ENPM667

Control of Robotic Systems

Final Project

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The diagram of the cart given to us consists of two pendulums suspended from the cables attached to the crane. The crane moves along a one-dimensional track and it behaves as a frictionless cart with mass M actuated by an external force F that constitutes the input of the system. There are two loads suspended from cables attached to the crane. The loads have mass m_1 and m_2, and the lengths of the cables are l_1 and l_2 respectively. The following figure depicts the crane and associated variables used throughout this project.

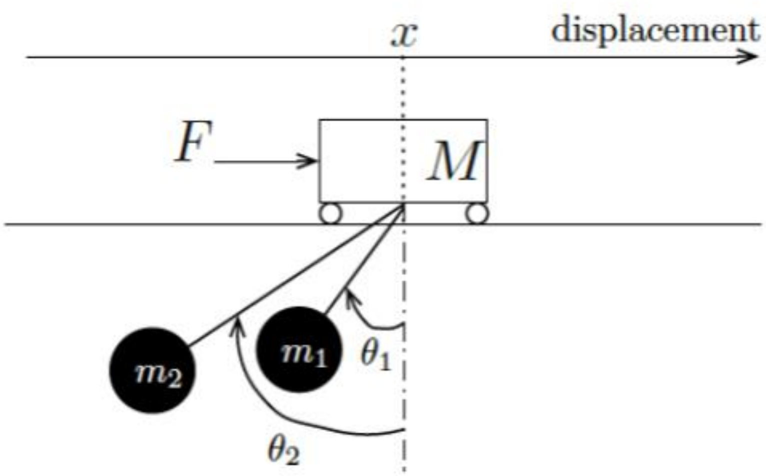


Figure 1 System diagram

A) Obtain the equations of motion for the system and the corresponding nonlinear state space representation.

<u>Equations of motion can be derived using Euler Lagrange's method:</u>

For the system provided we have three coordinates: x, θ_1, θ_2

This implies that the Euler Lagrange's methods should yield the following three equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = F$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0$$

The first step in computing the Euler Lagrange's equation of motion is to obtain the position vectors for the two pendulums having masses $m_1 and\ m_2$ suspended from the cart and then differentiating the position vectors with respect to time t to obtain the velocities of the moving body. For the given set up we assumed the reference to be a distance of l_2 below the cart in order to avoid any negative potential energy components.

Position vector for mass m_1 :

$$r_1 = (x - L_1 sin\theta_1)\ \hat{\imath} + (L_2 - L_1 cos\theta_1)\hat{\jmath}$$

Position vector for mass m_2:

$$r_2 = (x - L_2 sin\theta_2)\hat{\imath} + (L_2 - L_2 cos\theta_2)\hat{\jmath}$$

Differentiating the two position vectors obtained with respect to time t:

$$\dot{r_1} = \left(\dot{x} - L_1 cos\theta_1 \dot{\theta_1}\right)\hat{\imath} + \left(L_1 sin\theta_1 \dot{\theta_1}\right)\hat{\jmath}$$

$$\dot{r_2} = (\dot{x} - L_2 cos\theta_2 \dot{\theta_2})\hat{\imath} + \left(L_2 sin\theta_2 \dot{\theta_2}\right)\hat{\jmath}$$

Kinetic energy component due to cart = $\frac{1}{2}M\dot{x}^2$

Kinetic energy component due to the two suspended pendulums = $\frac{1}{2}m_1\left(\dot{x} - L_1 cos\theta_1 \dot{\theta_1}\right)^2 + \frac{1}{2}m_2\left(L_1 sin\theta_1 \dot{\theta_1}\right)^2 + \frac{1}{2}m_2\left(\dot{x} - L_2 cos\theta_2 \dot{\theta_2}\right)^2 + \frac{1}{2}m_2\left(L_2 sin\theta_2 \dot{\theta_2}\right)^2$

Total kinetic energy of the system:

$$KE = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m_1\left(\dot{x} - L_1 cos\theta_1\dot{\theta}_1\right)^2 + \frac{1}{2}m_2\left(L_1 sin\theta_1\dot{\theta}_1\right)^2 + \frac{1}{2}m_2\left(\dot{x} - L_2 cos\theta_2\dot{\theta}_2\right)^2 + \frac{1}{2}m_2\left(L_2 sin\theta_2\dot{\theta}_2\right)^2$$

Total potential energy of the system:

$$PE = MgL_2 + m_1 g(L_2 - L_1 cos\theta_1) + m_2 g(L_2 - L_2 cos\theta_2)$$

$$L = KE - PE$$

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m_1\left(\dot{x} - L_1 cos\theta_1\dot{\theta}_1\right)^2 + \frac{1}{2}m_1\left(L_1 sin\theta_1\dot{\theta}_1\right)^2 + \frac{1}{2}m_2\left(\dot{x} - L_2 cos\theta_2\dot{\theta}_2\right)^2 + \frac{1}{2}m_2\left(L_2 sin\theta_2\dot{\theta}_2\right)^2 - MgL_2 - m_1 g(L_2 - L_1 cos\theta_1) - m_2 g(L_2 - L_2 cos\theta_2)$$

Now using the Euler Lagrange's equations and simplifying we obtain:

$$\frac{\partial L}{\partial \dot{x}} = M\dot{x} + m_1\left(\dot{x} - L_1 cos\theta_1\dot{\theta}_1\right) + m_2(\dot{x} - L_2 cos\theta_2\dot{\theta}_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = M\ddot{x} + m_1(\ddot{x} + L_1 sin\theta_1\dot{\theta}_1\dot{\theta}_1 - L_1 cos\theta_1\ddot{\theta}_1) + m_2(\ddot{x} + L_2 sin\theta_2\dot{\theta}_2\dot{\theta}_2 - L_2 cos\theta_2\ddot{\theta}_2)$$

$$\frac{\partial L}{\partial x} = 0$$

$$L = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = F$$

$$\therefore F = (M + m_1 + m_2)\ddot{x} - m_1 L_1 cos\theta_1\ddot{\theta}_1 - m_2 L_2 cos\theta_2\ddot{\theta}_2 + m_1 L_1 \dot{\theta}_1{}^2 sin\theta_1 + m_2 L_2 sin\theta_2\dot{\theta}_2{}^2$$

-----[1]

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1\left(\dot{x} - L_1 cos\theta_1\dot{\theta}_1\right)(-L cos\theta_1) + m_1(L_1 sin\theta_1\dot{\theta}_1)(L_1 sin\theta_1)$$

$$= m_1 L_1(-\dot{x} cos\theta_1 + L_1\dot{\theta}_1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = m_1 L_1(-\ddot{x} cos\theta_1 + \dot{x} sin\theta_1\dot{\theta}_1 + L_1\ddot{\theta}_1)$$

$$\frac{\partial L}{\partial \theta_1} = m_1\left(\dot{x} - L_1 cos\theta_1\dot{\theta}_1\right)\left(l_1 sin\theta_1\dot{\theta}_1\right) + m_1\left(L_1 \sin\theta_1\dot{\theta}_1\right)\left(L_1 cos\theta_1\dot{\theta}_1\right) - m_1 g(L_1 sin\theta_1)$$

$$= \dot{\theta}_1 m_1 L_1[\dot{x} sin\theta_1] - m_1 g L_1 sin\theta_1$$

$$L = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0$$

$$\therefore m_1 L_1\left(-\ddot{x} cos\theta_1 + \dot{x} sin\theta_1\dot{\theta}_1 + L_1\ddot{\theta}_1\right) - \dot{\theta}_1 m_1 L_1[\dot{x} sin\theta_1] + m_1 g L_1 sin\theta_1 = 0$$

-----*[2]*

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2\left(\dot{x} - L_2 cos\theta_2\dot{\theta}_2\right)(-L cos\theta_2) + m_2(L_2 sin\theta_2\dot{\theta}_2)(L_2 sin\theta_2)$$

$$= m_2 L_2(-\dot{x} cos\theta_2 + L_2\dot{\theta}_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = m_2 L_2(-\ddot{x} cos\theta_2 + \dot{x} sin\theta_2\dot{\theta}_2 + L_2\ddot{\theta}_2)$$

$$\frac{\partial L}{\partial \theta_2} = m_2\left(\dot{x} - L_2 cos\theta_2\dot{\theta}_2\right)\left(l_2 sin\theta_2\dot{\theta}_2\right) + m_2\left(L_2 \sin\theta_2\dot{\theta}_2\right)\left(L_2 cos\theta_2\dot{\theta}_2\right) - m_2 g(L_2 sin\theta_2)$$

$$= \dot{\theta}_2 m_2 L_2[\dot{x} sin\theta_2] - m_2 g L_2 sin\theta_2$$

$$L = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0$$

$$\therefore m_2 L_2\left(-\ddot{x} cos\theta_2 + \dot{x} sin\theta_2\dot{\theta}_2 + L_2\ddot{\theta}_2\right) - \dot{\theta}_2 m_2 L_2[\dot{x} sin\theta_2] + m_2 g L_2 sin\theta_2 = 0$$

-----*[3]*

After obtaining the equations of motion for the system provided to us, we can obtain the corresponding non-linear state space representation.

