Macroeconomics II

Lesson 02 — Foundations, Basic RBC, Solution of RBC

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Foundations

Foundations

We use models to be precise in what we mean, in two senses:

- Which assumptions we rely on.
- What are the exact relationships between variables.

We will rely on

- Identical, price-taking firms.
- Representative and infinitely lived household. (We will give microfoundations for this last assumption)
- In the model, money plays no role. Explanations are based on *real* factors.

Foundations — Representative Household

Theorem 1 (Gorman's Aggregation Theorem)

Consider an economy with $N < \infty$ commodities and a set $\mathcal{H}$ of households. Suppose that the preferences of household $h \in \mathcal{H}$ can be represented by an indirect utility function of the form

$$v^h(p, w^h) = a^h(p) + b(p)w^h$$

and that each household $h \in \mathcal{H}$ has a positive demand for each commodity. Then, these preferences can be aggregated and represented by those of a representative household, with indirect utility

$$v(p, w) = a(p) + b(p)w$$

where $a \equiv \int_{h\in\mathcal{H}} a^h(p)dh$ and $w \equiv \int_{h\in\mathcal{H}} w^h dh$

Foundations — Representative Household

Remarks on Gorman's Aggregation Theorem (Theorem 1):

- Linear Engel curves and same slope across households for the same commodity. How realistic is this?
- Gorman preferences are necessary for the economy to admit a strong representative household.
- The aggregation is done with an integral. It is a Lebesgue integral so, when $\mathcal{H}$ is finite or countable, $\int_{h \in \mathcal{H}} w^h dh$ is equivalent to $\sum_{h \in \mathcal{H}} w^h$.
- These preferences usually imply max rep. household $\Leftrightarrow$ Pareto optimality [Acemoglu 2009, see Chapter 5]

Foundations — Infinitely Lived Household

How can we assume infinitely lived households? Blanchard 1985 model.

- People live finite lives but date of death is uncertain.
- People die with a constant probability $\nu > 0$. Strong simplifying assumption but implies individuals live on expectation $1/\nu$ periods.
- Each individual has felicity function $u : \mathbb{R}_+ \mapsto \mathbb{R}$ and a (pure) discount factor $\hat{\beta}$.
- Normalize $u(0) = 0$ to be the utility after death and consider an individual with a consumption plan $\{c_t\}_{t=0}^{\infty}$ conditional on living.

Foundations — Infinitely Lived Household

Expected utility at time $t = 0$ is given by

$$\begin{aligned} U_0(c_0, c_1, \ldots) &= u(c_0) + \hat{\beta}(1-\nu)u(c_1) + \hat{\beta}\nu u(0) \\ &\quad + \hat{\beta}^2(1-\nu)^2 u(c_2) + \hat{\beta}^2\nu^2 u(0) + \cdots \\ &= \sum_{t=0}^{\infty} (\hat{\beta}(1-\nu))^t u(c_t) \end{aligned}$$

Letting $\hat{\beta}(1-\nu) = \beta$ we have

$$\mathbb{E}_0(U_0) = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

Foundations — Infinitely Lived Households

- This model shows we can rationalize infinitely lived households as

$$\mathbb{E}_0(U_0) = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

- This shows people can behave as if they would live forever. Other models make the consumption profiles be age dependent but this is out of the scope of this part.

- This is a possible way of rationalizing this assumption.

A Simple Dynamic Optimization Problem

A Simple Dynamic Optimization Problem

- Suppose there are many agents with identical preferences and endowments.
- There are enough agents so that they behave as price-takers.
- Suppose agents live for two periods t and $t + 1$.
- The agent receives endowment of Y_t and Y_{t+1} deterministically.
- In period t borrowing or lending is allowed. B_t are bonds with cost q_t.
- If $B_t > 0$ the agent saves, if $B_t < 0$ the agent borrows.

A Simple Dynamic Optimization Problem

- The agent begins at time t and must die with zero bonds.

- The agent faces two flow budget constraints for each period:

$$\begin{aligned} C_t + q_t B_t &\leq Y_t \\ C_{t+1} &\leq Y_{t+1} + B_t \end{aligned}$$

- The agent maximizes expected present discount value of flow utility

$$\begin{aligned} \max_{C_t, B_t, C_{t+1}} \quad & \log(C_t) + \beta \log(C_{t+1}) \\ \text{subject to} \quad & C_t + q_t B_t \leq Y_t \\ & C_{t+1} \leq Y_{t+1} + B_t \end{aligned}$$

A Simple Dynamic Optimization Problem

- In this context, everything is differentiable and well-behaved, so we can set-up a Lagrangian.

$$\mathcal{L} = \log(C_t) + \beta \log(C_{t+1}) + \lambda_1(Y_t - C_t - q_t B_t) + \lambda_2(Y_{t+1} + B_t - C_{t+1})$$

- The necessary FOCs imply

$$\begin{aligned} \frac{1}{C_t} &= \lambda_1 \\ \beta \frac{1}{C_{t+1}} &= \lambda_2 \\ q_t \lambda_1 &= \lambda_2 \end{aligned}$$

- Combining these, we get the Euler equation

$$q_t \frac{1}{C_t} = \beta \frac{1}{C_{t+1}}$$

A Simple Dynamic Optimization Problem

- **Dynamic Problem:** relationship between consumption today and tomorrow.

- The Euler condition says that saving less today $(\downarrow B_t)$ increases current consumption by q_t reducing tomorrow's consumption. Tomorrow's utility from today's perspective is reduced by β / C_{t+1}.

- A first notion of equilibrium:
  - Agents behave optimally. (what does this imply?)
  - All markets clear. Which implies

$$B_t = 0 \qquad Y_t = C_t$$

- What if we introduce a stochastic component?

A Simple Dynamic Stochastic Optimization Problem

- Same set-up as before but now the future endowment is stochastic. You can either be lucky (state Y^1_{t+1}) or unlucky (state Y^2_{t+1}).

