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Recursive Estimation

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Problem Set 2:
Bayes' Theorem and Bayesian Tracking

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Notes:

- **Notation:** Unless otherwise noted, x, y, and z denote random variables, p_x denotes the probability density function of x, and $p_{x|y}$ denotes the conditional probability density function of x conditioned on y. Note that shorthand (as introduced in Lecture 1 and 2) and longhand notation is used. The expected value of x and its variance is denoted by $\mathrm{E}[x]$ and $\mathrm{Var}[x]$, and $\Pr(Z)$ denotes the probability that the event Z occurs. A normally distributed random variable x with mean μ and variance σ^2 is denoted by $x \sim \mathcal{N}(\mu, \sigma^2)$.

- Please report any errors found in this problem set to the teaching assistants (hofermat@ethz.ch or csferrazza@ethz.ch).

Problem Set

Problem 1

Mr. Jones has devised a gambling system for winning at roulette. When he bets, he bets on red, and places a bet only when the ten previous spins of the roulette have landed on a black number. He reasons that his chance of winning is quite large since the probability of eleven consecutive spins resulting in black is quite small. What do you think of this system?

Problem 2

Consider two boxes, one containing one black and one white marble, the other, two black and one white marble. A box is selected at random with equal probability and a marble is drawn at random with equal probability from the selected box. What is the probability that the marble is black?

Problem 3

In Problem 2, what is the probability that the first box was the one selected given that the marble is white?

Problem 4

Urn 1 contains two white balls and one black ball, while urn 2 contains one white ball and five black balls. One ball is drawn at random with equal probability from urn 1 and placed in urn 2. A ball is then drawn from urn 2 at random with equal probability. It happens to be white. What is the probability that the transferred ball was white?

Problem 5

Stores A, B and C have 50, 75, 100 employees, and respectively 50, 60 and 70 percent of these are women. Resignations are equally likely among all employees, regardless of sex. One employee resigns and this is a woman. What is the probability that she works in store C?

Problem 6

a) A gambler has in his pocket a fair coin and a two-headed coin. He selects one of the coins at random with equal probability, and when he flips it, it shows heads. What is the probability that it is the fair coin?

b) Suppose that he flips the same coin a second time and again it shows heads. What is now the probability that it is the fair coin?

c) Suppose that he flips the same coin a third time and it shows tails. What is now the probability that it is the fair coin?

Problem 7

Urn 1 has five white and seven black balls. Urn 2 has three white and twelve black balls. An urn is selected at random with equal probability and a ball is drawn at random with equal probability from that urn. Suppose that a white ball is drawn. What is the probability that the second urn was selected?

Problem 8

An urn contains b black balls and r red balls. One of the balls is drawn at random with equal probability, but when it is put back into the urn c additional balls of the same color are put in with it. Now suppose that we draw another ball at random with equal probability. Show that the probability that the first ball drawn was black, given that the second ball drawn was red is $b/(b+r+c)$.

Problem 9

Three prisoners are informed by their jailer that one of them has been chosen to be executed at random with equal probability, and the other two are to be freed. Prisoner A asks the jailer to tell him privately which of his fellow prisoners will be set free, claiming that there would be no harm in divulging this information, since he already knows that at least one will go free. The jailer refuses to answer this question, pointing out that if A knew which of his fellows were to be set free, then his own probability of being executed would rise from 1/3 to 1/2, since he would then be one of two prisoners. What do you think of the jailer's reasoning?

Problem 10

Let x and y be independent random variables. Let $g(\cdot)$ and $h(\cdot)$ be arbitrary functions of x and y, respectively. Define the random variables $v = g(x)$ and $w = h(y)$. Prove that v and w are independent. That is, functions of independent random variables are independent.

Problem 11

Let x be a continuous, uniformly distributed random variable with $x \in \mathcal{X} = [-5, 5]$. Let

$$\begin{aligned} z_1 &= x + n_1 \\ z_2 &= x + n_2, \end{aligned}$$

where n_1 and n_2 are continuous random variables with probability density functions

$$p_{n_1}(\bar{n}_1) = \begin{cases} \alpha_1 \left(1 + \bar{n}_1\right) & \text{for } -1 \le \bar{n}_1 \le 0 \\ \alpha_1 \left(1 - \bar{n}_1\right) & \text{for } \quad 0 \le \bar{n}_1 \le 1 \\ 0 & \text{otherwise}, \end{cases}$$

$$p_{n_2}(\bar{n}_2) = \begin{cases} \alpha_2 \left(1 + \frac{1}{2}\bar{n}_2\right) & \text{for } -2 \le \bar{n}_2 \le 0 \\ \alpha_2 \left(1 - \frac{1}{2}\bar{n}_2\right) & \text{for } \quad 0 \le \bar{n}_2 \le 2 \\ 0 & \text{otherwise}, \end{cases}$$

where α_1 and α_2 are normalization constants. Assume that the random variables x, n_1, n_2 are independent, i.e. $p_{x,n_1,n_2}(\bar{x}, \bar{n}_1, \bar{n}_2) = p_x(\bar{x})\, p_{n_1}(\bar{n}_1)\, p_{n_2}(\bar{n}_2)$.

