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Recursive Estimation

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Problem Set 5:
Particle Filtering

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Notes:

- **Notation:** Unless otherwise noted, x, y, and z denote random variables, p_x denotes the probability density function of x, and $p_{x|y}$ denotes the conditional probability density function of x conditioned on y. Note that shorthand (as introduced in Lecture 1 and 2) and longhand notation is used. The expected value of x and its variance is denoted by $\mathrm{E}[x]$ and $\mathrm{Var}[x]$, and $\Pr(Z)$ denotes the probability that the event Z occurs. A normally distributed random variable x with mean μ and variance σ^2 is denoted by $x \sim \mathcal{N}(\mu, \sigma^2)$.
- Please report any errors found in this problem set to the teaching assistants (hofermat@ethz.ch or csferrazza@ethz.ch).

Problem Set

Problem 1 (adapted from *D. Simon, Optimal State Estimation, 2006*)

Suppose you have a measurement $z(k) = x(k)^2 + w(k)$, where $w(k)$ has a triangular PDF:

$$p_{w(k)}(\bar{w}(k)) = \begin{cases} \frac{1}{2} + \frac{1}{4}\bar{w}(k) & \text{if } \bar{w}(k) \in [-2, 0] \\ \frac{1}{2} - \frac{1}{4}\bar{w}(k) & \text{if } \bar{w}(k) \in [0, 2] \\ 0 & \text{otherwise}. \end{cases}$$

Suppose that five particles $x_p^n(k)$, $n = 1, 2, 3, 4, 5$, are given as -2, -1, 0, 1, and 2, and that the measurement is obtained as $\bar{z}(k) = 1$. What are the weights β_n of the particles $x_p^n(k)$?

Problem 2 (adapted from *D. Simon, Optimal State Estimation, 2006*)

Suppose that five particles $x_p^n(k)$, $n = 1, 2, 3, 4, 5$, are found to have probabilities β_n of 0.1, 0.1, 0.1, 0.2, and 0.5 given a measurement at time k. The particles are resampled with the basic strategy covered in class, where for a total number of N samples, the following two steps are repeated N times:

– Generate a random number r that is uniformly distributed on $[0, 1]$.

– Pick particle $\bar{n}$ such that $\sum_{n=1}^{\bar{n}} \beta_n \geq r$, $\sum_{n=1}^{\bar{n}-1} \beta_n < r$.

a) What is the probability that the first particle will be chosen at least once during the resampling, i.e. $x_m^n(k) = x_p^1(k)$ for some $n \in \{1, 2, \ldots, 5\}$?

b) What is the probability that the fifth particle will be chosen at least once during the resampling, i.e. $x_m^n(k) = x_p^5(k)$ for some $n \in \{1, 2, \ldots, 5\}$?

c) What is the probability that the five particles $x_m^n(k)$, $n \in \{1, 2, \ldots, 5\}$, will be equal to the five particles $x_p^n(k)$, $n \in \{1, 2, \ldots, 5\}$, (disregarding order)?

Problem 3 (adapted from *D. Simon, Optimal State Estimation, 2006*)

In class, we introduced a roughening procedure that adds to each element $x_{m,i}^n(k)$, $i \in \{1, 2, \ldots, d\}$, of the particle $x_m^n(k) \in \mathbb{R}^d$ a random variable with a standard deviation of $KE_iN^{-1/d}$, where K is a tuning parameter and N is the number of particles. The value E_i represent the maximum difference between the particle elements before roughening, i.e.

$$E_i = \max_{n_1, n_2} \left| x_{m,i}^{n_1}(k) - x_{m,i}^{n_2}(k) \right|.$$

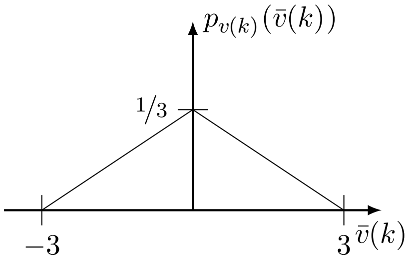
Suppose that you have five particles -1, -1, 0, 1, and 1. You want to use the described roughening procedure and add a uniformly distributed random variable with the given standard deviation. What range of K will give a probability of at least 1/8 that at least one of the roughened particles is less than -2?