- Probabilities of each state are p and $1 - p$ where $0 \leq p \leq 1$.

- Since future income is stochastic, so is consumption in period $t + 1$. We need to adapt our problem.

- The agents maximize **expected utility** conditional on today's information. We also assume rational expectations, which implies the agents *"know the model"*.

- Note further that there are two budget constraints for period $t + 1$ since they depend on which state of the world we are.

A Simple Dynamic Stochastic Optimization Problem

- The problem is then

$$\begin{aligned}
\max_{C_t, B_t, C^1_{t+1}, C^2_{t+1}} \quad & \log(C_t) + p\beta \log(C^1_{t+1}) + (1-p)\beta \log(C^2_{t+1}) \\
\text{subject to} \quad & C_t + q_t B_t \leq Y_t \\
& C^1_{t+1} \leq Y^1_{t+1} + B_t \\
& C^2_{t+1} \leq Y^2_{t+1} + B_t
\end{aligned}$$

- Settin-up the Lagrangian and computing the necessary FOCs as before, we get the Euler equation

$$q_t \frac{1}{C_t} = \beta \underbrace{\left(p \frac{1}{C^1_{t+1}} + (1-p) \frac{1}{C^2_{t+1}} \right)}_{\text{Expected Marginal Utility}} \Rightarrow q_t \frac{1}{C_t} = \beta \mathbb{E}_t \left(\frac{1}{C_{t+1}} \right)$$

A Simple Dynamic Stochastic Optimization Problem

- We could've written directly the problem in terms of the expectation operator as

$$\max_{C_t, B_t, C_{t+1}} \log(C_t) + \beta \mathbb{E}\left(\log(C_{t+1})\right) \tag{1}$$

$$\text{subject to} \quad C_t + q_t B_t \leq Y_t \tag{2}$$

$$\mathbb{E}_t\left(C_{t+1}\right) \leq \mathbb{E}_t\left(Y_{t+1}\right) + B_t \tag{3}$$

- This set-up is more convenient when we do not want to (or can't) specify all states of the world.

Time Consistency

Definition 1 (Time Consistency)

A solution $\{x_t\}_{t=0}^{T}$ (possibly $T \to \infty$) to a dynamic problem is **time consistent** if when $\{x_t\}_{t=0}^{T}$ is a solution starting at $t = 0$; $\{x_t\}_{t=\tau}^{T}$ is the solution to the continuation dynamic problem starting from $t = \tau > 0$.

- With exponential discounting and stationary utility functions, this definition is satisfied, however, with hyperbolic discounting things change $\Rightarrow$ **Problems of addiction**.

- Time consistency is a crucial property to find optimal plans.

Basic Real Business Cycle Model

Set-up of the Basic RBC Model

- Discrete-time Ramsey-Cass-Koopmans model (**dynamic**) with **stochastic** TFP shocks.
- We will also add endogenous labor supply.
- Infinitely-lived representative household.
- Identical price-taking firms.
- Seminal references Kydland and Prescott (1982) and Long and Plosser (1983)

Firms

Firms

- A single good Y_t is produced using capital K_t and labor H_t.
- The representative firm produces using a Cobb-Douglas technology

$$Y_t = A_t K_t^{\alpha} (X_t H_t)^{1-\alpha} \qquad 0 < \alpha < 1 \tag{4}$$

where

1. X_t is a deterministic labor-augmenting technical progress. $X_{t+1} = X_0(1+g)^t$ for $g > 0$.
2. A_t is a random productivity shock with persistence (AR(1) process).

$$\log(A_t) = (1-\rho)\log(A) + \rho \log(A_{t-1}) + \varepsilon_t \tag{5}$$

with $\rho \in (0,1)$, $A > 0$, and $\varepsilon \sim \mathcal{N}(0, \sigma_\varepsilon^2)$ is a purely transitory shock.

Firms

- Firms sell the output to households for consumption C_t and investment I_t purposes.

- The goods market is perfectly competitive thus firms take prices as given. By Walras' law we can normalize the price to 1 in every period.

- Labor and capital markets are also perfectly competitive.

- Firms hire labor and rent capital from households at the wage rate w_t and rental rate r_t^k.

- Given prices w_t and r_t^k, the representative firm chooses the demand for labor and capital by maximizing profits. Construct the profit function and necessary FOCs.

Households

Households — Structure and Notation

- In each period t there is a continuum of inifinitely lived identical households indexed by i over the interval $[0, 1]$.

- Each household consists of N_t members evolving exogenously according to

$$N_t = N_0(1+n)^t \quad n > 0$$

  where we normalize $N_0 = 1$ and let lower-case variables denote per capita terms.

- Each member is endowed with 1 unit of time that can use for work h_t or for leisure ℓ_t

$$h_t + \ell_t = 1$$

- Total labor supply $H_t = N_t h_t$.

Households — Preferences

- Agents enjoy leisure and consumption. We specify the following preferences

$$
\begin{aligned}
u(c_t, \ell_t) &= \log(c_t) + \xi \log(\ell_t) \\
&= \log(c_t) + \xi \log(1 - h_t) \\
&= u(c_t, h_t)
\end{aligned}
\tag{6}
$$

- We can specify preferences in terms of leisure or work, they are equivalent. Preferences are increasing in c_t and ℓ_t, but decreasing in h_t. Disutility of working.

- Since agents have identical preferences, the utility of a family is

$$N_t u(c_t, h_t) = (1+n)^t u(c_t, h_t)$$

Households — Resources

- Households save accumulating capital K_t over time but capital depreciates every period at the rate $\delta \in (0, 1)$.

- The law of motion for the capital stock is given by

$$K_{t+1} = (1 - \delta)K_t + I_t \tag{7}$$

- Notice that at time t, K_{t+1} will be known. For this reason it is sometimes called a **predetermined variable.**

- Households earn wages from labor, therefore, labor income is given by $w_t N_t h_t$ and from rented capital $r_t^k K_t$.

