Macroeconomics II

Lesson 01 — Preliminaries. Stylized Facts and Measurement of Business Cycles

Rafael Serrano-Quintero

Department of Economics
University of Barcelona

Preliminaries

The Course

- Macroeconomics II

- Check the syllabus

- Instructors: Alessandro Di Nola and Rafa Serrano-Quintero.

Me

- rafael.serrano@ub.edu

- Office hours: Send me an email and we can arrange a meeting.

Syllabus Highlights

What you have to do

1. Three problem sets
2. Midterm exam. 50% of the grade of the final exam. Other 50% will be Alessandro's part.

Materials

- In class notes and slides. Most of the work will be in the blackboard.
- Useful references:
  - Cooley (1995)
  - Judd (1998)
  - King and Rebelo (1999)
  - Ljungqvist and Sargent (2004)

Preliminary Concepts

Preliminary Concepts

Modern macroeconomics is

- **dynamic:** things change and are related over time.
- **stochastic:** there are random events. Expectations matter.
- We denote X_t the realization of variable X at time t.
- Models are stochastic and people try to guess what will happen in the future. We call that expectations.

Notation and Expectations

- If we do not know anything about the current state of the system, we call it **unconditional expectation** and express it as $\mathbb{E}(X_{t+1})$.

- If we condition on what we know at time t, we call the **conditional expectation** and we express it as $\mathbb{E}\left(X_{t+1}|\Omega_t\right)$ where Ω_t is what we know at time t. Usually, we shorten it by writing $\mathbb{E}_t(X_{t+1})$.

- Note that $\mathbb{E}_t(X_t) = X_t$ and $\mathbb{E}_t(X_{t-k}) = X_{t-k}$ for all $k > 0$.

Theorem 1 (Law of Iterated Expectations)

Let Y and Z be two arbitrary random variables, then

$$\mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(Y|Z))$$

i.e. the expected value of the conditional expected value of Y given Z is the same as the expected value of Y.

Notation and Expectations

- The LIE (Theorem 1) has the following implication

$$\mathbb{E}_t(\mathbb{E}_{t+1}(X_{t+2})) = \mathbb{E}_t(X_{t+2})$$

- Rational expectations [Muth 1961] is a stronger than simple expectations. Two conditions on expectations of future realizations of variables
  1. Correct on average.
  2. Unpredictable forecasting errors given current info.

- This implies the agents *“know the model and use it to act.”*

- Does this imply agents do not make mistakes? **No!!**

Stochastic Processes

Definition 1 (Markov Property)

A stochastic process has the Markov property if

$$\Pr\left[X_t = x | X_0, \ldots, X_{t-1}\right] \equiv \Pr\left[X_t = x | X_{t-1}\right]$$

i.e. that the conditional probability of future states depends upon the present state only. If you know X_{t-1} knowing $X_0, \ldots, X_{t-2}$ does not give you extra information.

Definition 2 (ARMA(p, q) Process)

An ARMA(p, q) process can be expressed as

$$X_t = c + \sum_{i=1}^{p} \rho_i X_{t-i} + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}$$

where c is a constant, ρ_i are the autorregressive parameters, and θ_i are the moving average parameters.

Stochastic Processes

- ARMA processes can sometimes be approximated with sufficiently long AR processes.

- An AR(p) process **does not** strictly have the Markov property. However

$$\begin{pmatrix} s_t \\ s_{t-1} \\ \vdots \\ s_{t-p+1} \end{pmatrix} = \begin{pmatrix} \rho_1 & \rho_2 & \cdots & \rho_p \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} s_{t-1} \\ s_{t-2} \\ \vdots \\ s_{t-p} \end{pmatrix} + \begin{pmatrix} \varepsilon_t \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

  which reduces to

$$\mathbf{s}_t = \Lambda \mathbf{s}_{t-1} + \varepsilon_t$$

Impulse Response Functions

If we give a shock to some exogenous variable. How do the endogenous variables react?

$$\text{IRF}(h) = \mathbb{E}_t(X_{t+h}|\varepsilon_t = \varepsilon) - \mathbb{E}_{t-1}(X_{t+h})$$

Example 1

Take the AR(1) process

$$X_t = \rho X_{t-1} + \varepsilon_t$$

suppose $\varepsilon \sim \mathcal{N}(0, 1)$ and suppose at time t there is a shock of $\varepsilon_t = 1$. Compute the IRF after h periods.

