MATH 5080 Practice Exam 2

The practice and actual exam will both cover materials in Lecture 14-25. The exam will be 90 minutes long, closed book, with only the provided cheat sheet allowed.

Problem 1: Let function $f \in R[a,b]$ and suppose that $f(x) \leq 0$, $\forall x \in [a,b]$. At $c \in [a,b]$, suppose that $\lim_{x \to c} f(x) = L < 0$. Prove that $\int_a^b f(x)\,dx < 0$.

Solution. From a theorem we learnt in class, $f \geq 0$ implies that $\int_a^b f(x)\,dx \geq 0$. Hence, by linearity, this immediately implies in our case ($f \leq 0$) that $\int_a^b f(x)\,dx \leq 0$. Same holds for integrals of f on any subinterval of $[a,b]$. Without loss of generality, assume $c \in (a,b)$.

Since $\lim_{x \to c} f(x) = L < 0$, let $\epsilon = -\frac{L}{2}$, one has by limit definition that there exists $\delta > 0$ such that for all $x \in N^*(c,\delta)$, $|f(x) - L| < -\frac{L}{2}$. Hence, $f(x) < \frac{L}{2} < 0$, $\forall x \in N^*(c,\delta)$.

Define a function g on $[c-\frac{\delta}{2}, c+\frac{\delta}{2}]$ such that $g(x) = f(x)$, $\forall 0 < |x-c| \leq \frac{\delta}{2}$, and $g(c) = \frac{L}{2}$. Then g and f differ only at a single point, hence g is also integrable and $\int_{c-\frac{\delta}{2}}^{c+\frac{\delta}{2}} g(x)\,dx = \int_{c-\frac{\delta}{2}}^{c+\frac{\delta}{2}} f(x)\,dx$ from a homework problem. Since $g(x) \leq \frac{L}{2}$ on the interval, one has that

$$\int_{c-\frac{\delta}{2}}^{c+\frac{\delta}{2}} f(x)\,dx = \int_{c-\frac{\delta}{2}}^{c+\frac{\delta}{2}} g(x)\,dx \leq \frac{L}{2} \cdot \delta < 0.$$

Hence, one concludes that

$$\int_a^b f(x)\,dx = \int_a^{c-\frac{\delta}{2}} + \int_{c-\frac{\delta}{2}}^{c+\frac{\delta}{2}} + \int_{c+\frac{\delta}{2}}^b \leq 0 + \frac{L\delta}{2} + 0 < 0.$$

□

Problem 2: Let $\sum_{n=1}^\infty a_n$ be a nonnegative series such that $\{a_n\}$ is monotone decreasing.

(a) Suppose $\sum_{n=1}^\infty a_{2n}$ converges. Prove that $\sum_{n=1}^\infty a_n$ also converges.

(b) Prove that $\sum_{n=1}^\infty a_n$ converges if and only if $\sum_{n=1}^\infty \sqrt{a_n a_{n+1}}$ converges. (*Hint: the geometric average $\sqrt{ab}$ is always between a and b.*)

Solution. (a) Since $\{a_n\}$ is decreasing, one has for all $n \in \mathbb{N}$ that $a_{2n+1} \leq a_{2n}$. Define a new series with $b_n := a_{2n+1}$. Then by Comparison test, $\sum_{n=1}^\infty b_n$ converges. Hence, by linearity, one also has the sum series $\sum_{n=1}^\infty (a_{2n+b_n}) = \sum_{n=1}^\infty (a_{2n} + a_{2n+1})$ converges.

It is easy to see that the series $\sum_{n=1}^\infty (a_{2n} + a_{2n+1})$ is a regrouping of the series $\sum_{n=2}^\infty a_n$. Since the series is nonnegative, convergence of any regrouping implies the convergence of the original series. Hence, $\sum_{n=2}^\infty a_n$. This further implies the convergence of the series $\sum_{n=1}^\infty a_n$, which only differs from the former by the value a_1 for all partial sums.

(b) First, assume that $\sum a_n$ converges. Then, since $a_{n+1} \leq \sqrt{a_n a_{n+1}} \leq a_n$, one has by Comparison test that $\sum \sqrt{a_n a_{n+1}}$ also converges.

