-time pad. The reason for going to all this extra effort is that it is how *all* proofs in this book are done. We use one-time pad as an opportunity to gently introduce the technique. Hybrid proofs have the advantage that it can be quite easy to justify that *adjacent* hybrids (e.g., $\mathcal{L}_{\text{hyb-}i}$ and $\mathcal{L}_{\text{hyb-}(i+1)}$) are interchangeable, so the method scales well even in proofs where the "endpoints" of the hybrid sequence are quite different.

Proof As described above, we will prove that

$$\mathcal{L}^{\text{OTP}}_{\text{ots-L}} \equiv \mathcal{L}_{\text{hyb-1}} \equiv \mathcal{L}_{\text{hyb-2}} \equiv \mathcal{L}_{\text{hyb-3}} \equiv \mathcal{L}_{\text{hyb-4}} \equiv \mathcal{L}^{\text{OTP}}_{\text{ots-R}},$$

for a particular sequence of $\mathcal{L}_{\text{hyb-}i}$ libraries that we choose. For each hybrid, we highlight the differences from the previous one, and argue why adjacent hybrids are interchangeable.

$\mathcal{L}^{\text{OTP}}_{\text{ots-L}}$:

$\mathcal{L}^{\text{OTP}}_{\text{ots-L}}$

QUERY($m_L, m_R \in \{0,1\}^\lambda$):
  $k \leftarrow \{0,1\}^\lambda$
  $c = k \oplus m_L$
  return c

As promised, the hybrid sequence begins with $\mathcal{L}^{\text{OTP}}_{\text{ots-L}}$. The details of one-time pad have been filled in and highlighted.

$\mathcal{L}_{\text{hyb-1}}$:

QUERY($m_L, m_R \in \{0,1\}^\lambda$):
  $c =$ QUERY$'(m_L)$
  return c

$\diamond$

$\mathcal{L}_{\text{otp-real}}$

QUERY$'(m \in \{0,1\}^\lambda)$:
  $k \leftarrow \{0,1\}^\lambda$
  return $k \oplus m$

Factoring out a block of statements into a subroutine makes it possible to write the library as a *compound* one, but does not affect its external behavior. Note that the new subroutine is exactly the $\mathcal{L}_{\text{otp-real}}$ library from Claim 2.3 (with the subroutine name changed to avoid naming conflicts). This is no accident!

$\mathcal{L}_{\text{hyb-2}}$:

QUERY($m_L, m_R \in \{0,1\}^\lambda$):
  $c =$ QUERY$'(m_L)$
  return c

$\diamond$

$\mathcal{L}_{\text{otp-rand}}$

QUERY$'(m \in \{0,1\}^\lambda)$:
  $c \leftarrow \{0,1\}^\lambda$
  return c

$\mathcal{L}_{\text{otp-real}}$ has been replaced with $\mathcal{L}_{\text{otp-rand}}$. From Claim 2.3 along with the chaining lemma Lemma 2.7, this change has no effect on the library's behavior.

$\mathcal{L}_{\text{hyb-3}}$:

QUERY($m_L, m_R \in \{0,1\}^\lambda$):
  $c =$ QUERY$'(m_R)$
  return c

$\diamond$

$\mathcal{L}_{\text{otp-rand}}$

QUERY$'(m \in \{0,1\}^\lambda)$:
  $c \leftarrow \{0,1\}^\lambda$
  return c

The argument to QUERY$'$ has been changed from m_L to m_R. This has no effect on the library's behavior since QUERY$'$ *does not actually use its argument* in these hybrids.

The previous transition is the most important one in the proof, as it gives insight into how we came up with this particular sequence of hybrids. Looking at the desired endpoints of our sequence of hybrids — $\mathcal{L}^{\text{OTP}}_{\text{ots-L}}$ and $\mathcal{L}^{\text{OTP}}_{\text{ots-R}}$ — we see that they differ only in swapping m_L for m_R. If we are not comfortable eyeballing things, we'd like a better justification for why it is "safe" to exchange m_L for m_R. However, the one-time pad rule (Claim 2.3) shows that $\mathcal{L}^{\text{OTP}}_{\text{ots-L}}$ in fact has the same behavior as a library $\mathcal{L}_{\text{hyb-2}}$ that doesn't use either of m_L or m_R. Now, in a program that doesn't use m_L or m_R, it is clear that we can switch them.

Having made this crucial change, we can now perform the same sequence of steps, but in reverse.

