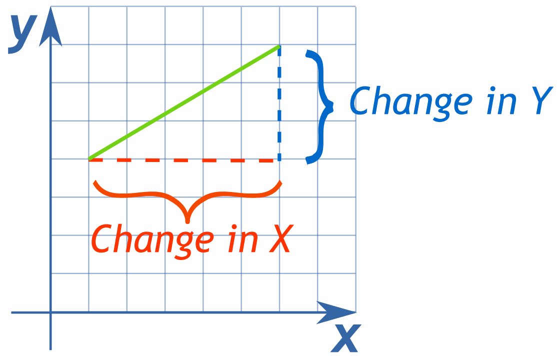
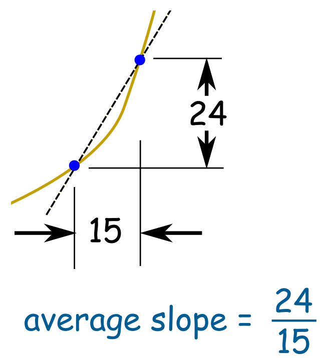
Introduction to Derivatives

It is all about slope!

$$\text{Slope} = \frac{\text{Change in Y}}{\text{Change in X}}$$

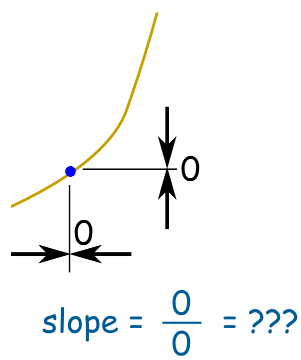


We can find an **average** slope between two points.



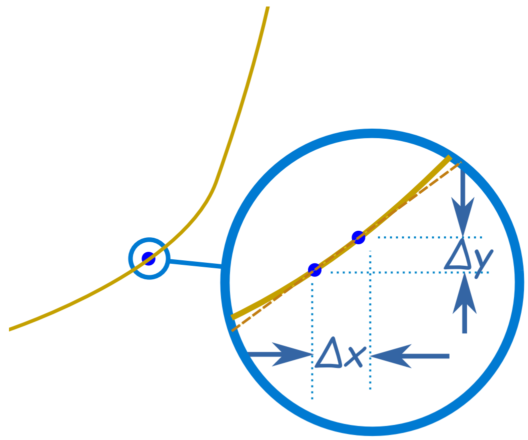
But how do we find the slope **at a point**?

There is nothing to measure!



But with derivatives we use a small difference ...

... then have it **shrink towards zero**.

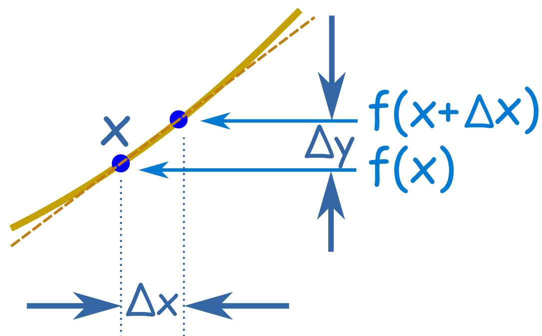


Let us Find a Derivative!

To find the derivative of a function $y = f(x)$ we use the slope formula:

$$\text{Slope} = \frac{\text{Change in Y}}{\text{Change in X}} = \frac{\Delta y}{\Delta x}$$

And (from the diagram) we see that:



x changes from x to $x+\Delta x$

y changes from $f(x)$ to $f(x+\Delta x)$

Now follow these steps:

- Fill in this slope formula: $\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$
- Simplify it as best we can
- Then make $\mathbf{\Delta x}$ shrink towards zero.

Like this:

Example: the function $\mathbf{f(x) = x^2}$

We know $\mathbf{f(x) = x^2}$, and we can calculate $\mathbf{f(x+\Delta x)}$:

Start with: $\mathbf{f(x+\Delta x) = (x+\Delta x)^2}$

Expand $(x + \Delta x)^2$: $\mathbf{f(x+\Delta x) = x^2 + 2x\,\Delta x + (\Delta x)^2}$

The slope formula is: $\dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

Put in **f(x+Δx)** and **f(x)**: $\dfrac{x^2 + 2x\,\Delta x + (\Delta x)^2 - x^2}{\Delta x}$

Simplify (x^2 and $-x^2$ cancel): $\dfrac{2x\,\Delta x + (\Delta x)^2}{\Delta x}$

Simplify more (divide through by Δx): $= 2x + \Delta x$

Then, **as Δx heads towards 0** we get: $= 2x$

Result: the derivative of $\mathbf{x^2}$ is **2x**

In other words, the slope at x is **2x**

We write **dx** instead of **"Δx heads towards 0"**.

