Linear Algebra

Chapter 1: Vectors and Matrices Algebra

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2020-2021

Summary

Euclidean Vector Algebra

The idea of using a number to locate a point on a line was known to the ancient Greeks. In 1637 Descartes extended this idea, using a pair of numbers (a_1, a_2) to locate a point in the plane, and a triple of numbers (a_1, a_2, a_3) to locate a point in space. The 19th Century mathematicians A. Cayley and H. G. Grassmann realized that there is no need to stop with three numbers. One can just as well consider a quadruple of numbers (a_1, a_2, a_3, a_4) or, more generally, an n-tuple of real numbers for any integer $n > 1$. Such an n-tuple is called an ***n*-dimensional point** or an ***n*-dimensional vector** and the set of all n-dimensional vectors are called the **Euclidian space**, the individual numbers $a_1, a_2, \cdots, a_n$ are called **coordinates** or **components** of the vector. The set of all n-dimensional vectors is called ***n*-space** and if components of vectors are real numbers, we denote this space by $\mathbb{R}^n$.

Euclidean Vector Algebra

In this chapter we shall usually denote vectors by capital letters $A, B, C, \cdots$ or bold small letters $\mathbf{a}, \mathbf{b}, \mathbf{c}, \cdots$ and components by the corresponding small letters $a, b, c, \cdots$. Therefore, we write

$$A = (a_1, a_2, \cdots, a_n).$$

Also, A can be written as

$$A = \langle a_1, a_2, \cdots, a_n \rangle \qquad \text{or} \qquad A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

Algebraically, two vectors $A = (a_1, a_2, \cdots, a_n)$ and $B = (b_1, b_2, \cdots, b_n)$ in n-space are called equal if and only if

$$a_1 = b_1, \quad , a_2 = b_2, \quad \cdots, \quad a_n = b_n.$$

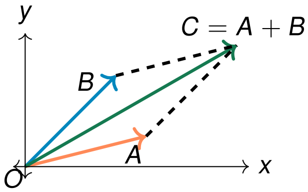
Euclidean Vector Algebra

There are two vector operations called **addition** and **multiplication by scalars**. The sum $A + B$ is defined to be the vector obtained by adding the corresponding components:

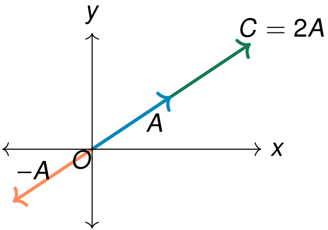
$$A + B = (a_1 + b_1, a_2 + b_2, \cdots, a_n + b_n).$$

If k is a scalar, we define kA or Ak to be the vector obtained by multiplying each component of A by k:

$$kA = (ka_1, ka_2, \dots, ka_n).$$



(a) Vector Addition



(b) Scalar Multiplication

Euclidean Vector Algebra

From this definition it is easy to verify the following properties of these operations:

- Vector addition is commutative,
$$A + B = B + A.$$
- Vector addition is associative,
$$A + (B + C) = (A + B) + C.$$
- Multiplication by scalars is associative,
$$c(dA) = (cd)A.$$
- Multiplication by scalars satisfies the two distributive laws
$$c(A + B) = cA + cB \qquad \text{and} \qquad (c + d)A = cA + dA.$$

Euclidean Vector Algebra

The vector whose all components are zeros is called the **zero vector** and is denoted by 0. It satisfies that $A + 0 = 0 + A = A$ for every vector A; in other words, 0 is the additive identity of vectors. Moreover, the vector denoted by $-A$ and is called the **negative** of A which satisfies $A + (-A) = 0$; it means that $-A$ is the additive inverse of A. The difference of vectors can be defined by the vector addition as $A - B = A + (-B)$.

Definition (Parallel Vectors)

Two vectors A and B in $\mathbb{R}^n$ are said to be **parallel** if $B = kA$ for some scalar k. Then A and B said to have the **same direction** if k is positive and the **opposite direction** if k is negative.

Example

Find the values of a and b which make the vectors $\begin{bmatrix} a \\ -2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ b \end{bmatrix}$ be parallel.

