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FERTILITY CHOICE IN A MODEL OF ECONOMIC GROWTH

BY ROBERT J. BARRO AND GARY S. BECKER[1]

Altruistic parents make choices of family size along with decisions about consumption and intergenerational transfers. We apply this framework to a closed economy, where the determination of interest rates and wage rates is simultaneous with the determination of population growth and the accumulation of capital. Thus, we extend the literature on optimal economic growth to allow for optimizing choices of fertility and intergenerational transfers. We use the model to assess the effects of child-rearing costs, the tax system, the conditions of technology and preferences, and shocks to the initial levels of population and the capital stock.

KEYWORDS: Economic growth, fertility, intergenerational transfers, population.

WE USE A FRAMEWORK where parents and children are linked through altruism to analyze a family's optimal choice of fertility. This choice is joint with decisions about consumption, intergenerational transfers, and the accumulation of capital. We apply this framework to a closed economy, where the determination of interest rates and wage rates is simultaneous with the determination of fertility, consumption, and saving. Thus, we extend the literature on optimal economic growth (e.g., Ramsey (1928), Cass (1965), and Koopmans (1965)) to allow for endogenous and optimizing choices of population growth and intergenerational transfers.

Section 1 derives the path of fertility, capital-labor ratios, interest rates, wage rates, and consumption per person. Although we pay particular attention to steady-state behavior, we also discuss dynamic paths.

Section 2 uses methods of comparative statics and dynamics to explore various implications of the model. We study changes in the initial capital-labor ratio, variations in the cost of raising children, different rates of technological progress, differences in the degree of altruism, shifts in productivity, and changes in the tax on capital. We also discuss the welfare implications of our analysis.

Section 3 sketches how the results would be modified to allow for life-cycle savings, along with the intergenerational savings considered in Sections 1 and 2. Section 4 summarizes the main conclusions.

Only a few papers have tried to integrate a model of fertility into a model of the economy. Willis (1985) adopts our framework of altruism to study whether fertility decisions are socially efficient. Eckstein and Wolpin (1982) consider normative and positive economic issues in a model where parents are not altruistic. Their model cannot consider bequests and investments in children's human capital because of the assumption that parents' utility depends on own

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consumption and on the number of children, but not on any measure of the quality of children. Razin and Ben Zion (1975) is an important early paper, but they make the unattractive assumption that the number of children and the utility per child are separable in parents' preferences. We appear to be the first to work out the dynamics and comparative statics of a model that combines a neoclassical production function with fertility choices based on parental altruism.

1. SETUP OF THE MODEL

The main analysis neglects life-cycle considerations and assumes that each person lives for two generations, childhood and adulthood. We abstract from marriage and the spacing of children, and assume that each person has children at the beginning of his adult period. A basic assumption is that parents are altruistic toward their children. Parents' utility depends on own consumption, and separably on the number of children and utility of each child.

If the utility of each child enters linearly, the utility of an adult in the ith generation is

$$(1) \qquad U_i = v(c_i) + a(n_i)n_iU_{i+1},$$

where c_i is own consumption, $v(c_i)$ is the utility from this consumption, n_i is the number of children, $a(n_i)$ measures the degree of altruism toward each child, and U_{i+1} is the utility attained by each child. For most of the discussion, we specialize the $a(n)$ function to the constant-elasticity form,

$$(2) \qquad a(n_i) = \alpha(n_i)^{-\varepsilon}, \quad \text{where} \quad 0 < \alpha < 1 \quad \text{and} \quad 0 < \varepsilon < 1.$$

This restriction on ε guarantees that, for given utility per child U_{i+1}, parental utility U_i increases, but at a diminishing rate, with the number of children n_i. By substituting out for U_{i+1} (and U_{i+2}, etc.) in equation (1), we get the dynastic utility function,

$$(3) \qquad U_0 = \sum_{i=0}^{\infty} \alpha^i (N_i)^{1-\varepsilon} v(c_i),$$

where $N_i = \prod_{j=0}^{i-1} n_j$ for $i = 1, 2, \ldots$ (and $N_0 = 1$) is the number of descendants in generation i. We also assume for most of our analysis that the elasticity of v with respect to c is constant—that is,

$$(4) \qquad v(c_i) = (c_i)^{\sigma},$$

where $\sigma < 1$.

Each adult earns the wage rate w_i on time supplied to the market. Since we define an adult's total time available per period to be 1 unit, w_i is also the amount of full labor income. Parents leave a bequest to each child of (nondepreciable) capital, k_{i+1}—possibly in the form of human capital—at the beginning of the child's adulthood. Capital k_i earns rent at the rate r_i. An adult in generation i spends his earnings and inheritance, $w_i + (1 + r_i)k_i$, on own consumption, c_i, on bequests to children, n_ik_{i+1}, and on costs of raising children. We assume that each child costs β_i (in units of real income), so that $n_i\beta_i$ is the total cost of raising children to adulthood. As we discuss later, this cost could represent purchases of goods and subtraction from the time available for work. (We assume a fixed amount of leisure that will be ignored.) The overall budget condition for an adult in generation i is

$$w_i + (1 + r_i)k_i = c_i + n_i(\beta_i + k_{i+1}). \tag{5}$$

The optimization problem as seen by the dynastic head is to maximize utility U_0 in equation (3), subject to the budget constraints in equation (5) and to the initial assets k_0.[2] In carrying out this maximization, each individual takes as given the path of wage rates, w_i, and interest rates, r_i. The chosen path of consumption per adult, $c_0, c_1, c_2, \dots$; capital stock per adult, $k_1, k_2, \dots$; and number of descendants, $N_1, N_2, \dots$, must be consistent with this maximization problem.[3]

Assuming that the attainable utility U_0 is bounded (see below), we can obtain the first-order conditions in the usual manner, with allowance for a Lagrange multiplier for each period that corresponds to each of the budget constraints in equation (5). People have no children in our model unless $\sigma + \varepsilon < 1$. This is a second-order condition for a maximum that we assume to hold. The two sets of first-order conditions are

$$(c_{i+1}/c_i)^{(1-\sigma)} = \alpha(1 + r_{i+1})/(n_i)^{\varepsilon} \qquad (i = 0, 1, \dots) \tag{6}$$

and

$$v(c_i)(1 - \varepsilon - \sigma) = v'(c_i)[\beta_{i-1}(1 + r_i) - w_i] \qquad (1 = 1, 2, \dots). \tag{7}$$

There is also the dynastic budget constraint, which equates the present value of all resources to the present value of all expenditures,[4]

$$k_0 + \sum_{i=0}^{\infty} d_i N_i w_i = \sum_{i=0}^{\infty} d_i(N_i c_i + N_{i+1}\beta_i), \tag{8}$$

where $d_i = \prod_{j=0}^{i} (1 + r_j)^{-1}$.[5]

Equation (6) is an arbitrage condition for shifting consumption from one generation to the next. Aside from the term that depends on n_i, this equation

[2] Consumption c_i and numbers of descendants N_i must also be nonnegative in each generation. However, we neglect integer restrictions on N_i. We can also allow for debt left to children ($k_{i+1} < 0$). Then Ponzi games, in which the debt grows forever as fast as or faster than the interest rate, would be ruled out if the present value (as of period 0) of debt must approach zero asymptotically. In our present model of a closed economy without life-cycle savings, $k_{i+1} > 0$ (positive bequests) is an equilibrium condition for the representative person.