$\mathcal{L}_{\text{hyb-4}}$:

```
QUERY(m_L, m_R ∈ {0,1}^λ):
  c = QUERY'(m_R)
  return c
```

◇

$\mathcal{L}_{\text{otp-real}}$

```
QUERY'(m ∈ {0,1}^λ):
  k ← {0,1}^λ
  return k ⊕ m
```

$\mathcal{L}_{\text{otp-rand}}$ has been replaced with $\mathcal{L}_{\text{otp-real}}$. Again, this has no effect on the library's behavior, due to Claim 2.3. Note that the chaining lemma is being applied with respect to a different common $\mathcal{L}^*$ than before (now m_R instead of m_L is being used).

$\mathcal{L}^{\text{OTP}}_{\text{ots-R}}$:

$\mathcal{L}^{\text{OTP}}_{\text{ots-R}}$

```
QUERY(m_L, m_R ∈ {0,1}^λ):
  k ← {0,1}^λ
  c = k ⊕ m_R
  return c
```

A subroutine call has been inlined, which has no effect on the library's behavior. The result is exactly $\mathcal{L}^{\text{OTP}}_{\text{ots-R}}$.

Putting everything together, we showed that $\mathcal{L}^{\text{OTP}}_{\text{ots-L}} \equiv \mathcal{L}_{\text{hyb-1}} \equiv \cdots \equiv \mathcal{L}_{\text{hyb-4}} \equiv \mathcal{L}^{\text{OTP}}_{\text{ots-R}}$. This completes the proof, and we conclude that one-time pad satisfies the definition of one-time secrecy. ■

Summary of the Hybrid Technique

We have now seen our first example of the hybrid technique for security proofs. The example illustrates features that are common to all security proofs used in this course:

- Proving security amounts to showing that two particular libraries, say $\mathcal{L}_{\text{left}}$ and $\mathcal{L}_{\text{right}}$, are interchangeable.
- To show this, we show a sequence of hybrid libraries, beginning with $\mathcal{L}_{\text{left}}$ and ending with $\mathcal{L}_{\text{right}}$. The hybrid sequence corresponds to a sequence of *allowable modifications* to the library. Each modification is small enough that we can easily justify why it doesn't affect the calling program's output probability.
- Simple things like factoring out & inlining subroutines, changing unused variables, consistently renaming variables, removing & changing unreachable statements, or unrolling loops are always "allowable" modifications in a hybrid proof. As we progress in the course, we will add to our toolbox of allowable modifications. For instance, if we want to prove security of some complicated system that uses a one-time-secret encryption Σ as one of its components, then we are allowed to replace $\mathcal{L}^{\Sigma}_{\text{ots-L}}$ with $\mathcal{L}^{\Sigma}_{\text{ots-R}}$ as one of the steps in the hybrid proof.

2.4 How to Demonstrate Insecurity with Attacks

We have seen an example of how to prove that an encryption scheme is secure. To show that a scheme is *insecure*, we just have to show that the two relevant libraries are *not* interchangeable. To show that two libraries are not interchangeable, all we have to do is

show *just one* calling program that behaves differently in the presence of the two libraries! To make the process sound more exciting, we refer to such a demonstration as an **attack**.

Below is an example of an insecure encryption scheme:

Construction 2.9

$\mathcal{K} = \left\{ \begin{array}{c} \textit{permutations} \\ \textit{of } \{1,\ldots,\lambda\} \end{array} \right\}$
$\mathcal{M} = \{0,1\}^\lambda$
$\mathcal{C} = \{0,1\}^\lambda$

KeyGen:
$k \leftarrow \mathcal{K}$
return k

Enc(k,m):
for $i := 1$ to λ:
  $c_{k(i)} := m_i$
return $c_1 \cdots c_\lambda$

Dec(k,c):
for $i := 1$ to λ:
  $m_i := c_{k(i)}$
return $m_1 \cdots m_\lambda$

To encrypt a plaintext m, the scheme simply rearranges its bits according to the permutation k.

Claim 2.10 *Construction 2.9 does **not** have one-time secrecy.*

Proof Our goal is to construct a program $\mathcal{A}$ so that $\Pr[\mathcal{A} \diamond \mathcal{L}^\Sigma_{\text{ots-L}} \Rightarrow 1]$ and $\Pr[\mathcal{A} \diamond \mathcal{L}^\Sigma_{\text{ots-R}} \Rightarrow 1]$ are different, where Σ refers to Construction 2.9. There are probably many "reasons" why this construction is insecure, each of which leads to a different distinguisher $\mathcal{A}$. We need only demonstrate one such $\mathcal{A}$, and it's generally a good habit to try to find one that makes the probabilities $\Pr[\mathcal{A} \diamond \mathcal{L}^\Sigma_{\text{ots-L}} \Rightarrow 1]$ and $\Pr[\mathcal{A} \diamond \mathcal{L}^\Sigma_{\text{ots-R}} \Rightarrow 1]$ as different as possible.