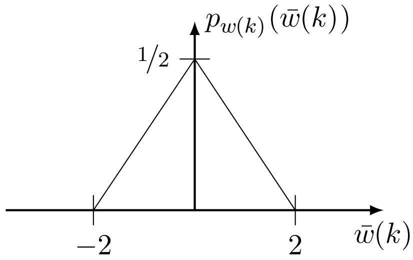
Problem 4 (adapted from Problem 5 of the final exam 2012)

Consider the discrete-time process and measurement model

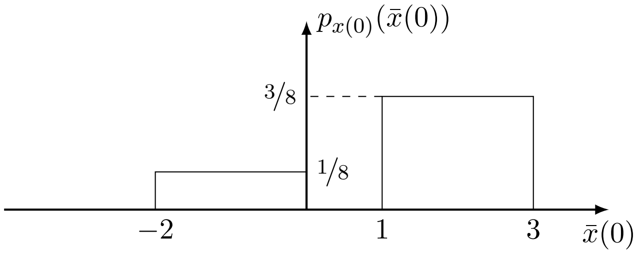
$$x(k) = x^3(k-1) + u(k-1) + v(k-1)$$
$$z(k) = x(k) + w(k)$$

where $x(k)$ is the scalar system state, $u(k)$ is a known control input, and $v(k)$ is process noise. The measurement $z(k)$ is corrupted by the measurement noise $w(k)$. Both $v(k)$ and $w(k)$ have a triangular PDF for all k:





The initial state $x(0)$ has the PDF



The random variables $\{v(\cdot)\}$, $\{w(\cdot)\}$, and $x(0)$ are all mutually independent.

Apply the basic PF algorithm discussed in class and perform the following calculations:

a) Initialize a PF with $N = 2$ particles, $x_m^1(0)$ and $x_m^2(0)$, from the given distribution of $x(0)$. For this calculation, you have access to a random number generator that generates mutually independent random samples r from a uniform distribution on the interval $[0, 1]$. For the calculation of $x_m^1(0)$, you obtained the sample $r_1 = 1/16$. For the calculation of $x_m^2(0)$, you obtained the sample $r_2 = 13/16$.

b) Step 1 of the PF: At $k = 2$, let $\bar{x}_m^1(2) = 1$ and $\bar{x}_m^2(2) = -1$ be the posterior particles. The control input is $u(2) = 2$. Calculate the prior particles at time $k = 3$, $x_p^1(3)$ and $x_p^2(3)$. You have access to the same random number generator as in part a). For the calculation of $x_p^1(3)$, you obtained the sample $r_1 = 7/8$. For the calculation of $x_p^2(3)$, you obtained the sample $r_2 = 2/9$.

c) Step 2 of the PF: At time $k = 7$, the prior particles are $\bar{x}_p^1(7) = -2$ and $\bar{x}_p^2(7) = -\frac{5}{2}$. The measurement is $\bar{z}(7) = -3$. Calculate the normalized particle weights β_1 for $x_p^1(7)$ and β_2 for $x_p^2(7)$. Calculate the posterior particles $x_m^1(7)$ and $x_m^2(7)$. Do not apply any roughening. You have access to the same random number generator as in part a). You obtained $r_1 = 3/8$ and $r_2 = 1/4$.

Problem 5

In this problem, you implement a PF for tracking the location of a person who got lost in a hilly terrain, which is shown in Figure 1. You know that the person is hiking roughly in the direction of the steepest descent, and you further roughly know where the person initially was.

