MATH 5080 Practice problems

Here are some practice problems corresponding to Lecture 23-25 on Metric spaces. No need to submit.

Problem 1: Let (X, d) be a compact metric space. Show that it has a countable dense subset. (This can help extend the proof of Arzelà-Ascoli theorem that we discussed to functions on any compact metric space.)

Solution. For every $n \in \mathbb{N}$, it's easy to see that $\{B(x, \frac{1}{n})\}_{x \in X}$ form an open cover of X. Since X is compact, there exists $x_1^n, \cdots, x_{k_n}^n \in X$ such that $X \subset \bigcup_{i=1}^{k_n} B(x_i^n, \frac{1}{n})$. Now denote $E = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{k_n} \{x_i^n\}$, we claim that E is a desired countable dense subset of X. It is obvious that E is countable, so it remains to prove that E is dense in X. For every $x \in X$ and every $\epsilon > 0$, let n be such that $\frac{1}{n} < \epsilon$, then there must exists some $x_i^n \in E$ such that $x \in B(x_i^n, \frac{1}{n})$ hence $d(x, x_i^n) < \epsilon$. The proof is complete. $\square$

Problem 2: Prove that every real valued continuous function on a compact metric space is bounded and attains maximum and minimum values.

Solution. It suffices to prove that continuous map maps compact set to compact set in an abstract metric space. (Recall that in any metric space, compact set is always bounded and closed, hence the image of the function is bounded and max and min can be attained.) To see that compactness is preserved by continuous map, we use the same argument as in the case of $\mathbb{R}$ (see Theorem 1 in section 4.3). The key property to use here is that continuous implies that the preimage of open sets are open. We omit the details. $\square$

Problem 3: Let (X, d) be a complete metric space. Prove that a set $S \subset X$ is closed if and only if all convergent sequences $\{a_n\} \subset S$ have their limit in S.

Solution. We showed in class that S is closed if and only if S is complete. If S is complete, then any convergent sequence $\{a_n\}$, which is Cauchy, must converge to a limit in S. On the other hand, if this property is satisfied by S, then for any Cauchy sequence $\{a_n\}$ in S, since it is also Cauchy in X which is complete, there is some $p \in X$ such that $\{a_n\}$ converges to p. By the uniqueness of limit, one has $p \in S$, hence S is complete (therefore is closed). $\square$

Problem 4: Let (X, d_X) be a metric space and (Y, d_Y) be a complete metric space. Define

$$C_b(X,Y) := \{f : X \to Y \mid f \text{ is continuous and bounded}\}$$

and define

$$d(f,g) := \sup_{x \in X} d_Y(f(x), g(x)), \quad \forall f, g \in C_b(X,Y).$$

(a) Prove that $(C_b(X,Y), d)$ is a metric space.

(b) Prove that this metric space is complete.

Solution. (a) It is easy to see that $d(f,g) \geq 0$. Suppose $d(f,g) = 0$, then by definition $d_Y(f(x), g(x)) = 0$ for all $x \in X$. Since d_Y is a metric, this implies that $f(x) = g(x)$ for all $x \in X$, hence $f = g$. It is also easy to see that $d(f,g) = d(g,f)$. For triangle inequality, we notice that

$$\begin{aligned} d(f,g) &= \sup_{x \in X} d_Y(f(x), g(x)) \leq \sup_{x \in X} [d_Y(f(x), h(x)) + d_Y(h(x), g(x))] \\ &\leq \sup_{x \in X} d_Y(f(x), h(x)) + \sup_{x \in X} d_Y(h(x), g(x)) = d(f,h) + d(h,g). \end{aligned}$$

(b) Let $\{f_n\}$ be a Cauchy sequence in $C_b(X,Y)$, i.e. for all $\epsilon > 0$, there is $N > 0$ such that $\sup_{x \in X} d_Y(f_n(x), f_m(x)) < \epsilon$ whenever $n, m > N$. Fix any $x \in X$, this implies that $d_Y(f_n(x), f_m(x)) < \epsilon$ for all $n, m > N$, hence $\{f_n(x)\}$ is a Cauchy sequence in Y. Since Y is complete, there is a point, denoted as $f(x)$, such that $f_n(x)$ converges to $f(x)$ in Y. Therefore, the sequence $\{f_n\}$ converges to f pointwisely on X.

From the same argument as in the Cauchy convergence criterion for sequences of functions, one can easily show that $d(f_n - f)$ converges to 0 as $n \to \infty$. So f is the limit of the sequence $\{f_n\}$. One can then mimic the same proof as in the case of $\mathbb{R}$ to show that f is continuous and bounded, hence $f \in C_b(X,Y)$. Therefore, this Cauchy sequence to a limit in the space which implies that the space is complete. $\square$

Problem 5: Let $X = (0, \infty)$ and $d(x,y) = |x - y| + |x^{-1} - y^{-1}|$. Show that (X, d) is a complete metric space.