Norms and Dot Products

Definition (The dot product)

If $A = (a_1, a_2, \cdots, a_n)$ and $B = (b_1, b_2, \cdots, b_n)$ are two n-dimensional vectors, their dot product is denoted by $A \cdot B$ is defined as:

$$A \cdot B = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

Obviously, to compute $A \cdot B$ we multiply the corresponding components of A and B and then add all the products. The dot product has the following algebraic properties for all vectors A, B, C and every scalar k:

1. $A \cdot B = B \cdot A$. (commutative law)
2. $A \cdot (B + C) = A \cdot B + A \cdot C$. (distributive law)
3. $k(A \cdot B) = (kA) \cdot B = A \cdot (kB)$. (homogeneity)
4. $A \cdot A > 0$ if $A \neq 0$ and $A \cdot A = 0$ if and only if $A = 0$. (positivity)

Norms and Dot Products

Definition (Length (Norm))

If A is a vector in $\mathbb{R}^n$, its **length** or **norm** is denoted by $||A||$ and is defined as

$$||A|| = \sqrt{A \cdot A}.$$

The definition means that if $A = (a_1, a_2, \cdots, a_n)$ in $\mathbb{R}^n$, then

$$||A|| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} = \left[\sum_{i=1}^{n} a_i^2\right]^{\frac{1}{2}}.$$

Theorem

If A is a vector in $\mathbb{R}^n$ and k is a scalar, we have the following properties:

1. $||A|| > 0$ *if* $A \neq 0$. *and* $||A|| = 0$ *if and only if* $A = 0$. *(positivity)*
2. $||kA|| = |k|\ ||A||$. *(homogeneity)*

Norms and Dot Products

Definition (Orthogonality)

Two vectors A and B in $\mathbb{R}^n$ are called **perpendicular** or **orthogonal** if $A \cdot B = 0$.

Example

Find the value of c making the vectors $A = (c, 1, -3)$ and $B = (c, -2c, 1)$ be orthogonal.

Solution

Indeed A and B are orthogonal. So $A \cdot B = 0$ and

$$(c, 1, -3) \cdot B = (c, -2c, 1).$$

Hence $c^2 - 2c - 3 = 0$ and $(c - 3)(c + 1) = 0$. Then $c = 3$ or $c = -1$.

Norms and Dot Products

Definition (Unit Vector)

A vector A in $\mathbb{R}^n$ such that $||A|| = 1$ is called a **unit vector**.

The concept of unit vectors is very useful. We often want to find a unit vector that has the same direction as a given vector A. One can see that this is the vector

$$\widehat{A} = \frac{A}{||A||}.$$

Example

Find a vector B of length 8 which in the opposite direction of the vector $A = \langle 3, 2, -6 \rangle$.

Norms and Dot Products

In physics and engineering, it is common to use the notation $i = (1, 0, 0)$, $j = (0, 1, 0)$ and $k = (0, 1, 1)$ in $\mathbb{R}^3$ instead. Every vector (a, b, c) in $\mathbb{R}^3$ can be expressed in the form

$$(a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1)$$

and simply

$$(a, b, c) = a\ i + b\ j + c\ k.$$

The two vectors $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0.1)$ which multiply the components a, b and c are called **unit coordinate vectors**. We now introduce the corresponding concept in $\mathbb{R}^n$.

Definition (Unit Coordinate)

In $\mathbb{R}_n$, the n vectors $e_1 = (1, 0, 0, \cdots, 0)$, $e_2 = (0, 1, 0, \cdots, 0)$, $\cdots$, $e_n = (0, 0, 0, \cdots, 1)$ are called the **unit coordinate vectors**. In other words, the k^{th} component of e_k is 1 and all other components are zeros.

Norms and Dot Products

The name ”unit vector”comes from the fact that each vector e_k has length 1. Note that these vectors are mutually orthogonal, i.e.

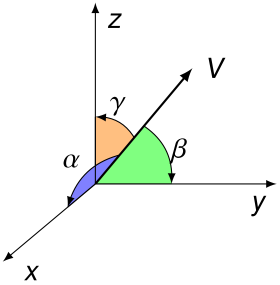
$$e_i \cdot e_j = \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases},$$

where δ_{ij} is called **Kronecker delta**.
Hence every vector $X = (x_1, x_2 < \cdots, x_n)$ in $\mathbb{R}^n$ can be expressed in the form

$$X = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n = \sum_{i=1}^{n} x_i e_i.$$

Moreover, this representation is unique.

The **direction angles** of a nonzero vector are the angles α, β and γ that makes with the positive *x*-, *y*-, and *z*-axes, respectively, as shown in the figure.



The cosines of these direction angles $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called the **direction cosines** of the vector and every ratio of this cosines are called the **direction ratio** of the vector.