Households — Budget Constraint

- In the model there is no borrowing, so households can consume only the amount they have in resources.

- Households spend their income in consumption or investment. We can use (7) to build the household's budget constraint

$$c_t N_t + K_{t+1} = w_t N_t h_t + \left(r_t^k + (1-\delta)\right) K_t$$

Can we write this in per capita terms?

- Consumption and leisure today depend on the returns from the predetermined capital stock, which was decided yesterday. In each period t, the household needs to make an **optimal plan** $\{c_t, h_t, k_{t+1}\}_{t=0}^{\infty}$

Households — Optimal Plans

- Notice that to evaluate the plan $\{c_t, h_t, k_{t+1}\}_{t=0}^{\infty}$ the instantaneous utility function is **not enough!**

- We need an objective that evaluates present and future utilities jointly, just like in the two periods model Problem (1). This is called the **inter-temporal utility function**.

- But we also have to take expectations since output is **uncertain!!** (recall Equations (4) and (5)). Then, the inter-temporal utility function is given by

$$U_t = \mathbb{E}_t \sum_{j=0}^{\infty} \tilde{\beta}^j (1+n)^j u(c_{t+j}, h_{t+j}) = \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j}, h_{t+j}) \tag{8}$$

where $\beta = \tilde{\beta}(1+n) < 1$ is the subjective discount factor.

Households — Optimal Plans

- It is crucial to define how the expectations operator works in $\mathbb{E}_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j}, h_{t+j})$. We will assume agents have **rational expectations**.

- As we mentioned in previous lectures, rational expectations [Muth 1961] imply they are
  - Correct on average.
  - Unpredictable forecasting errors given current info.

  So agents know the model and they know the structure of shocks over time.

- We are in a position to state the full household problem and the method we are going to use.

Households — Optimal Plans

- Recall the definition of Time Consistency (Def 1). An implication is that we can focus on the first period problem stated as (Can you see why?)

$$
\begin{aligned}
\max_{\{c_t, h_t, k_{t+1}\}} \quad & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t, h_t) \\
\text{subject to} \quad & c_t + (1+n)k_{t+1} = w_t h_t + (r_t^k + (1-\delta))k_t \quad \forall t \geq 0 \\
& k_0 > 0 \text{ given}
\end{aligned}
\tag{9}
$$

(This is also called the **sequential problem**) (Can you prove the problem is time-consistent?)

- Two standard approaches to solve this problem:
  1. Stochastic dynamic programming [Acemoglu 2009, see Chapter 16]
  2. Lagrangian multipliers.
- We will focus in the second approach since in our models everything is differentiable.

Households — Solving the Problem

- Let us define for simplicity the net rental rate of capital $r_t \equiv r_t^k - \delta$ and define the Lagrangian as

$$
\begin{aligned}
\mathcal{L} = \mathbb{E}_t \Bigg\{ \sum_{j=0}^{\infty} \beta^j \Bigg[u(c_{t+j}, h_{t+j}) + \lambda_{t+j} \Big(w_{t+j} h_{t+j} + \\
(1 + r_{t+j}) k_{t+j} - c_{t+j} - (1+n) k_{t+j+1} \Big) \Bigg] \Bigg\}
\end{aligned}
\tag{10}
$$

- Any optimal plan needs to satisfy the FOCs w.r.t. $\{c_t, h_t, k_{t+1}, \lambda_t\}$ and the transversality condition (TVC).

- The FOCs are necessary but for sufficency, we need to rule out explosive paths. That is achieved with the TVC.

Households — Solving the Problem

- **State** variables: $\{k_t, A_t\}$
- **Control** variables: $\{c_t, k_{t+1}, h_t\}$.
- Transition functions: equations (5) and (7).
- Return function: $u(c_t, h_t)$.
- Note k_{t+1} is **predetermined** since the state of the system and the choice of c_t uniquely identify k_{t+1}.

Households — Solving the Problem

The optimal plan $\{c_t^*, h_t^*, k_{t+1}^*\}_{t=0}^{\infty}$ is characterized by:

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Rightarrow \frac{1}{c_t^*} = \lambda_t^* \qquad \text{(Consumption FOC)}$$

$$\frac{\partial \mathcal{L}}{\partial h_t} = 0 \Rightarrow \frac{\xi}{1 - h_t^*} = \lambda_t^* w_t \qquad \text{(Intra-temporal Condition)}$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = 0 \Rightarrow \lambda_t^* = \frac{\beta}{1+n} \mathbb{E}_t \left((1 + r_{t+1}) \lambda_{t+1}^* \right) \qquad \text{(Capital FOC)}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_{t+1}} = 0 \Rightarrow c_t^* = w_t h_t^* + (1 + r_t) k_t^* - (1 + n) k_{t+1}^* \qquad \text{(Budget Constraint)}$$

$$\lim_{t \to \infty} \beta^t k_t^* \lambda_t^* = 0 \qquad \text{(Transversality Condition)}$$

Households — Solving the Problem

- Equation (Consumption FOC) relates marginal utility of consumption and the Lagrange multiplier.

- How much the household is willing to pay for an extra unit of savings in the present (**shadow-price**).

- Equations (Consumption FOC) and (Capital FOC) imply the Euler equation:

$$\frac{1}{c_t^*} = \frac{\beta}{1+n}\mathbb{E}_t\left((1+r_{t+1})\frac{1}{c_{t+1}^*}\right) \tag{11}$$

- As in the two-periods model, the Euler equation relates the trade-off between present and (expected) future consumption.

