Compsci 225 The University of Auckland

Assignment 3

Due: Tuesday, September 19th, by 4pm *Semester 2, 2017*

Hand your completed assignment in to the correct box in the Student Resource Centre (G38, Building 301) before the due date. Please use a cover sheet available from the Student Resource Centre. **Show all working**; problems that do not show their work will typically receive reduced or zero marks. Late assignments cannot be marked under any circumstances.

This assignment was written by Padraic Bartlett (`padraic.bartlett@auckland.ac.nz`), and covers chapters 5 and 6 in your coursebook (namely, relations and induction.) It has two multi-part problems, plus a pair of bonus questions. Have fun with it!

1. *(Relations.)*

   (a) Here's an exercise we started in class in week 5: consider the set C of people in this class. You can define lots of relations on C: things like "has the same hair color as," or "is taller than," or "went to the same high school as."

   A table is presented below, with an entry for each combination of truth values for the properties reflexive, symmetric, and transitive. Fill in these entries! That is, find a relation on C for each of these combinations. Make sure to explain why each of your claimed relations has the desired property.

refl?	T	T	T	T	F	F	F	F
symm?	T	T	F	F	T	T	F	F
trans?	T	F	T	F	T	F	T	F

   [16 marks]

   (b) Define a relation R on the integers $\mathbb{Z}$ as follows: for any $a, b \in \mathbb{Z}$, we say that aRb if and only if there is some $c \geq 2 \in \mathbb{Z}$ such that $a \equiv b \bmod c$.

   For example, $2R8$ because we can find an integer c such that $2 \equiv 8 \mod c$; namely, we have $2 \equiv 8 \bmod 3$, because $2 - 8 = -6$ is a multiple of 3.

   Prove or disprove: R an equivalence relation. [6 marks]

   (c) In a poset $P = (S, \leq)$, we say that a **chain** is any sequence $x_1, x_2, \ldots x_k$ of distinct elements in S, such that $x_1 \leq x_2 \leq \ldots \leq x_{k-1} \leq x_k$.

   For example, consider the poset $P = (\mathcal{P}(A), \subseteq)$, where $A = \{1, 2, \ldots n\}$. One example of a chain here could be the sequence $\emptyset, \{2,5\}, \{1,2,5\}, \{1,2,3,4,5\}$, because $\emptyset \subseteq \{2,5\} \subseteq \{1,2,5\} \subseteq \{1,2,3,4,5\}$.

   What is the **largest number** of elements you can have in a chain, in the poset $P = (\mathcal{P}(A), \subseteq)$ defined in this problem? Prove that your claim is correct. [6 marks]

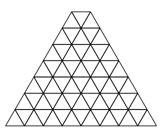
2. *(Induction.)*

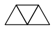
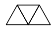
   (a) Come up with a proposition $P(n)$, depending on n, such that $P(1), P(2), \ldots P(999)$ are all true, but $P(1000)$ is false. [2 marks]

   (b) Prove that $17^n - 1$ is divisible by 16, for any $n \geq 1 \in \mathbb{N}$. [4 marks]

   (c) Let $A = \{1, 2, 3 \ldots n\}$. Prove **using induction** that the power set of A, $\mathcal{P}(A)$, contains 2^n elements. [4 marks]

   (d) Take an equilateral triangle with side length 2^n. Divide it up into side-length 1 equilateral triangles, and delete the top triangle. Call this shape T_n:



Take three side-length 1 equilateral triangles. Join them together to form the following tile:  Prove that you can tile[1] T_n with  tiles, for every $n \in \mathbb{N}$. [6 marks]

(e) Consider the following solitaire game:

- Take a piece of paper, and write as many distinct positive integers on it as you want.
- A **move** in this game consists of the following:
  - Pick any positive integer k on the page.
  - Erase it.
  - Write as many new positive integers less than k on the page as you want[2]. The only rule is that you cannot have any repeated positive integers on your page after you do this.

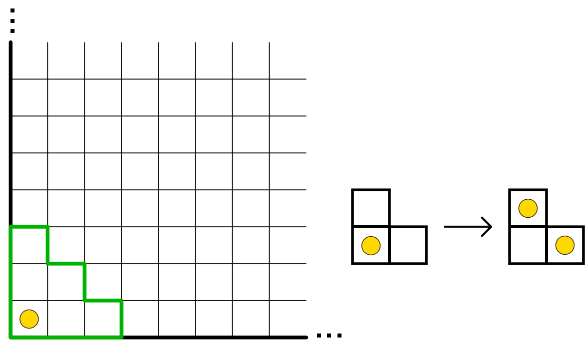
Prove that no matter what you do, this game eventually ends: that is, that you cannot play forever. [6 marks]

Bonus Problems! Possible Marks: 50

This problem is not worth any points. Do not hand it in with your normal work! Instead, if you do solve it, stop by my office during office hours (MTWThF 10-11) and talk to me about your solution. Chocolate fish will be awarded for correct solutions :D

1. Suppose you have a $\mathbb{N} \times \mathbb{N}$ grid of 1×1 squares. Consider the following game you can play on this board:

   - Starting configuration: put one coin on the square in the bottom-left-hand corner of our board.
   - Moves: suppose that there is a coin on the board such that the squares immediately to its north and east are empty. Then a valid move is the following: remove this coin from the board, and then put one new coin on the north square and another new coin on the east square.

   Is it possible to get all of the coins out of the green region? Or will there always be coins in that region, no matter what you do?

[1] That is, completely cover the shape without any overlaps or pieces hanging off. You are allowed to rotate the tiles when placing them.

[2] Writing nothing is a valid play.