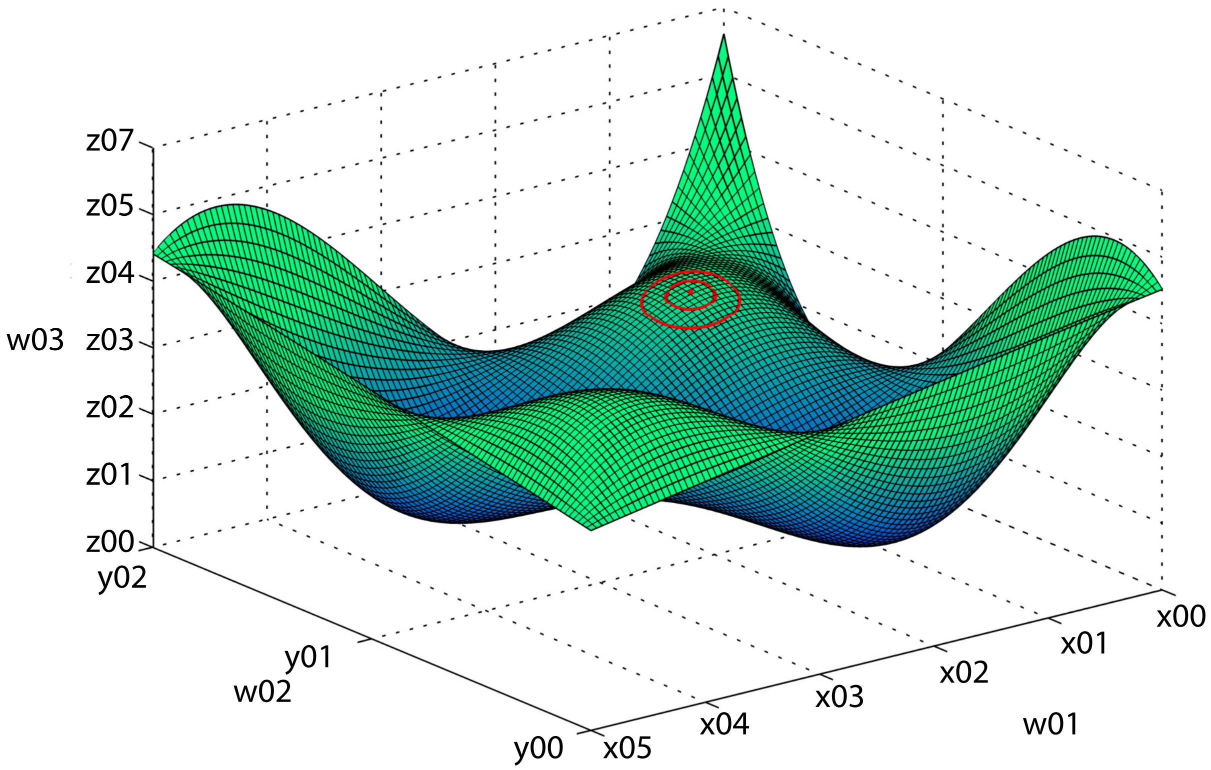


Figure 1: Plot of terrain. The red dot represents the mean starting location x_0 of the person, and the two circles represent one and two standard deviations σ_x of the Gaussian distribution of the initial location, respectively.

The coordinates of the person are given by x and y (both in kilometers). The altitude of the person is given by the terrain function $a = h(x, y)$ that maps the coordinates to an altitude a (in kilometers). Let $s(k) := (x(k), y(k))$ be the state of the person at discrete time k. The distribution of the initial state $s(0)$ is modeled as a Gaussian distribution with mean s_0 and variance P_0.

The simplified, discrete-time, time-invariant dynamics of the person in the terrain are

$$s(k) = q(s(k-1), v(k-1)) := s(k-1) - \theta \, \frac{\nabla h(x(k-1), y(k-1))}{\|\nabla h(x(k-1), y(k-1))\|} + v(k-1)$$

where θ is the constant horizontal discrete-time velocity of the person (in kilometers per time step); the process noise $v(k-1)$ is modeled as Gaussian distributed with zero mean and variance Q; $\nabla h(x(k-1), y(k-1))$ is the gradient

$$\nabla h(x(k-1), y(k-1)) := \begin{bmatrix} \frac{\partial h}{\partial x}(x(k-1), y(k-1)) \\ \frac{\partial h}{\partial y}(x(k-1), y(k-1)) \end{bmatrix}$$

and $\|\nabla h(x(k-1), y(k-1))\|$ is the Euclidean norm of the gradient vector.

At every time step $k > 1$ you receive reports of the current altitude of the person

$$z(k) = h(x(k), y(k)) + w(k)$$

which are corrupted by measurement noise $w(k)$ that has a Gaussian distribution with zero mean and variance R.

The function $h(x, y)$ and its gradient, as well as all numerical values for the relevant parameters stated above are defined in the Matlab template `RE_ProblemSet5Problem5_Template.m` available on the class website. Use this template and follow the instructions therein in order to implement the following problem parts.

a) Implement a simulation of the person's motion.

b) Implement the initialization of a PF for this problem using the given initial state distribution. Further implement the prior update of the PF and run your filter ignoring the measurements. Explore what happens if you choose smaller ($N \approx 5$) and larger ($N \approx 100$) numbers N of particles.

c) Implement the posterior update of the PF. Tune the number of particles by inspecting the tracking performance of the PF in the animation.

d) Add roughening to your implementation of the PF using the method shown in the lecture and tune the parameter K inspecting again the animation of the person and particles. Does the performance of the PF improve? Why? Why not?