Nonlinear state space representation is given by

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = h(x(t), u(t))$$

For the system provided we have six different states which are as follows:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \end{bmatrix} = \begin{bmatrix} X \\ \dot{X} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix}$$

$$[U] = [F]$$

$$\dot{X} = \begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \\ \dot{X}_4 \\ \dot{X}_5 \\ \dot{X}_6 \end{bmatrix} = \begin{bmatrix} \dot{X} \\ \dfrac{F + m_1 L_1 cos\theta_1 \ddot{\theta}_1 + m_2 L_2 cos\theta_2 \ddot{\theta}_2 - m_1 L_1 \dot{\theta}_1{}^2 sin\theta_1 - m_2 L_2 sin\theta_2 \dot{\theta}_2{}^2}{M + m_1 + m\_2} \\ \dot{\theta}_1 \\ \dfrac{1}{L_1}(-gsin\theta_1 + \ddot{x}cos\theta_1) \\ \dot{\theta}_2 \\ \dfrac{1}{L_2}(-gsin\theta_2 + \ddot{x}cos\theta_2) \end{bmatrix} * \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \end{bmatrix}$$

B) Obtain the linearized system around the given equilibrium points $\boldsymbol{x = 0}$, $\boldsymbol{\theta_1 = 0}$, $\boldsymbol{\theta_2 = 0}$. Write the state-space representation of the linearized system.

In order to linearize the nonlinear equations of motion we write equation [1],[2] and [3] in terms of $\ddot{x}$, $\ddot{\theta}_1$, $\ddot{\theta}_2$

$$\ddot{x} = \frac{F + m_1 L_1 cos\theta_1 \ddot{\theta}_1 + m_2 L_2 cos\theta_2 \ddot{\theta}_2 - m_1 L_1 \dot{\theta}_1{}^2 sin\theta_1 - m_2 L_2 sin\theta_2 \dot{\theta}_2{}^2}{M + m_1 + m\_2}$$

$$\ddot{\theta}_1 = \frac{1}{L_1}(-gsin\theta_1 + \ddot{x}cos\theta_1)$$

$$\ddot{\theta}_2 = \frac{1}{L_2}(-gsin\theta_2 + \ddot{x}cos\theta_2)$$

The state space representation for the linearized system with input $\vec{U}$ and output $\vec{Y}$ can be written as:

$$\dot{\vec{X}} = A\vec{X}(t) + B\vec{U}(t)$$
$$\vec{Y} = C\vec{X}(t) + D\vec{U}(t)$$

Here,

$$X = \begin{bmatrix} X \\ \dot{X} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \end{bmatrix}$$

$$\dot{X} = \begin{bmatrix} \dot{X} \\ \ddot{X} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \\ \dot{X}_4 \\ \dot{X}_5 \\ \dot{X}_6 \end{bmatrix}$$

Since we have six states, the A matrix will have dimension of 6x6 matrix. The A matrix for linearized system is obtained by computing the Jacobian with respect to all the states and then equating the Jacobian with the given equilibrium point $x = 0,\ \theta_1 = 0, \theta_2 = 0$ to obtain linear equations.

$$f_1 = \dot{X}$$

$$f_2 = \ddot{x} = \frac{F + m_1 L_1 cos\theta_1 \ddot{\theta}_1 + m_2 L_2 cos\theta_2 \ddot{\theta}_2 - m_1 L_1 \dot{\theta}_1{}^2 sin\theta_1 - m_2 L_2 sin\theta_2 \dot{\theta}_2{}^2}{M + m_1 + m\_2}$$

$$f_3 = \dot{\theta}_1$$

$$f_4 = \ddot{\theta}_1 = \frac{1}{L_1}(-g sin\theta_1 + \ddot{x} cos\theta_1)$$

$$f_5 = \dot{\theta}_2$$

$$f_6 = \ddot{\theta}_2 = \frac{1}{L_2}(-gsin\theta_2 + \ddot{x}cos\theta_2)$$

The A matrix for the linearized system obtained from non-linear system:

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial \dot{x}} & \frac{\partial f_1}{\partial \theta_1} & \frac{\partial f_1}{\partial \dot{\theta}_1} & \frac{\partial f_1}{\partial \theta_2} & \frac{\partial f_1}{\partial \dot{\theta}_1} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial \dot{x}} & \frac{\partial f_2}{\partial \theta_1} & \frac{\partial f_2}{\partial \dot{\theta}_1} & \frac{\partial f_2}{\partial \theta_2} & \frac{\partial f_2}{\partial \dot{\theta}_2} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial \dot{x}} & \frac{\partial f_3}{\partial \theta_1} & \frac{\partial f_3}{\partial \dot{\theta}_1} & \frac{\partial f_3}{\partial \theta_2} & \frac{\partial f_3}{\partial \dot{\theta}_2} \\ \frac{\partial f_4}{\partial x} & \frac{\partial f_4}{\partial \dot{x}} & \frac{\partial f_4}{\partial \theta_1} & \frac{\partial f_4}{\partial \dot{\theta}_1} & \frac{\partial f_4}{\partial \theta_2} & \frac{\partial f_4}{\partial \dot{\theta}_2} \\ \frac{\partial f_5}{\partial x} & \frac{\partial f_5}{\partial \dot{x}} & \frac{\partial f_5}{\partial \theta_1} & \frac{\partial f_5}{\partial \dot{\theta}_1} & \frac{\partial f_5}{\partial \theta_2} & \frac{\partial f_5}{\partial \dot{\theta}_2} \\ \frac{\partial f_6}{\partial x} & \frac{\partial f_6}{\partial \dot{x}} & \frac{\partial f_6}{\partial \theta_1} & \frac{\partial f_6}{\partial \dot{\theta}_1} & \frac{\partial f_6}{\partial \theta_2} & \frac{\partial f_6}{\partial \dot{\theta}_2} \end{bmatrix}$$

Computing all the required Jacobian and equating the Jacobian with the given equilibrium point $x = 0,\ \theta_1 = 0, \theta_2 = 0$ to obtain linear equations:

$$\frac{\partial f_1}{\partial x} = 0$$
$$\frac{\partial f_1}{\partial \dot{x}} = 1$$
$$\frac{\partial f_1}{\partial \theta_1} = 0$$
$$\frac{\partial f_1}{\partial \dot{\theta}_1} = 0$$
$$\frac{\partial f_1}{\partial \theta_2} = 0$$
$$\frac{\partial f_1}{\partial\ \dot{\theta}_2} = 0$$

$$\frac{\partial f_2}{\partial x} = 0$$
$$\frac{\partial f_2}{\partial \dot{x}} = 0$$
$$\frac{\partial f_2}{\partial \theta_1} = \frac{-m_1 g}{M}$$
$$\frac{\partial f_2}{\partial \dot{\theta}_1} = 0$$
$$\frac{\partial f_2}{\partial \theta_2} = \frac{-m_2 g}{M}$$
$$\frac{\partial f_2}{\partial \dot{\theta}_2} = 0$$

$$\frac{\partial f_3}{\partial x} = 0$$
$$\frac{\partial f_3}{\partial \dot{x}} = 0$$
$$\frac{\partial f_3}{\partial \theta_1} = 0$$
$$\frac{\partial f_3}{\partial \dot{\theta}_1} = 1$$
$$\frac{\partial f_3}{\partial \theta_2} = 0$$
$$\frac{\partial f_3}{\partial \dot{\theta}_2} = 0$$

$$\frac{\partial f_4}{\partial x} = 0$$
$$\frac{\partial f_4}{\partial \dot{x}} = 0$$
$$\frac{\partial f_4}{\partial \theta_1} = \frac{-(m_1 g + Mg)}{ML_1}$$
$$\frac{\partial f_4}{\partial \dot{\theta}_1} = 0$$
$$\frac{\partial f_4}{\partial \theta_2} = \frac{-m_2 g}{ML_1}$$

$$\frac{\partial f_5}{\partial x} = 0$$
$$\frac{\partial f_5}{\partial \dot{x}} = 0$$
$$\frac{\partial f_5}{\partial \theta_1} = 0$$
$$\frac{\partial f_5}{\partial \dot{\theta}_1} = 0$$
$$\frac{\partial f_5}{\partial \theta_2} = 0$$
$$\frac{\partial f_5}{\partial \dot{\theta}_2} = 1$$

$$\frac{\partial f_6}{\partial x} = 0$$
$$\frac{\partial f_6}{\partial \dot{x}} = 0$$
$$\frac{\partial f_6}{\partial \theta_1} = \frac{-m_1 g}{ML_2}$$
$$\frac{\partial f_6}{\partial \dot{\theta}_1} = 0$$
$$\frac{\partial f_6}{\partial \theta_2} = \frac{-(Mg + m_2 g)}{ML_2}$$
$$\frac{\partial f_6}{\partial \dot{\theta}_2} = 0$$

Hence, after computing the Jacobian and applying the equilibrium points, the linearized A matrix can be written as

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-m_1 g}{M} & 0 & \frac{-m_2 g}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-(Mg + m_1 g)}{ML_1} & 0 & \frac{-g m_2}{ML_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-m_1 g}{ML_2} & 0 & \frac{-(Mg + m_2 g)}{ML_2} & 0 \end{bmatrix}$$

Similarly, since we have six states, the B matrix will have dimension of 6x1 matrix. The B matrix for linearized system is obtained by computing the Jacobian with respect to all the inputs to the system and then equating the Jacobian with the given equilibrium point $x = 0,\ \theta_1 = 0, \theta_2 = 0$ to obtain linear equations.

