Compsci 225	The University of Auckland

Assignment 1

Due: Tuesday, August 8th, by 4pm *Semester 2, 2017*

Hand your completed assignment in to the correct box in the Student Resource Centre (G38, Building 301) before the due date. Please use a cover sheet available from the Student Resource Centre. **Show all working**; problems that do not show their work will typically receive reduced or zero marks. Late assignments cannot be marked under any circumstances.

This assignment was written by Padraic Bartlett (`padraic.bartlett@auckland.ac.nz`), and covers chapters 1 and 2 in your coursebook. It has three multi-part problems, plus a pair of bonus questions. Have fun with it!

1. *(Propositional logic and truth tables.)*

   (a) Show that the following two propositions are logically equivalent:
   - $A : \neg((\neg p) \wedge (\neg q))$
   - $B : p \vee q$ [4 marks]

   (b) In the above example, we were able to create something equivalent to the logical connective "or" by combining "and" and "not" together.

   Do this for the operations $p \to q$ and $p \leftrightarrow q$ as well: in other words, using just the two logical connectives $\neg$ and $\wedge$, create compound propositions that are logically equivalent to $p \to q$ and $p \leftrightarrow q$. [8 marks]

   (c) In parts (a) and (b), we showed that using just the connectives $\neg$ and $\wedge$, we could create propositions equivalent to all of the other logical connectives on page 2 of your coursebook.

   Suppose you started with only the two operations $\to$ and $\vee$ instead. Can you create compound propositions equivalent to $\neg p, p \wedge q$ and $p \leftrightarrow q$? Or is it impossible to create one of these propositions with just $\to$ and $\vee$? If you can, do so; if you cannot, explain why (i.e. prove) you cannot. [4 marks]

2. *(Proof techniques.)* You're a programmer! You've found yourself dealing with a program `mystery(n)` that has no comments in its code, and you want to know what it does. After some experimentation, you've found that `mystery(n)` takes in as input a natural number n, and does the following:

   (i) If n is either 0, 1, 2, or 3, output n and stop. Otherwise, go to (ii).

   (ii) If n is even, replace n with $n/2$ and go back to (i). Otherwise, go to (iii).

   (iii) Replace n with $n + 5$ and go to (i).

   Come up with the following proofs about `mystery(n)`:

   (a) Use the contrapositive to prove the claim "if this program outputs 3 on input n, then n is not a power[1] of 2." [4 marks]

   (b) Disprove the claim "Given any natural number n as input, this program will eventually stop" by finding a counterexample. [4 marks]

   (c) Prove by construction the claim "There is some input to this program that causes it to output 0." [4 marks]

   (d) Prove that "the input to this program is 1" and "the output of this program is 1" are equivalent statements (that is, do an $A \iff B$ proof.) [4 marks]

[1] A number n is a power of 2 if there is some integer $k \in \mathbb{Z}$ such that $n = 2^k$.

3. *(Divisibility.)*

   (a) Let n be the year in which you were born. Use the Euclidean algorithm to find the GCD of n and 2017. Show all of your working. [6 marks]

   (b) Let k be an odd number. Prove that $k^2 - 1$ is congruent to 0 modulo 8. [6 marks]

   (c) Let abc be a three-digit positive integer (where c is that number's ones' digit, b is its tens' digit, and a is its hundreds' digit.) Prove that the number $abcabc$ has at least three distinct prime factors. [6 marks]

Possible Marks: 50

Bonus Problems!

These problems are worth no points. Do not hand them in with your normal work! Instead, if you do solve either of them, stop by my office during office hours (MTWThF 10-11) and talk to me about your solution. Chocolate fish will be awarded for correct solutions :D

1. You've caught a leprechaun! He has 761 magical gold coins on him, and offers to play the following game with you:

   (a) At the start, all of his gold coins are in one single heap.

   (b) You can repeatedly perform the following operation: from any heap that contains at least three gold coins, you can remove exactly one gold coin and divide the remaining coins in that heap into two separate heaps. (These heaps do not have to be the same size, but both of them do need to be nonempty.)

   (c) You win if at any point, you can make it so that every heap on the table contains exactly three gold coins! In this case the leprechaun lets you have all of the gold coins.

   (d) You lose if you cannot win. In this case the leprechaun turns you into a newt.

   Should you play this game?

2. Let $g(n)$ denote the function that takes in as input a natural number n, and outputs the sum of the decimal digits of n. For example, $g(746) = 7 + 4 + 6 = 17$. Find

$$g\left(g\left(g\left(4444^{4444}\right)\right)\right).$$

   Hint: consider arithmetic mod 9!