Chapter 2

RANDOM VARIABLES
and
PROBABILITY DISTRIBUTIONS

2.1 Introduction

Our purpose is to develop mathematical models for describing the probabilities of occurrence of outcomes or events in a sample space. Since mathematical equations are expressed in terms of numerical values rather than heads, colors, or other properties, it is convenient to define a function, known as a random variable, which associates each outcome in the experiment with a real number. We can then express the probability model for the experiment in terms of this associated random variable. Of course, in many experiments the results of interest are already numerical quantities, and in that case the natural function to use as the random variable would be the identity function.

Definition 2.1: Random Variable

A random variable (abbreviated by r.v.), say **X**, is a function defined over a sample space, **S**, which associates a real number, **X**(e) = x, with each possible outcome e in **S**

Therefore a r.v. X is a function whose domain is S and whose range is the set of real numbers R^1 , i.e.

$$\mathbf{X : S \rightarrow R^1}$$

Capital letters, such as **X**, **Y**, **Z, …** will be used to denote random variables. The lower case letters x, y, z, ... will be used to denote possible values that the corresponding random variable can attain. For mathematical reasons, it will be necessary to restrict the types of functions that are considered as random variables.

2.2 Discrete Random Variables

Definition 2.2

If the set of all possible values of a r.v., **X**, is a countable set $x_1, x_2, \ldots, x_n$ or $x_1, x_2, \ldots$, then **X** is called a **discrete random variable**.

In most practical problems, discrete random variables represent count data, such as the number of defectives in a sample of k items, the number of customers in a shop, the number of successes of an experiment, the number of tosses of a coin until a head appear.

Definition 2.3

If **X** is a discrete r.v. with possible values $\mathbf{x_1, x_2, \ldots}$, the function defined by

$$\mathbf{f(x) = P[X = x] \quad , \quad x = x_1, x_2, \ldots} \qquad \mathbf{(2.1)}$$

that assigns the probability to each possible value x will be called the **discrete probability mass function (p.m.f.)** of **X**.

Sometimes a subscripted notation, $\mathbf{f_X(x)}$, is used. The following theorem, based on the axioms of probability, gives general properties that any discrete p.m.f. must satisfy.

Theorem 2.1

A function f(x) is a p.m.f. of a discrete r.v. **X** if and only if it satisfies the following conditions

1. $\mathbf{f(x) \geq 0}$ for all x, and
2. $\mathbf{\sum_{\text{all } x_i} f(x_i) = 1}$

Example 2.1

A lot of 12 television sets includes 3 with white cords. If 2 of the sets are chosen at random for shipment to a hotel, how many sets with white cords can the shipper expect to send to the hotel?

Solution

Let X be a random variable whose values x are the possible numbers of TV sets with white cords sanded to the hotel. Then x can be any of the numbers 0, 1, and 2. Now,

f(0) = P(X=0) = P(both sets are without white cords)

$$= P(\bar{W}_1 \cap \bar{W}_2) = \frac{9}{12}\frac{8}{11} = \frac{6}{11} = \frac{12}{22}$$

f(1) = P(X=1) = P(one set with and one without white cord)

$$= P(\bar{W}_1 \cap W_2) + P(W_1 \cap \bar{W}_2) = \frac{9}{12}\frac{3}{11} + \frac{3}{12}\frac{9}{11} = \frac{9}{22}$$

f(2) = P(X=2)=P(both sets are with white cords) = $P(W_1 \cap W_2) = \frac{3}{12}\frac{2}{11} = \frac{1}{22}$

or, in tabular form,

x	0	1	2
f(x)	6/11	9/22	1/22

There are many problems in which it is of interest to know the probability that the value of a r.v. **X** is less than or equal to some real number x, i.e. P(X ≤ x). This probability is denoted by **F(x)**, and refer to this function defined for all real numbers x as the ***cumulative distribution function*** (abbreviated by **CDF**).

Definition 2.4

If **X** is a discrete r.v. with p.m.f. **f(x)**, then the cumulative distribution function (CDF) of **X** is defined for any real x by

$$\mathbf{F(x) = P\,(X \le x) = \sum_{i:x_i \le x} f(x_i)}$$

where the summation is taken over all indices i such that $x_i \le x$.

The function **F(x)** is sometimes referred to simply as the ***distribution function*** of X, and the subscripted notation, $F_x(x)$, is sometimes used.

The CDF is a non-decreasing step function. The step-function form of F(x) is common to all discrete distributions, and the sizes of the steps or jumps in the graph of F(x) correspond to the values of f(x) at those points.