a) Calculate α_1 and α_2.

b) Use the change of variables formula from Lecture 2 to show that $p_{z_i|x}(\bar{z}_i|\bar{x}) = p_{n_i}(\bar{z}_i - \bar{x})$.

c) Calculate $p_{x|z_1,z_2}(x|0,0)$.

d) Calculate $p_{x|z_1,z_2}(x|0,1)$.

e) Calculate $p_{x|z_1,z_2}(x|0,3)$.

Problem 12

Consider the following estimation problem: an object B moves randomly on a circle with radius 1. The distance to the object can be measured from a given observation point P. The goal is to estimate the location of the object, see Figure 1.

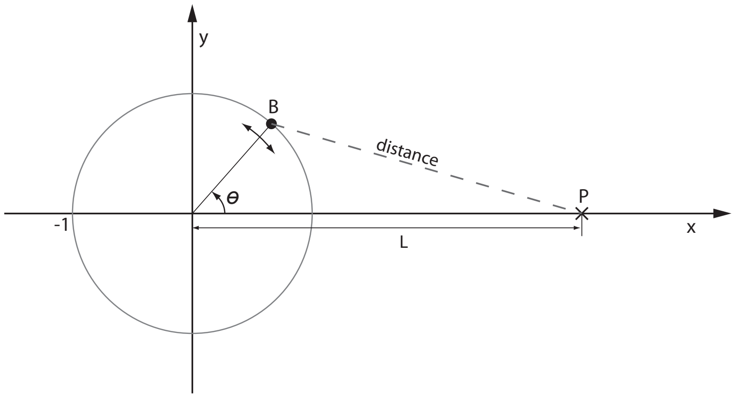


Figure 1: Illustration of the estimation problem.

The object B can only move in discrete steps. The object's location at time k is given by $x(k) \in \{0, 1, \ldots, N-1\}$, where

$$\theta(k) = 2\pi \frac{x(k)}{N}.$$

The dynamics are

$$x(k) = \mod(x(k-1) + v(k)\,,\, N), \qquad k = 1, 2, \ldots,$$

where $v(k) = 1$ with probability p and $v(k) = -1$ otherwise. Note that $\mod(N,\, N) = 0$ and $\mod(-1,\, N) = N - 1$. The distance sensor measures

$$z(k) = \sqrt{(L - \cos\theta(k))^2 + (\sin\theta(k))^2} + w(k),$$

where $w(k)$ represents the sensor error which is uniformly distributed on $[-e, e]$. We assume that $x(0)$ is uniformly distributed and $x(0)$, $v(k)$ and $w(k)$ are independent.

Simulate the object movement and implement a Bayesian tracking algorithm that calculates for each time step k the probability density function $p_{x(k)|z(1:k)}$.

a) Test the following settings and discuss the results: $N = 100$, $x(0) = \frac{N}{4}$, $e = 0.5$,

(i) $L = 2$, $p = 0.5$,
(ii) $L = 2$, $p = 0.5$,
(iii) $L = 2$, $p = 0.55$,
(iv) $L = 0.1$, $p = 0.55$,
(v) $L = 0$, $p = 0.55$.

b) How robust is the algorithm? Set $N = 100$, $x(0) = \frac{N}{4}$, $e = 0.5$, $L = 2$, $p = 0.55$ in the simulation, but use slightly different values for p and e in your estimation algorithm, $\hat{p}$ and $\hat{e}$, respectively. Test the algorithm and explain the result for:

(i) $\hat{p} = 0.45$, $\hat{e} = e$,
(ii) $\hat{p} = 0.5$, $\hat{e} = e$,
(iii) $\hat{p} = 0.9$, $\hat{e} = e$,
(iv) $\hat{p} = p$, $\hat{e} = 0.9$,
(v) $\hat{p} = p$, $\hat{e} = 0.45$.

