EXAM 2 CHEAT SHEET

- Infinite series
  - Given a series of real numbers $\sum_{n=1}^{\infty} a_n$, its *n-th partial sum* $S_n := \sum_{k=1}^{n} a_k$. We say the series *converges* to the sum S if the sequence $\{S_n\}$ converges to $S \in \mathbb{R}$. We say the series *diverges* if the sequence $\{S_n\}$ diverges.
  - We say $\sum_{n=1}^{\infty} a_n$ *converges absolutely* if $\sum_{n=1}^{\infty} |a_n|$ converges. We say $\sum_{n=1}^{\infty} a_n$ *converges conditionally* if $\sum_{n=1}^{\infty} |a_n|$ diverges and $\sum_{n=1}^{\infty} a_n$ converges.
  - The series $\sum_{n=1}^{\infty} a_n$ is called a *nonnegative series* if $a_n \geqslant 0$ for all n.
  - Given a series $\sum_{n=1}^{\infty} a_n$, a series $\sum_{k=1}^{\infty} b_k$ that can be obtained by adding parentheses to $\sum_{n=1}^{\infty} a_n$ is called a *regrouping* of $\sum_{n=1}^{\infty} a_n$.
  - A series $\sum_{n=1}^{\infty} a'_n$ is called a *rearrangement* of $\sum_{n=1}^{\infty} a_n$ if there exists a bijection $f : \mathbb{N} \to \mathbb{N}$ such that $a'_n = a_{f(n)}$, $\forall n \in \mathbb{N}$. It's called a *bounded rearrangement* if there exists $M > 0$ such that $|f(n) - n| \leqslant M$, $\forall n \in \mathbb{N}$.
  - Theorem: $\sum_{n=1}^{\infty} a_n$ converges if and only if for all $\epsilon > 0$, there is $N > 0$ such that $|\sum_{k=n}^{m} a_n| < \epsilon$ whenever $m \geqslant n > N$.
  - Corollary: If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.
  - Corollary: If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then it converges.
  - Proposition: If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge and $\alpha, \beta \in \mathbb{R}$, then $\sum_{n=1}^{\infty} \alpha a_n + \beta b_n$ converges.
  - Theorem: Let $\sum_{n=1}^{\infty} a_n$ be a nonnegative series. Then (1) the series converges if and only if its partial sums are bounded; (2) if the series diverges, then $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = \infty$.
  - Theorem: Let $\sum_{n=1}^{\infty} a_n$ be a series and $\sum_{k=1}^{\infty} b_k$ be a regrouping of it. Then (1) if $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{k=1}^{\infty} b_k$ converges and $\sum_{k=1}^{\infty} b_k = \sum_{n=1}^{\infty} a_n$; (2) if $\sum_{n=1}^{\infty} a_n$ is a nonnegative series, then it converges to A if and only if $\sum_{k=1}^{\infty} b_k$ converges to A.
  - Corollary: Let $\{S_n\}$ be the sequence of partial sums of the nonnegative series $\sum_{n=1}^{\infty} a_n$. If $\{S_n\}$ has a bounded (or equivalently, convergent) subsequence, then $\sum_{n=1}^{\infty} a_n$ converges.
  - Comparison test: Let $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ be nonnegative series and suppose that $a_n \leqslant b_n$ for all sufficiently large n. Then $\sum_{n=1}^{\infty} b_n$ converges implies that $\sum_{n=1}^{\infty} a_n$ converges.
  - Ratio test: Let $\sum_{n=1}^{\infty} a_n$ be a nonnegative series such that $a_n \neq 0$, $\forall n$. Then (1) if $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, then the series converges; (2) if $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$, then the series diverges.
  - Root test: Let $\sum_{n=1}^{\infty} a_n$ be a nonnegative series. Then (1) if $\limsup_{n\to\infty} \sqrt[n]{a_n} < 1$, then the series converges; (2) if $\limsup_{n\to\infty} \sqrt[n]{a_n} > 1$, then the series diverges.
  - Alternating series test: Let $\{a_n\}$ be a monotone sequence that converges to zero. Then the alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.
  - Dirichlet convergence test: Let $\sum_{n=1}^{\infty} a_n$ be a series with uniformly bounded partial sums, and $\{b_n\}$ be a monotone sequence that converges to zero. Then $\sum_{n=1}^{\infty} a_n b_n$ converges.
  - Theorem: Let $\sum_{n=1}^{\infty} a_n$ be a series such that $\lim_{n\to\infty} a_n = 0$ and that there is $N > 0$ and a convergent regrouping $\sum_{k=1}^{\infty} b_k$ of $\sum_{n=1}^{\infty} a_n$ such that each b_k is the sum of at most N terms of $\{a_n\}$. Then, $\sum_{n=1}^{\infty} a_n$ converges.