Equilibrium Concepts

Competitive Equilibrium

Competitive Equilibrium

Definition 2 (Competitive Equilibrium)

A competitive equilibrium is a price vector $\{w_t, r_t^k, r_t\}_{t=0}^{\infty}$ and an allocation $\{c_t, y_t, i_t, k_t, h_t, \ell_t\}_{t=0}^{\infty}$ such that given the initial capital stock $k_0 > 0$ and a sequence of stochastic productivity $\{A_t\}_{t=0}^{\infty}$ satisfying equation (5):

1. Households maximize the inter-temporal utility under (i) the period by period budget constraint; (ii) the constraint on the allocation of time endowment; and (iii) the capital law of motion.

2. Firms maximize profits under the technological constraint.

3. All markets clear, i.e. $\forall t > 0$

$$h_t^s = h_t^d = h_t \qquad \text{(Labor market)}$$

$$k_t^s = k_t^d = k_t \qquad \text{(Capital market)}$$

$$y_t = c_t + i_t \qquad \text{(Goods market)}$$

Competitive Equilibrium

Characterization of the Competitive Equilibrium (I)

$$\alpha \frac{y_t}{k_t} = r_t^k \tag{CE1}$$

$$(1-\alpha)\frac{y_t}{h_t} = w_t \tag{CE2}$$

$$\xi \frac{c_t}{1-h_t} = w_t \tag{CE3}$$

$$\frac{1}{c_t} = \frac{\beta}{1+n}\mathbb{E}_t\left(\frac{1+r_{t+1}}{c_{t+1}}\right) \tag{CE4}$$

$$r_t = r_t^k - \delta \tag{CE5}$$

Characterization of the Competitive Equilibrium (II)

$$(1+n)k_{t+1} = (1-\delta)k_t + i_t \tag{CE6}$$

$$h_t + l_t = 1 \tag{CE7}$$

$$\lim_{t\to\infty} \beta^t \frac{k_t}{c_t} = 0 \tag{CE8}$$

$$y_t = c_t + i_t \tag{CE9}$$

$$y_t = A_t k_t^{\alpha} (X_t h_t)^{1-\alpha} \tag{CE10}$$

$$\log(A_t) = (1-\rho)\log(A) + \rho\log(A_{t-1}) + \varepsilon_t \tag{CE11}$$

Competitive Equilibrium

Remark 1

- *X_t is exogenous and deterministic, we don't need to include it in the definition of the equilibrium.*
- *Market clearing conditions are implicit in the equilibrium conditions. There is no difference between supply and demand variables.*
- *The resource constraint (CE9) is implied by the budget constraint, we cannot use both conditions to solve for the equilibrium.*
- *Competitive markets + locally non-satiated preferences $\Rightarrow$ First Welfare Theorem holds.*

Competitive Equilibrium

An implication of the First Welfare Theorem is that we can solve equivalently

$$\max_{c_t, h_t, k_{t+1}} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t, h_t)$$

$$\text{subject to} \quad c_t + (1+n)k_{t+1} = A_t k_t^{\alpha} (X_t h_t)^{\alpha} + (1-\delta)k_t \ \forall\, t \geq 0$$

$$k_0 \text{ given}$$

This is usually called the **Social Planner Problem**. Show that the two problems are equivalent.

The Balanced Growth Path (BGP)

The Balanced Growth Path (BGP)

- Some aggregate ratios (e.g. capital-output ratio) have been historically constant.

- These are called the *Kaldor facts*. The original facts continue to hold but some new stylized facts emerged [Herrendorf, Rogerson, and Valentinyi 2019; Jones and Romer 2010].

- The basic RBC model was built around the following stylized facts

  - Constant labor income share [Karabarbounis and Neiman 2014].
  - Constant per capita hours of work [Boppart and Krusell 2020].
  - Constant consumption share
  - Constant investment share

The Balanced Growth Path (BGP)

Definition 3 (Balanced Growth Path Equilibrium)

A BGP is a competitive equilibrium in which all the endogenous variables grow at **constant rates** while the stochastic innovations stay constant at their mean values.

- Along the BGP $\varepsilon_t = 0 \; \forall t$

- In practice, we eliminate uncertainty and refer to an equilibrium with **perfect foresight**.

- For the BGP to exist, technology and preferences need to be restricted in the RBC.

Existence

Existence of BGP Equilibrium

- You have seen this with Luiz in Macroeconomics I but we briefly go through the proof.

- Define the gross growth rate of a variable as

$$\gamma_{\nu} = \frac{V_{t+1}}{V_t}$$

- To show the BGP equilibrium exists, we prove the equilibrium conditions are compatible with a situation in which all endogenous variables grow at constant rates.

Existence of BGP Equilibrium

- Since $\varepsilon_t = 0\ \forall t$ equation (CE11) implies $\lim_{t\to\infty} A_t = A$ or $\gamma_A = 1$.

- Using equation (CE6)

$$\gamma_k = \frac{k_{t+1}}{k_t} = \frac{1-\delta}{1+n} + \frac{1}{1+n}\frac{i_t}{k_t}$$

  so i_t/k_t is constant or $\gamma_i = \gamma_k$.

- From (CE9)

$$\gamma_y = \gamma_c \left(\frac{c_{t-1}}{y_{t-1}}\right) + \gamma_i \left(1 - \frac{c_{t-1}}{y_{t-1}}\right)$$

  which implies $\gamma_y = \gamma_c = \gamma_i = \gamma_k$

Existence of BGP Equilibrium

- Using equation (CE10) and using $\gamma_y = \gamma_k$ along the BGP

$$\gamma_y = \gamma_c = \gamma_i = \gamma_k = (1+g) \tag{14}$$

- Given that $h_t < 1$ by construction, the only possible way of ensuring a BGP is $\gamma_h = 1$ [Boppart and Krusell 2020]

- The two last results imply (**Show this!**)
  - Wage rate grows at $1+g$
  - r_t^k is constant
  - Total labor supply $H_t = N_t h_t$ grows at $1+n$
  - Leisure stays constant

Existence of BGP Equilibrium

- Check the TVC

$$\lim_{t\to\infty}\beta^t\frac{k_t}{c_t}=\frac{k_t}{c_t}\lim_{t\to\infty}\beta^t=0$$

- The intratemporal condition

$$\underbrace{\frac{\xi}{1-h_t}}_{\text{Constant}}=\underbrace{\frac{w_t}{c_t}}_{\text{Constant}}$$

- Euler equation

$$\gamma_c=\frac{c_{t+1}}{c_t}=\frac{\beta}{1+n}\underbrace{(1+r_{t+1})}_{\text{Constant}}$$

Remark 2

*This implies variables are not stationary but grow at constant rates. We look at the **deviations** with respect to long-run variables.*

Steady State Equilibrium

Definition 4 (Steady State Equilibrium)

A (deterministic) steady state equilibrium is a competitive equilibrium in which all the stochastic innovations are set to their mean values and all the endogenous variables stay constant over time.