Sample solutions

Problem 1

We introduce a discrete random variable x_i, representing the outcome of the ith spin with $x_i \in \{\text{red}, \text{black}\}$. We assume that both are equally likely and ignore the fact that there is a 0 or a 00 on a Roulette board.
Assuming *independence* between spins, we have

$$\Pr(x_k, x_{k-1}, x_{k-2}, \ldots, x_1) = \Pr(x_k)\ \Pr(x_{k-1}) \ldots \Pr(x_1).$$

The probability of eleven consecutive spins resulting in black is

$$\Pr(x_{11} = \text{black}, x_{10} = \text{black}, x_9 = \text{black}, \ldots, x_1 = \text{black}) = \left(\frac{1}{2}\right)^{11} \approx 0.0005.$$

This value is actually quite small. However, given that the previous ten were black, we calculate

$$\begin{aligned}
&\Pr(x_{11} = \text{black} | x_{10} = \text{black}, x_9 = \text{black}, \ldots, x_1 = \text{black}) \qquad \text{(by independence assumption)} \\
&\quad = \Pr(x_{11} = \text{black}) = \frac{1}{2} \\
&\quad = \Pr(x_{11} = \text{red} | x_{10} = \text{black}, x_9 = \text{black}, \ldots, x_1 = \text{black})
\end{aligned}$$

i.e. it is equally likely that ten black spins in a row are followed by a red spin, as that they are followed by another black spin (by the independence assumption). Mr. Jones' system is therefore no better than randomly betting on black or red.

Problem 2

We introduce two discrete random variables. Let $x \in \{1, 2\}$ represent which box is chosen (box 1 or 2) with probability $p_x(1) = p_x(2) = \frac{1}{2}$. Furthermore, let $y \in \{b, w\}$ represent the color of the drawn marble, where b is a black and w a white marble with probabilities

$$p_{y|x}(b|1) = p_{y|x}(w|1) = \frac{1}{2},$$

$$p_{y|x}(b|2) = \frac{2}{3}, \quad p_{y|x}(w|2) = \frac{1}{3}.$$

Then, by the total probability theorem, we find

$$p_y(b) = p_{y|x}(b|1)\ p_x(1) + p_{y|x}(b|2)\ p_x(2) = \frac{1}{2} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{2} = \frac{7}{12}.$$

Problem 3

$$p_{x|y}(1|w) = \frac{p_{y|x}(w|1)\ p_x(1)}{p_y(w)} = \frac{\frac{1}{2} \cdot \frac{1}{2}}{1 - p_y(b)} = \frac{\frac{1}{4}}{\frac{5}{12}} = \frac{3}{5}.$$

Problem 4

Let $x \in \{b, w\}$ represent the color of the ball drawn from urn 1, where $p_x(b) = \frac{1}{3}$, $p_x(w) = \frac{2}{3}$, and $y \in \{b, w\}$ be the color of the ball subsequently drawn from urn 2. Considering the different possibilities, we have

$$p_{y|x}(b|b) = \frac{6}{7} \qquad\qquad p_{y|x}(b|w) = \frac{5}{7}$$
$$p_{y|x}(w|b) = \frac{1}{7} \qquad\qquad p_{y|x}(w|w) = \frac{2}{7}.$$

We seek the probability that the transferred ball was white, given that the second ball drawn is white and calculate

$$\begin{aligned} p_{x|y}(w|w) &= \frac{p_{y|x}(w|w)\ p_x(w)}{p_y(w)} = \frac{\frac{2}{7} \cdot \frac{2}{3}}{p_{y|x}(w|b)\ p_x(b) + p_{y|x}(w|w)\ p_x(w)} \\ &= \frac{\frac{2}{7} \cdot \frac{2}{3}}{\frac{1}{7} \cdot \frac{1}{3} + \frac{2}{7} \cdot \frac{2}{3}} = \frac{4}{5}. \end{aligned}$$

Problem 5

We introduce two discrete random variables, $x \in \{A, B, C\}$ and $y \in \{M, F\}$, where x represents which store an employee works in, and y the sex of the employee. From the problem description, we have

$$\begin{aligned} p_x(A) &= \frac{50}{225} = \frac{2}{9} \\ p_x(B) &= \frac{75}{225} = \frac{1}{3} \\ p_x(C) &= \frac{100}{225} = \frac{4}{9}, \end{aligned}$$

and the probability that an employee is a woman is

$$\begin{aligned} p_{y|x}(F|A) &= \frac{1}{2} \\ p_{y|x}(F|B) &= \frac{3}{5} \\ p_{y|x}(F|C) &= \frac{7}{10}. \end{aligned}$$

We seek the probability that the resigning employee works in store C, given that it is a woman and calculate

$$p_{x|y}(C|F) = \frac{p_{y|x}(F|C)\ p_x(C)}{p_y(F)} = \frac{\frac{7}{10} \cdot \frac{4}{9}}{\sum\limits_{i \in \{A,B,C\}} p_{y|x}(F|i)\ p_x(i)} = \frac{\frac{7}{10} \cdot \frac{4}{9}}{\frac{1}{2} \cdot \frac{2}{9} + \frac{3}{5} \cdot \frac{1}{3} + \frac{7}{10} \cdot \frac{4}{9}} = \frac{1}{2}.$$

