Let any differential equation be,

$$\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = R(x)$$

$$\Rightarrow (D^2 + a_1 D + a_2) y = R(x)$$

$\therefore$ General solution will be,

$$y = \underbrace{y_c}_{\text{Complementary function}} + \underbrace{y_p}_{\text{Particular integral}}$$

Determination of particular integral:

$$\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = R(x)$$

$$\Rightarrow (D^2 + a_1 D + a_2) y = R(x)$$

$$\Rightarrow f(D) y = R(x)$$

$$\therefore y_p = \frac{1}{f(D)} R(x).$$

Formula:

$$(1-D)^{-1} = 1 + D + D^2 + D^3 + \cdots$$

$$(1+D)^{-1} = 1 - D + D^2 - D^3 + \cdots$$

$$(1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \cdots$$

$$(1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \cdots$$

If $R(x) = e^{ax}$ then $D = a$.

Ex-1:

Solve, $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = e^{5x}$ ——— (1)

Sol: Let $y = e^{mx}$

$$\frac{dy}{dx} = me^{mx}$$

$$\frac{d^2y}{dx^2} = m^2e^{mx}$$

For complementary function,

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0 \quad \text{——— (2)}$$

(2) =>

$$m^2e^{mx} - 3me^{mx} + 2e^{mx} = 0$$

$$\Rightarrow m^2 - 3m + 2 = 0$$

$$\Rightarrow m^2 - 2m - m + 2 = 0$$

$\Rightarrow m(m-2) - 1(m-2) = 0$

$\Rightarrow (m-2)(m-1) = 0$

$\therefore m_1 = 2 \; , \; m_2 = 1.$

$\therefore y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$= c_1 e^{2x} + c_2 e^{x}.$

For particular integral we can rewrite the equation ①.

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{5x}$$

$$\Rightarrow (D^2 - 3D + 2)y = e^{5x} \quad \text{——— ③}$$

$$\therefore y_p = \frac{1}{f(D)} R(x)$$

$$= \frac{1}{D^2 - 3D + 2} R(x)$$

$$= \frac{1}{5^2 - 3 \cdot 5 + 2} e^{5x} \quad [\because D = a = 5]$$

$$= \frac{1}{25 - 15 + 2} e^{5x} = \frac{1}{12} e^{5x}.$$

$$\therefore y = y_c + y_p = c_1 e^{2x} + c_2 e^{x} + \frac{1}{12} e^{5x}$$. Ans.

Ex-2:

Solve, $(D^2+2D+2)y = 2e^{-x}$

Solⁿ:

Given,

$$(D^2+2D+2)y = 2e^{-x} \quad \text{——} \quad (1)$$

For complementary function,

$$(D^2+2D+2)y = 0 \quad \text{——} \quad (2)$$

Let $y = e^{mx}$

$$\frac{dy}{dx} = me^{mx}$$

$$\frac{d^2y}{dx^2} = m^2 e^{mx}$$

(2) =>

$$m^2e^{mx} + 2me^{mx} + 2e^{mx} = 0$$

$$\Rightarrow m^2 + 2m + 2 = 0$$

$$\therefore m = \frac{-2 \pm \sqrt{2^2 - 4\cdot 1\cdot 2}}{2\cdot 1}$$

$$= \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm \sqrt{4i^2}}{2}$$

$$= \frac{-2 \pm 2i}{2} = -1 \pm i$$

$\therefore y_c = e^{-x}(c_1 \cos x + c_2 \sin x).$

$\therefore y_p = \frac{1}{f(D)} R(x)$

$= \frac{1}{D^2+2D+2} 2e^{-x}$

$= \frac{1}{(-1)^2+2(-1)+2} 2e^{-x} \quad [\because D = a = -1]$

$= \frac{1}{1} 2e^{-x} = 2e^{-x}$

$\therefore y = y_c + y_p = e^{-x}(c_1 \cos x + c_2 \sin x) + 2e^{-x}.$

Ans.

Ex-3:

Solve, $(D^2 - 2D + 1)y = e^x$

Soln:

Complementary function, $y_c = (c_1 + c_2 x)e^x$

$\therefore y_p = \frac{1}{D^2 - 2D + 1} e^x$

$= \frac{1}{1^2 - 2 \cdot 1 + 1} e^x$

$$= x \frac{1}{f'(D)} e^x$$

$$= x \frac{1}{2D-2} e^x$$

$$= x \frac{1}{2 \cdot 1 - 2} e^x$$

$$= x \cdot x \frac{1}{f''(D)} e^x$$

$$= x^2 \frac{1}{2} e^x = \frac{1}{2} x^2 e^x .$$

$$\therefore \; y = y_c + y_p$$

$$= (c_1 + c_2 x) e^x + \frac{1}{2} x^2 e^x .$$

Ans

Exercise :

i) Solve, $(D^2 + D) y = e^x$

ii) Solve, $(D^2 + 6D + 9) y = e^{-2x}$.

Ex - 4:

Solve, $(D^2 - 4) y = x^2$

Solⁿ:

$$y_c = c_1 e^{-2x} + c_2 e^{2x}$$

$$\therefore y_p = \frac{1}{D^2-4}x^2$$

$$= \frac{1}{-4+D^2}x^2$$

$$= \frac{1}{-4\left(1-\frac{D^2}{4}\right)}x^2$$

$$= \frac{-1}{4}\left(1-\frac{D^2}{4}\right)^{-1}x^2$$

$$= \frac{-1}{4}\left[1+\frac{D^2}{4}+\frac{D^4}{16}+\dots\right]x^2$$

$$= \frac{-1}{4}\left[x^2+\frac{1}{4}D^2(x^2)+\frac{1}{16}D^4(x^2)+\dots\right]$$

$$= \frac{-1}{4}\left[x^2+\frac{1}{4}\cdot 2+\frac{1}{16}\cdot 0\right]$$

$$= \frac{-1}{4}\left[x^2+\frac{1}{2}\right]$$

$$\therefore y_p = \frac{-1}{4}x^2-\frac{1}{8}$$

$$\therefore y = y_c + y_p$$

$$= c_1e^{-2x}+c_2e^{2x}-\frac{1}{4}x^2-\frac{1}{8}.$$ Ans.

Exercise:

iii) Solve, $\frac{d^2y}{dx^2}-9y=3x.$