  - Theorem: Let $\sum_{n=1}^{\infty} a_{f(n)}$ be a bounded rearrangement of $\sum_{n=1}^{\infty} a_n$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} a_{f(n)}$ converges. And if they do, they converge to the same sum.
  - Theorem: If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then all of its rearrangements also converge absolutely and to the same sum.
  - Theorem: Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge absolutely. Then the sum of all the terms in their product matrix, in any order, form an absolutely convergent series $\sum_{n=1}^{\infty} a_{k_n} b_{j_n}$, and there holds $\sum_{n=1}^{\infty} a_{k_n} b_{j_n} = \left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right)$.

- Riemann integration
  - A *partition* P of an interval $[a,b]$ is defined as a finite set of points $x_0, x_1, \cdots, x_n \in [a,b]$ such that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. We also write $\Delta x_i = x_i - x_{i-1}$ and define $\lambda(P) = \max_{1 \leqslant i \leqslant n}\{\Delta x_i\}$.
  - Given a function $f : [a,b] \to \mathbb{R}$. Suppose there exists $I \in \mathbb{R}$ such that $\forall \epsilon > 0$, $\exists \delta > 0$ such that for all partition P of $[a,b]$ with $\lambda(P) < \delta$, for all $\xi_i \in [x_{i-1}, x_i]$, $i = 1, \cdots, n$, there holds $|\sum_{i=1}^{n} f(x_i)\Delta x_i - I| < \epsilon$, then f is said to be *Riemann integrable* on $[a,b]$, denoted as $f \in R[a,b]$, and $I := \int_a^b f(x)\,dx$ is called the *definite integral* of f on $[a,b]$.
  - Let f be a bounded function on $[a,b]$ and P be a partition of $[a,b]$. Write $M_i = \sup_{x_{i-1} \leqslant x \leqslant x_i} f(x)$ and $m_i = \inf_{x_{i-1} \leqslant x \leqslant x_i} f(x)$, $\forall i = 1, \cdots, n$. Then $U(P,f) := \sum_{i=1}^{n} M_i \Delta x_i$ and $L(P,f) := \sum_{i=1}^{n} m_i \Delta x_i$ are called the *upper sum* and *lower sum* of f with respect to P.
  - We say a partition P^* is a *refinement* of P if $P^* \supset P$.
  - The *upper integral* $\overline{\int_a^b} f(x)\,dx := \inf_P U(P,f)$, the *lower integral* $\underline{\int_a^b} f(x)\,dx := \sup_P L(P,f)$.
  - Theorem: If $f \in R[a,b]$, then f is bounded.
  - Proposition: If $P^* \supset P$, then $U(P^*, f) \leqslant U(P,f)$ and $L(P^*, f) \geqslant L(P,f)$.
  - Proposition: For any partitions P_1 and P_2 of $[a,b]$, there holds $L(P_1, f) \leqslant U(P_2, f)$. Moreover, there holds $\underline{\int_a^b} f(x)\,dx \leqslant \overline{\int_a^b} f(x)\,dx$.
  - Theorem: Let f be bounded on $[a,b]$. Then the following are equivalent: (1) $f \in R[a,b]$; (2) $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$; (3) $\forall \epsilon > 0$, there exists a partition P of $[a,b]$ such that $U(P,f) - L(P,f) < \epsilon$.
  - Theorem: If f is continuous on $[a,b]$, then $f \in R[a,b]$.
  - Theorem: If f is monotone on $[a,b]$, then $f \in R[a,b]$.
  - Theorem: Let f be bounded on $[a,b]$. Suppose that f has only finitely many discontinuities on $[a,b]$, then $f \in R[a,b]$.
  - Theorem: If $f, g \in R[a,b]$, and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g \in R[a,b]$ and $\int_a^b \alpha f + \beta g = \alpha \int_a^b f + \beta \int_a^b g$.
  - Theorem: If $f, g \in R[a,b]$, then $fg \in R[a,b]$.
  - Theorem: If $f, g \in R[a,b]$ and that $f \geqslant g$ on $[a,b]$, then $\int_a^b f \geqslant \int_a^b g$.
  - Theorem: If $f \in R[a,b]$, then $|f| \in R[a,b]$ and $\left|\int_a^b f\right| \leqslant \int_a^b |f|$.
  - Theorem: If $m \leqslant f(x) \leqslant M$ on $[a,b]$ and $f \in R[a,b]$, then $m(b-a) \leqslant \int_a^b f \leqslant M(b-a)$.
  - Theorem: If $g \in R[a,b]$ and that $m \leqslant g(x) \leqslant M$ on $[a,b]$. Then, if f is continuous on $[m, M]$, there holds $f(g) \in R[a,b]$.