Sample solutions

Problem 1

The weights β_n are given by

$$\beta_n = \alpha \, p_{z(k)|x_p(k)}\left(\bar{z}(k)|\bar{x}_p^n(k)\right).$$

With the measurement equation $z(k) = x(k)^2 + w(k)$, we obtain

$$p_{z(k)|x_p(k)}\left(\bar{z}(k)|\bar{x}_p^n(k)\right) = p_{w(k)}\left(\bar{z}(k) - \left(\bar{x}_p^n(k)\right)^2\right).$$

Given that $\bar{z}(k) = 1$, we obtain for

$$\begin{aligned}
n = 1: &\quad p_{z(k)|x_p(k)}(1|-2) &&= p_{w(k)}\left(1-(-2)^2\right) &&= 0\\
n = 2: &\quad p_{z(k)|x_p(k)}(1|-1) &&= p_{w(k)}\left(1-(-1)^2\right) &&= 1/2\\
n = 3: &\quad p_{z(k)|x_p(k)}(1|0) &&= p_{w(k)}\left(1-0^2\right) &&= 1/4\\
n = 4: &\quad p_{z(k)|x_p(k)}(1|1) &&= p_{w(k)}\left(1-1^2\right) &&= 1/2\\
n = 5: &\quad p_{z(k)|x_p(k)}(1|2) &&= p_{w(k)}\left(1-2^2\right) &&= 0
\end{aligned}$$

From this we can calculate α,

$$\alpha = \left(\sum_{n=1}^{5} p_{z(k)|x_p(k)}\left(1|\bar{x}_p^n(k)\right)\right)^{-1} = \frac{4}{5}.$$

The factors β_n are then

$$\beta_1 = 0, \quad \beta_2 = \frac{2}{5}, \quad \beta_3 = \frac{1}{5}, \quad \beta_4 = \frac{2}{5}, \quad \beta_5 = 0.$$

Problem 2

a) Pr (first particle is chosen at least once)

$$\begin{aligned}
&= 1 - \Pr\,(\text{first particle is never chosen})\\
&= 1 - 0.9^5\\
&\approx 0.4095 = 40.95\%
\end{aligned}$$

b) Pr (fifth particle is chosen at least once)

$$\begin{aligned}
&= 1 - 0.5^5\\
&\approx 0.9688 = 96.88\%
\end{aligned}$$

c) $\Pr\left(\text{particles } \{x_m^n(k)\} \text{ equal particles } \{x_p^n(k)\}\right)$

$$\begin{aligned}
&= 0.1 \cdot 0.1 \cdot 0.1 \cdot 0.2 \cdot 0.5 \cdot 5! \qquad (5! = \text{all possible combinations})\\
&= 0.012 = 1.2\%
\end{aligned}$$

Problem 3

Here we have $d = 1$, $N = 5$ and $E_1 = 2$. According to the given formula, the standard deviation is

$$\sigma_1 = \frac{2K}{5}.$$

For a uniform distribution on the interval $[-\bar{s}, \bar{s}]$, as shown in Fig. 2, the standard deviation is

$$\sigma = \frac{1}{\sqrt{3}}\bar{s}.$$

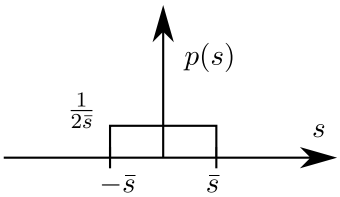


Figure 2: Uniform distribution.

Therefore,

$$\bar{s} = \frac{2\sqrt{3}}{5}K.$$

The particles after roughening, denoted by ${}^{r}x_m^n(k)$, are obtained from

$${}^{r}x_m^n(k) = x_m^n(k) + \Delta x^n(k),$$

where $\Delta x^n(k)$ is uniformly distributed. For the given particles,

$$\bar{x}_m^1(k) = -1, \quad \bar{x}_m^2(k) = -1, \quad \bar{x}_m^3(k) = 0, \quad \bar{x}_m^4(k) = 1, \quad \bar{x}_m^5(k) = 1,$$