Impulse Response Functions

Take a multivariate process and assume $\mathbf{X}$ is a 2×1 vector

$$\mathbf{X}_t = \mathbf{A}\mathbf{X}_{t-1} + \mathbf{B}\boldsymbol{\varepsilon}_t$$

Assume also that off-diagonals elements of $\mathbf{A}$ and $\mathbf{B}$ are not zero. The IRF is now a vector

$$\begin{aligned} \mathrm{IRF}_1(h) &= \mathbf{A}^{h-1}\mathbf{B}_1 \\ \mathrm{IRF}_2(h) &= \mathbf{A}^{h-1}\mathbf{B}_2 \end{aligned}$$

Quick Recap on Dynamic Optimization

Recap Dynamic Optimization

5 Ingredients of a Dynamic Model

1. **State variables:** what summarizes the information from the past.
2. **Control variables:** what the agent chooses.
3. **Return function:** what evaluates the sequence of choices.
4. **Transition function:** how the state variables change from one period to the next. (Also called law of motion)
5. **Planning/time horizon:** how long the agent needs to make decisions.

Recap Dynamic Optimization

State and Control Variables

State variables:

- The agents take the information from the states as given.

- We can distinguish between **endogenous** and **exogenous** state variables.
  1. **Endogenous** if they are determined by the decision of the agent in previous period(s).
  2. **Exogenous** if they are determined by the conditions of the problem but **not** by the decisions of the agent.

Control variables:

- These are the variables the agent actually chooses.

- Their choice will determine the future state of the **endogenous** state variables.

Recap Dynamic Optimization

Example

Consumption-Savings Problem

$$
\begin{aligned}
\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t u(c_t) && (1) \\
\text{subject to} \quad & a_{t+1} + c_t = (1+r)a_t && (2) \\
& a_{t+1} \geq 0 && (3)
\end{aligned}
$$

Consumption-Savings Problem

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$

$$\text{subject to} \quad a_{t+1} + c_t = (1+r)a_t \tag{2}$$

$$a_{t+1} \geq 0 \tag{3}$$

- **State variable(s):** $\{a_t\}$.

Consumption-Savings Problem

$$\begin{aligned}
\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t u(c_t) && (1)\\
\text{subject to} \quad & a_{t+1} + c_t = (1+r)a_t && (2)\\
& a_{t+1} \geq 0 && (3)
\end{aligned}$$

- **Control variable(s):** $\{c_t, a_{t+1}\}$.

Consumption-Savings Problem

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$

$$\text{subject to} \quad a_{t+1} + c_t = (1+r)a_t \tag{2}$$

$$a_{t+1} \geq 0 \tag{3}$$

- **Transition equation:** equation (2).

Consumption-Savings Problem

$$
\begin{aligned}
\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t u(c_t) && (1) \\
\text{subject to} \quad & a_{t+1} + c_t = (1+r)a_t && (2) \\
& a_{t+1} \geq 0 && (3)
\end{aligned}
$$

- **Return function:** $u(c_t)$.

Recap Dynamic Optimization

Solution through the Lagrangian

- A solution consists of a sequence $\{a_t, c_t\}_{t=0}^{\infty}$ that yields the maximum lifetime utility.

- We write the (current value) Lagrangian as

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t) - \lambda_t \left[c_t + a_{t+1} - (1+r)a_t \right] - \mu_t a_{t+1} \right\} \tag{3}$$

- What are μ_t and λ_t? They are sometimes called co-state variables or Lagrange multipliers

- The budget constraint must be satisfied **at all periods** so, in fact, we have a problem with infinitely many constraints (one for each t) and infinitely many Lagrange multipliers or co-state variables $\{\lambda_t, \mu_t\}_{t=0}^{\infty}$.

Recap Dynamic Optimization

Karush-Kuhn-Tucker Conditions

- At each $t \geq 0$ it must be satisfied:

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Rightarrow \lambda_t = u'(c_t) \tag{4}$$

$$\frac{\partial \mathcal{L}}{\partial a_{t+1}} = 0 \Rightarrow \lambda_t + \mu_t = (1+r)\beta\lambda_{t+1} \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = 0 \Rightarrow a_{t+1} + c_t = (1+r)a_t \tag{6}$$

$$\mu_t \, a_{t+1} = 0 \tag{7}$$

- Equation (4) is the first-order condition with respect to our control variable c_t.

