Chain Rule

13.5: Theorem-13.5.1, Theorem-13.5.2, Theorem-13.5.3, Related rates problems, Theorem-13.5.4, Theorem-13.5.5, Other versions of chain rule. Examples:1-8 Exercise Set 13.5: 1-10, 17-34, 41-44, 50-54

1–6 Use an appropriate form of the chain rule to find dz/dt.

1. $z = 3x^2y^3;\ x = t^4,\ y = t^2$
2. $z = \ln(2x^2 + y);\ x = \sqrt{t},\ y = t^{2/3}$
3. $z = 3\cos x - \sin xy;\ x = 1/t,\ y = 3t$
4. $z = \sqrt{1 + x - 2xy^4};\ x = \ln t,\ y = t$
5. $z = e^{1-xy};\ x = t^{1/3},\ y = t^3$
6. $z = \cosh^2 xy;\ x = t/2,\ y = e^t$

7–10 Use an appropriate form of the chain rule to find dw/dt.

7. $w = 5x^2y^3z^4;\ x = t^2,\ y = t^3,\ z = t^5$
8. $w = \ln(3x^2 - 2y + 4z^3);\ x = t^{1/2},\ y = t^{2/3},\ z = t^{-2}$
9. $w = 5\cos xy - \sin xz;\ x = 1/t,\ y = t,\ z = t^3$
10. $w = \sqrt{1 + x - 2yz^4x};\ x = \ln t,\ y = t,\ z = 4t$

17–22 Use appropriate forms of the chain rule to find $\partial z/\partial u$ and $\partial z/\partial v$.

17. $z = 8x^2y - 2x + 3y;\ x = uv,\ y = u - v$
18. $z = x^2 - y\tan x;\ x = u/v,\ y = u^2v^2$
19. $z = x/y;\ x = 2\cos u,\ y = 3\sin v$
20. $z = 3x - 2y;\ x = u + v\ln u,\ y = u^2 - v\ln v$
21. $z = e^{x^2y};\ x = \sqrt{uv},\ y = 1/v$
22. $z = \cos x\sin y;\ x = u - v,\ y = u^2 + v^2$

23–30 Use appropriate forms of the chain rule to find the derivatives.

23. Let $T = x^2y - xy^3 + 2;\ x = r\cos\theta,\ y = r\sin\theta$. Find $\partial T/\partial r$ and $\partial T/\partial\theta$.
24. Let $R = e^{2s-t^2};\ s = 3\phi,\ t = \phi^{1/2}$. Find $dR/d\phi$.
25. Let $t = u/v;\ u = x^2 - y^2,\ v = 4xy^3$. Find $\partial t/\partial x$ and $\partial t/\partial y$.
26. Let $w = rs/(r^2 + s^2);\ r = uv,\ s = u - 2v$. Find $\partial w/\partial u$ and $\partial w/\partial v$.
27. Let $z = \ln(x^2 + 1)$, where $x = r\cos\theta$. Find $\partial z/\partial r$ and $\partial z/\partial\theta$.
28. Let $u = rs^2\ln t,\ r = x^2,\ s = 4y + 1,\ t = xy^3$. Find $\partial u/\partial x$ and $\partial u/\partial y$.
29. Let $w = 4x^2 + 4y^2 + z^2,\ x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi$. Find $\partial w/\partial\rho$, $\partial w/\partial\phi$, and $\partial w/\partial\theta$.
30. Let $w = 3xy^2z^3,\ y = 3x^2 + 2,\ z = \sqrt{x - 1}$. Find dw/dx.
31. Use a chain rule to find the value of $\left.\dfrac{dw}{ds}\right|_{s=1/4}$ if $w = r^2 - r\tan\theta;\ r = \sqrt{s},\ \theta = \pi s$.
32. Use a chain rule to find the values of
$$\left.\frac{\partial f}{\partial u}\right|_{u=1,v=-2} \quad\text{and}\quad \left.\frac{\partial f}{\partial v}\right|_{u=1,v=-2}$$
if $f(x, y) = x^2y^2 - x + 2y;\ x = \sqrt{u},\ y = uv^3$.
33. Use a chain rule to find the values of
$$\left.\frac{\partial z}{\partial r}\right|_{r=2,\theta=\pi/6} \quad\text{and}\quad \left.\frac{\partial z}{\partial\theta}\right|_{r=2,\theta=\pi/6}$$
if $z = xye^{x/y};\ x = r\cos\theta,\ y = r\sin\theta$.
34. Use a chain rule to find $\left.\dfrac{dz}{dt}\right|_{t=3}$ if $z = x^2y;\ x = t^2,\ y = t + 7$.

41–44 Use Theorem 13.5.3 to find dy/dx and check your result using implicit differentiation.

41. $x^2y^3 + \cos y = 0$
42. $x^3 - 3xy^2 + y^3 = 5$
43. $e^{xy} + ye^y = 1$
44. $x - \sqrt{xy} + 3y = 4$

50. (a) Let $z = f(x^2 - y^2)$. Use the result in Exercise 49(a) to show that
$$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0$$
(b) Let $z = f(xy)$. Use the result in Exercise 49(a) to show that
$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 0$$
(c) Confirm the result in part (a) in the case where $z = \sin(x^2 - y^2)$.
(d) Confirm the result in part (b) in the case where $z = e^{xy}$.
51. Let f be a differentiable function of one variable, and let $z = f(x + 2y)$. Show that
$$2\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$$
52. Let f be a differentiable function of one variable, and let $z = f(x^2 + y^2)$. Show that
$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0$$
53. Let f be a differentiable function of one variable, and let $w = f(u)$, where $u = x + 2y + 3z$. Show that
$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = 6\frac{dw}{du}$$
54. Let f be a differentiable function of one variable, and let $w = f(\rho)$, where $\rho = (x^2 + y^2 + z^2)^{1/2}$. Show that
$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 = \left(\frac{dw}{d\rho}\right)^2$$