COMP1805A - Test 3 (Section B2 - 10:35am)

March 14, 2014

Student Name: **20 points total**

Student Number:

Instructions: This is a closed book exam. No calculators are allowed. All questions should be answered on this sheet in the space provided. Print the single initial of your last name in box on the top-left corner of the test. You have 40 minutes.

Sums: $\sum_{i=0}^{n} i^2 = n^3/3 + n^2/2 + n/6$, $\sum_{i=0}^{n} i^3 = n^4/4 + n^3/2 + n^2/4$, $\sum_{i=0}^{n} i^4 = n^5/5 + n^4/2 + n^3/3 - n/30$

1. (5pts) Prove or disprove that $f(n) = 2n^3 - 3n^2 + 5 \in \Omega(n^3)$

   Let $c = 1$ and $n_0 = 3$. Then, for all $n \geq n_0 = 3$, notice that

$$\begin{aligned}
& n \geq 3 \\
\Rightarrow\ & n^3 \geq 3n^2 \quad \text{(since } n \geq 1\text{)} \\
\Rightarrow\ & n^3 - 3n^2 \geq 0 \\
\Rightarrow\ & 2n^3 - 3n^2 \geq n^3 \quad \text{(add } n^3 \text{ to both sides)} \\
\Rightarrow\ & 2n^3 - 3n^2 + 5 \geq n^3 \quad \text{(since } 5 > 0\text{)} \\
\Rightarrow\ & 2n^3 - 3n^2 + 5 \geq c\,n^3 \quad \text{(since } c = 1\text{)} \\
\Rightarrow\ & f(n) \geq cn^5
\end{aligned}$$

   Therefore, by the definition of big-Omega, using the constants $c = 1$ and $n_0 = 3$, we have proven that $f(n) \in \Omega(n^3)$. ❑

2. (5pts) Compute $S(n) = \sum_{i=2}^{n}(3 - 2i)$.

$$\begin{aligned}
\sum_{i=2}^{n}(3 - 2i) &= \sum_{i=2}^{n}(3) + \sum_{i=2}^{n}(-2i) \quad \text{(split the sum)} \\
&= 3\sum_{i=2}^{n} 1 - 2\sum_{i=2}^{n} i \quad \text{(pull out constants)} \\
&= 3(n - 2 + 1) - 2\sum_{i=2}^{n} i \quad \text{(evaluate first sum)} \\
&= 3(n - 1) - 2\left(\sum_{i=1}^{n} i - \sum_{i=1}^{1} i\right) \quad \text{(adjust for sum limits)} \\
&= 3n - 3 - 2\frac{n(n+1)}{2} + 2(1) \quad \text{(evaluate sums)} \\
&= -n^2 + 2n - 1 \quad \text{(arithmetic)}
\end{aligned}$$

3. (5pts) Prove or disprove that if $f(n) \in \Theta(h(n))$ then $10000 \cdot f(n) \in O(h(n))$.

   Since $f(n) \in \Theta(h(n))$, we know that there exist constants $c_1 > 0$, $c_2 > 0$ and $n_0 \geq 0$, such that for all $n \geq n_0$

$$c_1\, h(n) \leq f(n) \leq c_2\, h(n).$$

   Now, let $c' = 10000c_2$ and $n_0' = n_0$. Then, for all $n \geq n_0' = n_0$, notice that

$$10000 \cdot f(n) \leq 10000 \cdot (c_2\, h(n)) = (10000c_2)\, h(n) = c' h(n).$$

   That is, for all $n \geq n_0'$ we have

$$10000 f(n) \leq c' h(n)$$

   Therefore, using the definition of big-O, with constants $c' = 10000c_2$ and $n_0' = n_0$, we have shown that $10000f(n) \in O(h(n))$. ❑

4. (5pts) Prove the following: For all integers, 5x - 4 is even if and only if 3x + 7 is odd. You may use the following results without proof:

   - n is an even integer iff $n = 2k$ for some integer k
   - n is an odd integer iff $n = 2k + 1$ for some integer k
   - n is a rational number if $n = a/b$ for some integers a, b, with $b \neq 0$
   - if a and b are integers, then $a \times b$, $a + b$ and $a - b$ are integers.

   Since we have a biconditional (if and only if) we'll prove each direction independently. First, notice that

$$\begin{aligned}
& 5x - 4 \text{ is even} \quad \text{(asumption)} \\
\Rightarrow\ & 5x - 4 = 2k \quad \text{(for some integer } k\text{)} \\
\Rightarrow\ & 5x - 4 + (-2x + 11) = 2k + (-2x + 11) \quad \text{(add } -2x + 11 \text{ to both sides)} \\
\Rightarrow\ & 3x + 7 = 2(k - x + 5) + 1 \quad \text{(where } k - x + 5 \text{ is an integer)} \\
\Rightarrow\ & 3x + 7 \text{ is an odd number}
\end{aligned}$$

   Next we prove the other direction:

$$\begin{aligned}
& 3x + 5 \text{ is odd} \quad \text{(assumption)} \\
\Rightarrow\ & 3x + 7 = 2k + 1 \quad \text{(for some integer } k\text{)} \\
\Rightarrow\ & 3x + 7 + (2x - 11) = 2k + 1 + (2x - 11) \quad \text{(add } 2x - 11 \text{ from both sides)} \\
\Rightarrow\ & 5x - 4 = 2k + 1 + 2x + 11 \\
\Rightarrow\ & 5x - 4 = 2(k + x + 6) \quad \text{(arithmetic)} \\
\Rightarrow\ & 5x - 4 \text{ is even}
\end{aligned}$$

   Since $A \Leftrightarrow B$ is equivalent to $A \Rightarrow B \wedge B \Rightarrow A$, we have thus prove that $5x - 4$ is even if and only if $3x + 7$ is odd. ❑