- Corollary: If $f \in R[a,b]$ and g is a function obtained by changing the value of f at finitely many points on $[a,b]$. Then, $g \in R[a,b]$ and $\int_a^b g = \int_a^b f$.

- Theorem, Let $a < c < b$, then $f \in R[a,b]$ if and only if $f \in R[a,c]$ and $f \in R[c,b]$. Moreover, when $f \in R[a,b]$, there holds $\int_a^b f = \int_a^c f + \int_c^b f$.

- Fundamental Theorem of Calculus (part I): Let $f \in R[a,b]$ and suppose that there is a differentiable function F on $[a,b]$ such that $F'(x) = f(x), \forall x \in [a,b]]$ (i.e. F is an antiderivative of f on $[a,b]$). Then, $\int_a^b f = F(b) - F(a)$.

- Fundamental Theorem of Calculus (part II): Let $f \in R[a,b]$. Define $F(x) := \int_a^x f(t)\,dt$, $\forall x \in [a,b]$. Then F is continuous on $[a,b]$. In addition, if f is continuous at $x_0 \in [a,b]$, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

- Theorem: Let f be continuous on $[a,b]$ and φ have integrable derivative on $[\alpha,\beta]$. Suppose $\varphi(\alpha) = a$, $\varphi(\beta) = b$, and $\varphi(t) \in [a,b]$, $\forall t \in [\alpha,\beta]$. Then $\int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)\,dt$.

- Theorem: Let F, G be differentiable on $[a,b]$, and that $F' = f \in R[a,b]$, $G' = g \in R[a,b]$. Then $\int_a^b Fg = F(b)G(b) - F(a)G(a) - \int_a^b fG$.

- Sequences and series of functions
  - Let $\{f_n\}$ be a sequence of functions defined on a set $D \subset \mathbb{R}$. Suppose for all $x \in D$ the sequence $\{f_n(x)\}$ converges to limit $f(x) \in \mathbb{R}$, then $f(x)$ is a function defined on D and that $\lim_{n\to\infty} f_n(x) = f(x)$, $\forall x \in D$. We say $\{f_n\}$ *converges on D pointwisely* to f. Similarly, if $\sum_{n=1}^\infty f_n(x)$ converges at every $x \in D$ and define $S(x) = \sum_{n=1}^\infty f_n(x)$, then $S(x)$ is called the sum of the series and we say the series converges to $S(x)$ pointwisely on D.

  - Let $\{c_n\} \subset \mathbb{R}$. Then $\sum_{n=0}^\infty c_n x^n$ is called a *power series*. The numbers $\{c_n\}$ are called the *coefficients* of the series. The number $R := \sup\{|x| : \sum_{n=0}^n c_n x^n \text{ converges}\}$ is called the *radius of convergence* of the series. The set on which the series converges is called the *interval of convergence*.

  - We say a sequence $\{f_n\}$ converges *uniformly* on D (denoted as $f_n \rightrightarrows f$) if for all $\epsilon > 0$, there is $N > 0$ such that $|f_n(x) - f(x)| < \epsilon$ for all $x \in D$ and $n > N$. Similarly, we say a series $\sum_{n=1}^\infty f_n(x)$ converges *uniformly* on D if its partial sum sequence $\{S_n(x)\}$ converges uniformly on D.