The B matrix for the linearized system obtained from non-linear system:

$$B = \begin{bmatrix} \frac{\partial f_1}{\partial F} \\ \frac{\partial f_2}{\partial F} \\ \frac{\partial f_3}{\partial F} \\ \frac{\partial f_4}{\partial F} \\ \frac{\partial f_5}{\partial F} \\ \frac{\partial f_6}{\partial F} \end{bmatrix}$$

Computing all the required Jacobian and equating the Jacobian with the given equilibrium point $x = 0,\ \theta_1 = 0, \theta_2 = 0$ to obtain linear equations:

$$\frac{\partial f_1}{\partial F} = 0$$

$$\frac{\partial f_2}{\partial F} = \frac{1}{M}$$

$$\frac{\partial f_3}{\partial F} = 0$$

$$\frac{\partial f_4}{\partial F} = \frac{1}{ML_1}$$
$$\frac{\partial f_5}{\partial F} = 0$$
$$\frac{\partial f_6}{\partial F} = \frac{1}{ML_2}$$

Hence, after computing the Jacobian and applying the equilibrium points, the linearized A matrix can be written as

$$B = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{ML_1} \\ 0 \\ \frac{1}{ML_2} \end{bmatrix}$$

Since we have only one input which is the force acting on cart with mass M the U matrix will have dimension of 1x1 and it can be represented as:

$$[U] = [F]$$

Now C matrix will also have a dimension of 6x6, and it will simply be an identity matrix:

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

For the given system D matrix is assumed to be a zero matrix:

$$D = [0]$$

C) Obtain conditions for which the linearized system is controllable.

First, we need to construct the controllability matrix and calculate the rank of the controllability matrix. If the matrix is full ranked, then the system will be controllable.

$$Controllability = [\ B\ AB\ A^2B\ A^3B\ A^4B\ A^5B\ ]$$

$$Controllablity = \begin{bmatrix} 0 & \frac{1}{M} & 0 & i_{14} & 0 & i_{16} \\ \frac{1}{M} & 0 & i_{23} & 0 & i_{25} & 0 \\ 0 & \frac{1}{ML_1} & 0 & i_{34} & 0 & i_{36} \\ \frac{1}{ML_1} & 0 & i_{43} & 0 & i_{45} & 0 \\ 0 & \frac{1}{ML_2} & 0 & i_{54} & 0 & i_{56} \\ \frac{1}{ML_2} & 0 & i_{63} & 0 & i_{65} & 0 \end{bmatrix}$$

i_{14}	$\frac{-g(L_1m_2 + L_2m_1)}{M^2L_1L_2}$
i_{16}	$\frac{g^2(L_1^2m_2^2 + L_2^2m_1^2 + 2L_1L_2m_1m_2 + M(L_1^2m_2 + L_2^2m_1))}{M^3L_1^2L_2^2}$
i_{23}	$-\frac{g(L_1m_2 + L_2m_1)}{M^2L_1L_2}$
i_{25}	$\frac{g^2(L_1^2m_2^2 + L_2^2m_1^2 + 2L_1L_2m_1m_2 + M(L_1^2m_2 + L_2^2m_1))}{M^3L_1^2L_2^2}$
i_{34}	$-\frac{g(ML_2 + L_1m_2 + L_2m_1)}{M^2L_1^2L_2}$
i_{36}	$\frac{g^2(M^2L_2^2 + M(L_1^2m_2 + L_1L_2m_2 + 2L_2^2m_1) + L_1^2m_2^2 + L_2^2m_1^2 + 2L_1L_2m_1m_2)}{M^3L_1^3L_2^2}$
i_{43}	$-\frac{g(ML_2 + L_1m_2 + L_2m_1)}{M^2L_1^2L_2}$

i_{45}	$\dfrac{g^2(M^2L_2^2 + M(L_1^2m_2 + 2L_2^2m_1 + L_1L_2m_2) + L_1^2m_2^2 + L_2^2m_1^2 + 2L_1L_2m_1m_2)}{M^3L_1^3L_2^2}$
i_{54}	$-\dfrac{g(ML_1 + L_1m_2 + L_2m_1)}{M^2L_1L_2^2}$
i_{56}	$\dfrac{g^2(M^2L_1^2 + M(2L_1^2m_2 + L_1L_2m_1L_2^2m_1) + L_1^2m_2^2 + L_2^2m_1^2 + 2L_1L_2m_1m_2)}{M^3L_1^2L_2^3}$
i_{63}	$-\dfrac{g(ML_1 + L_1m_2 + L_2M_1)}{M^2L_1L_2^2}$
i_{65}	$\dfrac{g^2(M^2L_1^2 + M(2L_1^2m_2 + L_1L_2m_1 + L_2^2m_1) + L_1^2m_2^2 + L_2^2m_1^2 + 2L_1L_2m_1m_2)}{M^3L_1^2L_2^3}$

$$Rank\ (Controllablity) = 6$$

Thus, the controllability matrix obtained is full ranked which implies that the system is controllable.

The system becomes uncontrollable if the system is not full ranked. The controllability matrix will lose its rank if its determinant is equal to zero.

Det (Controllability matrix) = 0 = $\frac{-(g^6*(L_1-L_2)^2)}{L_1^6*L_2^6*M^6}$

$$0 = (L_1 - L_2)$$

This implies that the controllability matrix loses its rank when $L_1 = L_2$.

D) Check that the system is controllable and obtain an LQR controller. Simulate the resulting response to initial conditions when the controller is applied to the linearized system and also to the original nonlinear system. Adjust the parameters of the LQR cost until you obtain a suitable response. Use Lyapunov's indirect method to certify stability (locally or globally) of the closed-loop system.1

Eigen values of A are-

$$
\begin{gathered}
0 + 0i \\
0 + 0i \\
0 + 0.7285i \\
0 - 0.7285i \\
0 + 1.043i \\
0 - 1.043i
\end{gathered}
$$

As real part of all eigen values are zero the Lyapunov's indirect method is inconclusive and cannot certify system's global or local stability.

Assuming some random Q and R values-

R = 0.0010

$$
Q = \begin{bmatrix} 800 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 50000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 50000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 50000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 50000 \end{bmatrix}
$$

Eigen values of (A-BK) after applying the control law are-

$$
\begin{gathered}
-2.4413 + 0.0000i \\
-0.4102 + 0.0000i \\
-0.1421 + 0.9534i \\
-0.1421 - 0.9534i \\
-0.1475 + 0.6639i \\
-0.1475 - 0.6639i
\end{gathered}
$$

As real part of all the Eigen values are negative, the system is stable.