the probabilities are

$$\begin{aligned}
\Pr\left({}^{r}x_m^1(k) < -2\right) &= \Pr\left(\Delta x^1(k) < -1\right) = \max\left(0, \int_{-\bar{s}}^{-1} \frac{1}{2\bar{s}}\, ds\right) = \max\left(0, \frac{\bar{s}-1}{2\bar{s}}\right)\\
\Pr\left({}^{r}x_m^2(k) < -2\right) &= \max\left(0, \frac{\bar{s}-1}{2\bar{s}}\right)\\
\Pr\left({}^{r}x_m^3(k) < -2\right) &= \max\left(0, \frac{\bar{s}-2}{2\bar{s}}\right)\\
\Pr\left({}^{r}x_m^4(k) < -2\right) &= \max\left(0, \frac{\bar{s}-3}{2\bar{s}}\right)\\
\Pr\left({}^{r}x_m^5(k) < -2\right) &= \max\left(0, \frac{\bar{s}-3}{2\bar{s}}\right).
\end{aligned}$$

We consider four different cases:

(1) $\bar{s} \leq 1$

(2) $1 < \bar{s} \leq 2$

(3) $2 < \bar{s} \leq 3$

(4) $\bar{s} > 3$.

We need to calculate

$$\begin{aligned}\Pr(\text{at least one } < -2) &= 1 - \prod_{n=1}^{5} \Pr\left({}^r x_m^n(k) \geq -2\right) \\ &= 1 - \prod_{n=1}^{5} \left(1 - \Pr\left({}^r x_m^n(k) < -2\right)\right).\end{aligned}$$

We do this for each case:

(1) $\Pr(\text{at least one } < -2) = 0$

(2) $\Pr(\text{at least one } < -2) = 1 - \left(1 - \frac{\bar{s}-1}{2\bar{s}}\right)^2 \quad \geq \frac{1}{8}$.
We solve the quadratic equation, and use the fact that $\bar{s} \in [1, 2]$:

$$\bar{s} \geq \frac{2 + \sqrt{14}}{5} \approx 1.15$$

(3) $\Pr(\text{at least one } < -2) = 1 - \left(1 - \frac{\bar{s}-1}{2\bar{s}}\right)^2 \left(1 - \frac{\bar{s}-2}{2\bar{s}}\right)$.
For $\bar{s} \in [2, 3]$, this is always larger than $\frac{1}{8}$.

(4) $\Pr(\text{at least one } < -2) = 1 - \left(1 - \frac{\bar{s}-1}{2\bar{s}}\right)^2 \left(1 - \frac{\bar{s}-2}{2\bar{s}}\right) \left(1 - \frac{\bar{s}-3}{2\bar{s}}\right)$.
For $\bar{s} > 3$, this is always larger than $\frac{1}{8}$.

To sum up:

For

$$\begin{aligned}\bar{s} \;\geq\; \frac{2+\sqrt{14}}{5} \qquad &\Leftrightarrow \quad \frac{2\sqrt{3}}{5}K \;\geq\; \frac{2+\sqrt{14}}{5} \\ &\Leftrightarrow \quad K \;\geq\; \frac{2+\sqrt{14}}{2\sqrt{3}} \quad \approx 1.66,\end{aligned}$$

the probability that at least one roughened particle is less than -2 is at least $\frac{1}{8}$.

Problem 4

a) We use the "sampling a distribution" algorithm for a CRV from Lecture 2 in order to draw samples from $f(x(0))$ for the two particles. We calculate the cumulative distribution function (CDF) for $x(0)$

$$F_{x(0)}(\bar{x}(0)) = \int_{-\infty}^{\bar{x}(0)} p_{x(0)}(\eta)\, d\eta = \begin{cases} 0 & \text{if } \bar{x}(0) < -2 \\ \frac{1}{8}(2 + \bar{x}(0)) & \text{if } -2 \leq \bar{x}(0) < 0 \\ \frac{1}{4} & \text{if } 0 \leq \bar{x}(0) < 1 \\ \frac{1}{4} + \frac{3}{8}(\bar{x}(0) - 1) & \text{if } 1 \leq \bar{x}(0) \leq 3 \\ 1 & \text{if } \bar{x}(0) > 3. \end{cases}$$

The first number obtained from the random number generator is $r_1 = 1/16$. Therefore, we find the first state sample $x_m^1(0)$ by solving

$$F_{x(0)}(\bar{x}_m^1(0)) = \frac{1}{16}$$

for $\bar{x}_m^1(0)$. The CDF is piece-wise linear on the intervals of $x(0)$ defined above. We first find the relevant interval of $x(0)$ for the given r_1. The interval is $-2 \leq x(0) \leq 0$ since $0 \leq r_1 \leq 1/4$ (find 0 and 1/4 by plugging in the interval bounds -2 and 0 into the respective functions of the CDF). We then solve

$$\frac{1}{16} = \frac{1}{8}(\bar{x}_m^1(0) + 2)$$

for $\bar{x}_m^1(0)$ and find $\bar{x}_m^1(0) = -3/2$. Analogously, we find $\bar{x}_m^2(0) = 5/2$ for $r_2 = 13/16$.

