Compsci 225	The University of Auckland

Assignment 2

Due: Tuesday, August 22nd, by 4pm *Semester 2, 2017*

Hand your completed assignment in to the correct box in the Student Resource Centre (G38, Building 301) before the due date. Please use a cover sheet available from the Student Resource Centre. **Show all working**; problems that do not show their work will typically receive reduced or zero marks. Late assignments cannot be marked under any circumstances.

This assignment was written by Padraic Bartlett (padraic.bartlett@auckland.ac.nz), and covers chapters 3 and 4 in your coursebook (namely, sets, quantifiers, and graph theory.) It has three multi-part problems, plus a pair of bonus questions. Have fun with it!

1. *(Set operations.)*
   (a) Let A be the set $\{1,2,3\}\times\{0,1,2\}$, and B be the set $\{(x,y) \mid x,y \in \mathbb{N} \text{ and } 0 \leq 2x-y \leq 6\}$. Prove that $A \subseteq B$. Is A equal to B? [4 marks]
   (b) Suppose that A and B are a pair of sets. Prove that the two sets $(A \setminus B) \cup (B \setminus A)$ and $(A \cup B) \setminus (A \cap B)$ are equal. Check this by hand, when A is the set of endemic birds on Tiritiri Matangi whose Māori name contains the letter 'k', and B is the set of endemic birds on Tiritiri Matangi whose scientific name ends with the letter 'a'. [4 marks]
   (c) Take a nonempty set A, and any element $a \in A$. Is it possible that $a \in \mathcal{P}(A)$? Or is this impossible? Either find an example set A where this is possible, or prove that this is impossible. [4 marks]

2. *(Quantifiers.)*
   (a) Without using any words of negation (i.e. without using the words not, no, nor, neither,...), write down a sentence that describes the negation of the following:

   "If a book on my bookshelf has a page with more than fifty words on it, then the first letter of every word on that page is a vowel."

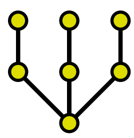
   [2 marks]
   (b) A famous quote of Abraham Lincoln goes as follows:

   "You can fool all the people some of the time, and some of the people all the time, but you cannot fool all the people all the time."

   Let P be the set of all people, T be the set of all times, and $f(p,t)$ a function that takes in a person p and time t, and outputs either "fooled by you" or "not fooled by you." Write this statement using quantifiers and logical/set theory notation. [2 marks]
   (c) Let S be a subset of $\mathbb{R}$. Consider the following two propositions:
   - A: There is a real number M such that for any $x \in S$, we have $|x| \leq M$.
   - B: For any $x \in S$, there is a real number M such that $|x| \leq M$.

   Write both of these statements using quantifiers. Is $A \Rightarrow B$ true for every $S \subseteq \mathbb{R}$? Is $B \Rightarrow A$ true for every $S \subseteq \mathbb{R}$? [4 marks]

3. *(Graph theory.)*
   (a) A sequence $d_1, d_2, \ldots d_n$ of integers is called **graphic** if there is a simple graph G on n vertices, such that the values $d_1, \ldots d_n$ are the degrees of the n vertices of G. For example, the sequence $3,2,2,2,1,1,1$ is graphic, because the following graph has one vertex with degree 3, three with degree 2, and three with degree 1:



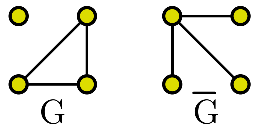
Determine whether any of the following sequences are graphic. If so, find an example graph with those degrees; if not, prove no such graph can exist.

i. $3, 3, 3, 3$ ii. $6, 2, 2, 2$. iii. $5, 3, 3, 2, 2, 2$. iv. $5, 3, 3, 3, 3, 2, 2$.

[12 marks]

(b) Given a graph G, its **complement** is the graph $\overline{G}$ formed as follows:

- The vertices of $\overline{G}$ are the same as the vertices of G.
- We connect two vertices in $\overline{G}$ with an edge if and only if they are not connected by an edge in G.



i. Draw two graphs on at least five vertices, along with their complements [4 marks]

ii. A graph is called **self-complementary** if it is isomorphic to its complement. Find a self-complementary graph that contains at least four vertices. [4 marks]

iii. Is there a self-complementary graph on six vertices? Either find such a graph, or prove that it is impossible. [4 marks]

iv. Prove or disprove: for any graph G, at least one of G or $\overline{G}$ is connected. [6 marks]

$\boxed{\text{Possible Marks: 50}}$

Bonus Problems!

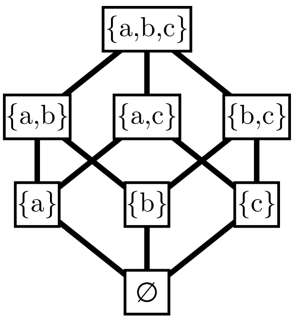
These problems are worth no points. Do not hand them in with your normal work! Instead, if you do solve either of them, stop by my office during office hours (MTWThF 10-11) and talk to me about your solution. Chocolate fish will be awarded for correct solutions :D

The two bonus problems on this set are about the following structure:

Take a set A. Consider the following graph G_A that we can make out of the subsets of A:

- Vertices: the elements of $\mathcal{P}(A)$.
- Edges: given any two subsets $X, Y \in \mathcal{P}(A)$, we draw an edge from X to Y if we can write $X = Y \cup \{a\}$, for some element $a \in A$; in other words, if Y is just the subset X with one more element in it.

For example, if our set was $A = \{a, b, c\}$, then G_A would be the following graph:



You can play a **game** on this graph, as follows: take two players, One and Two, and have them alternate turns with One going first. On a player's turn, they pick any subset that's not yet been removed from the board, and then remove it, along with all of the subsets of that subset that are also on the board. The goal of the game is to not be the player that takes the set A.

1. Let A be a set with five elements. Create the board for this game. Then, beat me at this game in office hours. (Limit one try per person. You may decide whether you go first or second. Computer aid is permissible.)

2. If you want to guarantee that you win this game, should you go first or second?