Example 2.2

Let X be a discrete r.v. with the following p.m.f.;

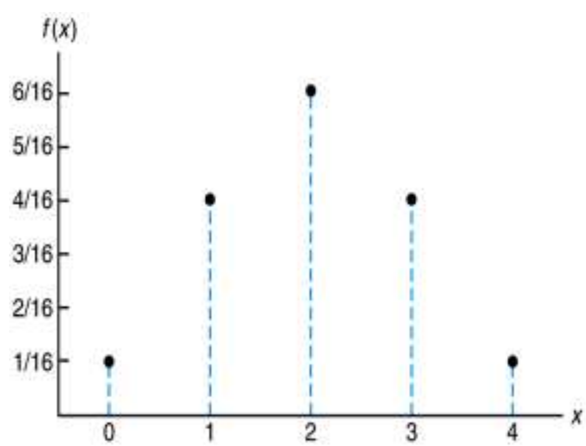
$$\mathbf{f(x) = \frac{1}{16}\binom{4}{x}, \quad for\ x = 0,1,2,3,4.}$$

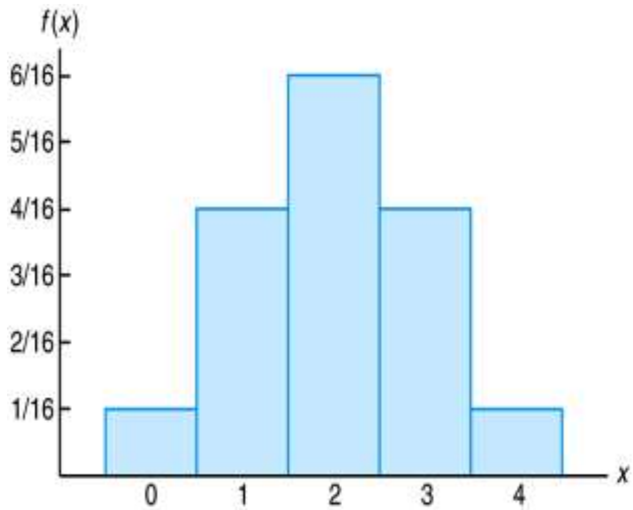
Find the cumulative distribution function of the X and plot both f(x) and F(x). Using F(x), verify that f(2) = 3/8.

Solution

Direct calculations of the probability distribution of X give

x	0	1	2	3	4
f(x)	1/16	1/4	3/8	1/4	1/16





Therefore,

F(0) = f(0) =1/16

F(l) = f(0) + f(1) = 5/16,

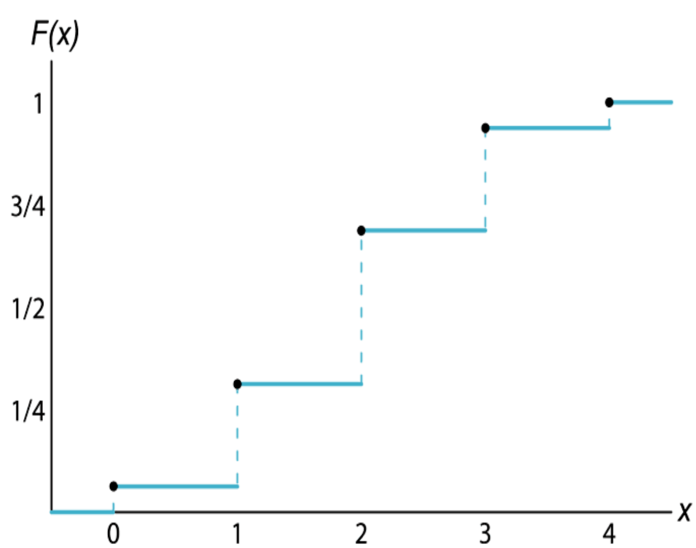
F(2) = f(0) + f(l) + f(2) = 11/16,

F(3) = f(0) + f(l) + f(2) + f(3) = 15/16

F(4) = f(0) + f(l) + f(2) + f(3) + f(4) = 1.

Hence

$$\mathbf{F(x)} = \begin{cases} \mathbf{0} & \mathbf{for\ \ x < 0,} \\ \mathbf{\dfrac{1}{16}} & \mathbf{for\ 0 \le x < 1,} \\ \mathbf{\dfrac{5}{16}} & \mathbf{for\ 1 \le x < 2,} \\ \mathbf{\dfrac{11}{16}} & \mathbf{for\ 2 \le x < 3,} \\ \mathbf{\dfrac{15}{16}} & \mathbf{for\ 4 \le x < 4,} \\ \mathbf{1} & \mathbf{for\ x \ge 4.} \end{cases}$$



Now,

$$f(2) = F(2) - F(1) = \frac{11}{16} - \frac{5}{16} = \frac{3}{8}.$$

The general relationship between F(x) and f(x) for a discrete distribution is given by the following theorem.