  - Given a set D and a function $f : D \to \mathbb{R}$, the *uniform norm* (or, *sup norm*) on D is defined as $\|f\|_{U(D)} := \sup_{x\in D} |f(x)|$.

  - We say a series $\sum_{n=1}^\infty f_n(x)$ converges *uniformly absolutely* on D if the series $\sum_{n=1}^\infty |f_n(x)|$ converges uniformly on D.

  - Suppose f is infinitely differentiable at x_0. Then $\sum_{n=0}^\infty \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$ is called the *Taylor series of f at x_0*. Suppose in addition that the series converges to $f(x)$ on some neighborhood U of x_0, then we say the series is the *Taylor expansion of f on U* centered at x_0. If for all $x_0 \in I$, there exists $\delta > 0$ such that f has a Taylor expansion on $(x_0 - \delta, x_0 + \delta)$ centered at x_0, then f is said to be *real analytic* on the interval I.

  - Let f be infinitely differentiable on $(x_0 - R, x_0 + R)$. Then the *remainder term of order n* of the Taylor series of f at x_0 is the function $R_n(x) := f(x) - \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$.

  - Let f be defined on an interval I. We say f can be *approximated by polynomials* on I if $\forall\epsilon > 0$, there exists a polynomial $p(x)$ such that $|f(x) - p(x)| < \epsilon$ for all $x \in I$.

  - Let (X,d) be a metric space. Then $S \subset X$ is *sequentially compact* if any sequence $\{a_n\} \subset S$ has a convergent subsequence whose limit is contained in S.

  - A family of functions $\{f_n\}$ are *equicontinuous* at $x_0 \in D$ if for all $\epsilon > 0$, there is $\delta > 0$ such that $|f_n(x_0) - f_n(y)| < \epsilon$ for all $n \in \mathbb{N}$, whenever $|x_0 - y| < \delta$. The family $\{f_n\}$ are called *uniformly equicontinuous* on D if for all $\epsilon > 0$, there is $\delta > 0$ such that $|f_n(x) - f_n(y)| < \epsilon$ for all $n \in \mathbb{N}$ and all $x, y \in D$ whenever $|x - y| < \delta$.

- Proposition: If $\sum_{n=0}^{\infty} c_n x^n$ converges at $x = x_0 \neq 0$, then it converges absolutely at all $x \in (-|x_0|, |x_0|)$.

- Theorem: Given a power series $\sum_{n=1}^{\infty} c_n x^n$, let $\alpha := \limsup_{n\to\infty} \sqrt[n]{|c_n|} \in [0, \infty]$. Then its radius of convergence is $R = \frac{1}{\alpha}$.

- Proposition: Suppose $f_n \rightrightarrows f$ on D. Then for all $D' \subset D$, there holds $f_n \rightrightarrows f$ on D'.

- Proposition: Suppose $f_n \rightrightarrows f$ on D_1 and D_2, then $f_n \rightrightarrows f$ on $D_1 \cup D_2$.

- Proposition: Suppose $f_n \rightrightarrows f$ and $g_n \rightrightarrows g$ on D and $\alpha, \beta \in \mathbb{R}$. Then $\alpha f_n + \beta g_n \rightrightarrows \alpha f + \beta g$ on D. If $\{f_n\}, \{g_n\}$ are further assumed to be uniformly bounded on D, then $f_n g_n \rightrightarrows fg$ on D.

- Theorem: $f_n \rightrightarrows f$ on D if and only if $\|f_n - f\|_{U(D)} \to 0$ as $n \to \infty$.

- Cauchy convergence criterion: $\{f_n\}$ converges uniformly on D if and only if $\{f_n\}$ form a Cauchy sequence under the sup norm (i.e. for all $\epsilon > 0$, there is $N > 0$ such that $|f_n(x) - f_m(x)| < \epsilon$ whenever $x \in D$ and $m, n > N$.)

- Corollary: $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on D if and only if for all $\epsilon > 0$, there is $N > 0$ such that $|\sum_{k=n}^{m} f_k(x)| < \epsilon$ whenever $x \in D$ and $m \geqslant n > N$.

- Corollary: Let $\{f_n\}$ be a sequence of continuous functions on $[a, b]$. Suppose that $\{f_n\}$ converges uniformly on (a, b), then it converges uniformly on $[a, b]$.

