

Introduction to Matlab

Lesson 03 — Optimization

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Optimization Preliminaries

Optimization Preliminaries

- **Agents optimize** is the foundational assumption of economic theory.
 - Firms minimize costs / maximize profits.
 - Consumers maximize utility / minimize expenditures.
- Not only in theory ... Econometricians use
 - Maximum likelihood, least squares, method of moments ...
 - All optimization problems.
- We will learn the basics of how to state and solve this type of problems in Matlab.

Optimization Preliminaries — Definition of the Problem

The most general definition of an optimization problem (Note that $\max f(x) = \min -f(x)$)

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{(Objective Function)}$$

$$\text{s.t. } g(x) = 0 \quad \text{(Equality Constraints)}$$

$$h(x) \leq 0 \quad \text{(Inequality Constraints)}$$

where

- the Objective Function $f : \mathbb{R}^n \mapsto \mathbb{R}$
- the m Equality Constraints $g : \mathbb{R}^n \mapsto \mathbb{R}^m$
- the l Inequality Constraints $h : \mathbb{R}^n \mapsto \mathbb{R}^l$

Optimization Preliminaries — Definitions

Let $f : \mathcal{D} \subseteq \mathbb{R}^n \mapsto \mathbb{R}$.

Definition 1

A **critical point** $x^* \in \mathcal{D}$ of f satisfies $\nabla f(x^*) \equiv \left(\frac{\partial f}{\partial x_1}(x^*), \dots, \frac{\partial f}{\partial x_n}(x^*) \right) = 0$.

Definition 2

A point $x^* \in \mathcal{D}$ is a **min** of f on \mathcal{D} if $f(x^*) \leq f(x) \forall x \in \mathcal{D}$. It is a **strict min** if $f(x^*) < f(x) \forall x \neq x^* \in \mathcal{D}$.

Definition 3

A point $x^* \in \mathcal{D}$ is a **local (or relative or weak) min** of f on \mathcal{D} if there is a ball $B_r(x^*)$ such that $f(x^*) \leq f(x) \forall x \in B_r(x^*) \cap \mathcal{D}$. It is a **strict local (or relative or weak) min** if $f(x^*) < f(x) \forall x \neq x^* \in B_r(x^*) \cap \mathcal{D}$.

Optimization Preliminaries — Definitions

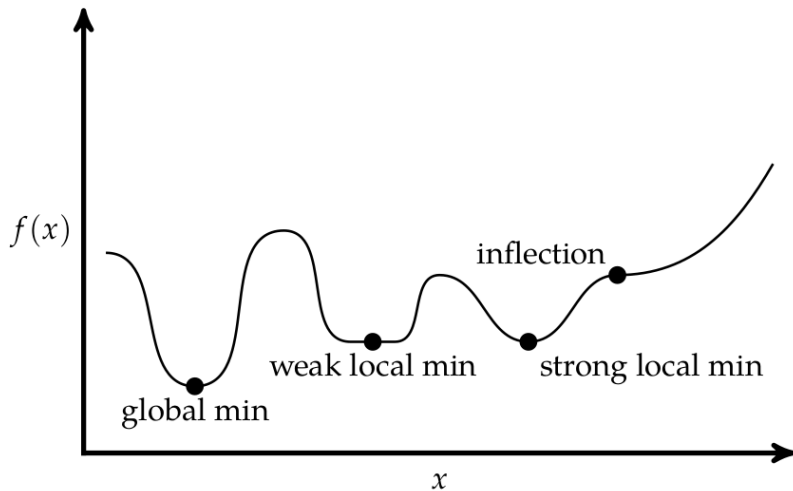


Figure 1: Classification of Critical Points. See [Kochenderfer and Wheeler 2019](#)

Optimization Preliminaries — First Order Conditions

- All points **need** that $f'(x^*) = 0$. Notice this is a **necessary condition** but not sufficient.
- The inflection point also has $f'(x^*) = 0$ (An inflection point is where the sign of the second derivative changes).