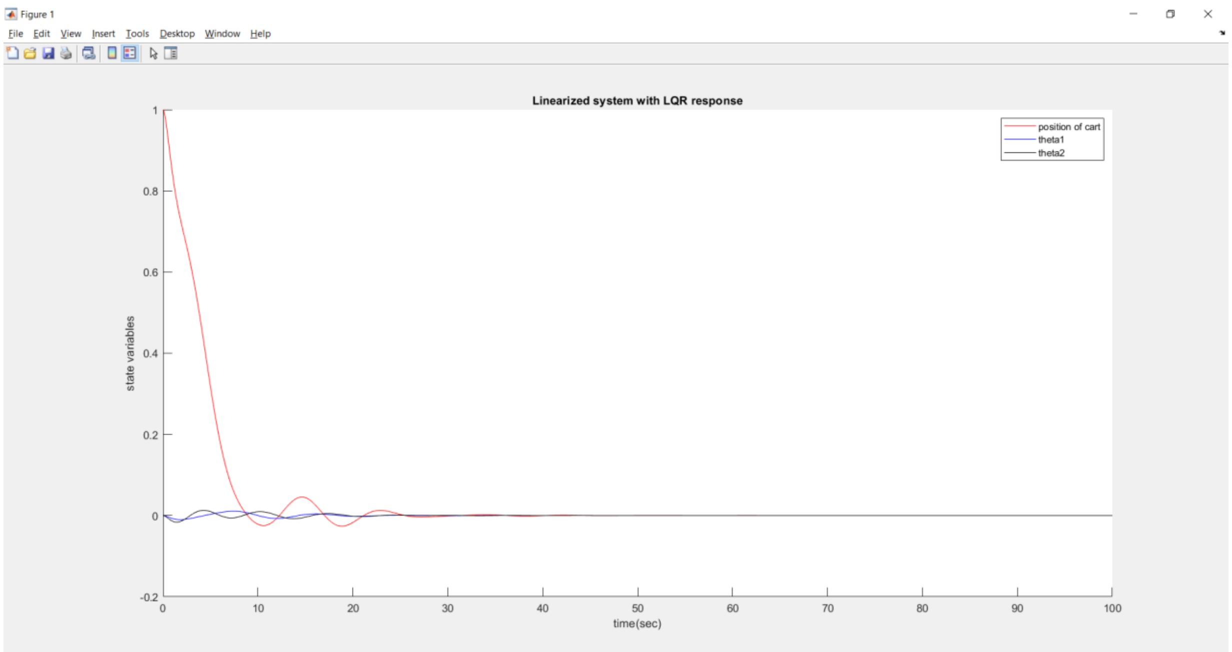


Figure 2 Response of Linearized system with LQR

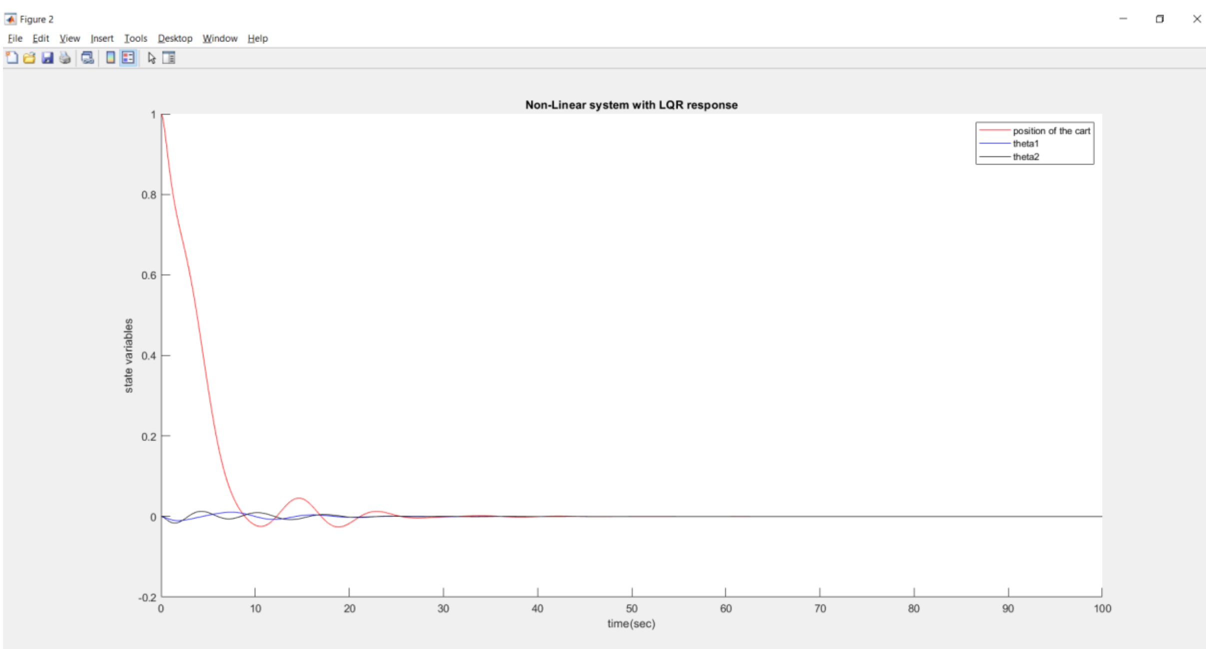


Figure 3 Response of non-Linearized system with LQR

E) Suppose that you can select the following output vectors: x(t), (θ_1(t), θ_2(t)), (x(t), θ_2(t)) or (x(t), θ_1(t), θ_2(t)). Determine for which output vectors the linearized system is observable.

- The observability matrix for vector x(t) has rank 6. Hence, the system is observable.
- The observability matrix for vector (θ_1(t), θ_2(t)) has rank 4. Hence, the system is not observable.
- The observability matrix for vector (x(t), θ_2(t)) has rank 6. Hence, the system is observable.
- The observability matrix for vector (x(t), θ_1(t), θ_2(t)) has rank 6. Hence, the system is observable.

F) Obtain your "best" Luenberger observer for each one of the output vectors for which the system is observable and simulate its response to initial conditions and unit step input. The simulation should be done for the observer applied to both the linearized system and the original nonlinear system.