Theorem 2.2

Let **X** be a discrete r.v. with pmf **f(x)** and CDF **F(x)**. If the possible values of **X** are indexed in increasing order $x_1 < x_2 < ... < x_n$ then

$f(x_1) = F(x_1)$,

$f(x_i) = F(x_i) - F(x_{i-1})$ for i = 2, 3, ...,n

2.3 Continuous Random Variables

When a r.v. can take on values on a continuous scale, it is called a **continuous random variable**. i.e. a r.v. **X** is continuous if its possible values are infinite and uncountable (often the possible values of a continuous r.v. are precisely the same values that are contained in the continuous sample space). In most practical problems, continuous random variable represent measured data, such as all possible heights, weights, temperature, distances, etc. Consequently, the probability distribution of the continuous r.v. cannot be given in tabular form, it can have a formula. Such a formula would necessarily be a function of the numerical values of the continuous variable **X** and denoted by **f(x)**. In dealing with continuous variables, f(x) is usually called the **probability density function** (abbreviated by **p.d.f.**)

Definition 2.5

A function **f(x)** defined over the set of all real numbers, is called **probability density function (pmf)** of a continuous r.v. **X** iff

$$P(a \le X \le b) = \int_a^b f(x)\,dx$$

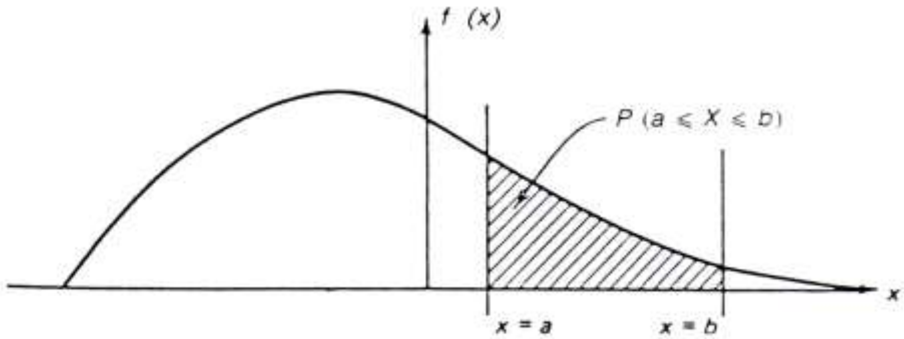
for any constants a and b with $a \le b$.

Note that f(c), the value of the p.d.f. of **X** at c, does not give P(X=c) as in the discrete case. In connection with continuous r.v.'s, probabilities are always associated with intervals and

$$P(X = c) = 0 \quad \text{for any real constant } c.$$

In view of this property, it does not matter whether we include the endpoints of the interval from a to b; symbolically,

$$P(a \le X \le b) = P(a \le X < b) = P(a < X \le b) = P(a < X < b)$$



Let us now state the following properties of the p.d.f., which again follow directly the axioms of probability.

Example 2.3

If X has the p.d.f.

$$f(x) = \begin{cases} k\ e^{-3x} & \text{for } x > 0 \\ 0 & \text{o.w.} \end{cases}$$

find k and P $(0.5 \le X \le 1)$.

Solution

To satisfy the second condition of theorem 2.3, we must have

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \quad \Rightarrow \quad \int_{0}^{\infty} k e^{-3x}\,dx = 1 \quad \Rightarrow \quad k\left.\frac{e^{-3x}}{-3}\right|_{0}^{\infty} = 1 \quad \Rightarrow \quad \frac{k}{3} = 1$$

and follows that k = 3. For the probability we get

$$P(0.5 \le X \le 1) = \int_{0.5}^{1} 3e^{-3x}\,dx = -e^{-3} + e^{1.5} = 0.173$$

Although the r.v. of the preceding example cannot take negative values, we artificially extended the domain of its p.d.f. to include all the real numbers.

As in the case, there are many problems in which it is of interest to know the probability that the value of a continuous random variable X is less than or equal to some real number x. Thus, let us make the following definition analogous to definition 2.4.

Definition 2.6

If X is a continuous r.v. with p.d.f. f(x), then its CDF is given by

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt \quad \text{for} \quad -\infty < x < \infty$$

The properties of CDF given in Theorem 2.2 hold also for the continuous case;

that is, **F(- ∞) = O**, **F(∞) = 1**, and **F(a) ≤ F(b)** when **a < b**. Furthermore, it follows directly from definition 2.6 that

Theorem 2.4

If f(x) and F(x) are the values of the p.d.f. and the CDF of X at x, then

$$P(a \le X \le b) = F(b) - F(a)$$

for any real constants a and b with $a \le b$, and

$$f(x) = \frac{dF(x)}{dx}$$

where the derivative exists.

Example 2.4

Find the CDF of the r.v. X of Example 2.3, and use it to reevaluate $P(0.5 \le X \le 1)$.