Recap Dynamic Optimization

Karush-Kuhn-Tucker Conditions

- At each $t \geq 0$ it must be satisfied:

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Rightarrow \lambda_t = u'(c_t) \tag{4}$$

$$\frac{\partial \mathcal{L}}{\partial a_{t+1}} = 0 \Rightarrow \lambda_t + \mu_t = (1+r)\beta\lambda_{t+1} \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = 0 \Rightarrow a_{t+1} + c_t = (1+r)a_t \tag{6}$$

$$\mu_t \, a_{t+1} = 0 \tag{7}$$

- Equation (5) is the first-order condition with respect to our second control a_{t+1} because our choice of c_t affects a_{t+1} which becomes next period's state.

Recap Dynamic Optimization

Karush-Kuhn-Tucker Conditions

- At each $t \geq 0$ it must be satisfied:

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Rightarrow \lambda_t = u'(c_t) \tag{4}$$

$$\frac{\partial \mathcal{L}}{\partial a_{t+1}} = 0 \Rightarrow \lambda_t + \mu_t = (1+r)\beta\lambda_{t+1} \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = 0 \Rightarrow a_{t+1} + c_t = (1+r)a_t \tag{6}$$

$$\mu_t \, a_{t+1} = 0 \tag{7}$$

- Equation (6) determines the feasibility of the choices.

Recap Dynamic Optimization

Karush-Kuhn-Tucker Conditions

- At each $t \geq 0$ it must be satisfied:

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Rightarrow \lambda_t = u'(c_t) \tag{4}$$

$$\frac{\partial \mathcal{L}}{\partial a_{t+1}} = 0 \Rightarrow \lambda_t + \mu_t = (1+r)\beta\lambda_{t+1} \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = 0 \Rightarrow a_{t+1} + c_t = (1+r)a_t \tag{6}$$

$$\mu_t \, a_{t+1} = 0 \tag{7}$$

- Equation (7) is the complementary slackness condition.

- Note that $c_t \geq 0$ implies that assets will never go to 0. Suppose that at some $t = \tau$ it happens that $a_\tau = 0$, this implies that $c_t = 0 \, \forall t \geq \tau$.

- Why is this not optimal? We have not said this, but typically, utility functions will satisfy

$$\lim_{c \to 0} u'(c) = +\infty$$

  This implies that you will consume at least some part of your wealth.

- By previous point, for all $t \geq 0$, we know that $a_{t+1} > 0$, but this implies from equation (7) that $\mu_t = 0$ for all $t \geq 0$.

- Therefore, it must be that
$$\lambda_t = \beta(1+r)\lambda_{t+1}$$
and using (4) we have
$$u'(c_t) = \beta(1+r)u'(c_{t+1}) \tag{8}$$
which is the **Euler equation**. This condition characterizes the optimal consumption path.

- Is that all?

Recap Dynamic Optimization

Remarks

- We are missing one last necessary condition for optimality! The **transversality condition**.

- Intuitively, we do not want to leave wealth unused for a period that will never arrive.

$$\lim_{T \to \infty} \beta^T \lambda_{T+1} a_{T+1} = 0 \tag{9}$$

- If time was finite (suppose T is the final period) it makes no sense to leave $a_{T+1} > 0$ since I could consume it and achieve higher utility at time $T-1$.

- The dynamic problem of the household is fully characterized by three conditions. The Euler equation (8), the budget constraint (2) that determines the law of motion for a_t, and the transversality condition (9).

Measurement and Stylized Facts of Business Cycles

Measurement

Trend and Cycle

- In Macro I you have studied the macroeconomics of the long-run. Why countries grow.
- Now we will focus on booms and busts or expansions and recessions.
- We first need to separate trend and cycle. Then, identify what are expansion and recession periods and define them formally.

