Review on Basic differential rules

① $y = c$ (Constant) $\longrightarrow y' = \frac{dy}{dx} = 0$

② $y = x^n \longrightarrow y' = n x^{n-1}$

③ $y = c f(x) \longrightarrow y' = c f'(x) = \frac{c\,df}{dx}$

where c is a constant

④ $y = f(x) \pm g(x) \longrightarrow y' = f'(x) \pm g'(x)$

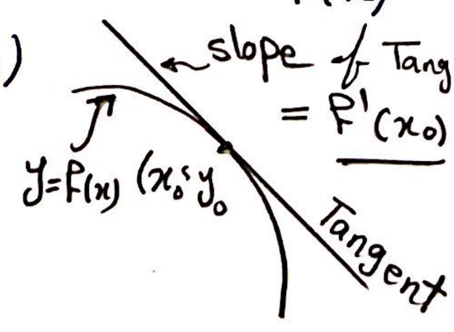
⑤ $y = [f(x)]^n \longrightarrow y' = n[f(x)]^{n-1} f'(x)$

⑥ $y = \sqrt{f(x)} \longrightarrow y' = \frac{f'(x)}{2\sqrt{f(x)}}$

⑦ $y = f(x) \cdot g(x) \longrightarrow y' = f(x) \cdot g'(x) + g(x) f'(x)$

⑧ $y = \frac{f(x)}{g(x)} \longrightarrow y' = \frac{g(x) f'(x) - f(x) g'(x)}{g^2(x)}$

- <u>Note</u>: The derivative of the function $f(x)$ at a point x_0 denoted by $f'(x_0)$ is the slope of the tangent to a curve of $f(x)$ at x_0



Examples:

Find $\frac{dy}{dx}$ for each of the following:

① $y = x^5 - \frac{4}{x^3} + \sqrt[4]{x^3} + 5x - 16$

$y = x^5 - 4x^{-3} + x^{3/4} + 5x - 16$

$y' = 5x^4 + 12x^{-4} + \frac{3}{4}x^{-1/4} + 5$

② $y = \frac{6}{\sqrt[3]{(x^2-5)^2}}$

$y = 6(x^2-5)^{-2/3}$

$y' = 6\left(\frac{-2}{3}\right)(x^2-5)^{-5/3}(2x).$

③ $y = (x^3-1)^5(2+7x^{-4})^7 + \left(\frac{x^2-3}{x^{-4}+2}\right)^{4/3}$

$y' = (x^3-1)^5 \cdot 7(2+7x^{-4})^6 \cdot (-28x^{-5})$

$+ (2+7x^{-4})^7 \cdot 5(x^3-1)^4 \cdot (3x^2)$

$+ \frac{4}{3}\left(\frac{x^2-3}{x^{-4}+2}\right)^{1/3} \cdot \left(\frac{(x^{-4}+2)(2x) - (x^2-3)(-4x^{-5})}{(x^{-4}+2)^2}\right.$

④ If $y=(x+\sqrt{x^2-1})^n$

Prove that $y' = \dfrac{ny}{\sqrt{x^2-1}}$

— Proof —

$$\frac{\sqrt{x^2-1}}{\sqrt{x^2-1}}$$

$$y' = n(x+\sqrt{x^2-1})^{n-1}\cdot\left(1+\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2-1}}\right)$$

$$= n(x+\sqrt{x^2-1})^{n-1}\left(\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}\right)$$

$$= n(x+\sqrt{x^2-1})^{n-1}\cdot\frac{(x+\sqrt{x^2-1})}{\sqrt{x^2-1}}$$

$$= \frac{n(x+\sqrt{x^2-1})^n}{\sqrt{x^2-1}}$$

$$= \frac{ny}{\sqrt{x^2-1}}$$

H.W

⑤ Find the slope of the tangent for the Curve $y=\dfrac{x}{x-2}$ at the point $(3,3)$

Derivative of Trigonometric Functions

Let $u = u(x)$, (u is a function of x)

① $y = \sin(u) \longrightarrow y' = \cos(u) \cdot u'$

② $y = \cos(u) \longrightarrow y' = -\sin(u) \cdot u'$

③ $y = \tan(u) \longrightarrow y' = \sec^2(u) \cdot u'$

④ $y = \cot(u) \longrightarrow y' = -\operatorname{cosec}^2(u) \cdot u'$

⑤ $y = \sec(u) \longrightarrow y' = \sec(u)\tan(u) \cdot u'$

⑥ $y = \operatorname{cosec}(u) \longrightarrow y' = -\operatorname{cosec}(u)\cot(u) \cdot u'$

Examples

Find y' for the following

① $y = \tan^2(\cos^3 x^2) + \sqrt{x \sec x + \sin\sqrt{x}}$

$$y' = 2\tan(\cos^3 x^2) \cdot \sec^2(\cos^3 x^2) \cdot (3\cos^2 x^2) \cdot (-\sin x^2) \cdot (2x)$$

$$+ \frac{x \sec x \tan x + \sec x \cdot (1) + \cos\sqrt{x} \cdot \left(\frac{1}{2\sqrt{x}}\right)}{2\sqrt{x \sec x + \sin\sqrt{x}}}$$