One immediate observation about the construction is that it only rearranges bits of the plaintext, without modifying them. In particular, encryption preserves (leaks) the number of 0s and 1s in the plaintext. By counting the number of 0s and 1s in the ciphertext, we know exactly how many 0s and 1s were in the plaintext. Let's try to leverage this observation to construct an actual distinguisher.

Any distinguisher must use the interface of the $\mathcal{L}_{\text{ots-}\star}$ libraries; in other words, we should expect the distinguisher to call the QUERY subroutine with *some* choice of m_L and m_R, and then do something based on the answer that it gets. If we are the ones writing the distinguisher, we must specify how these arguments m_L and m_R are chosen. Following the observation above, we can choose m_L and m_R to have a different number of 0s and 1s. An extreme example (and why not be extreme?) would be to choose $m_L = 0^\lambda$ and $m_R = 1^\lambda$. By looking at the ciphertext, we can determine which of m_L, m_R was encrypted, and hence which of the two libraries we are currently linked with.

Putting it all together, we define the following distinguisher:

$\mathcal{A}$
$c \leftarrow \text{QUERY}(0^\lambda, 1^\lambda)$
return $c \stackrel{?}{=} 0^\lambda$

Here is what it looks like when $\mathcal{A}$ is linked to $\mathcal{L}^{\Sigma}_{\text{ots-L}}$ (we have filled in the details of Construction 2.9 in $\mathcal{L}^{\Sigma}_{\text{ots-L}}$):

$\mathcal{A}$
$c \leftarrow \text{QUERY}(0^\lambda, 1^\lambda)$
return $c \stackrel{?}{=} 0^\lambda$

$\diamond$

$\mathcal{L}^{\Sigma}_{\text{ots-L}}$
QUERY(m_L, m_R):
$k \leftarrow$ {permutations of $\{1, \ldots, \lambda\}$}
for $i := 1$ to λ:
$c_{k(i)} := (m_L)_i$
return $c_1 \cdots c_\lambda$

.

We can see that m_L takes on the value 0^λ, so each bit of m_L is 0, and each bit of c is 0. Hence, the final output of $\mathcal{A}$ is 1 (true). We have:

$$\Pr[\mathcal{A} \diamond \mathcal{L}^{\Sigma}_{\text{ots-L}} \Rightarrow 1] = 1.$$

Here is what it looks like when $\mathcal{A}$ is linked to $\mathcal{L}^{\Sigma}_{\text{ots-R}}$:

$\mathcal{A}$
$c \leftarrow \text{QUERY}(0^\lambda, 1^\lambda)$
return $c \stackrel{?}{=} 0^\lambda$

$\diamond$

$\mathcal{L}^{\Sigma}_{\text{ots-R}}$
QUERY(m_L, m_R):
$k \leftarrow$ {permutations of $\{1, \ldots, \lambda\}$}
for $i := 1$ to λ:
$c_{k(i)} := (m_R)_i$
return $c_1 \cdots c_\lambda$

.

We can see that each bit of m_R, and hence each bit of c, is 1. So $\mathcal{A}$ will output 0 (false), giving:

$$\Pr[\mathcal{A} \diamond \mathcal{L}^{\Sigma}_{\text{ots-R}} \Rightarrow 1] = 0.$$

The two probabilities are different, demonstrating that $\mathcal{A}$ behaves differently (in fact, as differently as possible) when linked to the two libraries. We conclude that Construction 2.9 does not satisfy the definition of one-time secrecy. ■

Exercises

2.1. Show that the following libraries are interchangeable:

$\mathcal{L}_1$
QUERY$(m \in \{0,1\}^\lambda)$:
$x \leftarrow \{0,1\}^\lambda$
$y := x \oplus m$
return (x, y)

$\mathcal{L}_2$
QUERY$(m \in \{0,1\}^\lambda)$:
$y \leftarrow \{0,1\}^\lambda$
$x := y \oplus m$
return (x, y)

Note that x and y are swapped in the first two lines, but not in the return statement.