Trend and Cycle

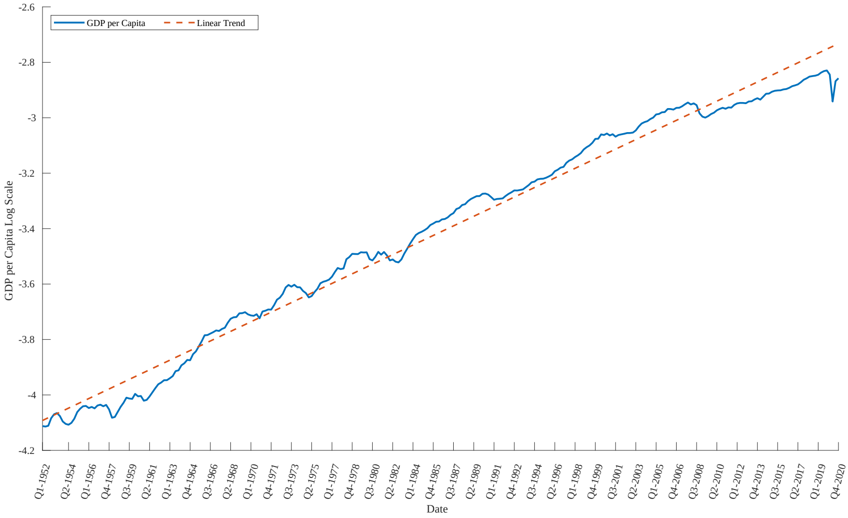

Figure 1: U.S. GDP per capita in Log Scale

Trend and Cycle

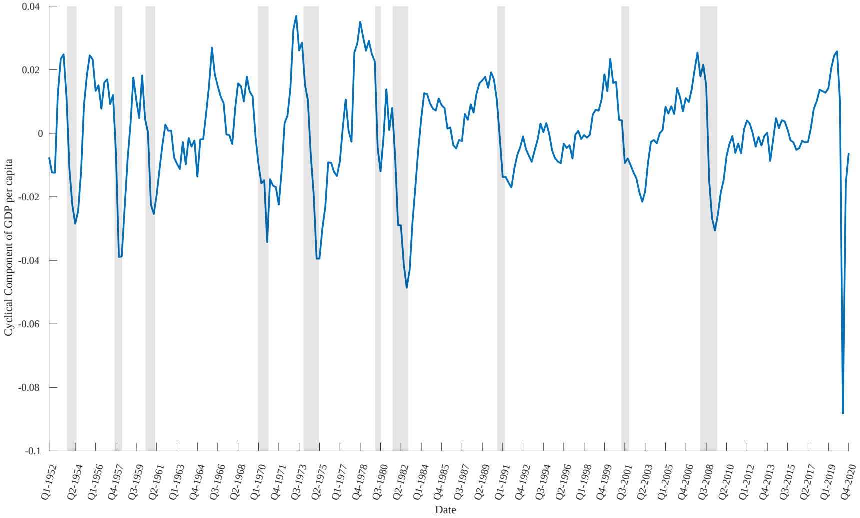

Figure 2: Cycles of U.S. GDP per capita in Log Scale

Trend and Cycle — Identification

- How can we go from trend to cycle? How do we remove this?

- Let y_t be GDP per capita and decompose it into a trend and a cycle component.

$$y_t = y_t^c + y_t^g$$

- How can we estimate these two? Several options:
  - Spectral analysis [Hamilton 1994, see Chapter 6]
    - Describe y_t as a weighted sum of periodic functions like $\cos(\omega t)$ and $\sin(\omega t)$. Identify peaks and troughs and call that a cycle.

  - Detrending methods.
    - Linear detrending, first differences…

    - Filters: band-pass [Baxter and King 1999; Hodrick and Prescott 1997].

Hodrick-Prescott Filter

The method separates the trend component from the solution of the problem

$$\min_{\{y_t^g\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} (y_t - y_t^g)^2 + \lambda \left[(y_{t+1}^g - y_t^g) - (y_t^g - y_{t-1}^g) \right]^2$$

where λ is a smoothing parameter. The cycle is thus

$$y_t^c = y_t - y_t^g$$

Widely used method but with many critiques. From Hamilton (2018):

1. Introduces spurious dynamic relations w/o basis in the underlaying DGP and filtered values at the end are very different from the middle.

2. Choosing λ from a statistical formalization implies $\neq$ to the typically applied.

3. Not good for prediction, double-sided.

Hodrick-Prescott Filter

Alternatives

- Hamilton (2018) proposes instead a regression approach.