Second, assume that $\sum \sqrt{a_n a_{n+1}}$ converges. One again has the inequality $a_{n+1} \leq \sqrt{a_n a_{n+1}} \leq a_n$. By Comparison test, this implies that the sequence $\sum_{n=1}^{\infty} a_{n+1}$ converges. Since the n-th partial sum of this series only differs from that of $\sum_{n=1}^{\infty} a_n$ by the value a_1, one deduces that $\sum a_n$ must also converge. $\square$

Problem 3: Let $f = k$ on $[a, b]$, where $k \in \mathbb{R}$ is a constant.

(a) Use the definition of Riemann integrability to show that f is integrable and find the value of $\int_a^b f(x)\, dx$.

(b) Let P be a partition of $[a, b]$, evaluate $U(P, f)$ and $L(P, f)$.

Solution. (a) We claim that $f \in R[a, b]$ and that $\int_a^b f(x)\, dx = k(b - a)$. For all $\epsilon > 0$, it suffices to find $\delta > 0$ such that if $\lambda(P) < \delta$, then all Riemann sums with this partition P satisfies

$$\left| \sum_{i=1}^{n} f(\xi_i)\Delta x_i - k(b-a) \right| < \epsilon.$$

To see this, let δ be any fixed number. For all partition P, we claim that the above estimate is always true for all choices of $\{\xi_i\}$. Indeed, fixing any choice of $\{\xi_i\}$, since $f(x) = k$ on $[a, b]$, it is obvious that $f(\xi_i) = k$, $\forall i$. Therefore, $\sum_{i=1}^{n} f(\xi_i)\Delta x_i = k \sum_{i=1}^{n} \Delta x_i = k(b - a)$, hence the desired inequality follows trivially.

(b) Let P be any partition of $[a, b]$, since f is a constant, one has that $m_i = M_i = k$ for all i. Therefore, $U(P, f) = L(P, f) = k(b - a)$. $\square$

Problem 4: Consider the series of functions $\sum_{n=1}^{\infty}(1 - x^3)x^n$.

(a) Find the largest domain D on which the series converges pointwisely.

Solution. Rewrite the series as $(1 - x^3)\sum_{n=1}^{\infty} x^n$. By the theory of geometric series, it is easy to see that if $|x| > 1$, then the series diverges.

If $x = 1$, then one has $1 - x^3 = 0$ and the series becomes the zero series, which converges.

If $x = -1$, then $1 - x^3 = 2$. Since $\sum(-1)^n$ diverges, one has that the series diverges.

If $|x| < 1$, then the geometric series $\sum x^n$ converges.

In conclusion, the largest domain D on which the series converges pointwisely is $(-1, 1]$. $\square$

(b) Does the series converge uniformly on D? Justify your answer.

Solution. We first find the sum function of the series on its domain of convergence. From the discussion in the above, if $x = 1$, then $S(x) = \sum_{n=1}^{\infty}(1-x^3)x^n = \sum_{n=1}^{\infty} 0 = 0$. If $|x| < 1$, one has

$$S(x) = \sum_{n=1}^{\infty}(1-x^3)x^n = (1-x^3)\sum_{n=1}^{\infty} x^n = (1-x^3)\frac{x}{1-x} = x(1+x+x^2).$$

In conclusion, one has $S(x) = \begin{cases} x(1+x+x^2), & x \in (-1,1), \\ 0, & x = 1. \end{cases}$ At $x = 1$, since $\lim_{x\to 1} S(x) = 3 \neq 0 = S(1)$, S is not continuous at $x = 1$. Since each summand $(1-x^3)x^n$ is obviously continuous on $(-1,1]$, this shows that the series doesn't converge uniformly on D. $\square$

Problem 5: (a) Is it true that every continuous f on $[a,b]$ must have an antiderivative? Justify your answer.

Solution. Yes. By the Fundamental theorem of calculus, if f is continuous on $[a,b]$, then the function $F(x) := \int_a^x f(t)\,dt$ is differentiable on $[a,b]$ and satisfies $F'(x) = f(x)$, $\forall x \in [a,b]$. By definition, F is an antiderivative of f on $[a,b]$. $\square$

(b) Find all continuous functions f on $[a,b]$ satisfying $\int_a^x f(t)\,dt = \int_x^b f(t)\,dt$, $\forall x \in [a,b]$.