Solution

For $x > 0$, $$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{O}^{x} 3e^{-3t}\,dt = -e^{-3t}\Big|_{O}^{x} = 1 - e^{-3x}$$

and since $F(x) = 0$ for $x \le 0$, we can write

$$F(x) = \begin{cases} 0 & \text{for } x \le 0 \\ 1 - e^{-3x} & \text{for } x > O \end{cases}$$

To determine the probability $P(0.5 \le X \le 1)$, we use the first part of theorem 2.4, getting

$$P(0.5 \le X \le 1) = F(1) - F(0.5) = (1 - e^{-1}) - (1 - e^{-0.5}) = 0.173$$

This agrees with the result obtained by using the p.d.f. directly in example 2.3.

Example 2.5

The demand for a certain commodity is a r.v. X specified by the following p.d.f.

$$f(x) = \begin{cases} kx & \text{for } 0 < x < 10 \\ k(20 - x) & \text{for } 10 \le x < 20 \\ 0 & \text{otherwise} \end{cases}$$

Find: **a-** The value of k and sketch the p.d.f.

b- The CDF of X and sketch it.

c- $P(X > 8)$ and $P(5 < X < 15)$.

Solution

a- Since f(x) is a p.d.f. then by theorem 2.3, we must have

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_{O}^{10} kx\,dx + \int_{10}^{20} k(20 - x)\,dx = 100\,k$$

Hence k = 0.01 and the p.d.f. now is given by

$$f(x) = \begin{cases} x/100 & \text{for } 0 < x < 10 \\ 0.2 - x/100 & \text{for } 10 \le x < 20 \\ 0 & \text{otherwise} \end{cases}$$

and its graph is shown in Figure 2.1

b- The **CDF** of **X** is given by

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \begin{cases} 0 & \text{if } x < 0 \\ \int_{0}^{x} 0.01\,t\,dt & \text{if } 0 \le x < 10 \\ \int_{0}^{10} .01\,t\,dt + \int_{10}^{x} (0.2 - .01\,t)\,dt & \text{if } 10 \le x < 20 \\ 1 & \text{if } 20 \le x \end{cases}$$

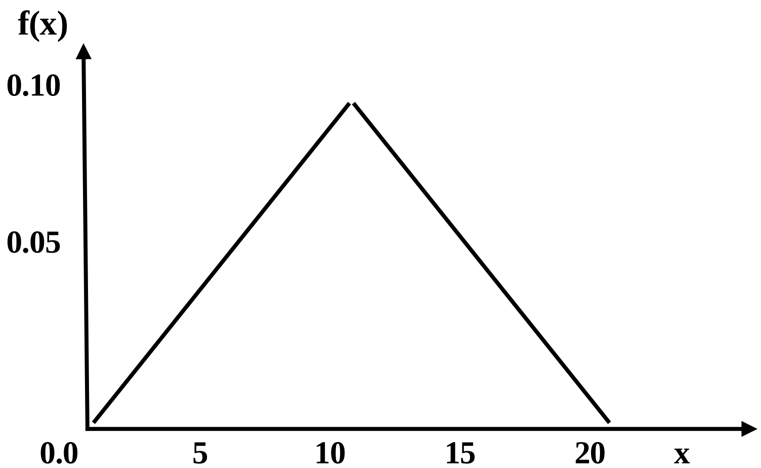


Fig. (2.1) The p.d.f. of the demand X of example 2.5

Hence;

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 0.005x^2 & \text{if } 0 \le x < 10 \\ 0.2\,x - 0.005x^2 - 1 & \text{if } 10 \le x < 20 \\ 1 & \text{if } 20 \le x \end{cases}$$

and its graph is shown in Figure 2.2.

c- $P(X > 8) = 1 - P(X \le 8) = 1 - F(8) = 1 - 0.32 = 0.68$

$P(5 < X < 15) = F(15) - F(5) = 0.875 - 0.125 = 0.75$

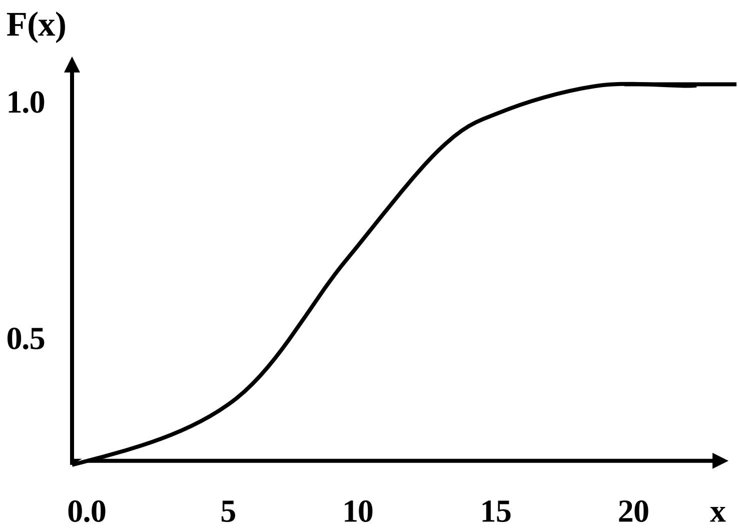


Fig. (2.2) The CDF of the demand X of example 2.3

2.4 Mathematical Expectation

The mathematical expectation of a discrete r.v. is the sum of the products obtained by multiplying each value of the r.v. by the corresponding probability. Referring to the mathematical expectation of a r.v. simply as its **expected value or mean value**, and extending the definition to the continuous case by replacing the operation of summation by integration. We thus have