- You know how to do this. Which equations from (CE1)-(CE11) need to be adjusted?

- We can now compute the steady state equilibrium.

Steady State Equilibrium

Characterization of the Steady State Equilibrium (I)

$$\alpha \frac{y_t}{k_t} = r_t^k \tag{SS1}$$

$$(1-\alpha)\frac{y_t}{h_t} = w_t \tag{SS2}$$

$$\xi \frac{c_t}{1-h_t} = w_t \tag{SS3}$$

$$\frac{1}{c_t} = \frac{\beta}{(1+g)(1+n)} \mathbb{E}_t \left(\frac{1+r_{t+1}}{c_{t+1}} \right) \tag{SS4}$$

$$r_t = r_t^k - \delta \tag{SS5}$$

Steady State Equilibrium

Characterization of the Steady State Equilibrium (II)

$$(1+g)(1+n)k_{t+1} = (1-\delta)k_t + i_t \tag{SS6}$$

$$h_t + l_t = 1 \tag{SS7}$$

$$\lim_{t\to\infty} \beta^t \frac{k_t}{c_t} = 0 \tag{SS8}$$

$$y_t = c_t + i_t \tag{SS9}$$

$$y_t = A_t k_t^{\alpha} h_t^{1-\alpha} \tag{SS10}$$

$$\log(A_t) = (1-\rho)\log(A) + \rho\log(A_{t-1}) + \varepsilon_t \tag{SS11}$$

Steady State Equilibrium

Remark 3

- *The steady state is a* ***static concept****. There are no more dynamics involved once we reach this point (i.e. the point is stationary).*

- *To see this, note that the law of motion for capital in the steady state is*

$$i = \left[(1+n)(1+g) - (1-\delta)\right] k$$

- *The role of investment is to replace depreciated capital while n and g appear because we deal with efficiency units.*

Solution of the RBC Model

Solution of the RBC Model

- We have the equilibrium conditions of the model but to **solve it** we need to find the sequence
$$\{c_t, h_t, k_{t+1}, i_t, y_t, r_t^k, w_t\}_{t=0}^{\infty}$$
that, given $\{A_t\}_{t=0}^{\infty}$ and $k_0 > 0$ solves the system (SS1)-(SS11).

- Several ways of finding this (local vs global solutions).
  - VFI, projection, policy function iteration...

- We are going to focus on **perturbation methods** (Here you can find a more in-depth lecture.)

- Why we need numerical methods? We have to solve a system of stochastic difference equations
$$x_t = \mathbb{E}_t(x_{t+1})$$
which are highly non-linear and usually impossible to solve analytically.

Solution to the RBC Model

Perturbation Method — Summary of the Method

1. Find the steady state equilibrium. (We will skip this)

2. Log-linearize the optimality conditions around the steady state. (Expectational linear equations, this is an **approximation!!**)

3. Solve the system of linear stochastic difference equations:
    3.1 Check unicity and stability (Blanchard-Khan Conditions)

    3.2 Find linear policy functions

Solution to the RBC Model

Perturbation Method — Remarks

- Perturbation methods are **local solution methods**.
- They work well *close* to the steady state.
- Advantages:
  - Low computational expense (fast)
  - Easily automated (Dynare)
- Drawbacks:
  - Accuracy decreases rapidly if away from the steady state.
  - Not suitable for models with strong non-linearities (e.g. kinks, ZLB)

Solution to the RBC Model

Perturbation Method — General Ideas

- Critical: **Taylor's theorem**.

- Suppose $f : \mathbb{R}^n \mapsto \mathbb{R}^m$. A linear approximation at $x = x_0$ is
$$f(x) = f(x_0) + f_x(x_0)(x - x_0) + \frac{1}{2}(x - x_0)' f_{xx}(x_0)(x - x_0) + \mathcal{R}$$

- $\mathcal{R}$ increases away of x_0 and is reduced with higher orders of expansion.

- The idea is to transform every variable u_t as a log-deviation from the steady state, i.e.
$$\hat{u}_t = \log(u_t) - \log(u)$$
where u is the steady state value of u_t.

- Then take the Taylor approximation of the transformed system around the steady state.

Solution to the RBC Model

Log-Linearization

- In practice, it is typically easier to replace any variable u_t with

$$u_t = ue^{\log(u_t) - \log(u)} = ue^{\hat{u}_t}$$

- Why? If we have products on both sides of an equation:

$$w_t = (1-\alpha)\frac{y_t}{h_t} \Rightarrow w_t e^{\hat{w}_t} = (1-\alpha)\frac{y}{h}e^{\hat{y}_t - \hat{h}_t}$$
$$e^{\hat{w}_t} = e^{\hat{y}_t - \hat{h}_t} \Rightarrow \hat{w}_t = \hat{y}_t - \hat{h}_t$$

- With additive conditions $e^{\hat{u}_t} \approx 1 + \hat{u}_t$ which implies

$$y_t = c_t + i_t \Rightarrow \hat{y}_t = \left(\frac{c}{y}\right)\hat{c}_t + \left(\frac{i}{y}\right)\hat{i}_t$$

- Define $\beta^* = \beta/((1+n)(1+g))$.