[3] We pretend that the dynastic head can pick the entire time path. However, since the objective function is time consistent, the descendants face a problem of the same form, and they have no incentive to deviate from the choices made initially. See Becker and Barro (1988) for further discussion of time consistency in this framework.

[4] The dynastic budget equation follows from the constraints for each period, as shown in equation (5), as long as the transversality condition is satisfied: that the present value of the future capital stock approaches zero asymptotically ($\lim_{i \to \infty} d_i N_i k_i = 0$). We also use the constraint on borrowing, which is discussed in note 2 above.

[5] We can think of equations (6) and (7) as coming from the maximization of U_0 in equation (3), subject to the present-value constraint in equation (8). Then equation (6) follows readily from maximization with respect to c_i, holding fixed c_j for $j \neq i$ and N_j for all j. Equation (7) follows from maximization with respect to N_i, holding fixed N_j for $j \neq i$ and c_j for all j.

expresses the familiar result that the utility rate of substitution between consumption in periods $i+1$ and i depends on the "time-preference" factor, α, and the interest-rate factor, $1+r_{i+1}$. The standard conclusion is that a rise in α or r_{i+1} increases c_{i+1} relative to c_i. In our modified arbitrage condition, an increase in fertility, n_i, lowers altruism per child, given by $a(n_i)$, and thereby increases the discount on future consumption. Lower altruism is equivalent to a higher rate of time preference. Therefore, higher fertility is associated with a reduction in c_{i+1} relative to c_i, for given values of r_{i+1} and α.

Equation (7) says that the marginal benefit of an additional child (or equivalently of an additional adult descendant for the next period) balances the marginal cost. The right side of the equation is the net lifetime cost of an additional adult in generation i. The lifetime earnings of each adult, w_i, subtract from the term, $\beta_{i-1}(1+r_i)$, which is the cost of rearing a child in generation $i-1$ when valued in goods of generation i. Note that $1+r_i$ is the real interest factor across generations, and would typically be a large number.

Since $1-\varepsilon-\sigma>0$, equation (7) implies that children are a net financial burden to altruistic parents: the cost of rearing an additional child, $\beta_i(1+r_{i+1})$, would exceed his lifetime earnings, w_{i+1}. Children may be a net burden in modern economies because β_i must include the sizable costs of acquiring human capital if w_{i+1} includes the earnings from human capital.

Using the form of $v(c_i)$ from equation (4), we can rewrite equation (7) as

$$(9) \qquad c_i = \frac{\sigma}{(1-\varepsilon-\sigma)}\left[\beta_{i-1}(1+r_i)-w_i\right] \qquad (i=1,2,\ldots).$$

Equation (9) shows that an increase in the net cost of creating a descendant in generation i, $\beta_{i-1}(1+r_i)-w_i$, leads to an increase in consumption per person, c_i, for that generation. When people are more costly to produce, it is optimal to endow each person produced with a higher level of consumption. In effect, it pays to raise the "utilization rate" (in the sense of a higher c_i) when costs of production of descendants are greater.

Equation (9) implies that consumption per person, c_i, would change across generations only if the net cost of creating descendants also changes. Hence, descendants have the same consumption if they are equally costly to produce. In contrast, the usual models of optimal consumption over time imply that consumption grows (or falls) over time if the interest rate exceeds (or is less than) the rate of time preference. It is the endogeneity of the discount rate in our analysis that makes the rate of growth between generations of consumption per person independent of the level of interest rates. For the same reason the growth rate does not depend on the degree of altruism or time preference, as summarized by the parameter α.

Changes in the level of interest rates or in the degree of altruism mainly affect fertility, n_i. We can use equations (6) and (9) to solve out for the fertility rate:

$$(10) \qquad (n_i)^{\varepsilon} = \left[\alpha(1+r_{i+1})\right]\left[\frac{\beta_{i-1}(1+r_i)-w_i}{\beta_i(1+r_{i+1})-w_{i+1}}\right]^{(1-\sigma)} \qquad (i=1,2,\ldots).$$

For example, if β_i, r_i, and w_i are the same for all generations, the fertility rate, n_i for $i = 1, 2, \ldots,$ would rise with increases in the interest rate or the rate of altruism, α. Higher interest rates or greater altruism motivate a family to have a larger number of descendants, and fertility is the rate of investment in these descendants. These responses satisfy the arbitrage condition for shifting consumption across generations (as given in equation (6)), even though the time path of consumption per person does not change. The change in the relative number of people across generations is a form of intertemporal substitution that replaces the usual response in the path of consumption per person.

In an open economy, where the interest rate is given exogenously by the rest of the world, a country's aggregate holding of assets, K_i, could differ from its stock of productive capital. That is, the country could be a net international creditor or debtor (and $K_i < 0$ is even possible). In this paper we assume a closed economy where the holding of assets equals the stock of productive capital. As in standard growth models of a closed economy, interest rates and wage rates, r_i and w_i, then depend—via conditions for production—on the economy's stock of productive capital.

In order to work out the full equilibrium, we now specify that production takes place in "firms" via a standard one-sector production function that exhibits constant returns to scale and Harrod-neutral technical progress. Hence,

$$Y_i = F\left[K_i, \quad (1+g)^i L_i\right], \tag{11}$$

where Y_i is output of goods, K_i is capital (goods accumulated prior to the start of generation i), L_i is labor, and g is the exogenous rate of technical progress. We assume the Harrod-neutral form of technological change in order to deal with steady-state situations where per capita variables all grow at the constant rate g. For simplicity, we neglect depreciation of capital. Letting $\hat{y}_i \equiv Y_i/[(1+g)^i L_i]$ denote output per effective worker, and $\hat{k}_i \equiv K_i/[(1+g)^i L_i]$ denote capital per effective worker, we also have

$$\hat{y}_i = f(\hat{k}_i), \qquad f' > 0, \quad f'' < 0. \tag{12}$$

The output from production is paid either as rentals to capital, $r_i K_i$, or as wages to workers, $w_i L_i$. A competitive market and profit maximization by firms guarantee that r_i equals the marginal product of capital and w_i equals the marginal product of labor,

$$r_i = f'(\hat{k}_i); \quad w_i = \left[f(\hat{k}_i) - \hat{k}_i f'(\hat{k}_i)\right](1+g)^i. \tag{13}$$

Hence, by $f'' < 0$, a higher value for $\hat{k}_i$ means a lower interest rate r_i and a higher wage rate w_i.