Definition 2.7

If X is a discrete r.v. with p.m.f. **f(x)**, the **expected value** of **X** is

$$\mathbf{E(X)} = \sum_i x_i\, f(x_i)$$

Correspondingly, if X is a continuous r.v. with p.d.f. **f(x)**, the **expected value** of X is

$$\mathbf{E(X)} = \int_{-\infty}^{\infty} x \,.\, f(x)\, dx$$

In this definition it is assumed, of course, that the sum or the integral exists; otherwise, the mathematical expectation is undefined.

Example 2.6

For the probability distribution of X given in example 2.1,

x	0	1	2
f(x)	6/11	9/22	1/22

Now,

$$E(X) = 0.\frac{6}{11} + 1.\frac{9}{22} + 2.\frac{1}{22} = \frac{1}{2}$$

and since half a set cannot possibly be shipped, it should be clear that the term "**expect**" is not used in its colloquial sense. Indeed, it should be interpreted as an **average** (or mean) pertaining to repeated shipments made under the given conditions.

Example 2.7

If X is the number of points rolled with a fair die, find the expected value of **g(X) = 2X² + 1**.

Solution

Since each possible outcome has the probability $\frac{1}{6}$**, we get**

$$E[g(X)] = \sum_{x=1}^{6} (2x^2+1).\frac{1}{6} = (2(1)^2+1).\frac{1}{6} + \cdots + (2(6)^2+1).\frac{1}{6} = \frac{94}{3}$$

Example 2.8

The p.d.f. of the r.v. X is given by

$$f(x) = \begin{cases} k\,x(1-x^2) & \text{for} \quad 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

a- Find the value of k and P(-0.2 < X < 0.4).

b- Find E(X) and E(X^2).

Solution

a- Since f(x) is a p.d.f., it must satisfy,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \quad \Rightarrow \quad k\int_0^1 x(1-x^2)\,dx = 1 \quad \Rightarrow k\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = 1 \Rightarrow k = 4$$

Thus

$$P(-0.2 \le X \le 0.4) = \int_{-02}^{0.4} f(x)\,dx = 4\int_0^{0.4} x(1-x^2)\,dx = 0.2944$$

b- $$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = 4\int_0^1 x.x(1-x^2)\,dx = 4\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{8}{15}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = 4\int_0^1 x^2.x(1-x^2)\,dx = 4\left[\frac{x^4}{4} - \frac{x^6}{6}\right]_0^1 = \frac{1}{3}$$

The determination of mathematical expectations can often be simplified by using its properties given by the following definition:

Definition 2.8

If c, c_1 and c_2 are constants, then

i- $E[\,c\,] = c$,

ii- $E[\,c\,g(X)\,] = c\,E[g(X)]$,

iii- $E[\,c_1\,g_1(X) + c_2\,g_2(X)\,] = c_1\,E[g_1(X)] + c_2\,E[g_2(X)]$

COROLLARY

If **a** and **b** are constants, then $E(aX + b) = aE(X) + b$

Example 2.9

Making use of the fact that

$$E(X^2) = (1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2).\frac{1}{6} = \frac{91}{6}$$

for the r.v. of example 2.5, rework that example.

Solution

$$E(2X^2 + 1) = 2E(X^2) + 1 = 2.\frac{91}{6} + 1 = \frac{94}{3}$$

Example 2.10

If the p.d.f. of X is given by

$$f(x) = \begin{cases} 2(1 - x) & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

a- show that $E(X^r) = \dfrac{2}{(r+1)\,(r+2)}$, and

b- use this result to evaluate $E[(2X + 1)^2]$.

Solution

$$\text{a-}E(X^r) = \int_0^1 x^r.2(1-x)\,dx = 2\int_0^1 (x^r - x^{r+1})\,dx$$

$$= 2\left(\frac{1}{r+1} - \frac{1}{r+2}\right) = \frac{2}{(r+1)\,(r+2)}$$

b- Since $E[(2X + 1)^2] = 4E(X^2) + 4E(X) + 1$, and substitution of $r = 1$ and $r = 2$ into the preceding formula yields

$$E(X) = \frac{2}{2.3} = \frac{1}{3} \quad \text{and} \quad E(X^2) = \frac{2}{3.4} = \frac{1}{6},$$

we get

$$E[\,(2X + 1)^2\,] = 4.\frac{1}{6} + 4.\frac{1}{3} + 1 = 3.$$

2.5 Mean and Variance

It is of interest to note that the term "moment" comes from the field of applied mathematics, if the quantities f(x) in the discrete case were point masses acting perpendicularly to the x-axis at distances x from the origin, μ_1 would be the x-coordinate of the center of gravity, namely, the first moment divided by $\Sigma f(x) = 1$, and μ_2 would be the moment of inertia. This also explains why the moments μ_r are called moments about the origin -in analogy to applied mathematics, the length of lever arm is in each case the distance from the origin. The analogy applies also in the continuous case.

