MATH 5080 Practice Exam 2

The practice and actual exam will both cover materials in Lecture 14-25. The exam will be 90 minutes long, closed book, with only the provided cheat sheet allowed.

Problem 1: Let function $f \in R[a,b]$ and suppose that $f(x) \leq 0$, $\forall x \in [a,b]$. At $c \in [a,b]$, suppose that $\lim_{x\to c} f(x) = L < 0$. Prove that $\int_a^b f(x)\,dx < 0$.

Problem 2: Let $\sum_{n=1}^{\infty} a_n$ be a nonnegative series such that $\{a_n\}$ is monotone decreasing.

(a) Suppose $\sum_{n=1}^{\infty} a_{2n}$ converges. Prove that $\sum_{n=1}^{\infty} a_n$ also converges.

(b) Prove that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$ converges. (*Hint: the geometric average $\sqrt{ab}$ is always between a and b.*)

Problem 3: Let $f = k$ on $[a,b]$, where $k \in \mathbb{R}$ is a constant.

(a) Use the definition of Riemann integrability to show that f is integrable and find the value of $\int_a^b f(x)\,dx$.

(b) Let P be a partition of $[a,b]$, evaluate $U(P,f)$ and $L(P,f)$.

Problem 4: Consider the series of functions $\sum_{n=1}^{\infty}(1-x^3)x^n$.

(a) Find the largest domain D on which the series converges pointwisely.

(b) Does the series converge uniformly on D? Justify your answer.

Problem 5: (a) Is it true that every continuous f on $[a,b]$ must have an antiderivative? Justify your answer.

(b) Find all continuous functions f on $[a,b]$ satisfying $\int_a^x f(t)\,dt = \int_x^b f(t)\,dt$, $\forall x \in [a,b]$.

Problem 6: Let $\{f_n\}$ be a sequence of continuous functions on a compact domain D. Suppose $\{f_n\}$ converges uniformly on D. Show that $\{f_n\}$ are uniformly equicontinuous on D.

Problem 7: (a) Find the Taylor series for the function e^x centered at 0. (You can use the fact that $(e^x)' = e^x$.)

(b) Prove that the series converges to e^x uniformly on any compact interval on $\mathbb{R}$.

Problem 8: Let (X, d) be a complete metric space that contains no isolated point. Prove that the space is uncountable. (*Hint: try to use the Baire Category Theorem.*)