Problem 6

a) Let $x \in \{F, U\}$ represent whether it is a fair (F) or an unfair (U) coin with $p_x(F) = p_x(U) = \frac{1}{2}$. We introduce $y \in \{h, t\}$ to represent how the toss comes up (heads or tails) with

$$p_{y|x}(h|F) = \frac{1}{2} \qquad\qquad p_{y|x}(t|F) = \frac{1}{2}$$
$$p_{y|x}(h|U) = 1 \qquad\qquad p_{y|x}(t|U) = 0\,.$$

We seek the probability that the drawn coin is fair, given that the toss result is heads and calculate

$$
\begin{aligned}
p_{x|y}(F|h) &= \frac{p_{y|x}(h|F)\ p_x(F)}{p_y(h)} \\
&= \frac{\frac{1}{2}\cdot\frac{1}{2}}{\underbrace{p_{y|x}(h|F)\ p_x(F)}_{\text{fair coin}} + \underbrace{p_{y|x}(h|U)\ p_x(U)}_{\text{unfair coin}}} = \frac{\frac{1}{4}}{\frac{1}{2}\cdot\frac{1}{2} + 1\cdot\frac{1}{2}} = \frac{1}{3}.
\end{aligned}
$$

b) Let y_1 represent the result of the first flip and y_2 that of the second flip. We assume conditional independence between flips (conditioned on x), yielding

$$
p(y_1, y_2|x) = p(y_1|x)\ p(y_2|x)\,.
$$

We seek the probability that the drawn coin is fair, given that the both tosses resulted in heads and calculate

$$
p_{x|y_1,y_2}(F|h,h) = \frac{p_{y_1,y_2|x}(h,h|F)\ p_x(F)}{p_{y_1,y_2}(h,h)} = \frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}}{\underbrace{\left(\frac{1}{2}\right)^2\cdot\frac{1}{2}}_{\text{fair coin}} + \underbrace{1^2\cdot\frac{1}{2}}_{\text{unfair coin}}} = \frac{1}{5}.
$$

c) Obviously, the probability then is 1, because the unfair coin cannot show tails. Formally, we show this by introducing y_3 to represent the result of the third flip and computing

$$
p_{x|y_1,y_2,y_3}(F|h,h,t) = \frac{p_{y_1,y_2,y_3|x}(h,h,t|F)\ p_x(F)}{p_{y_1,y_2,y_3}(h,h,t)} = \frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}}{\underbrace{\left(\frac{1}{2}\right)^3\cdot\frac{1}{2}}_{\text{fair coin}} + \underbrace{1^2\cdot 0\cdot\frac{1}{2}}_{\text{unfair coin}}} = 1.
$$

Problem 7

We introduce the following two random variables:

- $x \in \{1, 2\}$ represents which box is chosen (box 1 or 2) with probability $p_x(1) = p_x(2) = \frac{1}{2}$,
- $y \in \{b, w\}$ represents the color of the drawn ball, where b is a black and w a white ball with probabilities

$$
\begin{aligned}
p_{y|x}(b|1) &= \frac{7}{12} \qquad & p_{y|x}(w|1) &= \frac{5}{12} \\
p_{y|x}(b|2) &= \frac{12}{15} \qquad & p_{y|x}(w|2) &= \frac{3}{15}.
\end{aligned}
$$

We seek the probability that the second box was selected, given that the ball drawn is white and calculate

$$
p_{x|y}(2|w) = \frac{p_{y|x}(w|2)\, p_x(2)}{p_y(w)} = \frac{\frac{3}{15}\cdot\frac{1}{2}}{\frac{5}{12}\cdot\frac{1}{2} + \frac{3}{15}\cdot\frac{1}{2}} = \frac{12}{37}.
$$

Problem 8

Let $x_1 \in \{B, R\}$ be the color (black or red) of the first ball drawn with $p_{x_1}(B) = \frac{b}{b+r}$ and $p_{x_1}(R) = \frac{r}{b+r}$; and let $x_2 \in \{B, R\}$ be the color of the second ball with

$$p_{x_2|x_1}(B|R) = \frac{b}{b+r+c} \qquad p_{x_2|x_1}(R|R) = \frac{c+r}{b+r+c}$$
$$p_{x_2|x_1}(B|B) = \frac{b+c}{b+r+c} \qquad p_{x_2|x_1}(R|B) = \frac{r}{b+r+c}.$$

