**Second order differential equation with constant coefficient**

Let

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots \ldots \ldots + a_n y = F(x)$$

where, $a_0, a_1, \ldots, a_n$ are constant coefficients.

Then,

$\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0$ ...........(1)

Let $y = e^{mx}$

$\frac{dy}{dx} = me^{mx}$

$\frac{d^2y}{dx^2} = m^2 e^{mx}$

Substituting these values into (1) we get,

$m^2 e^{mx} + a_1 m e^{mx} + a_2 e^{mx} = 0$

$e^{mx}(m^2 + a_1 m + a_2) = 0 \Rightarrow m^2 + a_1 m + a_2 = 0$

$Say\ m = m_1, m_2$

**Case-1:**

If $m_1, m_2$ are different real roots then the general solution will be,

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

**Case-2:**

If $m_1 = m_2 = m$ then the general solution will be,

$$y = (c_1 + c_2 x)e^{mx}$$

**Case-3:**

If $m_1 = a + ib, m_2 = a - ib$ ; $m = a \pm ib$ then the general solution will be,

$$y = e^{ax}(c_1 \cos bx + c_2 \sin bx)$$

# Solve, $2\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = 0$

Sol: Let $y = e^{mx}$

$$\frac{dy}{dx} = me^{mx}$$

$$\frac{d^2y}{dx^2} = m^2e^{mx}$$

Substituting these values into we get,

$$2m^2 - 3m + 1 = 0 => (2m - 1)(m - 1) = 0$$

$$m_1 = 1\,;\, m_2 = \frac{1}{2}$$

$$y = c_1e^x + c_2e^{x/2}$$

# Solve, $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$

Sol: Let $y = e^{mx}$

$$\frac{dy}{dx} = me^{mx}$$

$$\frac{d^2y}{dx^2} = m^2e^{mx}$$

Substituting these values into we get,

$$m^2 - 4m + 4 = 0$$

$$m = 2$$

General solution is, $y = (c_1 + c_2x)e^{2x}$

# Solve, $(D^2 - 4D + 13)y = 0$

Sol: Given that,

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 0$$

Let $y = e^{mx}$

$$\frac{dy}{dx} = me^{mx}$$

$$\frac{d^2y}{dx^2} = m^2e^{mx}$$

Substituting these values we get,

$$m^2 - 4m + 13 = 0$$

$$m = \frac{-(-4) \pm \sqrt{(-4)^2 - 4.1.13}}{2.1}$$

$$m = 2 \pm 3i$$

The general solution is, $y = e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$

# Solve, $\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + \frac{dy}{dx} + 6y = 0$

Sol:

Let $y = e^{mx}$

$\frac{dy}{dx} = me^{mx}$

$\frac{d^2y}{dx^2} = m^2e^{mx}$

$\frac{d^3y}{dx^3} = m^3e^{mx}$

Substituting these values we get,

$$m^3 - 4m^2 + m + 6 = 0$$

$$=> m^3 + m^2 - 5m^2 - 5m + 6m + 6 = 0$$

$$=> m^2(m+1) - 5m(m+1) + 6(m+1) = 0$$

$$=> (m+1)(m^2 - 5m + 6) = 0$$

$$=> (m+1)(m-2)(m-3) = 0$$

$$m_1 = -1, m_2 = 2, m_3 = 3$$

$$\therefore \textit{The general solution is}, y = c_1e^{-x} + c_2e^{2x} + c_3e^{3x}$$

#Solve, $\frac{d^3y}{dx^3} + 8y = 0$

Sol: Let $y = e^{mx}$

$$\frac{dy}{dx} = me^{mx}$$

$$\frac{d^2y}{dx^2} = m^2e^{mx}$$

$$\frac{d^3y}{dx^3} = m^3e^{mx}$$

Substituting these values we get,

$$m^3 + 8 = 0$$

$$(m+2)(m^2 - 2m + 4) = 0$$

$$m = -2 \;;\; m = \frac{-(-2) \pm \sqrt{(-2)^2 - 4.1.4}}{2.1} = 1 \pm \sqrt{3}i$$

The general solution/ Complementary function ($y_c$) is,

$$y = c_1e^{-2x} + e^x(c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x)$$

#Solve, $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0\,; y(0) = -3, y'(0) = -1$

Sol: Let $y = e^{mx}$

$$\frac{dy}{dx} = me^{mx}$$

$$\frac{d^2y}{dx^2} = m^2e^{mx}$$

$$m = \frac{-(-6) \pm \sqrt{36 - 100}}{2} = \frac{6 \pm 8i}{2} = 3 \pm 4i$$

$y(x) = e^{3x}(c_1 \cos 4x + c_2 \sin 4x)$ general solution/complementary function, $y_c$

$y(0) = -3$

$$=> \; e^{3.0}(c_1 \cos 0 + c_2 \sin 0) = -3 => c_1 = -3$$

$y'(0) = -1$

$y'(x) = e^{3x}(-4c_1 \sin 4x + 4c_2 \cos 4x) + 3e^{3x}(c_1\, cos\, 4x + c_2\, sin\, 4x)$

$y'(0) = -1$

$=> \; e^0(-4.(-3)\, sin\, 0 + 4c_2\, cos\, 0) + 3e^0(-3\, cos\, 0 + c_2\, sin\, 0) = -1$

$=> c_2 = 2$

$\therefore y(x) = e^{3x}(-3 \cos 4x + 2 \sin 4x)$

25. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 12y = 0, \qquad y(0) = 3, \qquad y'(0) = 5.$

26. $\dfrac{d^2y}{dx^2} + 7\dfrac{dy}{dx} + 10y = 0, \qquad y(0) = -4, \qquad y'(0) = 2.$

27. $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 8y = 0, \qquad y(0) = 1, \qquad y'(0) = 6.$

28. $3\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} - 4y = 0, \qquad y(0) = 2, \qquad y'(0) = -4.$

29. $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 9y = 0, \qquad y(0) = 2, \qquad y'(0) = -3.$

30. $4\dfrac{d^2y}{dx^2} - 12\dfrac{dy}{dx} + 9y = 0, \qquad y(0) = 4, \qquad y'(0) = 9.$

31. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 0, \qquad y(0) = 3, \qquad y'(0) = 7.$

32. $9\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + y = 0, \qquad y(0) = 3, \qquad y'(0) = -1.$

33. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 29y = 0, \qquad y(0) = 0, \qquad y'(0) = 5.$