We assume that child rearing involves household production that entails partly an input of time and partly an input of goods. Specifically, we suppose that each child requires $a(1+g)^i$ units of goods and b units of parents' time. Since parents' time is valued at w_i, the child-rearing cost is then

$$\beta_i = a(1+g)^i + b w_i, \qquad a \geqslant 0 \quad \text{and} \quad 0 \leqslant b < 1. \tag{14}$$

The condition $0 \leqslant b < 1$ means that adults do not spend all their time on child rearing. The goods spent on child rearing rise along with the increase in the "quality" of children as measured by their prospective productivity as adults. Since w_i rises along with $(1+g)^i$ in equation (13), the goods cost would otherwise eventually become negligible relative to the time cost. We neglect any dependence of β_i on other dimensions of child quality, such as prospective consumption, c_{i+1}, wage rate, w_{i+1}, or inheritance, k_{i+1}. Finally, we define the wage rate per effective worker and the "effective cost" of raising a child as

$$\begin{aligned} \hat{w}_i &\equiv w_i/(1+g)^i = \left[f(\hat{k}_i) - \hat{k}_i f'(\hat{k}_i)\right], \\ \hat{\beta}_i &\equiv \beta_i/(1+g)^i = a + b\hat{w}_i = a + b\left[f(\hat{k}_i) - \hat{k}_i f'(\hat{k}_i)\right]. \end{aligned} \tag{15}$$

Steady-State Growth

The time allocated to the production of goods, L_i, depends on the number of adults, N_i, and the time spent on raising children. (Recall that we assume a fixed amount of leisure.) Since each adult spends bn_i units of time on child rearing, where b is the time required per child, the time allocated to goods is

$$L_i = (1 - bn_i)N_i.$$

Therefore, we have to multiply by $(1 - bn_i)$ in order to replace amounts expressed per adult (such as k_i) by amounts expressed per unit of labor input. After also dividing through by $(1+g)^i$ in equation (5), we get the revised budget constraint in terms of effective units:

$$\hat{w}_i + (1 - bn_i)(1 + r_i)\hat{k}_i = \hat{c}_i + n_i\left[\hat{\beta}_i + (1+g)(1 - bn_{i+1})\hat{k}_{i+1}\right], \tag{16}$$

where $\hat{w}_i$ and $\hat{\beta}_i$ are defined in equation (15), and $\hat{c}_i \equiv c_i/(1+g)^i$.

Equation (9) implies that $\hat{c}_i$ is

$$\hat{c}_i = \left(\frac{\sigma}{1 - \varepsilon - \sigma}\right)\left[\hat{\beta}_{i-1}\left(\frac{1 + r_i}{1 + g}\right) - \hat{w}_i\right] \qquad (i = 1, 2, \ldots). \tag{17}$$

Substituting into equation (16) gives the key relation for determining the dynamics of $\hat{k}_i$:

$$\begin{aligned} \hat{w}_i + (1 - bn_i)(1 + r_i)\hat{k}_i = &\left(\frac{\sigma}{1 - \varepsilon - \sigma}\right)\left[\hat{\beta}_{i-1}\left(\frac{1 + r_i}{1 + g}\right) - \hat{w}_i\right] \\ &+ n_i\left[\hat{\beta}_i + (1+g)(1 - bn_{i+1})\hat{k}_{i+1}\right] \\ &(i = 1, 2, \ldots). \end{aligned} \tag{18}$$

Recall that $\hat{w}_i$, $\hat{\beta}_i$, and r_i depend on $\hat{k}_i$ (see equation (15)). The final element is the formula for the fertility rate, n_i, which follows from equation (10) as

$$(n_i)^{\varepsilon} = \alpha(1 + r_{i+1})\left[\frac{1}{(1+g)}\left(\frac{\hat{\beta}_{i-1}(1 + r_i) - (1+g)\hat{w}_i}{\hat{\beta}_i(1 + r_{i+1}) - (1+g)\hat{w}_{i+1}}\right)\right]^{(1-\sigma)} \qquad (i = 1, 2, \ldots). \tag{19}$$

In a steady state $\hat{k}_i$—and hence n_i, $\hat{c}_i$, r_i, $\hat{w}_i$, and $\hat{\beta}_i$—do not change. The amounts per person—y_i, c_i, w_i—all grow at the rate g. Hence, equations (16)–(19) become

$$\hat{w} + (1+r)\hat{k} = f(\hat{k}) + \hat{k} = \hat{c} + n[\hat{\beta} + (1+g)\hat{k}] + bn\hat{k}[1 + r - n(1+g)], \tag{20}$$

$$\hat{c} = \left(\frac{\sigma}{1-\varepsilon-\sigma}\right)\left[\hat{\beta}\left(\frac{1+r}{1+g}\right) - \hat{w}\right], \tag{21}$$

$$f(\hat{k}) + \hat{k} = \left(\frac{\sigma}{1-\varepsilon-\sigma}\right)\left[\hat{\beta}\left(\frac{1+r}{1+g}\right) - \hat{w}\right] + n[\hat{\beta} + (1+g)\hat{k}] + bn\hat{k}[1 + r - n(1+g)], \tag{22}$$

$$n^{\varepsilon} = \frac{\alpha(1+r)}{(1+g)^{(1-\sigma)}}. \tag{23}$$

In equation (23) the term involving $(1+g)$ reflects the growth of per capita consumption, c, at rate g in the steady state. A higher rate of growth of consumption reduces fertility because, given r, a family wants a more rapid growth of consumption only if preference for the present is lower. A reduction in n lowers the effective rate of time preference. Thus, for a given r, higher steady-state per capita growth, g, goes along with a lower growth rate of population.

Existence and Uniqueness of the Steady State

We now consider whether equations (22) and (23)—along with the condition, $\hat{w} = f(\hat{k}) - \hat{k}f'(\hat{k})$—define a unique steady-state solution for $\hat{k}$, and therefore also for n, $\hat{c}$, r, $\hat{w}$, and $\hat{\beta}$. Attainable dynastic utility must be finite; otherwise the transversality condition would not hold, and the first-order conditions, which led to equations (6) and (7), would not maximize utility.

For steady-state values $\hat{c}$ and n, utility is given from equations (3) and (4) by

$$U_0 = (\hat{c})^{\sigma}\left[\frac{1}{1-\alpha n^{(1-\varepsilon)}(1+g)^{\sigma}}\right], \quad \text{if} \quad \alpha n^{(1-\varepsilon)}(1+g)^{\sigma} < 1. \tag{24}$$

If $\alpha n^{(1-\varepsilon)}(1+g)^{\sigma} \geqslant 1$, then utility is infinite. By using the equation for the fertility rate from equation (23) to substitute out for α, the condition for bounded utility becomes

$$(1+r) > n(1+g). \tag{25}$$

As usual, the steady-state real interest rate, r, must exceed the steady-state growth rate of real income due either to growth of population $(n-1)$ or to growth in real income per person (g).

Substituting for n from equation (23) into the inequality in (25) leads to an inequality that holds only if the steady-state interest rate is below a critical value, $\bar{r}$ (see equation (A.1) in the Appendix). Since n reacts more than in proportion to $1 + r$ in equation (23), $\bar{r}$ is an upper bound for r.

We can get another inequality condition for r by solving out for n and $\hat{c}$ from equations (20) and (21). Substituting this expression for n into $1 + r - n(1 + g) > 0$ leads to

$$(26) \qquad [1 + r - n(1 + g)] = (+) \cdot \left[\frac{\hat{\beta}(1 + r)}{(1 + g)} - \hat{w} \right] > 0,$$

where $(+)$ is a positive term (if $\hat{k} \geqslant 0$ and $1 - \varepsilon > \sigma$). Expression (26) guarantees that the steady-state $\hat{c}$ in equation (21) is positive.