We seek the probability that the first ball drawn was black, given that the second ball drawn is red and calculate

$$\begin{aligned} p_{x_1|x_2}(B|R) &= \frac{p_{x_2|x_1}(R|B)\ p_{x_1}(B)}{p_{x_2}(R)} \\ &= \frac{\frac{r}{b+r+c} \cdot \frac{b}{b+r}}{\underbrace{\frac{c+r}{b+r+c} \cdot \frac{r}{b+r}}_{\text{first red}} + \underbrace{\frac{r}{b+r+c} \cdot \frac{b}{b+r}}_{\text{first black}}} = \frac{b}{b+r+c}. \end{aligned}$$

Problem 9

We can approach the solution in two ways:

Descriptive solution: The probability that A is to be executed is $\frac{1}{3}$, and there is a chance of $\frac{2}{3}$ that one of the others was chosen. If the jailer gives away the name of one of the fellow prisoners who will be set free, prisoner A does not get new information about his own fate, but the probability of the remaining prisoner (B or C) to be executed is now $\frac{2}{3}$. The probability of A being executed is still $\frac{1}{3}$.

Bayesian analysis: Let x represent which prisoner is to be executed, where $x \in \{A, B, C\}$. We assume that it is a random choice, i.e. $p_x(A) = p_x(B) = p_x(C) = \frac{1}{3}$.

Now let $y \in \{B, C\}$ be the prisoner name given away by the jailer. We can now write the conditional probabilities:

$$p(y|x) = \begin{cases} 0 & \text{if } x = y \text{ (the jailer does not lie)} \\ \frac{1}{2} & \text{if } x = A \text{ (}A\text{ is to be executed, jailer mentions }B\text{ and }C\text{ with equal probability)} \\ 1 & \text{if } x \neq A \text{ (jailer is forced to give the name of the other prisoner to be set free).} \end{cases}$$

You could also do this with a table:

y	x	$p(y\|x)$
B	A	1/2
B	B	0
B	C	1
C	A	1/2
C	B	1
C	C	0

To answer the question, we have to compare $p_{x|y}(A|\bar{y})$, $\bar{y} \in \{B, C\}$, with $p_x(A)$:

$$p_{x|y}(A|\bar{y}) = \frac{p_{y|x}(\bar{y}|A)\,p_x(A)}{p_y(\bar{y})} = \frac{\frac{1}{2}\cdot\frac{1}{3}}{\sum\limits_{k\in\{A,B,C\}} p_{y|x}(\bar{y}|k)\cdot p_x(k)}$$

$$= \frac{\frac{1}{6}}{\underbrace{\tfrac{1}{2}}_{p_{y|x}(\bar{y}|A)}\cdot\frac{1}{3} + \underbrace{0}_{p_{y|x}(\bar{y}|\bar{y})}\cdot\frac{1}{3} + \underbrace{1}_{p_{y|x}(\bar{y}|\,\text{not}\,\bar{y})}\cdot\frac{1}{3}} = \frac{1}{3},$$

where (not $\bar{y}$) $= C$ if $\bar{y} = B$ and (not $\bar{y}$) $= B$ if $\bar{y} = C$. The value of the posterior probability is the same as the prior one, $p_x(A)$. The jailer is wrong: prisoner A gets no additional information from the jailer about his own fate!

See also Wikipedia: Three prisoners problem, Monty Hall problem.

Problem 10

Consider the joint cumulative distribution

$$\begin{aligned} F_{v,w}(\bar{v},\bar{w}) &= \Pr\left((v \le \bar{v}) \text{ and } (w \le \bar{w})\right) \\ &= \Pr\left((g(x) \le \bar{v}) \text{ and } (h(y) \le \bar{w})\right). \end{aligned}$$

We define the sets $\mathcal{A}_{\bar{v}}$ and $\mathcal{A}_{\bar{w}}$ as below:

$$\begin{aligned} \mathcal{A}_{\bar{v}} &= \left\{x \in \bar{\mathcal{X}} : g(x) \le \bar{v}\right\} \\ \mathcal{A}_{\bar{w}} &= \left\{y \in \bar{\mathcal{Y}} : h(y) \le \bar{w}\right\}. \end{aligned}$$