Solution to the RBC Model

Log-Linearized System (I)

$$\hat{y}_t - \hat{k}_t = \hat{r}_t^k \tag{LL1}$$

$$\hat{y}_t - \hat{k}_t = \hat{w}_t \tag{LL2}$$

$$\hat{h}_t = \left(\frac{1-h}{h}\right)(\hat{w}_t - \hat{c}_t) \tag{LL3}$$

$$\hat{c}_t = \mathbb{E}_t\left(\hat{c}_{t+1}\right) - (1-\beta^*)\mathbb{E}_t\left(\hat{r}_{t+1}^k\right) \tag{LL4}$$

$$\hat{r}_t = \hat{r}_t^k \left(\frac{1-(1-\delta)\beta^*}{1-\beta^*}\right) \tag{LL5}$$

Solution to the RBC Model

Log-Linearized System (II)

$$(1+g)(1+n)k_{t+1} = (1-\delta)k_t + i_t \tag{LL6}$$

$$\hat{h}_t = -\left(\frac{1-h}{h}\right)\hat{\ell}_t \tag{LL7}$$

$$\hat{y}_t = \frac{c}{y}\hat{c}_t + \frac{i}{y}\hat{i}_t \tag{LL8}$$

$$\hat{y}_t = \hat{A}_t + \alpha\hat{k}_t + (1-\alpha)\hat{h}_t \tag{LL9}$$

$$\hat{A}_t = \rho\hat{A}_{t-1} + \varepsilon_t \tag{LL10}$$

Blanchard-Khan Conditions

Blanchard-Khan Conditions

- We have transformed a nonlinear system of equations into a linearized one. Now we want to check for uniqueness (linear policy functions).

- The linear system can be reduced to

$$\mathbb{E}_t \begin{pmatrix} \hat{c}_{t+1} \\ \hat{h}_{t+1} \\ \hat{k}_{t+1} \\ \hat{A}_{t+1} \end{pmatrix} = \mathbf{A} \begin{pmatrix} \hat{c}_t \\ \hat{h}_t \\ \hat{k}_t \\ \hat{A}_t \end{pmatrix}$$

  where $\mathbf{A}$ is a 4×4 matrix of coefficients, $\hat{c}_t$ and $\hat{h}_t$ are *jump/forward looking* variables, while $\hat{k}_t$ and $\hat{A}_t$ are *state/predetermined variables*.

Blanchard-Khan Conditions

- Suppose $\mathbf{A}$ has 4 distinct eigenvalues (if not, use the Jordan canonical form). Then

$$\mathbf{A} = \mathbf{\Gamma\Lambda\Gamma}^{-1}$$

  where $\mathbf{\Gamma}$ contains the eigenvectors of $\mathbf{A}$ and $\mathbf{\Lambda}$ is a diagonal matrix with its eigenvalues.

- Order the eigenvalues from smalles to largest. In fact, partition $\mathbf{\Lambda}$ as

$$\mathbf{\Lambda} = \begin{pmatrix} \underset{Q\times Q}{\lambda_1} & 0 \\ 0 & \underset{B\times B}{\lambda_2} \end{pmatrix}$$

  where λ_1 is a diagonal matrix containing the stable eigenvalues and λ_2 the unstable ones. (Obviously, $Q + B = 4$)

Blanchard-Khan Conditions

- Let us define $u_t \equiv \{\hat{c}_t, \hat{h}_t, \hat{k}_t, \hat{A}_t\}$ and write the system as

$$\mathbb{E}_t u_{t+1} = \mathbf{A} u_t \tag{19}$$

- Multiplying (19) by $\mathbf{\Gamma}^{-1}$ and defining $\mathbf{Z}_t \equiv \mathbf{\Gamma}^{-1} u_t$

$$\mathbb{E}_t \mathbf{Z}_{t+1} = \mathbf{\Lambda} \mathbf{Z}_t$$

or

$$\mathbb{E}_t \begin{pmatrix} \underset{Q\times 1}{Z_{1,t+1}} \\ \underset{B\times 1}{Z_{2,t+1}} \end{pmatrix} = \begin{pmatrix} \underset{Q\times Q}{\lambda_1} & 0 \\ 0 & \underset{B\times B}{\lambda_2} \end{pmatrix} \begin{pmatrix} \underset{Q\times 1}{Z_{1,t}} \\ \underset{B\times 1}{Z_{2,t}} \end{pmatrix} \tag{20}$$

Blanchard-Khan Conditions

- The system (20) has two independent components that can be treated separately. In fact, by the LIE (Recall from first lecture)

$$\mathbb{E}_t\left(Z_{1,t+T}\right) = \lambda_1^T Z_{1,t}$$
$$\mathbb{E}_t\left(Z_{2,t+T}\right) = \lambda_2^T Z_{1,t}$$

- Note that since $|\lambda_1| < 1$ the first component is stable (i.e. $\lim_{T\to\infty} \lambda_1^T Z_{1,t} = 0$) while the second component diverges.

- **BUT!!!** The TVC would be violated in this case! Thus, we need

$$Z_{2,t} = 0$$

Blanchard-Khan Conditions

- What does $Z_{2,t} = 0$ imply? Define

$$\mathbf{\Gamma}^{-1} = \begin{pmatrix} \underset{Q\times 2}{G_{11}} & \underset{Q\times 2}{G_{12}} \\ \underset{B\times 2}{G_{21}} & \underset{B\times 2}{G_{22}} \end{pmatrix} \Rightarrow Z_{2,t} = G_{21} \begin{pmatrix} \hat{c}_t \\ \hat{h}_t \end{pmatrix} + G_{22} \begin{pmatrix} \hat{k}_t \\ \hat{A}_t \end{pmatrix}$$

- $Z_{2,t} = 0$ implies

$$G_{21} \begin{pmatrix} \hat{c}_t \\ \hat{h}_t \end{pmatrix} = -G_{22} \begin{pmatrix} \hat{k}_t \\ \hat{A}_t \end{pmatrix}$$

- Three possible cases (G_{21} is dimension $B \times 2$)
  - $B = 2 \Rightarrow G_{21}$ is invertible, thus only one non-explosive solution.
  - $B < 2$ there are infinitely many non-explosive solutions (**indeterminacy**) .
  - $B > 2$ the system is possible. No solution.