Solution. We first show that d is a metric. Obviously, positivity and symmetry follow directly from the definition. It's also easy to see that $d(x,y) = 0$ if and only if $x = y$. Moreover, from the standard triangle inequality, one has that

$$d(x,y) \leq |x - z| + |z - y| + |x^{-1} - z^{-1}| + |z^{-1} - y^{-1}| = d(x,z) + d(z,y).$$

Therefore, (X, d) is a metric space.

To see the space is complete, let $\{a_n\}$ be a Cauchy sequence in X. This in particular implies that $\{a_n\}$ is also Cauchy under the standard metric on $\mathbb{R}$. Since $\mathbb{R}$ is complete, there is $p \in \mathbb{R}$ such that $|a_n - p| \to 0$ as $n \to \infty$. We claim that $p \in X$ and that $d(a_n, p) \to 0$.

Since $|a_n - p| \to 0$ and $a_n > 0$, we know that $p \geq 0$. It suffices to show that $p \neq 0$. (Indeed, if $p > 0$, then $p \in X$. And from the fact that $\frac{1}{x}$ is a continuous function on $(0, \infty)$, one concludes from $|a_n - p| \to 0$ that $|a_n^{-1} - p^{-1}| \to 0$, hence $d(a_n, p) \to 0$.) Suppose $p = 0$, then we have that $\{a_n^{-1}\}$ diverges to infinity. However, from the assumption that $\{a_n\}$ is Cauchy in metric d, one should have that $\{a_n^{-1}\}$ is Cauchy in the standard metric on $\mathbb{R}$. From the completeness of $\mathbb{R}$, $\{a_n^{-1}\}$ should converge to some real number $L \in \mathbb{R}$, which is a contradiction. $\Box$

Problem 6: Let (X, d) be a metric space and $S \subset X$. Prove that the following are equivalent:

(a) S is nowhere dense.

(b) S doesn't have a nonempty interior.

(c) S^c is dense in X.

Solution. First, if S is nowhere dense, we claim that S doesn't have a nonempty interior. Suppose not, then there is some point $p \in S$ that is an interior point. That means there exists some $\epsilon > 0$ such that $N(p, \epsilon) \subset S$, which is a contradiction to the definition of nowhere dense.

Second, suppose S doesn't have a nonempty interior, we want to show that S^c is dense in X. To see this, for any $x \in X$ and any $\epsilon > 0$, since x cannot be an interior point of S, $N(x, \epsilon)$ is not contained in S. Therefore, there is some $y \in S^c \cap N(x, \epsilon)$. Obviously, this shows that there is a point $y \in S^c$ such that $d(x, y) < \epsilon$, which implies that S^c is dense in X.

Last, suppose S^c is dense in X. Then for all $x \in X$ and all $\epsilon > 0$, we claim that $N(x, \epsilon)$ cannot be contained in S (hence S is nowhere dense). Indeed, from density of S^c, there is some $y \in S^c$ such that $d(x, y) < \epsilon$. Hence, $N(x, \epsilon)$ is not contained in S. $\Box$

Problem 7: Let (X, d) be a metric space whose completion is $(\bar{X}, \bar{d})$. Show that X is dense in $\bar{X}$.

Solution. Suppose not, then there exists a point $x_0 \in \bar{X}$ and $\epsilon > 0$ such that $N(x_0, \epsilon) \cap X = \emptyset$. Now define $Y = \bar{X} \setminus N(x_0, \epsilon)$. Since Y is a closed subset of a complete metric space, from the proposition in class, we know that $(Y, \bar{d})$ is a complete metric space too. Since $X \subset Y$ and the metric $\bar{d}$ agrees with d. This contradicts the fact that $\bar{X}$ is the completion of X (as by definition, one should have $\bar{X} \subset Y$). $\Box$

Problem 8: Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed vector spaces. In class, we defined their product space $X \times Y = \{(x, y) : x \in X, y \in Y\}$. This is obviously a vector space. Define the following norms on it:

$$\|(x,y)\|_1 = \|x\|_X + \|y\|_Y, \quad \|(x,y)\|_2 = \max\{\|x\|_X, \|y\|_Y\}, \quad \|(x,y)\|_3 = (\|x\|_X^2 + \|y\|_Y^2)^{1/2}.$$

Show that these three norms are equivalent.

Solution. It is easy to see that $\|(x,y)\|_1 \leq 2\|(x,y)\|_2$ and $\|(x,y)\|_2 \leq \|(x,y)\|_1$. So the first two norms are obviously equivalent. Also, one can see that $\|(x,y)\|_2 \leq \|(x,y)\|_3$ and that $\|(x,y)\|_3 \leq (2\|(x,y)\|_2^2)^{1/2} = \sqrt{2}\|(x,y)\|_2$, so the second and the last norm are also equivalent. By transitivity, the first and the third norm are equivalent too. $\square$

Problem 9: Let X be the space of all compactly supported Riemann integrable functions on $\mathbb{R}$, i.e.

$$X = \{f : \mathbb{R} \to \mathbb{R} \big| \text{ there exists } [a,b] \subset \mathbb{R} \text{ such that } f \in R[a,b] \text{ and } f = 0 \text{ outside } [a,b]\}.$$

(a) Show that X is a vector space (over $\mathbb{R}$).