b) We start by calculating the process noise samples for the prior update of the two particles. The CDF of the process noise distribution is:

$$F_v(\bar{v}) = \int_{-\infty}^{\bar{v}} p_v(\eta)\, d\eta = \begin{cases} 0 & \text{if } \bar{v} < -3 \\ \frac{(\bar{v}+3)^2}{18} & \text{if } -3 \leq \bar{v} < 0 \\ \frac{1}{2} + \frac{1}{3}\bar{v} - \frac{\bar{v}^2}{18} & \text{if } 0 \leq \bar{v} \leq 3 \\ 1 & \text{if } \bar{v} > 3. \end{cases}$$

The first number obtained from the random number generator is $r_1 = 7/8$. Therefore, we find the sample $v^1(2)$ for the prior update of the first particle by solving

$$F_v(\bar{v}^1(2)) = \frac{7}{8}$$

for $\bar{v}^1(2)$. The CDF is piece-wise quadratic on the intervals of v defined above and we first seek the relevant interval of v for the given r_1. The interval is $0 \leq v \leq 3$ since $1/2 \leq r_1 \leq 1$. We then solve

$$\frac{7}{8} = \frac{1}{2} + \frac{1}{3}\bar{v}^1(2) - \frac{(\bar{v}^1(2))^2}{18}$$

for $\bar{v}^1(2)$ and find the two solutions

$$\bar{v}_{1,2}^1(2) = 3 \pm \frac{3}{2}.$$

Since for $r_1 = 7/8$, $\bar{v}^1(2) \in [0, 3]$, we find $\bar{v}^1(2) = 3/2$. Analogously, we find $\bar{v}^2(2) = -1$ from $r_2 = \frac{2}{9}$. Finally

$$\begin{aligned} \bar{x}_p^1(3) &= (\bar{x}_m^1(2))^3 + u(2) + \bar{v}^1(2) = (1)^3 + 2 + \frac{3}{2} = \frac{9}{2} \\ \bar{x}_p^2(3) &= (\bar{x}_m^2(2))^3 + u(2) + \bar{v}^2(2) = (-1)^3 + 2 - 1 = 0. \end{aligned}$$

c) The measurement is $\bar{z}(7) = -3$. We obtain the measurement likelihood $p(z|x)$ from a change of variables:

$$p_{z|x}(\bar{z}|\bar{x}) = p_w(\bar{z} - \bar{x})\,.$$

Therefore, the measurement likelihoods are

$$\begin{aligned} &p_{z(7)|x(7)}(-3| - 2) = p_w(-3 + 2) = \frac{1}{4}, \quad (\bar{x}_p^1(7) = -2), \\ &p_{z(7)|x(7)}(-3| - 5/2) = p_w(-3 + 5/2) = \frac{3}{8}, \quad (\bar{x}_p^2(7) = -5/2). \end{aligned}$$

It follows that the normalized particle weights are

$$\beta_1 = \frac{\frac{1}{4}}{\frac{1}{4} + \frac{3}{8}} = \frac{2}{5}$$
$$\beta_2 = \frac{3}{5}.$$

We then apply the resampling algorithm discussed in class and obtain

$$\bar{x}_m^1(7) = -2$$
$$\bar{x}_m^2(7) = -2.$$

since β_1 is larger than both random samples $r_1 = 3/8$ and $r_2 = 1/4$.

Problem 5

You can download a sample implementation of the PF for all the above parts from the class website (`RE_ProblemSet5Problem5_Solution.m`). Follow the instructions in this file to compare your implementation to the solution.

Observations and comments to selected problem parts:

b) When you use too few particles, it may happen that all particles end up in the wrong local minimum (i.e. valley). With larger numbers this is less likely to happen: the approximation of the person's PDF without considering measurements is better. The resulting PDF is bi-modal, as expected due to the two local minima.

d) In part c), we can observe how sometimes all particles end up in the wrong local minimum. Roughening can reduce this problem since it causes the particles to be spread out further in addition to the similar effect of the process noise. This results in the particles "exploring" a larger area of possible locations, which, in this specific example, increases the PF performance.