Blanchard-Khan Conditions

Definition 5

Consider the system of linear rational expectations (19). If the number of eigenvalues of $\mathbf{A}$ outside the unit circle is

1. Equal to the number of forward looking variables, then there exists a unique solution.
2. Less than the number of forward looking variables, then there exist infinitely many solutions.
3. Larger than the number of forward looking variables, then no solution exists that does not violate the non-explosive condition.

Blanchard-Khan Conditions

- When the solution is unique, the policy functions can be expressed as

$$\hat{c}_t = \theta_{ck}\hat{k}_t + \theta_{ca}\hat{A}_t$$
$$\hat{h}_t = \theta_{hk}\hat{k}_t + \theta_{ha}\hat{A}_t$$

- That is, all endogenous variables can be expressed as functions of the states $\zeta_t \equiv \hat{k}_t, \hat{A}_t$. Let Y_t denote the vector of control variables, then the system can be expressed in **state-space** representation

$$\zeta_{t+1} = \Theta_\zeta \zeta_t + \Theta_\varepsilon \varepsilon_{t+1}$$
$$Y_t = \Theta_Y \zeta_t$$

Blanchard-Khan Conditions

- To *actually* solve for the endogenous variables, there are several solution methods. Among them, the method of undetermined coefficients can be used to find the coefficients analytically as functions of the deep parameters of the model.

- No big gains from solving analytically. These parameters are complicated functions of structural parameters and do not provide clear-cut insights on the relationships between variables.

- How to solve then? Numerical analysis!
  - Assign values to structural parameters (calibrate/estimate).
  - Solve numerically the linearized system.
  - Simulate the model to assess properties around the steady state.

Calibration

Calibration — Preliminaries

- We have seen we cannot solve our general model by hand, how can we solve then the model?

- Also, the model depends on a set of (structural) parameters for which (up to now) we do not know their *true* values.

$$\{\alpha, A, X_0, n, g, \delta, \rho, \sigma_\varepsilon, \tilde{\beta}, \xi\}$$

- This is **economics** so we need to choose **meaningful and realistic** parameter values.

- What does realistic mean? In line with the available data.

Calibration — Preliminaries

- How do we make it *in line with the data*? Match targets $\rightarrow$ moments.

- Choose particular parameters so that the consumption share in the model is equal to 0.7 (target) which matches the average value of consumption to GDP ratio in the data (moment).

- More sophisticated alternatives:
  - GMM Estimation, SMM Estimation, Bayesian Estimation, Classical Likelihood Estimation...

- Which moments to target? Those directly related to the data. In the basic RBC:
  - Targets related to the steady state.

Calibration — The Nuts and Bolts

- Two things to specify before getting in the details
  - The economy (country, region, city. . .) from which collect the calibration targets.
  - Time units (years, months, days. . .) which will affect rates.
- Usually we look at countries, but this is changing. The U.S. is a great choice because of data availability.
- We usually define business cycles in quarters, so the time unit usually is a quarter (3 months).

Calibration — The Nuts and Bolts

The Suppply Side — Production Function Parameters

- We need to calibrate $\{A, X_0, \alpha\}$

- A and X_0 will affect the scale of the endogenous variables $\Rightarrow$ normalize them (i.e. $A = X_0 = 1$).

- The firm's FOC for labor (CE2) implies

$$(1-\alpha) = \frac{wh}{y}$$

  which is the long-run labor income share.

- In the U.S. this value is about $2/3 \Rightarrow \alpha = 1/3$.

Calibration — The Nuts and Bolts

Growth Rates

- We need to calibrate $\{n, g\}$

- With data from 1969 to 2019 the yearly growth rate is 1.0014%. But every period is a quarter, so

$$n = (1 + 0.010014)^{1/4} - 1 = 0.0025$$

- Recall $(1 + g)$ denotes the (gross) growth rate of per capita GDP along the BGP. From 1947 to 2019 average GDP per capita growth was 1.9% which implies

$$g = (1 + 0.0190)^{1/4} - 1 = 0.0047$$

Calibration — The Nuts and Bolts

Capital Depreciation Rate

- Our model definition of capital includes all tangible capital:
  - Plant and equipment stocks, consumer durables, and housing.

- Depreciation is **not constant** across different types of capital. An assumption of $\delta \approx 10\%$ is the standard. See Koh, Santaeulàlia-Llopis, and Zheng (2020) for an analysis of the decline in the labor share and a discussion on the different depreciation rates across capital products. Consequently, we choose

$$\delta = (1.10)^{1/4} - 1 = 0.025;$$

Calibration — The Nuts and Bolts

Shock Process

- Taking logs in our production function in intensive form:

$$\underbrace{\hat{A}_t + (1-\alpha)\log(X_t)}_{\text{Solow Residual}} = \log(y_t) - (1-\alpha)\log(h_t) - (1-\alpha)\log(k_t)$$

- With a calibrated value for α and data on $\{y_t, h_t, k_t\}$ we can compute a series for the Solow Residual (remember growth accounting?)

$$SR_t = \theta_0 + \theta_1 t + \hat{A}_t$$
$$\hat{A}_t = \rho \hat{A}_{t-1} + \varepsilon_t$$

King and Rebelo (1999) report $\rho = 0.979$ and $\sigma_\varepsilon = 0.0072$. Note $\theta_0 \equiv (1-\alpha)\log(X_0)$ and $\theta_1 \equiv \log(1+g)$

Calibration — The Nuts and Bolts

Preferences

- The Euler equation at the steady state implies

$$\beta = \frac{(1+g)(1+n)}{1+r}$$

- We can use target values for r in the steady state to recover $\tilde{\beta}$. King and Rebelo 1999 report an average return on the S&P 500 Index of 6.5% per annum. Thus

$$\beta = \frac{1.0072}{(1.065)^{1/4}} = 0.992$$

- The intratemporal condition pins-down parameter ξ

$$\xi = (1-\alpha)\left(\frac{1-h}{h}\right)\frac{y}{c}$$

Calibration — The Nuts and Bolts

Preferences

- We need targets for h and the ratio c/y. A target for $h = 0.2$ is consistent with an individual devoting 20% of her time to working activities. (8 hours/day, 5 days/week $\approx$ 0.24, taking into account part-time workers. . .)