The inequality in (26) implies a critical value, $\underline{r}$, that the steady-state r must exceed. The value of $\underline{r}$ is defined by $\hat{\beta}[\hat{w}(\underline{r})] \cdot (1 + \underline{r}) = \hat{w}(\underline{r}) \cdot (1 + g)$. Putting this result together with the previous one, we find that $\underline{r} < r < \bar{r}$ must apply for the attainable utility to be bounded. Using $\hat{\beta} = a + b\hat{w}$, we show in the Appendix that the condition $\underline{r} < \bar{r}$ corresponds to

$$(27) \qquad \left(\frac{a + b\hat{w}}{\hat{w}} \right)^{(1 - \varepsilon)} > \alpha(1 + g)^{\sigma},$$

where the expression on the left is evaluated at the point where $r = \underline{r}$. If $a = 0$, the left side becomes $b^{(1 - \varepsilon)}$, and this inequality depends only on the parameters b, ε, g, α, and σ.

The Appendix shows that, if the inequality in (27) holds, then (i) there exists at least one steady state for $\hat{k}$ that satisfies equations (22) and (23) and has finite utility; (ii) there is no steady state with infinite utility; (iii) the steady state is unique if $b = 0$, or if b is positive and not "too large;" however, we do not have precise bounds on the admissible range for b.

During the remainder of the paper we assume that (27) holds.

Figure 1 depicts graphically the determination of the steady-state values for r and n (where the value for $\hat{k}$ follows from $f'(\hat{k}) = r$). The curve denoted n_1 shows the combinations of r and n that satisfy the intertemporal-substitution condition in equation (23). A higher n implies a greater effective rate of time preference, which means a higher r. Therefore, the n_1 curve has a positive slope.

The curve denoted n_2 shows the combinations of r and n that satisfy equation (22), given that $f'(\hat{k}) = r$. Values along this curve accord with the steady-state budget constraint in equation (20), where the value for $\hat{c}$ satisfies equation (21). The slope of the curve is negative, as shown, if a higher value of $\hat{k}$ (hence a lower value of r) is associated with a higher n—that is, if people spend part of an increase in wealth on having more children. This relation is ambiguous because a higher $\hat{k}$ also induces a substitution effect against children through a higher $\hat{w}$, and hence, a higher $\hat{\beta}$.

The limiting values for n_1 and n_2 as $r \to 0$ or $r \to \infty$ are discussed in the Appendix. In particular, $\lim_{r \to 0}(n_2) > \lim_{r \to 0}(n_1)$, as shown in Figure 1, as

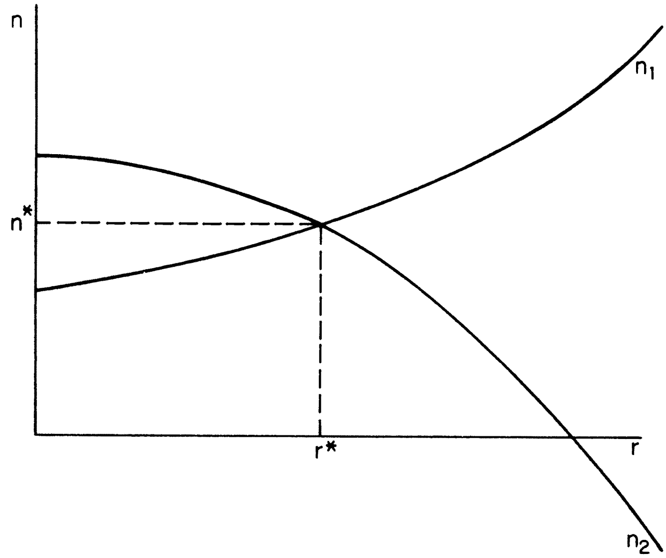

FIGURE 1.—Determination of steady-state values.

Note: The curve n_1 shows the combinations of the interest rate, r, and the fertility rate, n, that satisfy the intertemporal-substitution condition in equation (23). The curve n_2 shows the combinations that satisfy equation (22), which combines the intertemporal budget constraint from equation (20) with the determination of consumption from equation (21).

long as (27) holds. This condition ensures that the curves intersect at least once. However, since the n_2 curve need not slope downward throughout—and, more importantly, may have a slope that is more positive than that of the n_1 curve—there may be multiple intersections (see Figure 2). If the parameter b is small enough, multiple intersections can be ruled out. That is because a small b limits the effect of a change in the value of time on the demand for children, and therefore ensures that the n_2 curve has a negative slope.

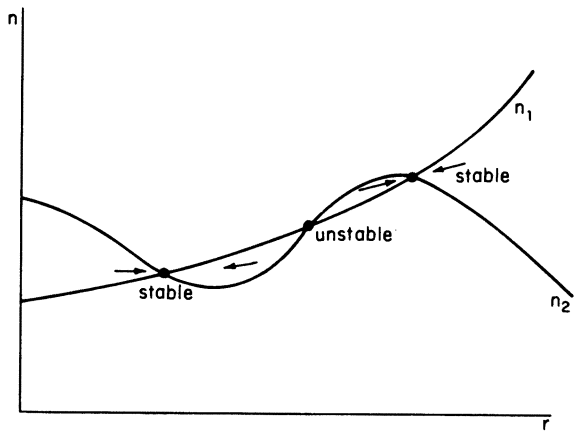

FIGURE 2.—Steady-state values with multiple steady states.

Note: The n_2 curve is modified from Figure 1 to allow for multiple intersections with the n_1 curve.

Dynamics and Stability

The dynamics of $\hat{k}_i$ and the other variables comes from equations (18) and (19). If $b = 0$, the economy converges in a single generation to the steady-state values of $\hat{k}, n, \ldots$. For suppose that the economy started with a low capital intensity, so that $\hat{k}_0 < \hat{k}$. All variables would attain their steady-state values from generation 1 onward because these values satisfy equations (17) and (18) (derived from the first-order conditions) for $i = 1, 2, \ldots$; that is, these equations do not depend on $\hat{k}_0$ for $i \geqslant 1$. The condition $\hat{k}_1 = \hat{k}$ would be attained by choosing low values of initial fertility ($n_0 < n$) and consumption per person ($\hat{c}_0 < \hat{c}$). The values n_0 and $\hat{c}_0$ come from the dynastic budget constraint (equation (8)) and the first-order condition from equation (6) for $i = 0$.

Becker and Barro (1988) allowed for a fuller dynamic adjustment in open economies by introducing a dependence of child-rearing costs, $\hat{\beta}_i$, on the number of children, n_i. In a closed economy we get a similar dynamics from the positive relation between $\hat{\beta}_i$ and $\hat{w}_i$. For example, if $\hat{k}_0 < \hat{k}$, then $\hat{w}_0$ is below the steady-state value $\hat{w}$, and hence $\hat{\beta}_0 < \hat{\beta}$. Children are relatively cheap because the opportunity cost of parents' time is relatively low. This substitution effect counters the wealth effect of a low $\hat{k}_0$, and works against choosing a low value of n_0 and a high value of $\hat{k}_1$. Therefore, $\hat{k}_1$ would be smaller than the steady-state value $\hat{k}$. If the steady state is stable (which requires $\hat{k}_1 > \hat{k}_0$), the economy's approach to the steady state would be stretched out across several generations.