When **r = 1**, we have $\boldsymbol{\mu_1 = E(X)}$, which is just the expected (or mean) value of the r.v. **X**.

Definition 2.9

The **mean** of **X** or the **mean** of the distribution of X, denoted by $\boldsymbol{\mu}$ is given by

$\boldsymbol{\mu = E(X)}$

The mean of a r.v. X is of special importance in statistics because it describes where the probability distribution is centered.

Definition 2.10

The **variance** of the distribution of X, or simply the **variance of X**, denoted by $\boldsymbol{\sigma^2}$, **var(X)**, or **V(X);** is defined as

$$\sigma^2 = \mathrm{var}(X) = E[(X - \mu)^2] \qquad \textbf{(2.6)}$$

the positive square root of the variance, $\boldsymbol{\sigma}$ is called the **standard deviation**.

The quantity $\boldsymbol{x - \mu}$ in definition 2.10 is called the **deviation of an observation from its mean**. Since these deviations are being squared and then averaged, σ^2 will be much smaller for a set of x values that are close to μ than it would be for a set of values that vary considerably from μ.

An alternative and preferred formula for finding $\boldsymbol{\sigma^2}$, which often simplifies the calculations, is given in the following theorem.

Theorem 2.5

$$\sigma^2 = E[X^2] - \mu^2 \qquad \textbf{(2.7)}$$

Example 2.11

The weekly demand for Pepsi, in thousands of liters, from a local chain of efficiency stores, is a continuous r.v. X having the p.d.f.

$$f(x) = \begin{cases} 2(x-1) & \text{for} \quad 1 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Find the mean and variance of X.

Solution

$$\mu = E(X) = 2\int_1^2 x(x-1)\,dx = \frac{5}{3}$$

and

$$E(X^2) = 2\int_1^2 x^2(x-1)\,dx = \frac{17}{6}$$

Therefore,

$$\sigma^2 = \frac{17}{6} - \left(\frac{5}{3}\right)^2 = \frac{1}{18}.$$

Theorem 2.6

If a and b are constants, then

$\boldsymbol{\sigma^2_{aX+b} = \text{var}(aX + b) = a^2\sigma^2}$

Proof

By definition,

$$\text{var}(aX+b) = E[(aX+b) - \mu_{aX+b}]^2$$

Now,

$$\mu_{aX+b} = E[aX + b] = a\mu + b$$

Therefore,

$$\text{var}(aX + b) = E[(\{aX+b\} - \{a\mu+b\})]^2 = a^2 E[(X - \mu)^2] = a^2\sigma^2.$$

This theorem states that the variance is unchanged if a constant is added to or subtracted from a r.v. The addition or substraction of a constant simply shifts the values of X to the right or to the left but does not change their variability. However, if a r.v. is multiplied or divided by a constant, then the variance is multiplied or divided by the square of the constant.

2.6 Moments and Moment Generating Function

Definition 2.11

The **r**th moment about the origin (zero) or simply the **r**th moment of the r.v. X, denoted $\boldsymbol{\mu_r}$ is the expected value of $\boldsymbol{X^r}$; symbolically,

Clearly we have,

$\boldsymbol{\mu_1 = E[X]} = \boldsymbol{\mu}$, which is just the expected (or mean) value of the r.v. X and

$\sigma^2 =$ **var(X) = E[X²] - μ² = μ₂ - μ²**

Another type of moments which is of special importance in statistics are known as the **central moments**. These moments serve to describe the shape of the graph of its p.m.f. or p.d.f.

Definition 2.12

The **r**th **central moment** (about the mean) of the r.v. X, denoted C_r is the expected value of $(X - \mu)^r$; symbolically,

$$C_r = E[(X-\mu)^r] = \begin{cases} \sum_i (x_i - \mu)^r . f(x_i) & \text{if } X \text{ is disc.} \\ \int_{-\infty}^{\infty} (x-\mu)^r f(x)\,dx & \text{if } X \text{ is cont.} \end{cases} \quad \text{for } r = 1, 2, \ldots\ldots \tag{2.9}$$

Note that $C_1 = 0$ for any r.v. for which μ exists.

The most important measure of variability of a r.v. X is the second central moment C_2, since we have;

$$\sigma^2 = var(X) = E[(X - \mu)^2] = C_2$$

Although the moments of most distributions can be determined directly by evaluating the necessary integrals or sums, there is an alternative procedure which sometimes is more easier. this technique utilizes **Moment Generating Function** (abbreviated by M.G.F.).