2.2. Show that the following libraries are **not** interchangeable. Describe an explicit distinguishing calling program, and compute its output probabilities when linked to both libraries:

$\mathcal{L}_{\text{left}}$
$\underline{\text{QUERY}(m_L, m_R \in \{0,1\}^\lambda)}$:
$k \leftarrow \{0,1\}^\lambda$
$c := k \oplus m_L$
return (k, c)

$\mathcal{L}_{\text{right}}$
$\underline{\text{QUERY}(m_L, m_R \in \{0,1\}^\lambda)}$:
$k \leftarrow \{0,1\}^\lambda$
$c := k \oplus m_R$
return (k, c)

★ 2.3. In abstract algebra, a (finite) **group** is a finite set $\mathbb{G}$ of items together with an operator $\otimes$ satisfying the following axioms:

- **Closure:** for all $a, b \in \mathbb{G}$, we have $a \otimes b \in \mathbb{G}$.
- **Identity:** there is a special *identity element* $e \in \mathbb{G}$ that satisfies $e \otimes a = a \otimes e = a$ for all $a \in \mathbb{G}$. We typically write "1" rather than e for the identity element.
- **Associativity:** for all $a, b, c \in \mathbb{G}$, we have $(a \otimes b) \otimes c = a \otimes (b \otimes c)$.
- **Inverses:** for all $a \in \mathbb{G}$, there exists an *inverse* element $b \in \mathbb{G}$ such that $a \otimes b = b \otimes a$ is the identity element of $\mathbb{G}$. We typically write "a^{-1}" for the inverse of a.

Define the following encryption scheme in terms of an arbitrary *group* $(\mathbb{G}, \otimes)$:

$\mathcal{K} = \mathbb{G}$	$\underline{\text{KeyGen:}}$	$\underline{\text{Enc}(k, m)\text{:}}$	$\underline{\text{Dec}(k, c)\text{:}}$
$\mathcal{M} = \mathbb{G}$	$k \leftarrow \mathbb{G}$	return $k \otimes m$	??
$C = \mathbb{G}$	return k		

(a) Prove that $\{0,1\}^\lambda$ is a group with respect to the XOR operator. What is the identity element, and what is the inverse of a value $x \in \{0,1\}^\lambda$?

(b) Fill in the details of the Dec algorithm and prove (using the group axioms) that the scheme satisfies correctness.

(c) Prove that the scheme satisfies one-time secrecy.

2.4. Suppose we modify one-time pad to add a few 0 bits to the end of every ciphertext:

$\mathcal{K} = \{0,1\}^\lambda$	$\underline{\text{KeyGen:}}$	$\underline{\text{Enc}(k, m)\text{:}}$	$\underline{\text{Dec}(k, c)\text{:}}$
$\mathcal{M} = \{0,1\}^\lambda$	$k \leftarrow \{0,1\}^\lambda$	$c := k \oplus m$	remove last 2 bits of c
$C = \{0,1\}^{\lambda+2}$	return k	return $c \| 00$	$m := k \oplus c$
			return m

(In Enc, "||" refers to concatenation of strings.) Show that the resulting scheme still satisfies one-time secrecy.

2.5. Show that the following encryption scheme does **not** have one-time secrecy, by constructing a program that distinguishes the two relevant libraries from the one-time secrecy definition.

$\mathcal{K} = \{1, \ldots, 9\}$	$\underline{\text{KeyGen:}}$	$\underline{\text{Enc}(k, m)\text{:}}$
$\mathcal{M} = \{1, \ldots, 9\}$	$k \leftarrow \{1, \ldots, 9\}$	return $k \times m \,\%\, 10$
$C = \mathbb{Z}_{10}$	return k	

2.6. Consider the following encryption scheme. It supports plaintexts from $\mathcal{M} = \{0,1\}^\lambda$ and ciphertexts from $C = \{0,1\}^{2\lambda}$. Its keyspace is:

$$\mathcal{K} = \left\{k \in \{0,1,\_\}^{2\lambda} \mid k \text{ contains exactly } \lambda \text{ "\_" characters}\right\}$$

To encrypt plaintext m under key k, we "fill in" the _ characters in k using the bits of m.

Show that the scheme does **not** have one-time secrecy, by constructing a program that distinguishes the two relevant libraries from the one-time secrecy definition.