Theorem 1

Let $f : \mathcal{D} \subseteq \mathbb{R}^n \mapsto \mathbb{R}$ be a C^1 function. If x^ is a local min or max of f in \mathcal{D} and x^* is an interior point of \mathcal{D} , then*

$$\nabla f(x^*) = 0$$

Optimization Preliminaries — Second Order Conditions

Theorem 2

Let $f : \mathcal{D} \subseteq \mathbb{R}^n \mapsto \mathbb{R}$ be a \mathcal{C}^2 function. Suppose x^* is a critical point of f .

1. If $H_f(x)$ is positive (negative) definite, then x^* is a strict local min (max) of f .
2. If $H_f(x)$ is indefinite, then x^* is neither a local min or max of f .

Definition 4

Let $f : \mathcal{D} \subseteq \mathbb{R}^n \mapsto \mathbb{R}$ be a \mathcal{C}^2 function. The Hessian matrix H_f is a square $n \times n$ matrix whose (i, j) —th entry is defined by $(H_f)_{i,j} = \frac{\partial^2 f}{\partial x_i \partial x_j}$.

Optimization Preliminaries — Recap

- This has been a very brief recap on basic optimization.
- For a refresher, you can take a look at [Simon and Blume 1994](#), Chapters 17-19.
- We will cover the very basics of optimization and implementation in Matlab.
- All numerical optimization methods:
 - Search for **feasible choices**
 - Generate a sequence of **guesses**
 - Try to make the sequence **converge** to the true solution.

Pretty similar to root finding algorithms...right?

Optimization in One Dimension

The Simplest Optimization Problem

The simplest optimization problem is **unconstrained optimization** in one dimension

$$\min_{x \in \mathbb{R}} f(x)$$

where $f : \mathbb{R} \mapsto \mathbb{R}$. Why focusing on one dimension?

1. Illustrate techniques in a clear way.
2. Many multivariate methods boil down to solving a sequence of one-dimensional problems.

Optimization Methods — Categories

Four general categories for optimization methods:

1. Use derivatives.
2. Do not use derivatives.
3. Mixed methods.
4. Simulation-based methods.

We will look at basic methods in the first two categories.

Bracketing Method

Bracketing Method — The Intuition

- Suppose $f : \mathbb{R} \mapsto \mathbb{R}$ is **unimodal**.
- Let $a < b < c \in \mathbb{R}$ be three points such that

$$f(a), f(c) > f(b) \quad (1)$$

- Then, we know a minimum exists in $[a, c]$.

- How to find the optimum?

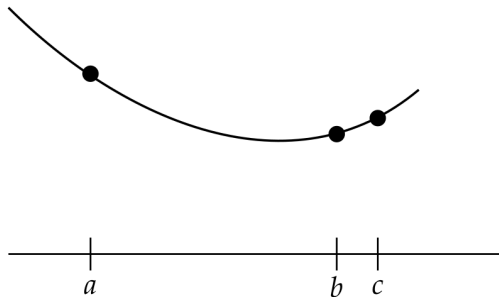


Figure 2: See [Kochenderfer and Wheeler 2019](#)

Bracketing Method — Remarks

- Slow. Similar to bisection.
- The stopping criterion is clear. If the length of the interval $[a, c]$ is sufficiently small, for practical terms, we have found the optimum.
- The method finds a **local minimum**, depending on the starting triplet (a, b, c) the method will converge to one minimum or another. This is a fairly common problem in many methods.
- If we know there is only one solution, the method always converges.
- Note we need three points in each iteration, depending on how costly it is to compute f this might be a problem.

Newton-Raphson Method

Newton-Raphson Method

- Familiar? Yes! It is very closely related to the root finding algorithm!
- Given an initial x_0 , compute a second order Taylor expansion around x_0 :

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2$$

- Minimizing this approximation, the FOC we get is

$$f'(x_0) + f''(x_0)(x^* - x_0) = 0$$

solving for x^*

$$x^* = x_0 - \frac{f'(x_0)}{f''(x_0)}$$

- Which is the **same iteration scheme** that we saw previously!