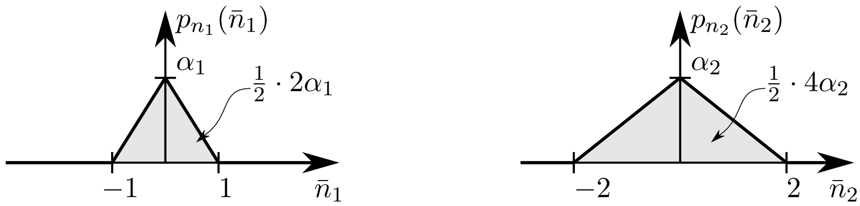
Now we can deduce

$$\begin{aligned} F_{v,w}(\bar{v},\bar{w}) &= \Pr\left((x \in \mathcal{A}_{\bar{v}}) \text{ and } (y \in \mathcal{A}_{\bar{w}})\right) && \forall\, \bar{v},\bar{w} \\ &= \Pr\left(x \in \mathcal{A}_{\bar{v}}\right)\Pr\left(y \in \mathcal{A}_{\bar{w}}\right) && \text{(by independence assumption)} \\ &= \Pr\left(g(x) \le \bar{v}\right)\Pr\left(h(y) \le \bar{w}\right) \\ &= \Pr\left(v \le \bar{v}\right)\Pr\left(w \le \bar{w}\right) \\ &= F_v(\bar{v})\,F_w(\bar{w}). \end{aligned}$$

Therefore v and w are independent.

Problem 11

a) By definition of a PDF, the integrals of the PDFs p_{n_i}, $i = 1, 2$ must evaluate to one, which we use to find the α_i. We integrate the probability density functions, see the following figure:



For n_1, we obtain

$$\int_{-\infty}^{\infty} p_{n_1}(\bar{n}_1)\, d\bar{n}_1 = \alpha_1 \left(\int_{-1}^{0} (1+\bar{n}_1)\, d\bar{n}_1 + \int_{0}^{1} (1-\bar{n}_1)\, d\bar{n}_1 \right)$$
$$= \alpha_1 \left(1 - \frac{1}{2} + 1 - \frac{1}{2}\right) = \alpha_1 .$$

Therefore $\alpha_1 = 1$. For n_2, we get

$$\int_{-\infty}^{\infty} p_{n_2}(\bar{n}_2)\, d\bar{n}_2 = \alpha_2 \left(\int_{-2}^{0} \left(1+\frac{1}{2}\bar{n}_2\right) d\bar{n}_2 + \int_{0}^{2} \left(1-\frac{1}{2}\bar{n}_2\right) d\bar{n}_2 \right)$$
$$= \alpha_2 (2-1+2-1) = 2\alpha_2 .$$

Therefore $\alpha_2 = \frac{1}{2}$.

b) The goal is to calculate $p(z_i|x)$ from $z_i = g(n_i, x) := x + n_i$ and the given PDFs p_{n_i}, $i = 1, 2$. We apply the change of variables formula for CRVs with conditional PDFs:

$$p(z_i|x) = \frac{p(n_i|x)}{\frac{\partial g}{\partial n_i}(n_i, x)} \quad \text{(just think of } x \text{ as a parameter that parametrizes the PDFs).}$$

The proof for this formula is analogous to the proof in the lecture notes for a change of variables for CRVs with unconditional PDFs. We find that

$$\frac{\partial g}{\partial n_i}(n_i, x) = 1, \quad \text{for all } n_i, x$$

and, therefore, the fraction in the change of variables formula is well-defined for all values of n_i and x. Due to the independence of n_i and x, $p(n_i|x) = p(n_i)$:

$$p(z_i|x) = \frac{p(n_i|x)}{\frac{\partial g}{\partial n_i}(n_i, x)} = p(n_i)\, .$$

Substituting $n_i = z_i - x$, we finally obtain

$$p_{z_i|x}(\bar{z}_i|\bar{x}) = p_{n_i}(\bar{n}_i) = p_{n_i}(\bar{z}_i - \bar{x})\, .$$

c) We use Bayes' theorem to calculate

$$p(x|z_1, z_2) = \frac{p(z_1, z_2|x)\, p(x)}{p(z_1, z_2)}. \tag{1}$$

First, we calculate the prior PDF of the CRV x, which is uniformly distributed:

$$p(x) = \begin{cases} \frac{1}{10} & \text{for } -5 \le x \le 5 \\ 0 & \text{otherwise} \end{cases} .$$

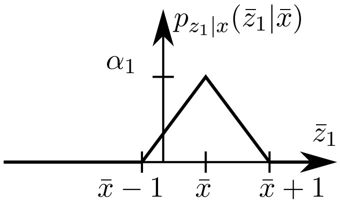
In b), we showed by change of variables that $p_{z_i|x}(\bar{z}_i|\bar{x}) = p_{n_i}(\bar{z}_i - \bar{x})$. Since n_1, n_2 are independent, it follows that z_1, z_2 are conditionally independent given x. Therefore, we can rewrite the measurement likelihood as

$$p(z_1, z_2|x) = p(z_1|x)\, p(z_2|x)\, .$$

We calculate the individual conditional PDFs $p_{z_i|x}$:

$$p_{z_1|x}(\bar{z}_1|\bar{x}) = p_{n_1}(\bar{z}_1 - \bar{x}) = \begin{cases} \alpha_1\,(1 - \bar{z}_1 + \bar{x}) & \text{for } 0 \leq \bar{z}_1 - \bar{x} \leq 1 \\ \alpha_1\,(1 + \bar{z}_1 - \bar{x}) & \text{for } -1 \leq \bar{z}_1 - \bar{x} \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