The process of adjustment is locally stable and monotonic if the parameter b is small enough (see the Appendix). As $b \to 0$, the results approach the case where the economy jumps to the steady state in one generation. When b gets large, the dynamics can be unstable. For example, starting from a small $\hat{k}_0$, the low wage rate makes it relatively cheap to produce children (if b is large), which can make $\hat{k}_1 < \hat{k}_0$—that is, $\hat{k}$ can move in a destabilizing direction. Stability requires that the child-producing sector not be too labor intensive in its methods of production (see the discussion in Tamura (1986)).

We conjecture (but have not yet proven) that a steady state is locally stable if and only if the n_2 curve is steeper than the n_1 curve at that steady state (see the first and third steady states in Figure 2 and the only one in Figure 1). If so, when b is sufficiently small to ensure uniqueness, it also ensures stability. Moreover, the behavior of the n_1 and n_2 curves as $r \to 0$ and $r \to \infty$ implies an odd number of steady states, where the number of stable steady states exceeds the number of unstable ones. For example, with three steady states, the middle one would be unstable, and the economy would approach either a state with low r (high $\hat{k}$) and low n, or a state with high r (low $\hat{k}$) and high n (see Figure 2). The following analysis of comparative statics and dynamics is restricted to local variations around one of the stable steady states.

Robustness of the Results

The most important assumption in our model is that parents' utility is linearly related to each child's utility. This assumption, together with the assumption of separability between parents' consumption and children's utility, is essentially equivalent to assuming that dynastic utility is time consistent and additively separable in the per capita consumptions of each generation. Our main conclusions do not change if the current period utility, v, is concave in n, as well as c, or if the altruism function $a(n)$ in equation (1) does not have a constant elasticity (but retains a negative slope). The arbitrage condition for a steady state would still imply the positive relation between n and r shown by the curve n_1 in Figures 1 and 2. The relation between n and r derived from the budget constraint and the determination of consumption would still be negatively sloped (as the curve n_2 is in Figure 1), if higher values of $\hat{k}$ induce larger values of n.

Steady states are still stable if a higher $\hat{k}$ does not reduce n by much, if at all, but it does become possible for n and $\hat{k}$ to cycle as they approach their steady-state values. If $a(n)$ becomes less elastic (in absolute value) as n increases, an increase in $\hat{k}$ raises n by a greater percentage than when $a(n)$ has a constant elasticity. A larger increase in n induces a greater fall in $\hat{k}$ through the quality-quantity interaction. If the cost of rearing children does not depend on the wage rate ($b = 0$), $\hat{k}$ returns to its steady-state value in one generation when the elasticity of $a(n)$ is constant. Therefore, if $b = 0$, $\hat{k}$ must overshoot its steady-state value if the magnitude of the elasticity declines as n increases (see the proof of this result in Benhabib and Nishimura (1986)). Since fertility overshoots its steady-state value when $\hat{k}$ does, n and $\hat{k}$ oscillate around the steady state as they converge toward that state.

Fertility decisions by a dynastic family require at least one generation for implementation. Consequently, these oscillations may produce long cycles in fertility and per capita income of the type claimed by Kondratieff (1935), Kuznets (1958), and Easterlin (1968). One caution, however, is that the aggregation across families may smooth the long cycles that we get from the present model.

2. COMPARATIVE STATICS AND DYNAMICS

Changes in Initial Conditions

Many economic disturbances amount to shifts in the initial conditions, as represented by $\hat{k}_0$, without changes in the other parameters. For example, natural disasters, wars, or epidemics might shift the existing amounts of capital stock or population—and hence the capital-labor ratio—without affecting the parameters for altruism, (α, ε), growth, g, or child-rearing costs, (a, b).

Suppose that an epidemic, such as the Black Death, eliminates part of the population, while leaving the physical capital intact. Then if the economy were initially at a steady state, the new capital-labor ratio, $\hat{k}_0$, would exceed the unchanged steady-state value $\hat{k}$. The model implies a transition from $\hat{k}_0$ to $\hat{k}$, which features temporarily high consumption per person ($\hat{c}_0 > \hat{c}$), and also fertility ($n_0 > n$) if the income effect on the demand for children dominates the negative substitution effect from the higher wage rate. (Because wage rates are temporarily high, $n_0 > n$ would not apply if the parameter b were large enough.)

The approach to the steady state is generally gradual, so that high fertility and consumption per person are eliminated gradually. Fertility rates apparently were higher for awhile in England after the Black Death (see Postan (1975, p. 43)).

The economy eventually approaches the same steady state as before in terms of capital per effective worker, $\hat{k}$, interest and wage rates, r and $\hat{w}$, fertility rate, n, etc. However, the disturbance has permanent effects on the levels of all quantities —population, capital stock, consumption, and so on. Since the transition features relatively low saving per capita (corresponding to high per capita consumption and high expenditures per adult on child rearing), the stock of capital falls permanently relative to the path that would have been followed in the absence of the initial shock to population. Correspondingly, the transitional rise in fertility, if it occurs at all, makes up only for part of the initial fall in the level of population. As in some other theories of endogenous growth (see the discussion in King and Rebelo (1986)), the levels of various quantities do not return to a fixed trend line even when the disturbances are temporary. The empirical analysis by Nelson and Plosser (1982) and Campbell and Mankiw (1987) supports this type of result, since they indicate that macroeconomic time series typically do not have a tendency to return to a fixed trend line.

An adverse shock that impinges on physical capital, but not on population, leads to a low value of $\hat{k}_0$. Then the transition exhibits relatively low consumption per person ($\hat{c}_0 < \hat{c}$), and low fertility ($n_0 < n$) if the income effect from a lower $\hat{k}$ is dominant. As before, the economy returns eventually to the same steady-state values, $\hat{k}$, $\hat{c}$, n, etc. But, if n falls when $\hat{k}$ does, there are again permanent declines in the levels of capital, consumption, and population, as compared to the paths that would have been followed without the initial shock. The permanent fall in population is clear if fertility is lower during the transition. With a permanent fall in population, the capital stock must also be permanently lower because the steady-state ratio of capital to the effective labor input does not change.

Our results on long-run effects to the levels of variables contrast with those delivered by growth models of the Cass (1965)-Koopmans (1965) type. In those models a shock to the initial capital stock has no long-run effect on the levels of capital, output, population, etc. However, in models where population is exogenous, a one-time displacement to the path of population—with no change to subsequent growth rates of population—has, by assumption, a long-run effect on the levels of capital, output, population, etc.

Differences in the Cost of Raising Children

Consider the effects from a shift in the child-rearing cost, $\hat{\beta}$, which reflects changes in the parameters a or b in the formula $\hat{\beta} = a + b\hat{w}$. Notice first from equation (23), which corresponds to the curve n_1 in Figure 1, that the steady-state fertility rate, n, does not depend directly on $\hat{\beta}$. That is, for given values of α, g, and σ, and for a given interest rate r, a permanent change in $\hat{\beta}$ has no effect on population growth. This result is surprising in that a higher cost of raising children is generally thought to reduce fertility. But this effect shows up on the overall level of descendants, rather than on fertility rates. The latter determines the relative number of descendants in different generations—the rate of population growth—whereas the cost of children determines levels of population.