Newton-Raphson Method — Remarks

- Newton-Raphson's method tries to find **critical points**.
- We **must check** $f''(x^*)$ to check what we have found.
- Problems:
 - Convergence is not ensured.
 - $f''(x)$ might be difficult to compute. If we rely on finite difference methods, we will be adding errors.
 - Very sensitive to initial conditions.
- **From Fernández-Villaverde's slides:** *If you do not know where you're going, at least go slowly.*

Newton-Raphson Method — An Example

Example 1

Apply the Newton-Raphson method to solve

$$\min_{x \in \mathbb{R}^n} f(x) = \frac{x^2}{2} - \log(x^2)$$

- The true solution is $x^* = \pm\sqrt{2} \approx \pm 1.4142 \dots$
- Your choice of starting point x_0 will determine to which minimum you converge.

Newton-Raphson Method — An Example

Solution 1

First, we need the first and second derivatives of $f(x)$ which are

$$f'(x) = x - \frac{2}{x} \text{ and } f''(x) = 1 + \frac{2}{x^2}.$$

We define the functions as anonymous functions as

```
1 fun = @(x) (x.^2) ./ 2 - log(x.^2);  
2 fpr = @(x) x - (2./x);  
3 fppr = @(x) 1 + 2./(x.^2);
```

Finally, we need to set up a standard loop checking for terminal conditions and use the iteration scheme

```
1 x1 = x0 - fpr(x0) ./ fppr(x0);
```

Optimization in Matlab

Unconstrained Optimization

Unconstrained Optimization in Matlab

- We have covered two basic methods of unconstrained optimization.
- We are going to see now how to solve **(un)constrained optimization** problems using Matlab's routines.
- We start with the unconstrained optimization routine **fminunc**.
- This solves unconstrained multivariate optimization problems.
- It can use two types of algorithms. Both based on Newton-Raphson methods.
 - **BFGS** which is a quasi-Newton method.
 - **Trust-region methods**. Approximates the objective function in a subset, if this approximates well the function, it extends the region, otherwise, it contracts the region.

Unconstrained Optimization — `fminunc`

The basic syntax of `fminunc` takes as inputs a function `fun` and an initial point `x0`

```
1 [x, fval] = fminunc(fun, x0)
```

The basic output is x^* and $f(x^*)$

- This computes the derivatives numerically
- You can also supply the gradient and the Hessian.
- To supply gradient and Hessian, you need to write it in the function script.

Unconstrained Optimization — fminunc Example

Example 2

Let's solve previous example

$$\min_{x \in \mathbb{R}^n} f(x) = \frac{x^2}{2} - \log(x^2)$$

- The true solution is $x^* = \pm\sqrt{2} \approx \pm 1.4142 \dots$
- Using `fminunc` the initial choice will crucially determine which optimum we achieve.

Unconstrained Optimization — fminunc Example

Solution 2

Since it is a simple function, we will use an anonymous function to optimize it.

```
1 fmin = @(x) (x.^2 ./ 2) - log(x.^2)
2
3 % Find the minimum
4 x0 = 1;
5 [xmin1, fval1] = fminunc(@(x) fmin(x), x0);
6
7 % Changing the initial guess, will change the minimum
8 x0 = -1
9 [xmin2, fval2] = fminunc(@(x) fmin(x), x0);
```

Unconstrained Optimization — fminunc Multivariate

Example 3

Let's move to multivariate optimization and minimize the **Rosenbrock function**.

$$f(x_1, x_2) = 100 (x_2 - x_1^2)^2 + (1 - x_1)^2$$

The gradient is

$$\nabla f(x) = \begin{pmatrix} -400(x_2 - x_1^2)x_1 - 2(1 - x_1) \\ 200(x_2 - x_1^2) \end{pmatrix}$$

We will write $f(x_1, x_2)$ as a function script that will give as outputs the value of f and the value of the gradient. Check out **nargout** and **nargin**