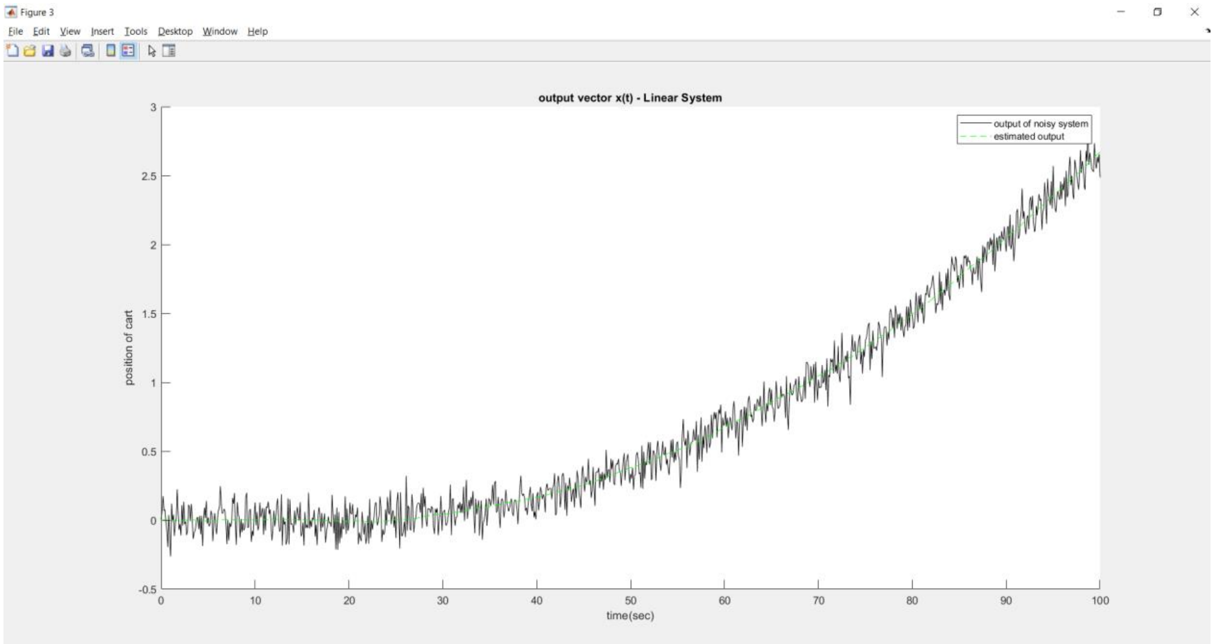


Figure 4 Output vector x(t) - Linear System

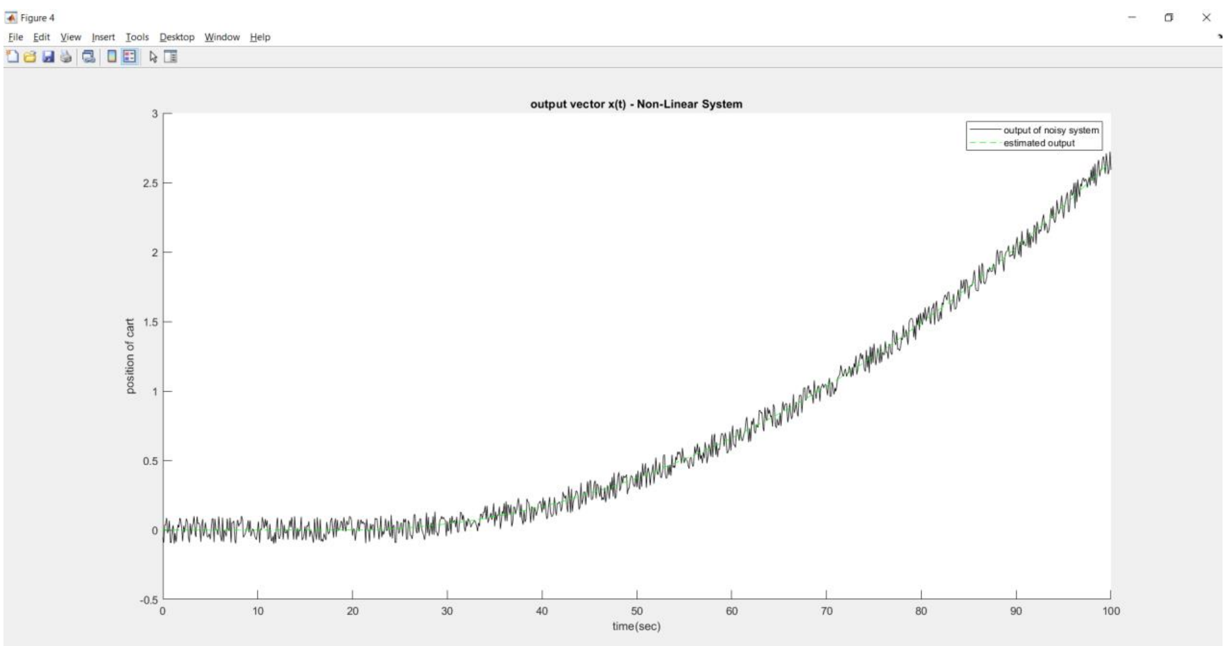


Figure 5 Output vector x(t) - Non-Linear System

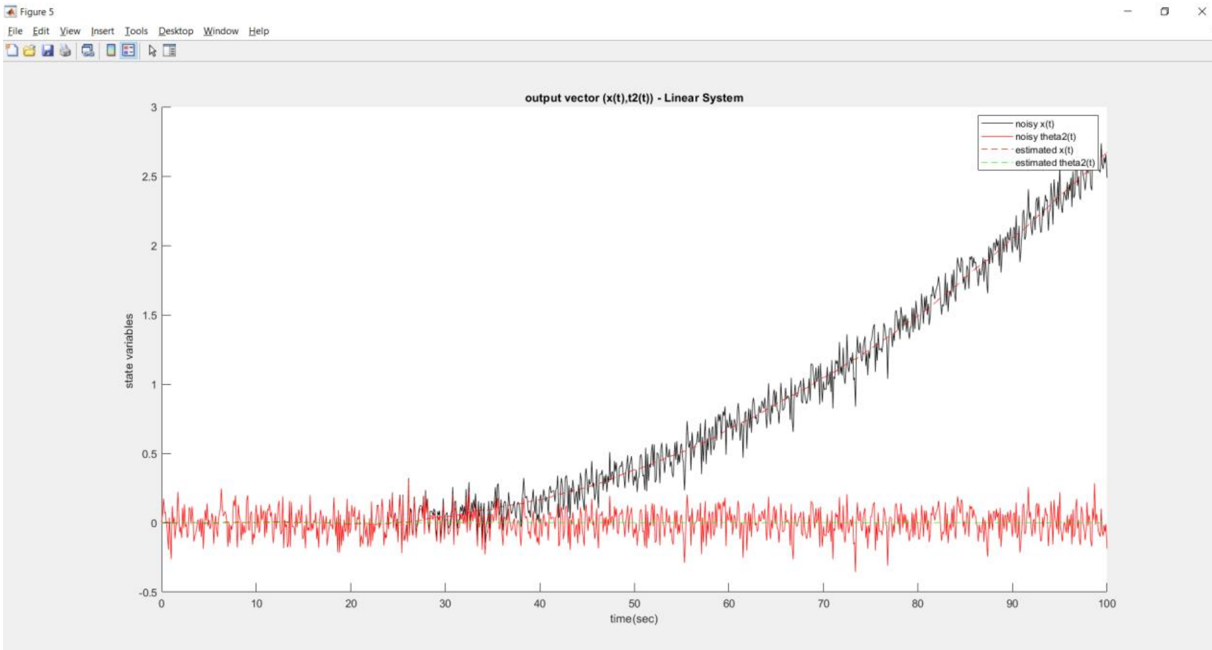


Figure 6 Output vector x(t), t2(t) - Linear System

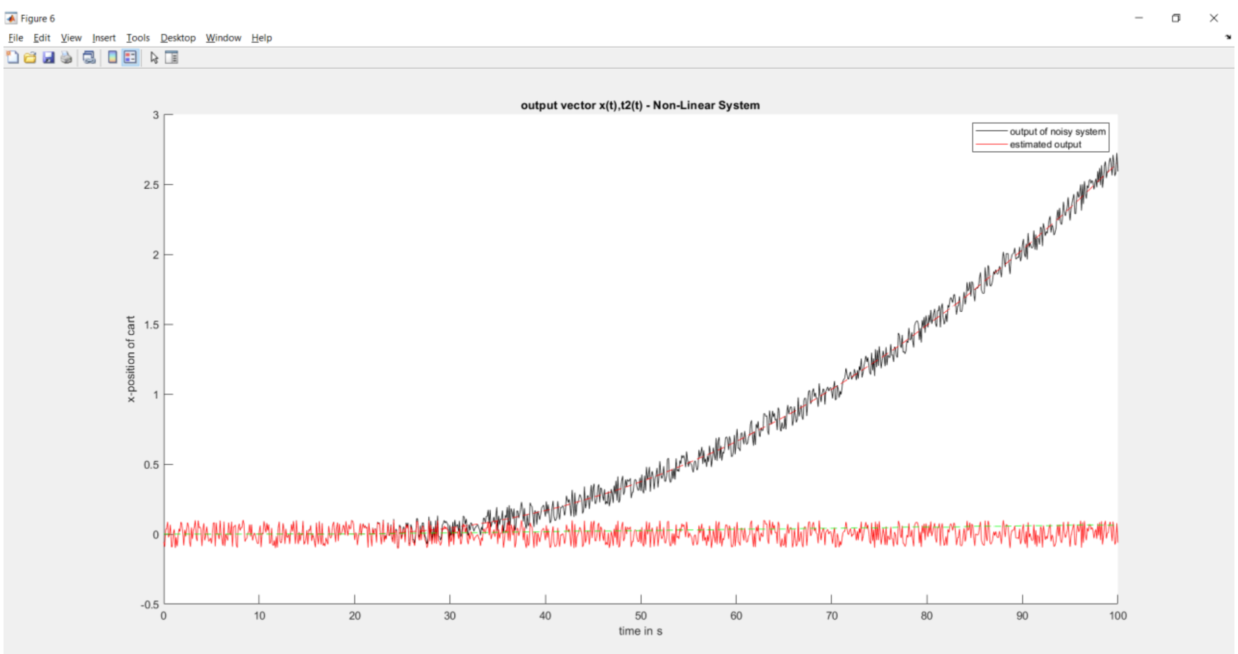


Figure 7 Output vector x(t), t2(t) - Non-Linear System

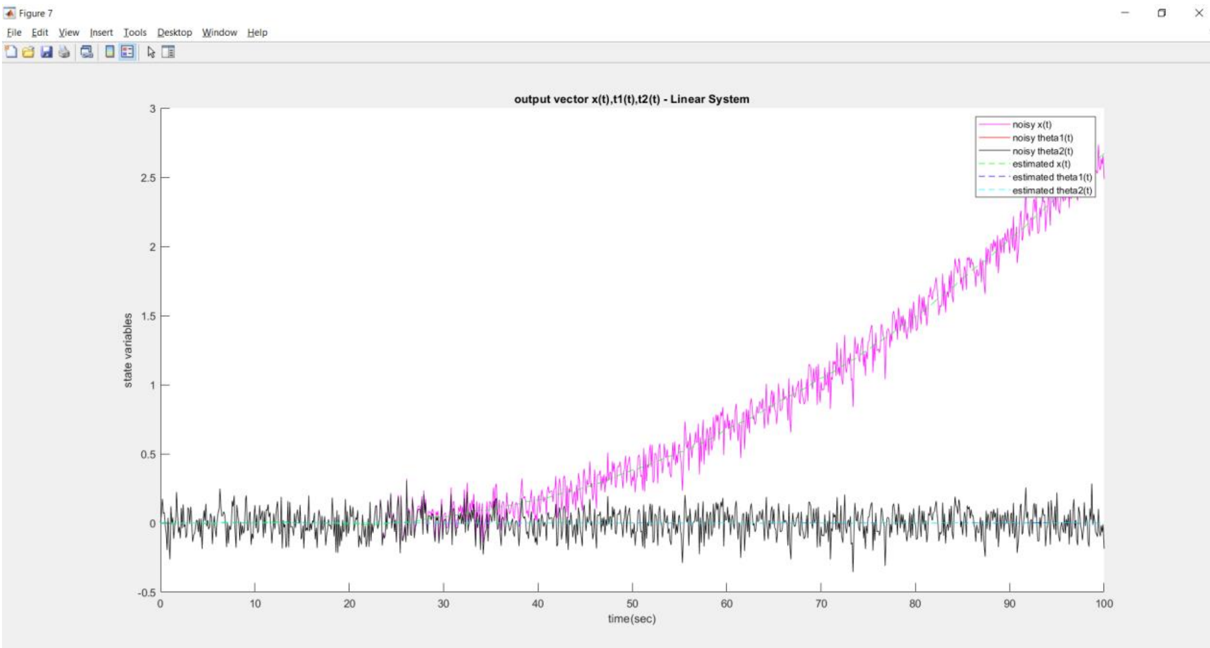


Figure 8 Output vector x(t), t1(t), t2(t) - Linear System

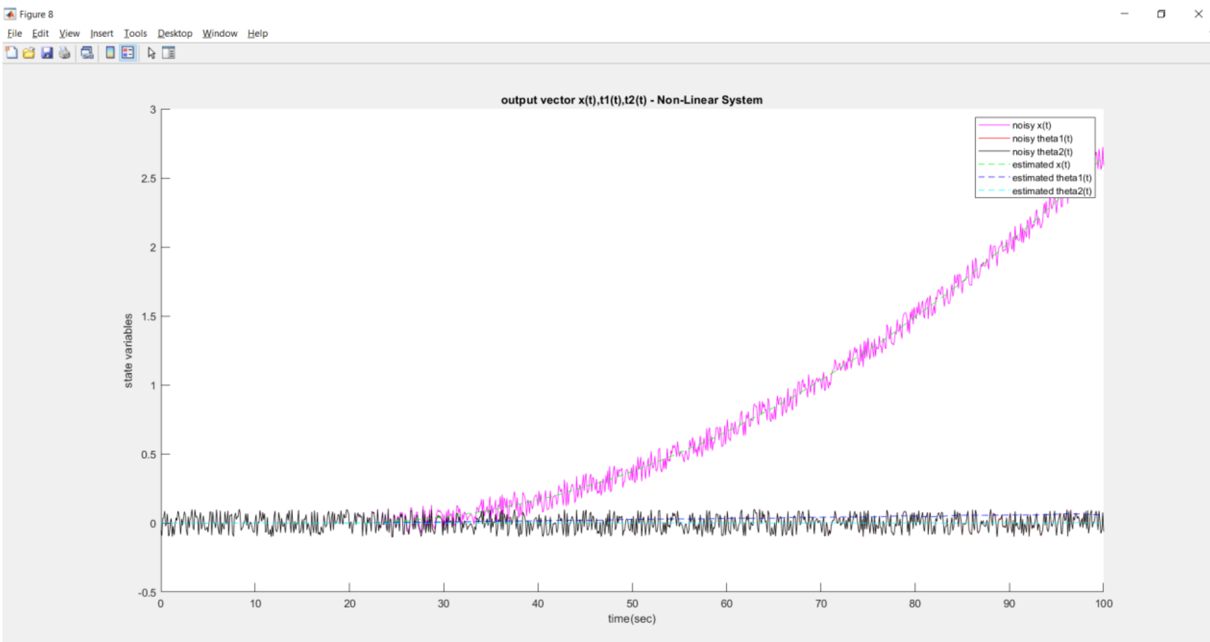


Figure 9 Output vector x(t), t1(t), t2(t) - Non-Linear System

G) Design an output feedback controller for your choice of the "smallest" output vector. Use the LQG method and apply the resulting output feedback controller to the original nonlinear system. Obtain your best design and illustrate its performance in simulation. How would you reconfigure your controller to asymptotically track a constant reference on x? Will your design reject constant force disturbances applied on the cart?