Definition 2.13

The M.G.F. of the r.v. X, when it exits, is given by

$$M_X(t) = E[e^{Xt}] = \begin{cases} \sum_i e^{t x_i} . f(x_i) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x)\,dx & \text{if } X \text{ is continuous} \end{cases} \tag{2.10}$$

where t is arbitrary real valued parameter. If the M.G.F. exits for all $|t| \leq T$, say, then all moments exist.

To demonstrate how the moments μ_r of a r.v. X can be generated or determined from the M.G.F. $M_X(t)$, let us substitute for e^{tx} its Maclaurin's expansion, namely,

$$e^{tX} = 1 + tX + \frac{t^2 X^2}{2!} + \frac{t^3 X^3}{3!} + \ldots + \frac{t^r X^r}{r!} + \ldots$$

Hence, using the properties of expectations, we obtain

$$M_X(t) = E[e^{tX}] = 1 + tE[X] + \frac{t^2}{2!}E[X^2] + \frac{t^3}{3!}E[X^3] + \ldots + \frac{t^r}{r!}E[X^r] + \ldots$$

and therefore,

$$M_X(t) = 1 + \mu_1 \frac{t}{1!} + \mu_2 \frac{t^2}{2!} + \mu_3 \frac{t^3}{3!} + \ldots + \mu_r \frac{t^r}{r!} + \ldots$$

Thus, if we are given the form of M(t) for a particular r.v. we may determine the moments up to any order by expanding M(t) as a power series in t and then obtain μ_r , as the coefficient of $\frac{t^r}{r!}$, $r = 1, 2, \ldots$

Example 2.12

Find the M.G.F. of the r.v. X whose p.d.f. is given by

$$f(x) = \begin{cases} e^{-x} & \text{for} \quad x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

and use it to find an expression for μ_r.

Solution.

By definition,

As is well known, when $|t|<1$ the Maclaurin's series for this M.G.F. is

$$M_X(t) = 1 + t + t^2 + t^3 + \ldots + t^r + \ldots$$

$$= 1 + 1!\frac{t}{1!} + 2!\frac{t^2}{2!} + 3!\frac{t^3}{3!} + \ldots + r!\frac{t^r}{r!} + \ldots$$

and hence, $\mu_r = r!$ for $r = 1, 2, 3, \ldots$.

The main difficulty in using the Maclaurin's series of a M.G.F. to determine the moments of a r.v. is usually not that of finding the M.G.F. , but that of expanding it into a Maclaurin's series. Alternatively, if the Maclaurin's series of M(t) is not readily obtained, the moments could be determined by using the following theorem

Theorem 2.9

$$\mu_r = \left.\frac{d^r M_X(t)}{dt^r}\right|_{t=0} , \quad r = 1, 2, 3, \ldots$$

This follows immediately by differentiating the Maclaurin's series of M(t) r times then putting t=0. In particularly, the first two moments are given by

Example 2.13

Given that X has the p.m.f.

$$f(x) = \frac{1}{8}\binom{3}{x} \quad \text{for} \quad x = 0, 1, 2, 3$$

find the M.G.F. of X and use it to find μ_1 and μ_2.

Solution

In accordance with definition 2.13 for discrete r.v.'s,

$$M_X(t) = E[e^{tX}] = \frac{1}{8}.\sum_{x=0}^{3} e^{tx}\binom{3}{x} = \frac{1}{8}(1 + 3e^t + 3e^{2t} + e^{3t}) = \frac{1}{8}(1 + e^t)^3$$

Then by theorem 2.9,

$$\mu_1 = M'_X(0) = \left[\frac{3}{8}(1 + e^t)^2 e^t\right]_{t=0} = \frac{3}{2}$$

and

$$\mu_2 = M''_X(0) = \left[\frac{3}{4}(1 + e^t)e^{2t} + \frac{3}{8}(1 + e^t)^2 e^t\right]_{t=0} = 3$$

Often the work involved in using M.G.F.'s can be simplified by making use of the following theorem.

Theorem 2.10

If a and b are constants, then

$$M_{aX+b}(t) = e^{bt}.M_X(at)$$

In particular, if a=1, then $M_{X\pm b}(t) = e^{\pm bt}.M_X(t)$,

if b = 0, then $M_{aX}(t) = M_X(at)$ **and**

if a = 1/σ and b = -μ/σ, then $M_{\frac{X-\mu}{\sigma}}(t) = e^{-\frac{\mu t}{\sigma}}.M_X\left(\frac{t}{\sigma}\right)$

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EXERCISES

[1] For each of the following, determine whether the given function can serve as the probability mass function of a r.v. with the given range.

a- $f(x) = (x-2)/5$, for x = 1, 2, 3, 4, 5;

b- $f(x) = x^2/30$, for x = 0, 1, 2, 3, 4;

c- $f(x) = 1/5$, for x = 0, 1, 2, 3, 4, 5.