Unconstrained Optimization — fminunc Multivariate

- The function script

```
1 function [f, fgrad] = rosenbrock(x)
2   % Not necessary, but for clarity we unpack the two inputs
3   x1 = x(1);
4   x2 = x(2);
5
6   % Compute f
7   f = 100.*(x2 - x1.^2).^2 + (1 - x1).^2;
8
9   % Compute gradient (if necessary)
10  if nargin > 1
11      % Notice this is a vector, and the order MATTERS!
12      fgrad = [-400*(x2 - x1.^2).*x1 - 2*(1 - x1);
13              200*(x2 - x1.^2)];
14  end
15 end
```

Unconstrained Optimization — fminunc Multivariate

- The optimization call

```
1 % Initial point
2 x0 = [14, 4];
3 % Optimization call
4 [x, fval] = fminunc(@(x)rosenbrock(x),x0);
```

- This **does not** tell fminunc that we must use the gradient.

- We need to use an options parser. Check out **optimoptions**.

```
1 % Add options
2 opts = optimoptions('fminunc','Algorithm','trust-region'
    ,...
    'SpecifyObjectiveGradient', true);
3
4 [xg, fvalg] = fminunc(@(x)rosenbrock(x),x0,opts);
```

Unconstrained Optimization — fminunc Multivariate

- The optimization without gradient yields $f(x_1, x_2) = 1.4045$
- The optimization **with** gradient yields $f(x_1, x_2) = 1.459 \times 10^{-11}$. Quite a change!
- Note however the initial guess is pretty bad $x = (14, 4)$ when it should be close to $(1, 1)$.
- Improving the guess reduces the differences. Numerical derivatives work well in this case.
- A derivative free solver for unconstrained optimization is **fminsearch**.

Constrained Optimization

Constrained Optimization

- Matlab offers several options.
- **fminbnd** — Finds a minimum of a **single-variable** function $f(x)$ in a given interval. The constraints are of the type $a \leq x \leq b$.
- **fmincon** — It is a multivariate constrained optimization command. It accepts constraints of the type $g(x) \leq 0$ and $h(x) = 0$.
- There are others that you can **check out here**.

Constrained Optimization — `fmincon`

- Let's focus on `fmincon` which is quite general for the type of problems you will most likely encounter.

- The general declaration of the function is

```
x = fmincon(fun,x0,A,b,Aeq,beq,lb,ub,nonlcon,options)
```

- We know `options`, `fun`, and `x0` from before.
- `A` and `b` are a matrix and a vector respectively denoting linear constraints such as $Ax \leq b$. `Aeq` and `beq` denote $Ax = b$.
- `lb` and `ub` are lower and upper bounds respectively for each variable (i.e. $a \leq x \leq b$).

Constrained Optimization — `fmincon`

- `nonlcon` are the nonlinear constraints that are supplied in function scripts. These should take the form $h(x) = 0$.
- To optimize the Rosenbrock function in a unit circle, we add the constraint $(x_1 - 1)^2 + (x_2 - 1)^2 - 1 = 0$.
- The argument `nonlcon` must be

```
1 ucircle = @(x) c = (x(1)-1)^2 + (x(2)-1)^2 - 1;  
2 [x, fval] = fmincon(@(x)rosenbrock(x),x0  
    ,[],[],[],[],[],[],@ucircle);
```

- The empty brackets `[]` denote empty arguments.

Optimization — General Tips

Optimization — General Tips I

- $g(x) \leq 0$ is easier to solve than $g(x) = 0$. Recall tolerances.
- A good initial guess is extremely important when optimizing nonlinear functions.
- Normalize variables, unit free problems are typically easier.
- Simplify the computations as much as you can.

Optimization — General Tips II

- Good approaches to solving complex problems:
 - Solve an easier version of the problem to get a good guess.
 - Change of variables.
 - Combine local and global solution methods.

Details on Optimization Methods

Details on Bracketing

Bracketing Method — The Algorithm

1. Define h as the step size, a constant $\alpha > 1$, and a given initial x_0 and compute

$$f(x_0), f(x_0 \pm \alpha h), f(x_0 \pm \alpha^2 h), \dots$$

until we find a triplet satisfying (1). Choose a stopping criterion ε .