Solution. Since f is continuous, it is integrable on $[a,b]$ and the second part of the FTC applies. Hence, $\int_a^x f(t)\,dt$ is an antiderivative of f on $[a,b]$. On the other hand, $\int_x^b f(t)\,dt = \int_a^b f(t)\,dt - \int_a^x f(t)\,dt$. Therefore, one has from the assumption that

$$\int_a^x f(t)\,dt = \int_x^b f(t)\,dt = \int_a^b f(t)\,dt - \int_a^x f(t)\,dt,$$

which implies

$$\int_a^x f(t)\,dt = \frac{1}{2}\int_a^b f(t)\,dt.$$

Differentiating both sides, one has that $f(x) = 0$ on $[a,b]$. $\square$

Problem 6: Let $\{f_n\}$ be a sequence of continuous functions on a compact domain D. Suppose $\{f_n\}$ converges uniformly on D. Show that $\{f_n\}$ are uniformly equicontinuous on D.

Solution. For all $\epsilon > 0$, by the uniform convergence assumption, there is some $N > 0$ such that $|f_n(x) - f_m(x)| < \frac{\epsilon}{3}$, $\forall m, n > N$. Since D is compact, all functions f_n are uniformly continuous on D. So there exists $\delta_n > 0$, for every n, such that $|f_n(x) - f_n(y)| < \frac{\epsilon}{3}$ whenever $|x - y| < \delta_n$.

Let $\delta := \min\{\delta_1, \cdots, \delta_N, \delta_{N+1}\} > 0$. We claim that if $|x - y| < \delta$, then $|f_n(x) - f_n(y)| < \epsilon$ for all $n \in \mathbb{N}$, which would imply that the functions are uniformly continuous on D.

To see the claim, for $n \leq N$, by definition of δ, the claim automatically follows. For $n > N$, one has by triangle inequality that

$$|f_n(x) - f_n(y)| \leq |f_n(x) - f_{N+1}(x)| + |f_{N+1}(x) - f_{N+1}(y)| + |f_{N+1}(y) - f_n(y)|.$$

If $|x-y| < \delta \leq \delta_{N+1}$, by definition, the middle terms $|f_{N+1}(x) - f_{N+1}(y)| < \frac{\epsilon}{3}$. Since $n, N+1 > N$, by the Cauchy condition, one has that both the first term and the third term in the above are bounded by $\frac{\epsilon}{3}$. Hence the proof is complete. $\square$

Problem 7: (a) Find the Taylor series for the function e^x centered at 0. (You can use the fact that $(e^x)' = e^x$.)

(b) Prove that the series converges to e^x uniformly on any compact interval on $\mathbb{R}$.

Solution. (a) Since $(e^x)' = e^x$, writing $f(x) = e^x$, one has that $f^{(n)}(0) = e^0 = 1$ for all $n \in \mathbb{N}$. Therefore, its Taylor series centered at 0 is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

(b) We first claim that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ on $\mathbb{R}$. To see this, it suffices to show that the convergence holds on any given interval (a, b). For fixed (a, b), let $M = \max\{|a|, |b|\}$. Then one has that $f^{(n)}(x) = e^x \leq e^M$ for all $x \in (a, b)$. By the theorem we learnt in class, this shows that f is real analytic on (a, b) and its Taylor series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges to e^x for all $x \in (a, b)$.

Now that we have proved that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ on $\mathbb{R}$, we know that the radius of convergence of the series is infinity. Let $[a, b]$ be any given compact interval, then there exists $C > 0$ such that $[-C, C] \supset [a, b]$. Since the series converges at $x = C + 1$, it must converge uniformly on any closed interval contained in $(-(C+1), C+1)$, which includes $[a, b]$. By the uniqueness of limit, the limit function of the uniform convergent series must also be e^x. $\square$

Problem 8: Let (X, d) be a complete metric space that contains no isolated point. Prove that the space is uncountable. (*Hint: try to use the Baire Category Theorem.*)

Solution. Suppose X is countable. For every $x \in X$, define a set $O_x := X \setminus \{x\}$. First, it's easy to see that O_x is open. (Indeed, for every $y \in O_x$, let $\epsilon = \frac{d(x,y)}{2} > 0$, then $N(y, \epsilon) \subset O_x$.) Also, since x is not isolated, for every $\epsilon > 0$, there must exists some $y \in O_x$ such that $d(x, y) < \epsilon$, hence O_x is dense in X. Therefore, $\{O_x\}_{x \in X}$ is a countable sequence of open dense subsets of X. By BCT, their intersection is also dense and in particular nonempty. This means that there is some point $x_0 \in \bigcap O_x$. However, this implies that $x_0 \neq x$, $\forall x \in X$, contradiction. $\square$