However, a change in $\hat{\beta}$ affects the steady-state ratio of capital to effective worker, $\hat{k}$, and therefore also the steady-state interest rate, $r = f'(\hat{k})$. A higher $\hat{\beta}$ motivates parents to endow each child with more consumption ($\hat{c}$ in equation (21)),[6] whereas the higher direct outlay on child-rearing would force a reduction in consumption per person for a given value of $\hat{k}$ (equation (20)). Therefore, people are motivated to raise $\hat{k}$. In terms of Figure 1, the n_2 curve shifts leftward because $\hat{k}$ is higher (hence r is lower) for a given n. Since the n_1 curve does not shift, it follows that r and n decline.[7] Thus, a permanently higher cost of raising children does lead to a lower steady-state rate of population growth, but only through an indirect channel that involves lower interest rates.

A number of empirically interesting disturbances amount to changes in the cost of rearing children. For example, a reduced rate of child mortality lowers the expected cost of raising a surviving child. Therefore, if $\hat{\beta}$ now refers to the cost of surviving children, our results imply that a lower rate of child mortality leads to an increase in r that raises the steady-state number of surviving children per adult. That is, a lower rate of child mortality raises the steady-state rate of population growth.

Taxes or subsidies on children also shift $\hat{\beta}$. An example is the formal inducements and other policies in China that raise the cost to parents of having more than one child. Another example comes from the positive effect of a larger social security program on the cost of children. The increase in the net tax on younger persons lowers the net lifetime incomes of children and thereby raises the net cost of children. Our model has the novel implication that an expanded social security program *raises* capital intensity, $\hat{k}$, and lowers r through its effect on the cost of children. Hence, an expanded program would lower steady-state fertility and the rate of population growth, even if social security does not replace old-age support from children.

Thus far we have assumed that families are homogeneous with respect to child-rearing costs. However, we can also treat variations in $\hat{\beta}$ as representing heterogeneity in child-rearing costs either across families within a single closed economy, or across closed economies. In the former case, each family faces the same interest rate, r, at each date. If they also have the same wage rate, $\hat{w}$, then families with larger values of $\hat{\beta}$ have higher steady-state values of consumption and wealth per person, $\hat{c}$ and $\hat{k}$. In the steady state the higher value of $\hat{k}$ exactly offsets the higher $\hat{\beta}$ to motivate each family to select the same steady-state growth rate in the number of descendants, n, independently of the value of $\hat{\beta}$.

[6] This result is analogous to the proposition that people invest more in each child's "quality" when the quantity of kids is smaller. See Becker and Lewis (1973).

[7] The result is the same if the n_2 curve slopes upward as long as the slope is smaller than that of the n_1 curve in a neighborhood of the steady state. We believe that this condition holds if the steady state is locally stable (see the discussion in Section 1).

Therefore, when families share the same capital and labor markets, those with high costs of reproduction, $\hat{\beta}$, do not tend to disappear over time.

If we consider different countries with their own capital and labor markets, then countries with high values of $\hat{\beta}$ have relatively low interest rates. Consequently, they have relatively low steady-state growth rates of population. In other words, there is a process of "natural selection" whereby countries with high costs of reproduction become smaller in population relative to those with low costs. If international migration is possible, countries with high values of $\hat{\beta}$ and low rates of native fertility would attract population from other countries because a high $\hat{\beta}$ leads to more capital and higher wages per worker.

Differences in the Rate of Technical Progress

A higher growth rate g means that consumption per person, c_i, rises at a faster rate along the steady-state path. For a given r, people choose a higher growth rate of consumption per person only if time preference for the present falls. In our analysis this fall requires a decline in fertility. Therefore, a higher growth rate, g, lowers n for a given r, as shown in equation (23). This effect on fertility arises for most forms of technical progress. We focus on the Harrod-neutral form only to illustrate the results with simple steady-state formulas.

A higher per capita growth rate, g, also affects the steady-state values of $\hat{k}$ and r. We can show from equations (20) and (23) that a rise in g leads to a lower $\hat{k}$ and hence to a higher r.[8] The higher r raises n, but the net effect on n would be negative if a lower $\hat{k}$ reduces n.

As with the cost of children, variations in per capita growth rates, g, could refer to heterogeneity across families or countries. For families in the same country or for open economies with the same capital market, those with higher g have lower steady-state growth rates of population, since the only effect is through the higher growth rate of consumption (see equation (23)). That is, population in more productive families and countries would *decline* relative to that in less productive families and countries. Of course, people would try to migrate toward economies with higher rates of technological progress, so that rates of growth in population in these economies would tend to exceed their "natural" rates.

Fertility and the rate of technological progress may also be negatively related among closed economies. However, the positive association between g and r in our model weakens and may eliminate the inverse association between g and n. We do not know of empirical evidence on whether countries with high rates of per capita growth have high real interest rates. Perhaps international capital markets have become sufficiently "perfect" in recent decades to allow only small persisting deviations in real interest rates across countries.

[8] In Figure 1 the n_1 curve shifts downward. The shift in the n_2 curve is ambiguous. However, as long as the steady state is stable, we can show that the overall change in r must be positive.

The results of Kormendi and Meguire (1985, equation (4)) with data spanning about 30 years suggest that a higher per capita growth rate does go along with a lower rate of population growth. They fit an equation with post-World War II data from 47 countries, where the dependent variable is the average growth rate of aggregate real GNP. They find a regression coefficient for the average growth rate of population (treated as exogenous) that is positive, but significantly below unity. However, other studies have not always found a negative relation between the growth rate of per capita income and the growth rate of population (see, e.g., the discussion in Kuznets (1967)).

Differences in Altruism

A higher altruism coefficient, α, amounts to a lower rate of time preference. Equation (22) does not depend on α, so the n_2 curve in Figures 1 and 2 does not shift. However, equation (23) implies that n increases for a given r, which means that the n_1 curve shifts upward in the figures. Consequently, if n increases as $\hat{k}$ increases, so that the n_2 curve is negatively sloped in the vicinity of the steady state,[9] a higher degree of altruism induces each generation to save more in the form of numbers of descendants as well as in the form of capital per descendant.

If the altruism coefficient differs across families in the same economy or across open economies, the more altruistic units (higher α's) would have higher steady-state growth rates of population. Therefore, more altruistic families or societies tend to grow in size relative to less altruistic ones. In other words, altruism is a trait that is "culturally" selected through its effect on fertility.

Shifts in Productivity

To allow for shifts in the level of productivity, we modify the production function from equation (11) to

$$Y_i = F\left[K_i, \quad \theta(1+g)^i L_i\right], \tag{28}$$

where the positive variable θ picks up Harrod-neutral changes in the technology.