Example: Below is an example encryption of $m = 1101100001$.

$$\begin{aligned} k &= 1\_\_0\_\_11010\_1\_0\_0\_\_\_ \\ m &= \ \ 11\ 01\ \ \ \ \ 1\ 0\ 0\ 001 \\ \Rightarrow \mathsf{Enc}(k,m) &= 1110011101010110000001 \end{aligned}$$

2.7. Suppose we modify the scheme from the previous problem to first permute the bits of m (as in Construction 2.9) and then use them to fill in the "_" characters in a template string. In other words, the key specifies a random permutation on positions $\{1,\ldots,\lambda\}$ as well as a random template string that is 2λ characters long with λ "_" characters.

Show that even with this modification the scheme does not have one-time secrecy.

2.8. Prove that if an encryption scheme Σ has $|\Sigma.\mathcal{K}| < |\Sigma.\mathcal{M}|$ then it cannot satisfy one-time secrecy. Try to structure your proof as an explicit attack on such a scheme.

2.9. Let Σ denote an encryption scheme where $\Sigma.C \subseteq \Sigma.\mathcal{M}$ (so that it is possible to use the scheme to encrypt its own ciphertexts). Define Σ^2 to be the following **nested-encryption** scheme:

$\mathcal{K} = (\Sigma.\mathcal{K})^2$
$\mathcal{M} = \Sigma.\mathcal{M}$
$C = \Sigma.C$

KeyGen:
$k_1 \leftarrow \Sigma.\mathcal{K}$
$k_2 \leftarrow \Sigma.\mathcal{K}$
return (k_1,k_2)

$\mathsf{Enc}((k_1,k_2),m)$:
$c_1 := \Sigma.\mathsf{Enc}(k_1,m)$
$c_2 := \Sigma.\mathsf{Enc}(k_2,c_1)$
return c_2

$\mathsf{Dec}((k_1,k_2),c_2)$:
$c_1 := \Sigma.\mathsf{Dec}(k_2,c_2)$
$m := \Sigma.\mathsf{Dec}(k_1,c_1)$
return m

Prove that if Σ satisfies one-time secrecy, then so does Σ^2.

2.10. Let Σ denote an encryption scheme and define Σ^2 to be the following **encrypt-twice** scheme:

$\mathcal{K} = (\Sigma.\mathcal{K})^2$
$\mathcal{M} = \Sigma.\mathcal{M}$
$C = \Sigma.C$

KeyGen:
$k_1 \leftarrow \Sigma.\mathcal{K}$
$k_2 \leftarrow \Sigma.\mathcal{K}$
return (k_1,k_2)

$\mathsf{Enc}((k_1,k_2),m)$:
$c_1 := \Sigma.\mathsf{Enc}(k_1,m)$
$c_2 := \Sigma.\mathsf{Enc}(k_2,m)$
return (c_1,c_2)

$\mathsf{Dec}((k_1,k_2),(c_1,c_2))$:
$m_1 := \Sigma.\mathsf{Dec}(k_1,c_1)$
$m_2 := \Sigma.\mathsf{Dec}(k_2,c_2)$
if $m_1 \neq m_2$ return err
return m_1

Prove that if Σ satisfies one-time secrecy, then so does Σ^2.

2.11. Formally define a variant of the one-time secrecy definition in which the calling program can obtain two ciphertexts (on chosen plaintexts) encrypted under the same key. Call it two-time secrecy.

(a) Suppose someone tries to prove that one-time secrecy implies two-time secrecy. Show where the proof appears to break down.

(b) Describe an attack demonstrating that one-time pad does not satisfy your definition of two-time secrecy.

2.12. In this problem we consider modifying one-time pad so that the key is not chosen uniformly. Let $\mathcal{D}_\lambda$ denote the probability distribution over $\{0,1\}^\lambda$ where we choose each bit of the result to be 0 with probability 0.4 and 1 with probability 0.6.

Let Σ denote one-time pad encryption scheme but with the key sampled from distribution $\mathcal{D}_\lambda$ rather than uniformly in $\{0,1\}^\lambda$.

(a) Consider the case of $\lambda = 5$. A calling program $\mathcal{A}$ for the $\mathcal{L}^{\Sigma}_{\text{ots-}\star}$ libraries calls QUERY(01011, 10001) and receives the result 01101. What is the probability that this happens, assuming that $\mathcal{A}$ is linked to $\mathcal{L}_{\text{ots-L}}$? What about when $\mathcal{A}$ is linked to $\mathcal{L}_{\text{ots-R}}$?

(b) Turn this observation into an explicit attack on the one-time secrecy of Σ.