The conditional PDF of z_1 given x is illustrated in the following figure:



Analogously

$$p_{z_2|x}(\bar{z}_2|\bar{x}) = p_{n_2}(\bar{z}_2 - \bar{x}) = \begin{cases} \alpha_2\left(1 - \frac{1}{2}\bar{z}_2 + \frac{1}{2}\bar{x}\right) & \text{for } 0 \leq \bar{z}_2 - \bar{x} \leq 2 \\ \alpha_2\left(1 + \frac{1}{2}\bar{z}_1 - \frac{1}{2}\bar{x}\right) & \text{for } -2 \leq \bar{z}_2 - \bar{x} \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathrm{num}(\bar{x})$ be the numerator of the Bayes' rule fraction (1). Given the measurements $z_1 = 0$, $z_2 = 0$, we find

$$\mathrm{num}(\bar{x}) = \frac{1}{10} p_{z_1|x}(0|\bar{x})\, p_{z_2|x}(0|\bar{x})\,.$$

We consider four different intervals of x: $[-5,-1]$, $[-1,0]$, $[0,1]$ and $[1,5]$. Evaluating $\mathrm{num}(\bar{x})$ for these intervals results in:

- for $\bar{x} \in [-5,-1]$ or $\bar{x} \in [1,5]$,

  $$\mathrm{num}(\bar{x}) = 0$$

- for $\bar{x} \in [-1,0]$,

  $$\mathrm{num}(\bar{x}) = \frac{1}{10}\alpha_1\,(1+\bar{x})\,\alpha_2\left(1 + \frac{\bar{x}}{2}\right) = \frac{1}{20}\,(1+\bar{x})\left(1 + \frac{\bar{x}}{2}\right)$$

- for $\bar{x} \in [0,1]$,

  $$\mathrm{num}(\bar{x}) = \frac{1}{10}\alpha_1\,(1-\bar{x})\,\alpha_2\left(1 - \frac{\bar{x}}{2}\right) = \frac{1}{20}\,(1-\bar{x})\left(1 - \frac{\bar{x}}{2}\right).$$

Finally, we need to calculate the denominator of the Bayes' rule fraction (1), the normalization constant, which can be calculated using the total probability theorem:

$$p_{z_1,z_2}(\bar{z}_1,\bar{z}_2) = \int_{-\infty}^{\infty} p_{z_1,z_2|x}(\bar{z}_1,\bar{z}_2|\bar{x})\,p_x(\bar{x})\,d\bar{x} = \int_{-\infty}^{\infty} \mathrm{num}(\bar{x})\,d\bar{x}.$$

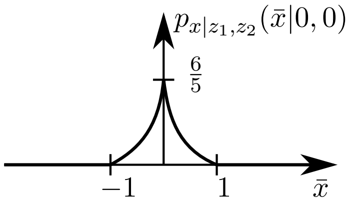
Evaluated at $z_1 = z_2 = 0$, we find

$$\begin{aligned} p_{z_1,z_2}(0,0) &= \frac{1}{20}\left(\int_{-1}^{0}(1+\bar{x})\left(1+\frac{\bar{x}}{2}\right)d\bar{x} + \int_{0}^{1}(1-\bar{x})\left(1-\frac{\bar{x}}{2}\right)d\bar{x}\right) \\ &= \frac{1}{20}\left(\frac{5}{12} + \frac{5}{12}\right) = \frac{1}{24}. \end{aligned}$$

Finally, we obtain the posterior PDF

$$
\begin{aligned}
p_{x|z_1,z_2}(\bar{x}|0,0) &= \frac{1}{p_{z_1,z_2}(0,0)}\text{num}(\bar{x}) = 24 \cdot \text{num}(\bar{x}) \\
&= \begin{cases} 0 & \text{for } -5 \leq \bar{x} \leq -1,\ 1 \leq \bar{x} \leq 5 \\ \frac{6}{5}\left(1+\bar{x}\right)\left(1+\frac{\bar{x}}{2}\right) & \text{for } -1 \leq \bar{x} \leq 0 \\ \frac{6}{5}\left(1-\bar{x}\right)\left(1-\frac{\bar{x}}{2}\right) & \text{for } 0 \leq \bar{x} \leq 1 \end{cases}
\end{aligned}
$$

which is illustrated in the following figure:



The posterior PDF is symmetric about $\bar{x} = 0$ with a maximum at $\bar{x} = 0$: both sensors "agree".