(b) Show that $\|f\|_{L^1} := \int_a^b |f(x)|\, dx$ is a norm on X (where $[a,b]$ is the support of f).

(The space X is not Banach under this norm, and we will denote its completion by $L^1(\mathbb{R})$.)

Solution. (a) If $f, g \in X$, then since their supports are both compact, there exists a bounded interval $[a,b]$ such that $f+g=0$ outside $[a,b]$. Therefore, $f+g \in X$. Similarly, for all $\alpha \in \mathbb{R}$, one has that $\alpha f \in X$. So X is closed under addition and scalar multiplication. It is pretty straightforward to check that the required conditions for vector space are all satisfied. We only note that the zero element of X is the zero function.

(b) Since $|f| \geq 0$, one has that $\|f\|_{L^1} \geq 0$. The only function that can make $\int |f| = 0$ is the constant zero function. Homogeneity follows immediately from the definition of the norm. Lastly, to see the triangle inequality, one just uses the standard triangle inequality to get

$$\|f+g\|_{L^1} = \int |f+g| \leq \int |f| + |g| = \int |f| + \int |g| = \|f\|_{L^1} + \|g\|_{L^1}.$$

$\square$

Problem 10: Show that the continuous functions of compact support (i.e. zero outside some bounded interval) are dense in $L^1(\mathbb{R})$.

Solution. Let $f \in L^1(\mathbb{R})$. For all $\epsilon > 0$, since $L^1(\mathbb{R})$ is the completion of compactly supported Riemann integrable functions, we know that there is some $g \in R[a,b]$ that is supported on $[a,b]$ such that $\|f-g\|_{L^1} < \frac{\epsilon}{2}$. From HW 8 problem #9, we know that there exists some continuous h on $[a,b]$ such that $\|g-h\|_{L^1} < \frac{\epsilon}{2}$. By triangle inequality, this implies that

$$\|f-h\|_{L^1} \leq \|f-g\|_{L^1} + \|g-h\|_{L^1} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

$\square$

Problem 11: Show that if X is a finite dimensional normed space on $\mathbb{R}$. Then all norms on X are equivalent.

Solution. Let $\{e_1, \cdots, e_n\}$ be a basis of X. We first define a norm on X via

$$\|\sum_{i=1}^{n} c_i e_i\|_1 := \max_{1\le i\le n} |c_i|.$$

It's easy to see that this is indeed a norm on X. Now let $\|\cdot\|_2$ be any other norm on X, our goal is to show that it is equivalent to $\|\cdot\|_1$. By triangle inequality, one has for all $x = \sum_{i=1}^{n} c_i e_i \in X$ that

$$\|x\|_2 \le \sum_{i=1}^{n} |c_i|\|e_i\|_2 \le \|x\|_1 \sum_{i=1}^{n} \|e_i\|_2.$$

Let $C := \sum_{i=1}^{n} \|e_i\|_2$, then $\|x\|_2 \le C\|x\|_1$ for all $x \in X$.

This also implies that $f(x) := \|x\|_2$ is a continuous function on $(X, \|\cdot\|_1)$. We claim that $B = \{x \in X : \|x\|_1 \le 1\}$ is compact. (To see this, let $\{x_k\}_k$ be a sequence in B, then the sequence of its first coordinates is bounded in $\mathbb{R}$. By the theorem we learnt, there is a subsequence $\{x_{k_j}\}_j$ whose first coordinates sequence converges to some number $p_1 \in \mathbb{R}$. Then consider the second coordinates sequence, using boundedness again and the result in $\mathbb{R}$, one can pass to another subsequence to make sure that the second coordinates converge to some number $p_2 \in \mathbb{R}$. Repeating this argument for all n coordinates, one can eventually find a subsequence of $\{x_k\}$ such that all its coordinates converge to some $p_i \in \mathbb{R}$ satisfying $|p_i| \le 1$. Then obviously this implies that the subsequence converges, in norm $\|\cdot\|_1$, to the point $\sum_{i=1}^{n} p_i e_i \in B$. Therefore the set B is sequentially compact.)

Since the set $S = \{x \in X : \|x\|_1 < 1\}$ is open, one has that $B \setminus S = \{x \in X : \|x\|_1 = 1\}$ is closed and compact. Since a continuous map on a compact set attains its max and min, we know that there is some point $x_0 \in B \setminus S$ such that $\|x\|_2 = f(x) \ge f(x_0) = \|x_0\|_2 > 0$ for all $\|x\|_1 = 1$. Denote $C' = \|x_0\|_2$. Then we claim that $\|x\|_1 \le C'\|x\|_2$ for all $x \in X$.

To see this, just note that for every nonzero $x \in X$, $y := \frac{x}{\|x\|_1}$ satisfies $\|y\|_1 = 1$. Hence, $\|y\|_2 \ge \|x_0\|_2 = C' = C'\|y\|_1$. By homogeneity, this implies that $\|x\|_1 \le C'\|x\|_2$. $\square$