For a given, constant value of θ, the form of the previous analysis goes through as long as all variables are in "effective" units through division by the term $\theta(1+g)^i$. In other words, $\hat{k}_i = K_i/\theta(1+g)^i L_i$, and so on. Suppose that the economy starts at a steady state, and that a once-and-for-all improvement in productivity implies a permanent increase in θ. This change implies an immediate decline in the effective capital-labor ratio, $\hat{k}_0$. If we assume for the moment that the steady-state value $\hat{k}$ does not change, then the economy would be placed in a position where $\hat{k}_0 < \hat{k}$. In this case, the transition typically involves values for fertility and consumption per *effective* person that fall below the steady-state

[9]If the n_2 curve slopes upward, then r still decreases, but n now also decreases because the overall effect of $\hat{k}$ on n is negative.

values—that is, $n_0 < n$ and $\hat{c}_0 < \hat{c}$. (Since θ increases, a fall in consumption per effective person need not imply a fall in consumption per person.)

Whether a permanent shift in θ affects the steady-state $\hat{k}$ depends on the effects on the costs of rearing children. We have assumed that the rearing of a child requires b units of parents' time and $a(1+g)^i$ units of goods, so that $\beta_i = a(1+g)^i + bw_i$. If the parameter b does not change, and if the number of goods required depends on the productivity of workers as measured by θ as well as by $(1+g)^i$, then $\beta_i = a\theta(1+g)^i + bw_i$, where a and b are constants. In that case, changes in θ would not affect the steady-state value of $\hat{\beta}$, which is the value of β in effective units. Then the steady-state values of $\hat{k}$, n, and $\hat{c}$, would also be unaffected by a change in θ, which has been the assumption so far.

Taxes on Capital

Standard growth models with infinite horizons imply that the after-tax real rate of return is pegged in the steady state by the exogenous rate of time preference. Hence, in the long run, a tax on capital is shifted fully to labor. In our model, a tax on capital affects the steady-state fertility rate and thereby the effective rate of time preference. Hence the tax alters the after-tax real rate of return in the steady state, which means that the tax shifting typically departs from 100% in the long run.

Suppose that the proportional tax rate τ applies to the rental income from capital, $f'(\hat{k}) \cdot \hat{k}$. Further, assume that the receipts from this tax are rebated in a lump-sum manner to households. Then, if $r = (1-\tau) \cdot f'(\hat{k})$ denotes the after-tax real interest rate, the steady-state equations (22) and (23) remain valid. (The left side of equation (22) is now the sum of wage income, after-tax rental income, and the rebate of tax receipts.)

Consider an increase in the tax rate, τ. Equation (23) and hence the curve n_1 in Figures 1 and 2 are unaffected. For a given r, a rise in τ implies an increase in the gross rental rate, $f'(\hat{k})$, and therefore a decrease in $\hat{k}$. The effect on the curve n_2 depends on whether people have fewer or additional children when $\hat{k}$ falls. If they have fewer, the n_2 curve shifts downward and n and r decrease. In other words, the decrease in $\hat{k}$ and consequent rise in $f'(\hat{k})$ are not sufficient in this case to shift the tax fully from capital to labor in the long run. The long-run decline in r is possible because the decline in n causes a fall in the effective rate of time preference.

However, if people react to a fall in $\hat{k}$ by raising n, the resulting upward shift in the n_2 curve increases n and r. Therefore, a tax on capital may be shifted to labor by more than 100% in the long run. The higher effective rate of time preference—due to the increase in n—requires the steady-state r to be higher.

Social Welfare Across Generations

Since our model has no externalities or public goods, the path of consumption and fertility determined by utility maximization and competitive markets is

Pareto optimal. In particular, the utility of any generation cannot increase without lowering the utility of other generations. Stated differently, if a social planner's preferences with regard to population and consumption in various generations are the same as those of a representative person in the current generation (see equation (3)), then the social planner and the representative person would make the same choices for fertility, consumption, and investment (see Willis (1985)).

Parents and children clash in our model even though parents are altruistic toward their children. The conflict arises because children want larger bequests than parents are willing to give. A social planner who is more sympathetic than the present generation to children and other future generations will place relatively more weight on numbers of people and consumption in the future. We interpret "more weight" to mean that the planner uses a higher "altruism" coefficient, α, to discount the welfare of children and other future generations. Although decisions of the current generation are still Pareto optimal, they no longer maximize "social welfare," as judged by the planner's preferences.

Our previous discussion of the effects of changes in the degree of altruism, α, indicates the nature of the discrepancy between privately optimal and "socially" optimal resource allocation. A social planner who places more weight than the present generation on the welfare of future generations would choose higher values for steady-state per capita capital, $\hat{k}$, and fertility, n (assuming that the n_2 curve slopes downward). That is, such a social planner would increase the number of people and the utility of each person in the future by reducing the consumption and utility of the present generation.

3. LIFE-CYCLE SAVINGS

It is straightforward to extend the family's optimization problem to include choices of consumption and saving over the life cycle. The pattern of consumption during a person's lifetime depends as usual on the interest rate and on parameters of time preference. Outside of the steady state, the dynamics of aggregate consumption depends on demographic dynamics, as well as on intergenerational forces.

For present purposes, the inclusion of life-cycle elements has three important implications. First, since some assets may be held for life-cycle purposes, the assets held for intergenerational purposes (which corresponds to the amount of bequests) need no longer be positive (see note 2 above). We now have to assume that the parameters for altruism, etc., generate an equilibrium with positive bequests. Some support for this assumption comes from the results of Darby (1979) and Kotlikoff and Summers (1981). They find that intergenerational elements are more important than life-cycle forces in accounting for asset holding in the United States (but see Modigliani's criticism (1986, Section D)).

The second point is that the amount of assets held for life-cycle purposes tends to rise with the rate of interest. This effect modifies the various comparative-static exercises that we carried out. In each case where the interest rate changes, we have to take account of the response in the same direction of life-cycle savings. For example, suppose that altruism toward children increases but that the rate of time preference over the life cycle does not change. Then our previous analysis predicted an increase in the steady-state capital per effective worker, $\hat{k}$, and a fall in the steady-state interest rate, r. The new element is that the fall in r reduces life-cycle capital accumulation. Consequently, r will fall by less than previously, which means that the effect on fertility is greater than before.

A third effect from introducing life-cycle elements involves the spreading of births over many ages. All age-specific fertility rates may not change by the same proportion when say the cost of rearing children changes; the spacing of children may also change. Demographers have shown how shifts in age-specific fertility rates alter steady-state age distributions and rates of population growth. They have also shown that age distributions and growth rates of population differ from their steady-state values during possibly extended transitional periods.[10] These demographic transitional periods interact with "economic" transitional periods in capital accumulation, levels of fertility, consumption, etc. to produce a more complete and complicated dynamics. The analysis of this more complete dynamics is beyond the scope of this paper.

4. SUMMARY AND CONCLUSIONS

This paper analyzes population growth, capital accumulation, and income growth in closed economies. The principal novelty is the determination of population growth through endogenous fertility decisions. We assume that the utility function of parents depends on their consumption, the number of their children, and the utility of each child. Associated with each parental utility function is a dynastic utility function that depends on the number of descendants and the consumption of each descendant in all generations.

Other ingredients of the analysis are conventional. Production takes place in competitive firms that have constant returns to scale. All productivity change is exogenous and all factor markets are perfectly competitive.