[2] For each of the following, determine the constant c so that the function can serve as the probability mass function of a r.v. with the given range.

a- $f(x) = c$, for x = 1, 2, 3, 4, 5;

b- $f(x) = c\binom{5}{x}$, for x = 0, 1, 2, 3, 4, 5;

c- $f(x) = c^2$, for x = 1, 2, 3,..., n;

d- $f(x) = c(1/4)^x$, for x = 1, 2, 3, ...

[3] For what values of k can $f(x) = (1 - k)\, k^x$ serve as the p.m.f. of a r.v. with countably infinite range x= 0, 1, 2, 3, ...?

[4] If the discrete r.v. X has the CDF,

$$F(x) = \begin{cases} 0 & \text{for} \quad x < -1 \\ 1/4 & \text{for} \ -1 \le x < 1 \\ 1/2 & \text{for} \ \ 1 \le x < 3 \\ 3/4 & \text{for} \ \ 3 \le x < 5 \\ 1 & \text{for} \ \ 5 \le x \end{cases}$$

Find: **a-** $P(-0.4<X<3.6)$ **b-** $P(X=5)$ **c-** the p.m.f. of X.

[5] The p.d.f. of the r.v. X is given by,

$$f(x) = \begin{cases} x & \text{for} \ \ 0 < x < 1 \\ 2 - x & \text{for} \ \ 1 \le x < c \\ 0 & \text{otherwise} \end{cases}$$

Find: **a-** The value of c, **b-** The CDF of X,

c- $P(0.8 < X < 1.2)$, **d-** Mean and variance of X.

[6] The CDF of the r.v. Y is given by

$$F(y) = \begin{cases} 1 - (1+y)e^{-y} & \text{for} \ \ y > 0 \\ 0 & \text{for} \ \ y \le 0 \end{cases}$$

Find $P(Y \le 2)$, $P(1 < Y < 3)$, $P(Y > 4)$ and the p.d.f. of Y.

[7] The p.d.f. of the r.v. X is given by

$$f(x)=\begin{cases} kx & \text{for } 0<x\le 1 \\ k & \text{for } 1<x\le 2 \\ k(3-x) & \text{for } 2<x\le 3 \\ 0 & \text{otherwise} \end{cases}$$

a- Find the value of k and the CDF of **X.**

b- Find E[**X**], var(1-2**X**) and P(0.2 < **X** < 0.8).

c- Find the median M of X.

(Hint: M is defined by P(X > M) = P(X < M) = 0.5)

d- Find the coefficients of skewness and kurtosis of X.

[8] The p.d.f. of the r.v. **X** is given by

$$f(x)=\begin{cases} kx(1-cx^2) & , \ 0\le x\le 1 \\ 0 & , \ \text{o.w.} \end{cases}$$

i- Show that $c < 1$ and that $k = 4/(2-c)$

ii- For the case c=1, find the E[X], var(2X-3) and P(0.2<X<0.8).

[9] Explain why there can be no r.v. for which $M_X(t)=\dfrac{t}{1-t}$

[10] Find the M.G.F. of the discrete r.v. X which has the p.m.f.

$$f(x)=2(1/3)^x \quad \text{for } x = 1,\ 2,\ 3,\ \ldots$$

and use it to determine the mean and variance of X.

[9] Multiple choices: Circle the correct answer from each of the following multiple choice questions:

1-The p.m.f. (probability mass function) of a r.v. X is given by

$$f(x)=k(0.6)^x\ ,\ x=0,1,2,\ldots$$

then the constant k is approximately:

a. 0.2 **b.** 0.4 **c.** 0.6 **d.** None of the above

2- The p.m.f. (probability mass function) of a r.v. X is given by:

f(x) = 1/k, x = -1, 0, 2, 4, 6 then the constant k equals:

a. 5 **b.** 1/ 5 **c.** 1/ 6 **d.** None of the above

3-The following table gives the distribution of a r.v. X ,where α and β are constants,

x	-1	0	1	β
P(X=x)	α	0.3	0.2	0.1

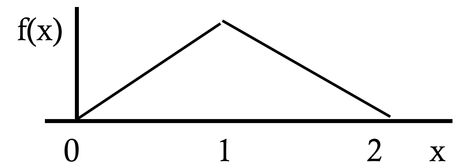
If E(X) = 0, then var(X) is

a. 1.0 **b.** 1.2 **c.** 1.5 **d.** None of the above

4- The probability density function of a r.v. X is given in the figure below, the $P(0.5 \le X \le 1)$ has value;

a- 0.125 **b-** 0.25

c- 0.375 **d-** 0.5



5-The p.m.f. (probability mass function) of a r.v. X is given by

$$f(x) = k(0.6)^x \ , \ x = 0,1,2,...$$

then the constant k is approximately:

a. 0.2 **b. 0.4** **c. 0.6** **d. None of the above**

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