2. If $b - a < c - b$, set $d = (b + c)/2$, otherwise, $d = (a + b)/2$. Compute $f(d)$.
3. If $d < b$ and $f(d) > f(b)$, replace (a, b, c) with (d, b, c) . If $d < b$ and $f(d) < f(b)$, replace (a, b, c) with (a, d, b) . If $d > b$ and $f(d) < f(b)$, replace (a, b, c) with (b, d, c) . Otherwise, replace (a, b, c) with (a, b, d) .
4. If $c - a < \varepsilon$, stop. Otherwise, go to step 2.

Bracketing Method — An Example

Let's minimize

$$f(x) = \frac{x^2}{2} - x$$

- The function has a global minimum in $x = 1$.
- Let us divide the code into two blocks:
 1. Initial bracketing.
 2. Refining given the bracketing.

Bracketing Method — Initial Bracketing

1. Start from guess x_0 and compute $x_1 = x_0 + \alpha h$.
2. Evaluate $f(x_0)$ and $f(x_1)$. If $f(x_1) < f(x_0)$ keep increasing until $f(x_2) > f(x_1)$
3. Otherwise, change direction and increase interval until $f(x_2) > f(x_0)$.
4. Increase α in each step to make the interval larger.

Bracketing Method — Initial Bracketing

- Start from guess $x_0 = -5$ (why not?) and compute increment.

```
1  % Parameters of bracketing method
2  h = 1e-2;
3  alp = 1.1;
4
5  % Step 1 - Initial bracketing
6  x0 = -5;
7  fx0 = fun(x0);
8  x1 = x0 + alp*h;
9  fx1 = fun(x1);
10 % If function is increasing in this direction, change
    direction
11 if fx1 > fx0
12     h = -h;
13 end
```

Bracketing Method — Initial Bracketing

- Establish the outer loop.

```
1  condition = true;  
2  it = 1;  
3  while condition  
4  % Outer loop  
5  end
```

- Suppose $h < 0$

```
1  x2 = x0 + alp*h;  
2  fx2 = fun(x2);  
3  % We found the function increases!  
4  if fx2 > fx0  
5      a = x2;  
6      b = x0;  
7      c = x1;  
8      condition = false;  
9  end
```

Bracketing Method — Initial Bracketing

- Suppose $h > 0$

```
1  x2 = x1 + alp*h;  
2  fx2 = fun(x2);  
3  % We found the function increases!  
4  if fx2 > fx1  
5      a = x0;  
6      b = x1;  
7      c = x2;  
8      condition = false;  
9  end
```

- If we have not found the function increases, update α

```
1  alp = alp*2;
```

- Then, simply put all together in nested if-else statements.

Bracketing Method — Refining Bracketing

- Initialize loop

```
1 while (difference > tol) && (it < maxit)
2     % Stuff goes here
3 end
```

- Define d and compute $f(d)$

```
1 % Step 2: define d and compute f(d)
2 if (b - a) > (c - b)
3     d = (a + b)/2;
4 else
5     d = (b + c)/2;
6 end
7 fd = fun(d);
```

Bracketing Method — Refining Bracketing

- Refine the interval if $d < b$

```
1 % Step 3: refine the
  interval
2 if d < b
3     if fd > fb
4         a1 = d;
5         b1 = b;
6         c1 = c;
7     else
8         a1 = a;
9         b1 = d;
10        c1 = b;
11    end
12 else
```

- Refine the interval if $d > b$

```
1 % Step 3: refine the
  interval
2 else
3     if fb < fd
4         a1 = a;
5         b1 = b;
6         c1 = d;
7     else
8         a1 = b;
9         b1 = d;
10        c1 = c;
11    end
12 end
```

Rename a_1 by a , compute $(c - a)$, and update iteration counter.

Details on Newton-Raphson

Newton-Raphson Method — The Algorithm

1. Choose initial guess x_0 and stopping parameters $\delta, \varepsilon > 0$.

2. Use the iteration scheme

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}$$

3. If

$$\frac{|x_k - x_{k+1}|}{1 + |x_k|} < \varepsilon \text{ and } |f'(x_k)| < \delta$$

stop. Otherwise, go to step 1.