② $y = \sqrt{\dfrac{1-\sin 2x}{1+\cot x}} + \sec^3(\sqrt{\cos x})$

$\to (\;\;)^{1/2}$ $\qquad \to \left(\sec(\sqrt{\cos x})\right)^3$

$$y' = \frac{1}{2}\left(\frac{1-\sin 2x}{1+\cot x}\right)^{-1/2} \cdot \left(\frac{(1+\cot x)(-2\cos 2x) - (1-\sin 2x)(-\csc^2 x)}{(1+\cot x)^2}\right)$$
$$+ 3(\sec\sqrt{\cos x})^2 \cdot \sec\sqrt{\cos x}\,\tan\sqrt{\cos x} \cdot \frac{-\sin x}{2\sqrt{\cos x}}$$

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③ $y = \sqrt{x^5 - \tan^2 x + x\cos^2 x}$

$$y' = \frac{5x^4 - 2\tan x \cdot \sec^2 x + x \cdot (2\cos x)(-\sin x) + \cos^2 x \cdot (1)}{2\sqrt{x^5 - \tan^2 x + x\cos^2 x}}$$

Derivative of exponential function

Let $u \equiv u(x)$, u is a function of x

$$y = e^u \longrightarrow y' = e^u \cdot u'$$

Ex: Find y' for the following

① $y = e^{\cot 2x}$

$$y' = e^{\cot(2x)} \cdot \left(-2\csc^2(2x)\right)$$

② $y = e^{\sqrt{x^2+1}} \tan(x^2)$

$$y' = e^{\sqrt{x^2+1}} \sec^2(x^2)\cdot(2x) + \tan(x^2)\cdot e^{\sqrt{x^2+1}} \cdot \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}$$

③ If $y = \dfrac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}$

Prove that $y' = \dfrac{8}{(e^{2x} + e^{-2x})^2}$

Proof

$$y' = \frac{(e^{2x} + e^{-2x})(2e^{2x} + 2e^{-2x}) - (e^{2x} - e^{-2x})(2e^{2x} - 2e^{-2x})}{(e^{2x} + e^{-2x})^2}$$

$$y' = \frac{2(e^{2x} + e^{-2x})^2 - 2(e^{2x} - e^{-2x})^2}{(e^{2x} + e^{-2x})^2}$$

$$y' = \frac{2(e^{4x} + 2 + e^{-4x}) - 2(e^{4x} - 2 + e^{-4x})}{(e^{2x} + e^{-2x})^2}$$

$$y' = \frac{\cancel{2e^{4x}} + 4 + \cancel{2e^{-4x}} - \cancel{2e^{4x}} + 4 - \cancel{2e^{-4x}}}{(e^{2x} + e^{-2x})^2}$$

$$y' = \frac{8}{(e^{2x} + e^{-2x})^2}$$

Derivative of Logarithmic Function:

Let $u \equiv u(x)$, (u is a function of x)

$y = \ln(u) \longrightarrow y' = \frac{u'}{u}$

$y = \log_a(u) \longrightarrow y' = \frac{1}{\ln a} \cdot \frac{u'}{u}$

where $a > 0$, $a \neq 1$, $\log_a u = \frac{\ln u}{\ln a}$

Examples

Find $\frac{dy}{dx}$ for the following

① $y = \ln(1 - 2x^2)$

$$\frac{dy}{dx} = \frac{-4x}{1 - 2x^2}$$

② $y = \log_2(\sin x^2)$

$$y' = \frac{1}{\ln 2} \frac{\cos(x^2) \cdot (2x)}{\sin(x^2)}$$

③ $y = \log(\tan(e^{3x}))$

$$y' = \frac{1}{\ln 10} \frac{\sec^2(e^{3x}) \cdot 3e^{3x}}{\tan(e^{3x})}$$

④ If $y = \ln(\sec x + \tan x)$

Prove that $y'' = \sec x \tan x$

—— proof ——

$$y' = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$$

$$y' = \frac{\sec x \, \cancel{(\tan x + \sec x)}}{\cancel{\sec x + \tan x}}$$

$$y' = \sec x$$

$$y'' = \sec x \tan x$$

⑤ If $y = \ln(\cos x)$

Prove that $y'' + e^{-2y} = 0$

—— proof ——

$$y' = \frac{-\sin x}{\cos x} = -\tan x$$

$$y'' = -\sec^2 x \rightarrow ①, \quad e^{-2y} = e^{-2\ln(\cos x)}$$

$$= e^{\ln(\cos x)^{-2}}$$

$$= (\cos x)^{-2} = \frac{1}{\cos^2 x} = \sec^2 x \rightarrow ②$$

From ① , ②

$$y'' + e^{-2y} = -\cancel{\sec^2 x} + \cancel{\sec^2 x} = 0$$

H.W

⑥ Find y' if

(i) $y = x^4 \ln(x^2+1) + e^{\sin x}$.

(ii) $y = x \sin x + e^{x^2} \ln(4x^3+1)$.