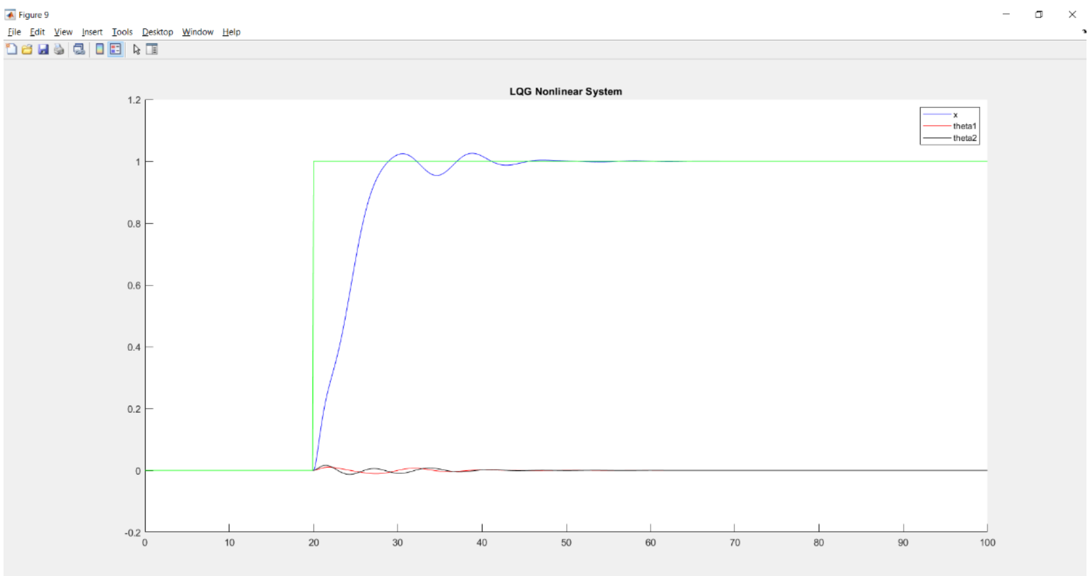


Figure 10 Response of LQG system for unit step function

The asymptotically tracking of a reference x is a trajectory tracking problem. PD and PID can be incorporated in the controller to track the constant reference on x. If we incorporate a PID controller, this will lead to the change in input to the system. Input(to the system) = [proportional gain X error] + [Derivate gain X derivative of error] + [Integral Gain X integral of error] .We can now design the controller to reduce the tracking error $\theta_d(t) - \theta_m(t)$.

No, our design won't reject constant force disturbances. The disturbances will be treated as additional input to the system and the constant force disturbance will result in increase of the number of input states. Now if the closed loop is stable and A is invertible, the static state can be computed as follows:

$$\lim_{t\to\infty}\left(\overrightarrow{X_c}\ (t)\right)^1 = -A_c^{-1}B_c\overrightarrow{u_s}$$

Appendix

Code and Simulink screenshots: -
main.m

```
clear
clc
parameters.M  = 1000;        % M      mass of cart
parameters.m1 = 100;         % m1     mass of ball 1
parameters.m2 = 100;         % m2     mass of ball 2
parameters.l1 = 20;          % l1     length of first pendulum
parameters.l2 = 10;          % l2     length of second pendulum
parameters.g  = 9.81;        % g      acceleration due to gravity

[sDot,A,B] = linearModel(parameters);
A = double(A);
B = double(B);
C = [B A*B (A^2)*B (A^3)*B (A^4)*B (A^5)*B];
r = rank(C);

if r == 6
   disp('The System is controllable as th rank of controlability matrix C ==
6')
end

lambda = eig(A)
disp("Since, poles i.e. Eigen values of A matrix lie on the imaginary axis so
Lyapunov's indirect method is inconclusive and cannot certify if the system
is locally or globaly stable.")

%% LQR
R=0.001;
Q = diag([800,5000,50000,500000,50000,50000]);

[K,P,e] = lqr(double(A),double(B),Q,R);
lambdaAfterLQR = eig(A-B*K)
disp("Eigen value of closed loop system is in the left half plane. Hence the
system is stable.");
%% simulation
s0 = [1; 0; 0; 0; 0; 0];

tSpan = 0:0.1:100;
[t,y1] = ode45(@(t,y)cartsim(y,t,parameters,A,B,K),tSpan,s0);

figure;
hold on
plot(t,y1(:,1),'r')
plot(t,y1(:,3),'b')
plot(t,y1(:,5),'k')
ylabel('state variables')
xlabel('time(sec)')
title('Linearized system with LQR response')
legend('position of cart','theta1','theta2')

%nonlinear system
```

```
[t1,y2] = ode45(@(t,y)cartSimNonLinear(y,t,parameters,-K*y),tSpan,s0);
figure;
hold on
plot(t1,y2(:,1),'r')
plot(t1,y2(:,3),'b')
plot(t1,y2(:,5),'k')
ylabel('state variables')
xlabel('time(sec)')
title('Non-Linear system with LQR response')
legend('position of the cart','theta1','theta2')


%% observablity

C1 = [1 0 0 0 0 0];              %x(t)
C2 = [0 0 1 0 0 0;...            %t1(t),t2(t)
      0 0 0 0 1 0];
C3 = [1 0 0 0 0 0;...            %x(t),t2(t)
      0 0 0 0 1 0];
C4 = [1 0 0 0 0 0;...            %x(t),t1(t),t2(t)
      0 0 1 0 0 0;...
      0 0 0 0 1 0];

%Checking Observability
ob1 = [C1;C1*A;C1*A^(2);C1*A^(3);C1*A^(4);C1*A^(5)];
ob2 = [C2;C2*A;C2*A^(2);C2*A^(3);C2*A^(4);C2*A^(5)];
ob3 = [C3;C3*A;C3*A^(2);C3*A^(3);C3*A^(4);C3*A^(5)];
ob4 = [C4;C4*A;C4*A^(2);C4*A^(3);C4*A^(4);C4*A^(5)];

rank(ob1)
disp("System is observable for x(t) as rank is 6")
rank(ob2)
disp("System is not observable for theta1(2), theta2(t) as rank is 4")
rank(ob3)
disp("System is observable for x(t), theta2(t) as rank is 6")
rank(ob4)
disp("System is observable for x(t), theta1(2), theta2(t) as rank is 6")


Bd = 0.01.*eye(6);
Bn = 0.1;
Bn1 = 0.1;
Bn3 = 0.1*[0,1;0,1];
Bn4 = 0.1*[0,0,1;0,0,1;0,0,1];
%% Obtaining Luenberger Observers using Kalman Bucy Filter (Lqe)

[L1,P,E] = lqe(A,Bd,C1,Bd,Bn);

[L3,P,E] = lqe(A,Bd,C3,Bd,Bn*eye(2));
[L4,P,E] = lqe(A,Bd,C4,Bd,Bn*eye(3));
P = [-24 -25 -26 -27 -28 -29];
L1p = place(A',C1',P)'

%% Arguments
uD = randn(6,size(tSpan,2));
uN = randn(size(tSpan));
u = 0*tSpan;
```

```
u(200:length(tSpan)) = 1;
u1 = [u; Bd*Bd*uD; uN];
Be = [B,Bd,zeros(size(B))];

%% Luenberger Observer for x(t)
%%
sysLO1 = ss(A-L1*C1,[B L1],C1,zeros(1,2));

De1 = [0,0,0,0,0,0,0,Bn1];

sys1 = ss(A,Be,C1,De1);
[y1,t] = lsim(sys1,u1,tSpan);

[x1,t] = lsim(sysLO1,[u; y1'],tSpan);

figure();
hold on
plot(t,y1(:,1),'k')
plot(t,x1(:,1),'g--')
ylabel('position of cart')
xlabel('time(sec)')
legend('output of noisy system','estimated output')
title('output vector x(t) - Linear System')
hold off

opt = simset('solver','ode45','SrcWorkspace','Current');
[tout2]=sim('nonLinerObserver',tSpan,opt);
figure();
hold on
plot(tout2,Output1(:,1),'k')
plot(tout2,stateVariables1(:,1),'g--')
ylabel('position of cart')
xlabel('time(sec)')
legend('output of noisy system','estimated output')
title('output vector x(t) - Non-Linear System')
hold off



%% Luenberger Observer for x(t),t2(t)
%%
sysLO3 = ss(A-L3*C3,[B L3],C3,zeros(2,3));

De3 = [zeros(size(C3)),Bn3];
sys3 = ss(A,Be,C3,De3);
[y3,t] = lsim(sys3,u1,tSpan);

[x3,t] = lsim(sysLO3,[u; y3'],tSpan);

figure();
hold on
plot(t,y3(:,1),'k')
plot(t,y3(:,2),'r')
plot(t,x3(:,1),'r--')
plot(t,x3(:,2),'g--')
ylabel('state variables ')
xlabel('time(sec)')
```

```
legend('noisy x(t)','noisy theta2(t)','estimated x(t)','estimated theta2(t)')
title('output vector (x(t),t2(t)) - Linear System')
hold off

opt = simset('solver','ode45','SrcWorkspace','Current');
[tout3]=sim('nonLinearObserver3',tSpan,opt);
figure();
hold on
plot(tout3,Output3(:,1),'k')
plot(tout3,Output3(:,2),'r')
plot(t,statesVariable3(:,1),'r--')
plot(t,statesVariable3(:,2),'g--')
ylabel('x-position of cart')
xlabel('time in s')
legend('output of noisy system','estimated output')
title('output vector x(t),t2(t) - Non-Linear System')
hold off

%% Luenberger Observer with State Estimator for C4: For output
(x(t),t1(t),t2(t)) - Kalman Filter
%%
sysLO4 = ss(A-L4*C4,[B L4],C4,zeros(3,4));

De4 = [zeros(3,5),Bn4];

sys4 = ss(A,Be,C4,De4);
[y4,t] = lsim(sys4,u1,tSpan);

[x4,t] = lsim(sysLO4,[u;y4'],tSpan);

figure();
hold on
plot(t,y4(:,1),'m')
plot(t,y4(:,2),'r')
plot(t,y4(:,3),'k')
plot(t,x4(:,1),'g--')
plot(t,x4(:,2),'b--')
plot(t,x4(:,3),'c--')
ylabel('state variables')
xlabel('time(sec)')
legend('noisy x(t)','noisy theta1(t)','noisy theta2(t)','estimated
x(t)','estimated theta1(t)','estimated theta2(t)')
title('output vector x(t),t1(t),t2(t) - Linear System')
hold off

opt = simset('solver','ode45','SrcWorkspace','Current');
[tout4]=sim('nonLinearObserver4',tSpan,opt);

figure();
hold on
plot(tout4,Output4(:,1),'m')
plot(tout4,Output4(:,2),'r')
plot(tout4,Output4(:,3),'k')
plot(tout4,stateVariables4(:,1),'g--')
plot(tout4,stateVariables4(:,2),'b--')
plot(tout4,stateVariables4(:,3),'c--')
ylabel('state variables')
```

```
xlabel('time(sec)')
legend('noisy x(t)','noisy theta1(t)','noisy theta2(t)','estimated
x(t)','estimated theta1(t)','estimated theta2(t)')
title('output vector x(t),t1(t),t2(t) - Non-Linear System')
hold off



%% LQG Controller for Output Vector C1 = [1,0,0,0,0,0]
%%
Ac = A-L1p*C1;
Bc = [B L1p];
Cc = eye(6);
Dc = 0*[B L1p];

opt = simset('solver','ode45','SrcWorkspace','Current');
sim('nonLinearLQG',tSpan,opt);


figure();
hold on
plot(tout,stateVariables(:,1),'b')
plot(tout,stateVariables(:,3),'r')
plot(tout,stateVariables(:,5),'k')
plot(tout,inputToLQG(:,1),'g')

title('LQG Nonlinear System')
legend('x','theta1','theta2')
hold off
```