- Weierstrass M-test: Let $f_n : D \to \mathbb{R}$, $\forall n \in \mathbb{N}$. Suppose for each n, there is $M_n > 0$ such that $|f_n(x)| \leqslant M_n$, $\forall x \in D$, and suppose that $\sum M_n$ converges. Then $\sum f_n(x)$ converges uniformly absolutely on D.

- Theorem: Suppose $f_n \rightrightarrows f$ on D and $x_0 \in D'$. For each $n \in \mathbb{N}$, suppose $\lim_{x\to x_0} f_n(x) = a_n \in \mathbb{R}$. Then $\{a_n\}$ converges and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \lim_{x\to x_0} f_n(x) = \lim_{x\to x_0} \lim_{n\to\infty} f_n(x) = \lim_{x\to x_0} f(x)$.

- Corollary: Suppose $f_n \rightrightarrows f$ on D and every f_n is continuous on D. Then f is continuous on D.

- Proposition: Suppose power series $\sum c_n x^n$ converges at $x = x_0 \neq 0$, then it converges uniformly absolutely on any closed subinterval of $(-|x_0|, |x_0|)$. Moreover, its sum function is continuous on $(-|x_0|, |x_0|)$.

- Theorem: Suppose $f_n \rightrightarrows f$ on $[a, b]$ and that $f_n \in R[a, b]$, $\forall n \in \mathbb{N}$. Then $f \in R[a, b]$ and $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.

- Corollary: Suppose the series $\sum f_n(x)$ converges uniformly to $S(x)$ on $[a, b]$ and that $f_n \in R[a, b]$, $\forall n \in \mathbb{N}$. Then $S \in R[a, b]$ and $\int_a^b S = \sum \int_a^b f_n$.

- Corollary: Suppose the radius of convergence of the power series $\sum c_n x^n$ is $R \in (0, \infty]$. Then for any $t_1, t_2 \in (-R, R)$, $\int_{t_1}^{t_2} \sum_{n=0}^{\infty} c_n x^n \, dx = \sum_{n=0}^{\infty} \int_{t_1}^{t_2} c_n x^n \, dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(t_2^{n+1} - t_1^{n+1})$.

- Theorem: Let f_n be differentiable on $[a, b]$, $\forall n \in \mathbb{N}$. Suppose that $\{f_n\}$ converges at some point $x_0 \in [a, b]$ and that for some function g there holds $f_n' \rightrightarrows g$ on $[a, b]$. Then $\{f_n\}$ converges uniformly to some function f on $[a, b]$ and $f'(x) = \lim_{n\to\infty} f_n'(x)$ on $[a, b]$.

- Corollary: Let f_n be differentiable on $[a, b]$, $\forall n \in \mathbb{N}$. Suppose that the series $\sum f_n(x)$ converges at some point $x_0 \in [a, b]$ and that $\sum f_n'(x)$ converges uniformly on $[a, b]$. Then $\sum f_n(x)$ converges uniformly to some function $S(x)$ on $[a, b]$ and that $S'(x) = \sum f_n'(x)$ on $[a, b]$.

- Corollary: Suppose the radius of convergence of the power series $\sum c_n x^n$ is $R \in (0, \infty]$. Then for all $x \in (-R, R)$, f is infinitely differentiable at x and $f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) c_n x^{n-k}$ for all $k \in \mathbb{N}$.

- Proposition: If $f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$ on $(x_0 - R, x_0 + R)$, then $c_n = \frac{f^{(n)}(x_0)}{n!}$ for all $n \geqslant 0$ and hence the power series must be the Taylor expansion of f.

- Theorem: Let f be infinitely differentiable on (a, b). If there is $M > 0$ such that $|f^{(n)}(x)| \leqslant M^n$ for all $x \in (a, b)$ and all n, then f is real analytic on (a, b), and for any $x_0 \in (a, b)$, one has that f can be Taylor expanded on

(a, b) centered at x_0.

- Weierstrass approximation theorem: Let f be continuous on $[a, b]$. Then f can be approximated by polynomials on $[a, b]$.