- The ratio c/y can be recovered from the resource constraint

$$\frac{c}{y} = 1 - \frac{i}{k}\frac{k}{y} = 1 - [(1+g)(1+n) - (1-\delta)]\left(\frac{\alpha}{r+\delta}\right)$$

- This implies $\xi = 3.6097$

Calibration — The Nuts and Bolts

Preferences — Intertemporal Elasticity of Substitution

- The utility function is a separable from leisure CRRA

$$u(c_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma}$$

  with $\sigma = 1 \Rightarrow$ log-utility. What is σ capturing? Consider our two-period model with general CRRA (i.e. $\sigma \neq 1$).

$$u(c_t, c_{t+1}) = \frac{c_t^{1-\sigma} - 1}{1-\sigma} + \beta \frac{c_{t+1}^{1-\sigma} - 1}{1-\sigma}$$

- Parameter σ controls the curvature of $u(\cdot)$

Calibration — The Nuts and Bolts

Preferences — Intertemporal Elasticity of Substitution

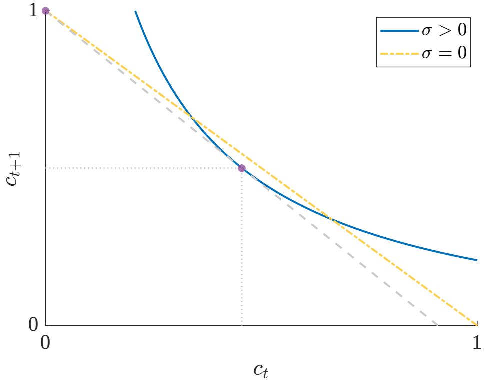

Figure 1: Consumption Smoothing

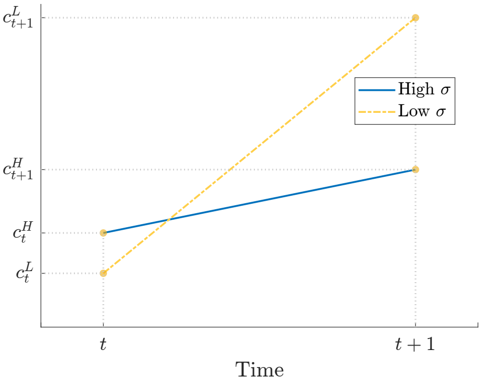

Figure 2: Consumption Path

Definition 6 (Intertemporal Elasticity of Substitution)

Assuming discrete time and that households have instantaneous preferences defined by the felicity function $u(c_t) : \mathbb{R}_+ \mapsto \mathbb{R}$, the intertemporal elasticity of substitution can be defined as

$$IES = -\frac{d\log(c_{t+1}/c_t)}{d\log(u'(c_{t+1})/u'(c_t))}$$

In general

$$IES = \frac{d\log(c_{t+1}/c_t)}{dR}$$

where R is the net real interest rate. Under CRRA preferences $IES = \frac{1}{\sigma}$.

Calibration — The Nuts and Bolts

Preferences — Intertemporal Elasticity of Substitution

- The IES measures how much individuals are willing to substitute consumption today for tomorrow.

- Recall the Euler equation with these preferences is

$$c_t^{-\sigma} = \frac{\beta}{(1+n)(1+g)} \mathbb{E}_t \left[(1+r_{t+1}) c_{t+1}^{-\sigma} \right]$$

- This implies consumption between t and $t+1$ is determined by
  1. **Patience**. How big is β.
  2. **Interest rate**. A higher r_{t+1} makes more profitable saving today.
  3. **Consumption smoothing**. The lower σ, the higher the willingness to substittue and the steeper the consumption profile between two periods.

Definition 7 (Frisch Elasticity of Labor Supply)

The Frisch elasticity of labor supply measures the percentage change in hours worked in response to the percentage change in wages holding constant the marginal utility of wealth (i.e. the Lagrange multiplier associated with the budget constraint λ_t):

$$\varepsilon_{h,w} = \left.\frac{dh_t/h_t}{dw_t/w_t}\right|_{\lambda_t=\lambda}$$

The Frisch elasticity is a measure of the substitution effect. With separable preferences

$$\varepsilon_{h,w} = \frac{1-h_t}{h_t}$$

Notice that with general preferences

$$u(c_t, h_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma} + \xi \frac{(1-h_t)^{1-\eta} - 1}{1-\eta}$$

the Frisch elasticity of labor supply is given by

$$\varepsilon_{h,w} = \frac{1-h_t}{\eta h_t}$$

Calibration — The Nuts and Bolts

Preferences — Frisch Elasticity

- A target of $h = 0.2$ implies

$$\varepsilon_{h,w} = \frac{0.8}{0.2} = 0.4$$

- Realistic? Micro estimates and macro estimates differ. See Chetty et al. (2011) for a review of the evidence.

- Mismatch between intensive (how many hours to work) and extensive (whether to work or not) margins. (We will see a model with indivisible labor supply.)

- Chetty et al. (2011) suggest elasticities no larger than 0.5 should be used for macro models with representative agents.

- With our calibrated model, we turn now to the actual solution in **Dynare**.