linearModel.m

```
function [sDot,A,B] = linearModel(parameters)
%% nonlinear model
syms a b c d e f F g x1
s = [a,b,c,d,e,f];

x = s(1);
xDot = s(2);
t1 = s(3);
t1Dot = s(4);
t2 = s(5);
t2Dot = s(6);

%sDot = zeros(6,1);
sDot(1) = xDot;
sDot(2) = (F - parameters.m1*parameters.g*sin(2*t1)/2 -
parameters.m2*parameters.g*sin(2*t2)/2 -
parameters.m1*parameters.l1*sin(t1)*(t1Dot)^2 -
parameters.m2*parameters.l2*sin(t2)*(t2Dot)^2)/(parameters.M +
parameters.m1*(sin(t1)^2) + parameters.m2*(sin(t2)^2));
sDot(3) = t1Dot;
sDot(4) = (1/parameters.l1)*(cos(t1)*(F -
parameters.m1*parameters.g*sin(2*t1)/2 -
parameters.m2*parameters.g*sin(2*t2)/2 -
```

```
parameters.m1*parameters.l1*sin(t1)*(t1Dot)^2 -
parameters.m2*parameters.l2*sin(t2)*(t2Dot)^2)/(parameters.M +
parameters.m1*(sin(t1)^2) + parameters.m2*(sin(t2)^2)) -
parameters.g*sin(t1));
sDot(5) = t2Dot;
sDot(6) = (1/parameters.l2)*(cos(t2)*(F -
parameters.m1*parameters.g*sin(2*t1)/2 -
parameters.m2*parameters.g*sin(2*t2)/2  -
parameters.m1*parameters.l1*sin(t1)*(t1Dot)^2 -
parameters.m2*parameters.l2*sin(t2)*(t2Dot)^2)/(parameters.M +
parameters.m1*(sin(t1)^2) + parameters.m2*(sin(t2)^2)) -
parameters.g*sin(t2));

f1x1 = diff(sDot(2),x1);
f1t1 = diff(sDot(2),t1);
f1t2 = diff(sDot(2),t2);
f2x1 = diff(sDot(4),x1);
f2t1 = diff(sDot(4),t1);
f2t2 = diff(sDot(4),t2);
f3x1 = diff(sDot(6),x1);
f3t1 = diff(sDot(6),t1);
f3t2 = diff(sDot(6),t2);

f2F = diff(sDot(2),F);
f4F = diff(sDot(4),F);
f6F = diff(sDot(6),F);

A11 = subs(f1x1, {c,d,e,f}, {0,0,0,0});
A12 = subs(f1t1, {c,d,e,f}, {0,0,0,0});
A13 = subs(f1t2, {c,d,e,f}, {0,0,0,0});
A21 = subs(f2x1, {c,d,e,f}, {0,0,0,0});
A22 = subs(f2t1, {c,d,e,f}, {0,0,0,0});
A23 = subs(f2t2, {c,d,e,f}, {0,0,0,0});
A31 = subs(f3x1, {c,d,e,f}, {0,0,0,0});
A32 = subs(f3t1, {c,d,e,f}, {0,0,0,0});
A33 = subs(f3t2, {c,d,e,f}, {0,0,0,0});

b1 = subs(f2F,{c,d,e,f},{0,0,0,0});
b2 = subs(f4F,{c,d,e,f},{0,0,0,0});
b3 = subs(f6F,{c,d,e,f},{0,0,0,0});

%% linearised model

A = [0 1 0 0 0 0;A11 0 A12 0 A13 0;0 0 0 1 0 0;A21 0 A22 0 A23 0;0 0 0 0 0
1;A31 0 A32 0 A33 0];
B = [0;b1;0;b2;0;b3];
```

cartsim.m

```
function sDot = cartsim(s,t,parameters,A,B,K)
%% Linear Model
sDot = (A-B*K)*s;
end
```