- How different is the value at date $t+h$ from the **expected value** at t?

- Suggests to estimate by OLS

$$y_{t+h} = \beta_0 + \beta_1 y_t + \beta_2 y_{t-1} + \beta_3 y_{t-2} + \beta_4 y_{t-3} + v_{t+h}$$

- The residuals

$$\hat{v}_{t+h} = y_{t+h} - \hat{\beta}_0 - \hat{\beta}_1 y_t - \hat{\beta}_2 y_{t-1} - \hat{\beta}_3 y_{t-2} - \hat{\beta}_4 y_{t-3}$$

Filtering Techniques

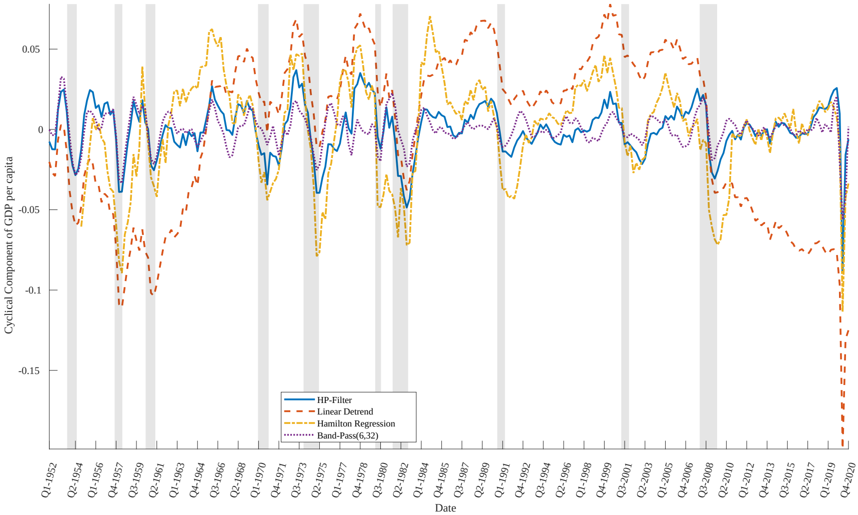

Figure 3: Cycles of U.S. GDP per capita in Log Scale

Stylized Facts

Business Cycle Statistics

When we study empirical properties of business cycles we typically look at three moments of the data

1. **Volatility:** Relative standard deviation $sd(x)/sd(gdp)$. Amplitude of fluctuations of x_t relative to GDP.

2. **Cyclicality:** the correlation between x_t and GDP $\rho(x_t, GDP)$
   - If $\rho(x_t, GDP) > 0$ then x_t is **procyclical**.
   - If $\rho(x_t, GDP) < 0$ then x_t is **countercyclical**.
   - If $\rho(x_t, GDP) = 0$ then x_t is **acyclical**.

3. **Persistence:** autocorrelation $\rho(x_t, x_{t-1})$

Business Cycle Statistics

Table 1: Business Cycle Statistics — Sims (2017)

Series	Std. Dev.	Rel. Std. Dev.	Corr w/y_t	Autocorr	Corr w/Y_{t-4}	Corr w/Y_{t+4}
Output	0.017	1.00	1.00	0.85	0.07	0.11
Consumption	0.009	0.53	0.76	0.79	0.07	0.22
Investment	0.047	2.76	0.79	0.87	-0.10	0.26
Hours	0.019	1.12	0.88	0.90	0.29	-0.03
Productivity	0.011	0.65	0.42	0.72	-0.50	0.35
Wage	0.009	0.53	0.10	0.73	-0.10	0.10
$1+$ Interest Rate	0.004	0.24	0.00	0.42	0.27	-0.25
Price Level	0.009	0.53	-0.13	0.91	0.09	-0.41
TFP	0.012	0.71	0.76	0.75	-0.34	0.34

Business Cycle Stylized Facts

Volatility:

- Investment $\gg$ GDP. Consumption $\ll$ GDP . Hours $\approx$ GDP. Prices $\ll$ GDP.

Cyclicality:

- Consumption, investment, hours are **procyclical**.

- Real wages and real interest rate **acyclical**. If used other expectations, real interest rate is slightly **countercyclical**. Moder macro models struggle to match this feature of the data.

Persistence: typically, all variables are strongly persistent.