We show that the steady-state rate of population growth is positively related to the degree of altruism toward children and to the steady-state long-term interest rate, and it is negatively related to the rate of growth between generations in per capita consumption. Most types of increases in the rate of technological progress raise the rate of growth in per capita consumption. If progress is Harrod neutral, and if a rise in income increases fertility, we show that more rapid progress lowers fertility and the population growth rate even though it also raises the steady-state interest rate. However, if fertility falls when income increases, more rapid progress may raise fertility and the population growth rate.

A permanent increase in the cost of raising children, due for example to a tax on children or to an expanded social security system, lowers the steady-state

[10] As with our discussion, the demographic literature on stable populations assumes a single sex that reproduces (see Coale and Demeny (1966)). Demographic dynamics are considerably more complicated when marriage and reproduction involve two sexes (see Pollak (1986)).

interest rate. The positive relation between fertility and interest rates means that an increase in the cost of children lowers steady-state rates of population growth and raises steady-state per capita capital stocks.

Since the rate of population growth is endogenous, even temporary changes in parameters have permanent effects on population levels, aggregate incomes, and aggregate amounts of capital. Although an epidemic, such as the Black Death, does not affect the steady-state growth rates of population and per capita income, it does lower the levels of population and aggregate income. The higher fertility after the epidemic is not sufficient to make up for the deaths due to the epidemic.

Economists have become reluctant to bring endogenous fertility and population change into models of economic growth and macroeconomic performance. They lost confidence in the Malthusian system because it has several unattractive implications, such as a positive relation between fertility rates and levels of per capita income. The new household economics has stimulated a modest revival of models that integrate endogenous population change into an analysis of economic growth. Our paper is a contribution to this revival. Its implications are much more in line with the behavior of modern economies than are those of the Malthusian system.

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APPENDIX

Equation (23) gives a relation between n and r. Let n_1 be the solution for n corresponding to a given r from this equation. Then we have $dn_1/dr > 0$,

$$\lim_{r\to 0}(n_1) = \left[\frac{\alpha}{(1+g)^{1-\sigma}}\right]^{1/\varepsilon}, \qquad \lim_{r\to\infty}(n_1) = \infty.$$

Equation (22) implies another relation between n and r. Let n_2 be the solution for n for a given r from this equation. Then we have $\lim_{r\to 0}(n_2) > (1/(1+g))$ (if $bn < 1$, which must hold in our model), $\lim_{r\to\infty}(n_2) = -\infty$ (assuming the limits as $r \to 0$ are $\hat{k} \to \infty$, $f(\hat{k})/\hat{k} \to 0$, $\hat{w}/\hat{k} \to 0$, and the limits as $r \to \infty$ are $\hat{k} \to 0$, $f(\hat{k})/\hat{k} \to \infty$, $\hat{w}/\hat{k} \to \infty$).

The condition for bounded utility, $(1+r) > n(1+g)$, implies from the relation for n_1 in equation (23),

$$r < \bar{r} = \left[\frac{(1+g)^{(1-\sigma-\varepsilon)}}{\alpha}\right]^{1/(1-\varepsilon)} - 1.$$

From equation (22), the condition $(1+r) > n_2\ (1+g)$ corresponds to $\hat{\beta}(1+r) > \hat{w}(1+g)$. It can be shown that $\hat{\beta}(1+r) = \hat{w}(1+g)$ has a unique solution for r, denoted $\underline{r}$, with $\hat{\beta}(1+r) \gtreqless \hat{w}(1+g)$ as $r \gtreqless \underline{r}$. Hence $r > \underline{r}$ ensures $(1+r) > n_2(1+g)$.

In a solution where $n = n_1 = n_2$, the two conditions, $(1+r) > n_1(1+g)$ and $(1+r) > n_2(1+g)$, require $\underline{r} < r < \bar{r}$, which can hold only if $\underline{r} < \bar{r}$. Let $\underline{\beta}$ and $\underline{w}$ correspond to $\underline{r}$, so that $\underline{\beta}(1+\underline{r}) =$

$\underline{w}(1+g)$. Then the condition $\underline{r} < \bar{r}$ is equivalent to

$$(\underline{w}/\underline{\beta})(1+g) < 1 + \bar{r} = \left[\frac{(1+g)^{(1-\sigma-\varepsilon)}}{\alpha} \right]^{1/(1-\varepsilon)},$$

which implies

$$\text{(A.1)} \qquad (\underline{\beta}/\underline{w})^{(1-\varepsilon)} > \alpha(1+g)^{\sigma}.$$

Condition (A.1), which says that the cost of raising children must be bigger than something, is necessary for the existence of a steady state with bounded utility.

If (A.1) holds, it follows that $\lim_{r \to 0}(n_1) < \lim_{r \to 0}(n_2)$. Therefore, since n_1 and n_2 are each continuous functions of r, there must exist at least one steady state—that is, a point where $n = n_1 = n_2$.

$r > \bar{r}$ implies $(1+r) < n_1(1+g)$, and—if (A.1) holds—$(1+r) > n_2(1+g)$; therefore $n_1 \neq n_2$. Similarly, $r < \underline{r}$ implies $(1+r) < n_2(1+g)$, and—if (A.1) holds—$(1+r) > n_1(1+g)$, so that $n_1 \neq n_2$. Therefore, $n_1 = n_2 = n$ can hold only if $\underline{r} < r < \bar{r}$, which implies $1 + r > n(1+g)$. It follows that (given A.1) all steady states—of which at least one must exist—exhibit bounded utility.

If $b = 0$, it can be shown that $dn_2/dr < 0$ holds in the region where $1 + r < n(1+g)$. Therefore, if $b = 0$ (and A.1 still holds), the steady state is unique and has bounded utility.

The general condition for a unique steady state is that the slope, dn_1/dr, exceed the slope, dn_2/dr, at any point where $n_1 = n_2$. This condition holds for small enough b. We have not obtained an expression for the upper bound of b that is consistent with a unique steady state. However, the limiting properties of the functions for n_1 and n_2 imply that there must always be an odd number of steady-state solutions.

We have carried out a dynamic analysis for an approximate form of the model—equations (18) and (19)—when the terms, $1 - bn_i$ and $1 - bn_{i+1}$, in equation (18) can be replaced by unity. Hence the approximation is satisfactory for small values of b. Given this approximation, the linearized form of the model is a second-order difference equation in k_i. (Otherwise it would be a third-order equation.) The two roots of the second-order equation are real and nonnegative. Hence the solution does not involve oscillatory behavior.

At $b = 0$ (where the approximation above is exact), one root is 0 and the other exceeds one. The second root implies explosive behavior for k_i, which violates a transversality condition. Therefore, the only solution, corresponding to the 0 root, features immediate convergence to the steady state.

As b becomes positive, the smaller root becomes positive, which means that the convergence to the steady state is no longer immediate. However, for small b, the system is locally stable with a monotonic approach to the steady state.

For large enough b, the smaller root of the difference equation exceeds unity, which indicates instability. It is unclear whether $b < 1$ is sufficient to rule out this possibility. Also the approximation made initially becomes unsatisfactory for a large value of b.

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