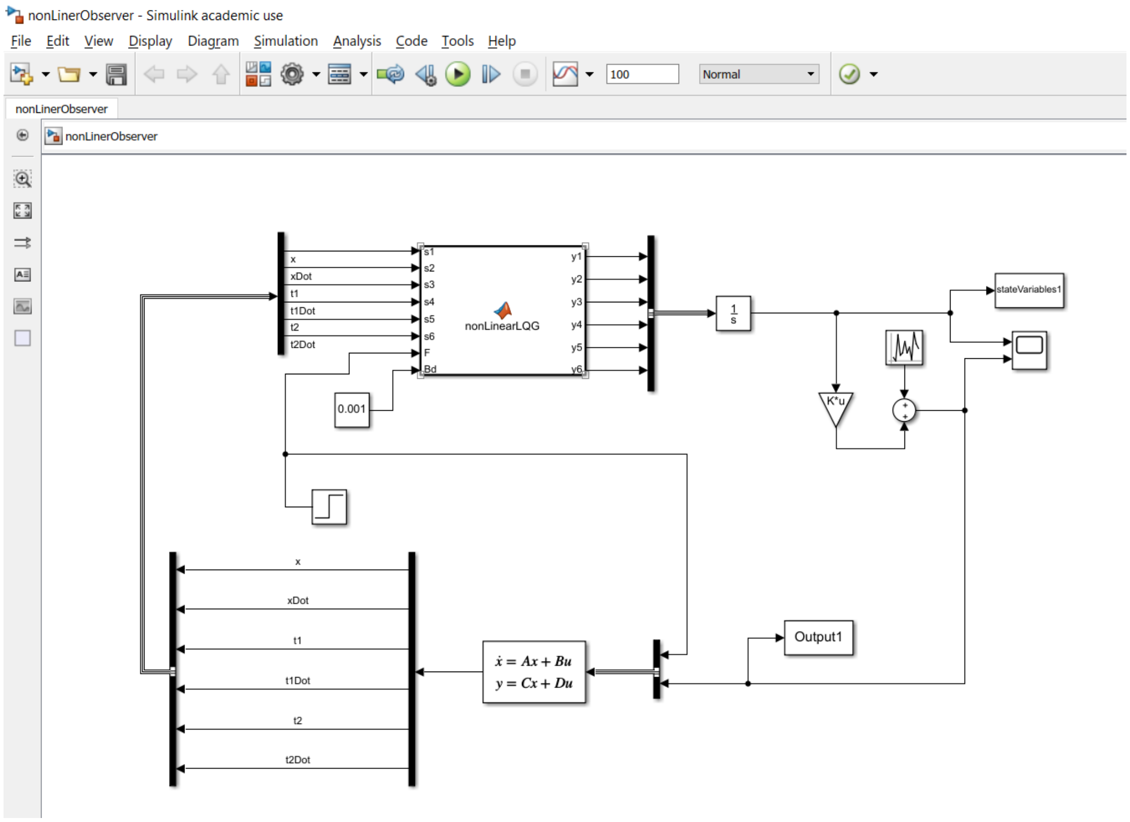
cartSimNonLinear.m

```
function sDot=cartSimNonLinear(s,t,parameters,F)
x = s(1);
xDot = s(2);
t1 = s(3);
t1Dot = s(4);
t2 = s(5);
t2Dot = s(6);
sDot=zeros(6,1);
sDot(1) = xDot;
sDot(2) = (F-
parameters.m1*(parameters.g*sin(t1)*cos(t1)+parameters.l1*sin(t1)*t1Dot^2)-
parameters.m2*(parameters.g*sin(t2)*cos(t2)+parameters.l2*sin(t2)*t2Dot^2))/(
parameters.M+parameters.m1*sin(t1)^2+parameters.m2*sin(t2)^2);
sDot(3) = t1Dot;
sDot(4) = (cos(t1)/parameters.l1)*((F-
parameters.m1*(parameters.g*sin(t1)*cos(t1)+parameters.l1*sin(t1)*t1Dot^2)-
parameters.m2*(parameters.g*sin(t2)*cos(t2)+parameters.l2*sin(t2)*t2Dot^2))/(
parameters.M+parameters.m1*sin(t1)^2+parameters.m2*sin(t2)^2))-
(parameters.g*sin(t1)/parameters.l1);
sDot(5) = t2Dot;
sDot(6) = (cos(t2)/parameters.l2)*((F-
parameters.m1*(parameters.g*sin(t1)*cos(t1)+parameters.l1*sin(t1)*t1Dot^2)-
parameters.m2*(parameters.g*sin(t2)*cos(t2)+parameters.l2*sin(t2)*t2Dot^2))/(
parameters.M+parameters.m1*sin(t1)^2+parameters.m2*sin(t2)^2))-
(parameters.g*sin(t2)/parameters.l2);
end
```

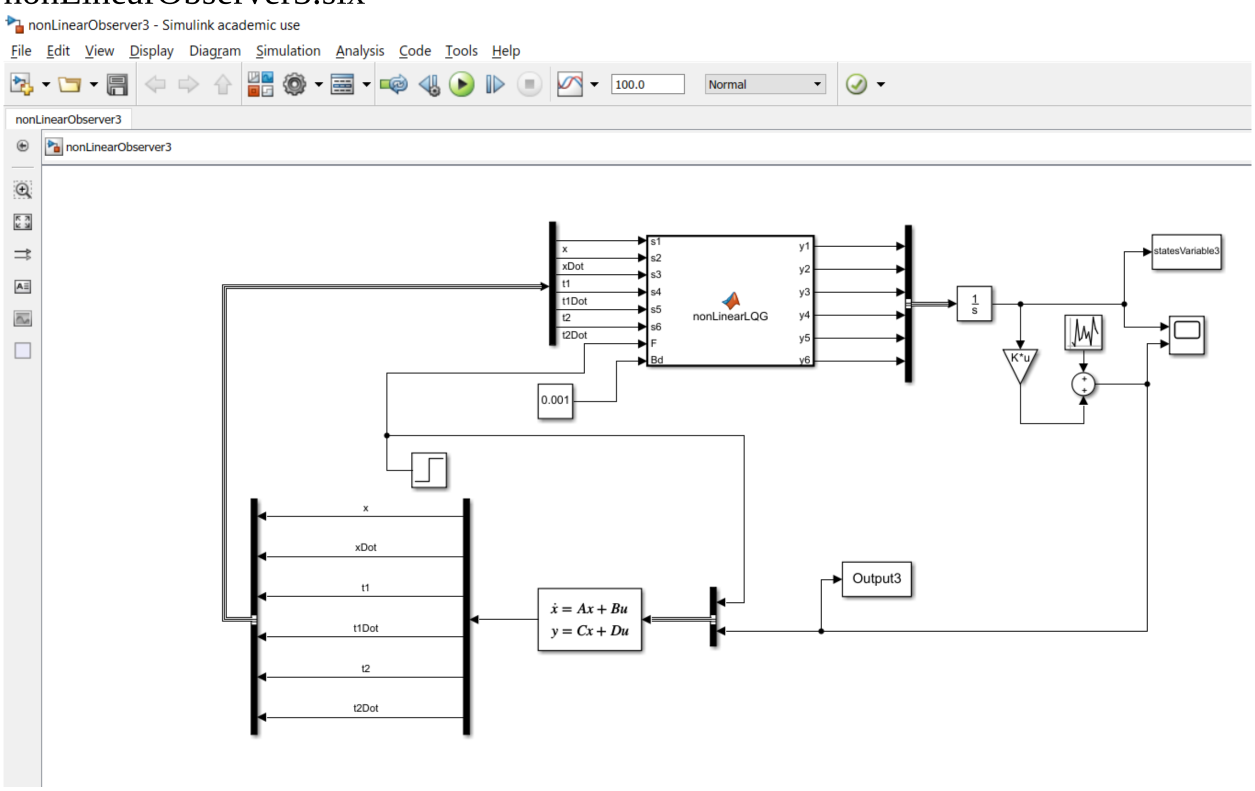
Uncontrolability.m

```
clear
clc
syms M m1 m2 l1 l2 g
parameters.M  = M;
parameters.m1 = m1;
parameters.m2 = m2;
parameters.l1 = l1;
parameters.l2 = l2;
parameters.g  = g;

[sDot,A,B] = linearModel(parameters);

C = simplify([B A*B (A^2)*B (A^3)*B (A^4)*B (A^5)*B])
simplify(det(C))
```

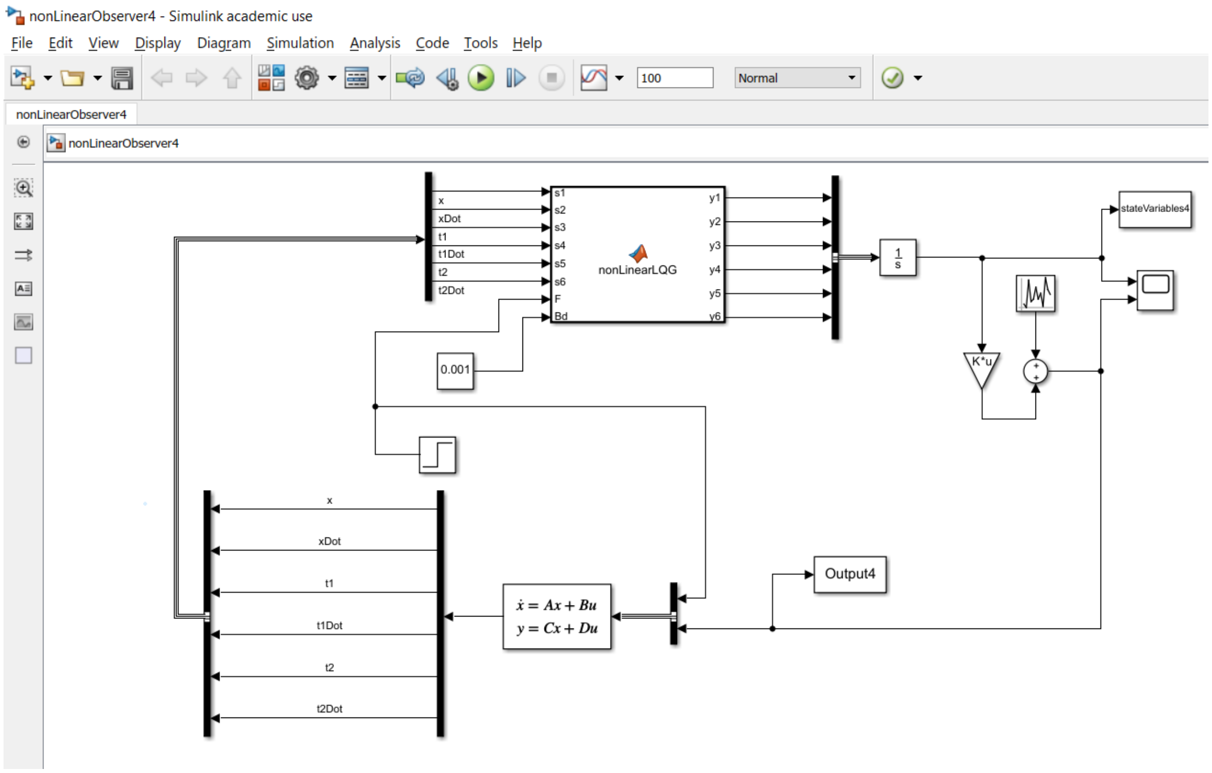
nonLinerObserver.slx



nonLinearObserver3.slx



nonLinearObserver4.slx



nonLinearLQG.m