- Arzelà-Ascoli theorem: Let $D \subset \mathbb{R}$ be compact. Then $\mathcal{F} \subset C(D)$ is compact (hence equivalently sequentially compact) if and only if $\mathcal{F}$ is closed, bounded, and uniformly equicontinuous.

• Metric spaces revisited

- Given two metric spaces (X, d_X), (Y, d_Y) and a map $f : X \to Y$, we say $\lim_{x \to p} f(x) = y$ for some $p \in X'$ and $y \in Y$, if for all $\epsilon > 0$, there is $\delta > 0$ such that $d_Y(f(x), y) < \epsilon$ whenever $d_X(x, p) < \delta$. We say f is *continuous* at p if for all $\epsilon > 0$, there is $\delta > 0$ such that $d_Y(f(x), f(p)) < \epsilon$ whenever $d_X(x, p) < \delta$.

- A sequence $\{a_n\}$ in metric space (X, d) *converges to* $L \in X$ if for all $\epsilon > 0$, there is $N > 0$ such that $d(a_n, L) < \epsilon$ whenever $n > N$.

- A sequence $\{a_n\}$ in metric space (X, d) is called a *Cauchy sequence* if for all $\epsilon > 0$, there is $N > 0$ such that $d(a_n, a_m) < \epsilon$ whenever $n, m > N$.

- The space (X, d) is called *complete*, if every Cauchy sequence in the space converges to some limit in X.

- Let (X, d) be a metric space. A set $S \subset X$ is *dense* in X, if for every $x \in X$ and every $\epsilon > 0$, there is $s \in S$ such that $d(x, s) < \epsilon$. A set $E \subset X$ is called *nowhere dense* if $\forall x \in X$ and $\forall \epsilon > 0$, $N(x, \epsilon)$ is not contained in E.

- Let X be a vector space (over $\mathbb{R}$ or $\mathbb{C}$). A *norm* in X is a real valued function on X, denoted as $\|\cdot\|$, satisfying the following properties: (1) $\|x\| \geqslant 0$ for all $x \in X$ and equality holds if and only if $x = 0$; (2) $\|x+y\| \leqslant \|x\| + \|y\|$, for all $x, y \in X$; (3) for all scalar α and all $x \in X$, $\|\alpha x\| = |\alpha|\|x\|$.

- Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ defined on the same vector space X are called *equivalent* if there is a constant C such that $C\|x\|_1 \leqslant \|x\|_2 \leqslant C^{-1}\|x\|_1$ for all $x \in X$.

- A *Banach space* is a complete normed vector space.

- Proposition: Given $f : X \to Y$ where X, Y are metric spaces, f is continuous on X if and only if $f^{-1}(O)$ is open in X for all open set O in Y.

- Proposition: A subset of a complete metric space is complete if and only if it is closed.

- Proposition: Every compact metric space is complete.

- Theorem: Given metric space (X, d), there is a complete metric space $(\bar{X}, \bar{d})$, called the *completion* of (X, d), satisfying: (1) $X \subset \bar{X}$ and $\bar{d}(x, y) = d(x, y)$, $\forall x, y \in X$; (2) for any other complete metric space (Y, d_Y) satisfying property (1), there holds $\bar{X} \subset Y$ and $d_Y(x, y) = \bar{d}(x, y)$, $\forall x, y \in \bar{X}$.

- Baire Category Theorem: Let (X, d) be a complete metric space. Then any countable family of open dense subset of X has a dense intersection.

- Osgood Theorem: Let (X, d) be a complete metric space. Let $\{f_n\}$ be a sequence of continuous functions on X and suppose that for each $x \in X$, $\{f_n(x)\}_n$ is a bounded set in $\mathbb{R}$. Then, there is a nonempty open set $V \subset X$ such that $\{f_n\}$ are uniformly bounded on V.

- Contraction Mapping Principle: Let (X, d) be a complete metric space and $f : X \to X$. Suppose there is some $\rho \in [0, 1)$ such that $d(f(x), f(y)) \leqslant \rho d(x, y)$ for all $x, y \in X$, then f has a unique fixed point (i.e. a point $x_0 \in X$ such that $f(x_0) = x_0$.)

- Theorem: The completion $\bar{X}$ of a normed vector space X under the metric $\|x - y\|_X$ has a natural vector space structure that makes $\bar{X}$ a Banach space.