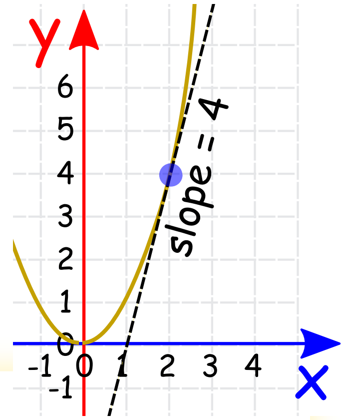
And "the derivative of" is commonly written $\dfrac{d}{dx}$ like this:

$$\frac{d}{dx}x^2 = 2x$$

"The derivative of $\mathbf{x^2}$ *equals* ***2x****"*

or simply *"d dx of* $\mathbf{x^2}$ *equals* ***2x****"*

So what does $\frac{d}{dx}x^2 = 2x$ mean?



It means that, for the function x^2, the slope or "rate of change" at any point is **2x**.

So when **x=2** the slope is **2x = 4**, as shown here:

Or when **x=5** the slope is **2x = 10**, and so on.

Note: $f'(x)$ can also be used for "the derivative of":

$$f'(x) = 2x$$

"The derivative of f(x) equals 2x"
or simply *"f-dash of x equals 2x"*

Let's try another example.

Example: What is $\frac{d}{dx}x^3$?

We know $\mathbf{f(x) = x^3}$, and can calculate $\mathbf{f(x+\Delta x)}$:

Start with: $\mathbf{f(x+\Delta x) = (x+\Delta x)^3}$

Expand $(x + \Delta x)^3$: $\mathbf{f(x+\Delta x) = x^3 + 3x^2\,\Delta x + 3x\,(\Delta x)^2 + (\Delta x)^3}$

The slope formula: $\frac{f(x+\Delta x) - f(x)}{\Delta x}$

Put in **f(x+Δx)** and **f(x)**: $\frac{x^3 + 3x^2\,\Delta x + 3x\,(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x}$

Simplify (x^3 and $-x^3$ cancel):	$\frac{3x^2\ \Delta x + 3x\ (\Delta x)^2 + (\Delta x)^3}{\Delta x}$
Simplify more (divide through by Δx):	$3x^2 + 3x\ \Delta x + (\Delta x)^2$
Then, **as Δx heads towards 0** we get:	$3x^2$

Result: the derivative of $\mathbf{x^3}$ is $\mathbf{3x^2}$

Have a play with it using the Derivative Plotter.

Derivatives of Other Functions

We can use the same method to work out derivatives of other functions (like sine, cosine, logarithms, etc).

But **in practice** the usual way to find derivatives is to use:

Derivative Rules

Example: what is the derivative of sin(x) ?

On Derivative Rules it is listed as being cos(x)

Done.

But using the rules can be tricky!

Example: what is the derivative of cos(x)sin(x) ?

We get a **wrong** answer if we try to multiply the derivative of cos(x) by the derivative of sin(x) ... !

Instead we use the "Product Rule" as explained on the Derivative Rules page.

And it actually works out to be $\cos^2(x) - \sin^2(x)$

So that is your next step: learn how to use the rules.

Notation

"Shrink towards zero" is actually written as a limit like this:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

"The derivative of **f** equals
the limit as Δx goes to zero of f(x+Δx) - f(x) over Δx"

Or sometimes the derivative is written like this (explained on Derivatives as dy/dx):

$$\frac{dy}{dx} = \frac{f(x+dx) - f(x)}{dx}$$

The process of finding a derivative is called "differentiation".

You **do** differentiation ... to **get** a derivative.

Where to Next?

Go and learn how to find derivatives using Derivative Rules, and get plenty of practice:

Question 1 Question 2 Question 3 Question 4 Question 5
Question 6 Question 7 Question 8 Question 9 Question 10