d) Similar to the above. Given $\bar{z}_1 = 0$, $\bar{z}_2 = 1$, we write the numerator as

$$
\text{num}(\bar{x}) = \frac{1}{10} p_{z_1|x}(0|\bar{x})\, p_{z_2|x}(1|\bar{x})\,. \tag{2}
$$

Again, we consider the same four intervals:

- for $\bar{x} \in [-5,-1]$ or $\bar{x} \in [1,5]$,

$$
\text{num}(\bar{x}) = 0
$$

- for $\bar{x} \in [-1,0]$,

$$
\text{num}(\bar{x}) = \frac{1}{10}\alpha_1\left(1+\bar{x}\right)\alpha_2\left(\frac{1}{2}+\frac{1}{2}\bar{x}\right) = \frac{1}{40}\left(1+\bar{x}\right)^2
$$

- for $\bar{x} \in [0,1]$

$$
\text{num}(\bar{x}) = \frac{1}{10}\alpha_1\left(1-\bar{x}\right)\alpha_2\left(\frac{1}{2}+\frac{1}{2}\bar{x}\right) = \frac{1}{40}\left(1-\bar{x}^2\right).
$$

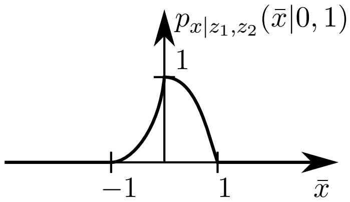
Normalizing yields

$$
p_{z_1,z_2}(0,1) = \int_{-1}^{1} \text{num}(\bar{x})\, dx = \frac{1}{120} + \frac{2}{120} = \frac{1}{40}.
$$

The solution is therefore

$$
p_{x|z_1,z_2}(\bar{x}|0,1) = \begin{cases} 0 & \text{for } -5 \leq \bar{x} \leq -1,\ 1 \leq \bar{x} \leq 5 \\ \left(1+\bar{x}\right)^2 & \text{for } -1 \leq \bar{x} \leq 0 \\ 1-\bar{x}^2 & \text{for } 0 \leq \bar{x} \leq 1. \end{cases}
$$

The posterior PDF is depicted in the following figure:



The probability values are higher for positive $\bar{x}$ values because of the measurement $\bar{z}_2 = 1$.

e) We start in the same fashion: given $\bar{z}_1 = 0$ and $\bar{z}_2 = 3$,

$$\text{num}(\bar{x}) = \frac{1}{10} p_{z_1|x}(0|\bar{x})\, p_{z_2|x}(3|\bar{x})\,.$$

However, the intervals of positive probability of $p_{z_1|x}(0|\bar{x})$ and $p_{z_2|x}(3|\bar{x})$ do not overlap, i.e.

$$\text{num}(\bar{x}) = 0 \quad \forall\, \bar{x} \in [-5, 5]\,.$$

In other words, given our noise model for n_1 and n_2, there is no chance to measure $\bar{z}_1 = 0$ and $\bar{z}_2 = 3$. Therefore, $p_{x|z_1,z_2}(\bar{x}|0,3)$ is *not defined*.

Problem 12

The MATLAB code is available on the class webpage. We notice the following:

a)
- (i) The PDF remains bimodal for all times. The symmetry in measurements and motion makes it impossible to differentiate between the upper and lower half circle.
- (ii) The PDF is initially bimodal, but the bias in the particle motion causes one of the two modes to have higher values after a few time steps.
- (iii) We note that this sensor placement also works. Note that the resulting PDF is not as 'sharp' because more positions explain the measurements.
- (iv) The resulting PDF is uniform for all $k = 1, 2, \ldots$. With the sensor in the center, the distance measurement provides no information on the particle position (all particle positions have the same distance to the sensor).

b)
- (i) The PDF has higher values at the position mirrored from the actual position, because the estimator model uses a value $\hat{p}$ that biases the particle motion in the opposite direction.
- (ii) The PDF remains bimodal, even though the particle motion is biased in one direction. This is caused by the estimator assuming that the particle motion is unbiased, which makes both halves of the circle equally likely.
- (iii) The incorrect assumption on the motion probability $\hat{p}$ causes the estimated PDF to drift away from the real particle position until it is forced back by measurements that cannot be explained otherwise.
- (iv) The resulting PDF is not as 'sharp' because the larger value of $\hat{e}$ implies that more states are possible for the given measurements.
- (v) The Bayesian tracking algorithm fails after a few steps (division by zero in the measurement update step). This is caused by a measurement that is impossible to explain with the present particle position PDF and the measurement error distribution defined by $\hat{e}$. Note that $\hat{e}$ underestimates the possible measurement error. This leads to measurements where the true error (with uniform distribution defined by e) contains an error larger than is assumed possible in the algorithm.