A line L in the xy-plane is determined when a point (x_0, y_0) on the line and the direction of the line m (its slope or angle of inclination) are given. The equation of the line can then be written using the point-slope form.

$$L \;:\; y - y_0 = m(x - x_0).$$

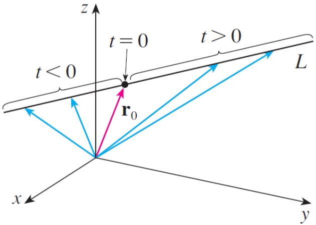
Likewise, a line L in three-dimensional space is determined when we know a point (x_0, y_0, z_0) on and the direction of L. In three dimensions, the direction of a line is a direction ratio of a parallel vector, say a, b, c. The equation of linea in space can be represent by several forms as the following form which is called the **symmetric form**:

$$L \;:\; \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

The equal expressions in the previous can be equated by a free scalar (parameter), say t. So we can get the following:

$$L \; : \; x = x_0 + ta, \qquad y = y_0 + tb, \qquad z = z_0 + tc.$$

where t is arbitrary real number.

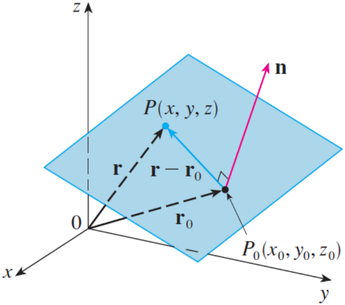


These equations are called **parametric form** of the line L. Shortly, we can write the equation of the line passing through the point (x_0, y_0, z_0); that is $r_0 = \langle x_0, y_0, z_0 \rangle$ as a vector, and parallel to the vector $v = \langle a, b, c \rangle$ as:

$$r = r_0 + tv,$$

where $t \in \mathbb{R}$ and $r = \langle x, y, x, z \rangle$ are the position vector for an arbitrary point on the line L. These form is called the **vector form**.

Although a line in space is determined by a point and a direction, a plane in space is more difficult to describe. A single vector parallel to a plane is not enough to convey the "direction" of the plane, but a vector perpendicular to the plane does completely specify its direction. Thus a plane in space is determined by a point (x_0, y_0, z_0) (or $r_0 = \langle x_0, y_0, z_0 \rangle$) in the plane and a vector $n = \langle a, b, c \rangle$ that is orthogonal to the plane. This orthogonal vector n is called a **normal vector**. Let r be an arbitrary point in the plane, and r_0 be the position vectors of a fixed point on the plane.



As shown in the figure, we have

$$P \ : \ n \cdot (r - r_0) = 0;$$

that is called a **vector equation of the plane**.

Example

Find an equation of the plane through the point $(2, 4, -1)$ with normal vector $n = \langle 2, 3, 4 \rangle$. Find a scaler equation the plane.

Solution

Putting

$$\langle 2, 3, 4 \rangle \cdot \langle x - 2, y - 4, z + 1 \rangle = 0$$

and consequently

$$2(x - 2) + 3(y - 4) + 4(z + 1) = 0,$$

we get

$$2x + 3y + 4z = 12.$$

As we mentioned before, a single vector parallel to a plane is not enough to determine the direction of the plane. But we can determine its direction using two non-parallel vectors such that each one of them is parallel to the plane. If the P passing thorough the point r_0 and parallel to the non-parallel vectors u and v, then its equation is given by:

$$P \ : \ r = r_0 + tu + sv,$$

where $t, s \in \mathbb{R}$ are patenters. Therefore, this form is called the **parametric form** of the plane.

Example

Find the standard form equation of the plane P passing through the point $(1, 2, -3)$ and parallel to the vectors $u = \langle 1, -1, 2 \rangle$ and $v = \langle 1, 0, 1 \rangle$.

Matrix Theory

Definition (Matrix)

A **matrix** is a rectangular array of elements (entries) which is enclosed by brackets arranged in horizontal rows and vertical columns.

The elements of a matrix are almost numbers or functions. A matrix is **real-valued** if its elements are real numbers and **complex-valued** if its element are complex numbers. Matrices are notated by capital latter $A, B, \cdots$ and their elements subscripted small latter; this subscription ij is called the **index** of the element. For example, the element a_{35} of the matrix A is the element lies in the third row and fourth column. Note that the matrix A is also written as $A = [a_{ij}]$ where $a_{ij} = (A)_{ij}$ denotes the general element of A which is in the ith row and jth column.

Matrix Theory

If a matrix has m rows and n columns, then the **size**, or **order**, of the matrix is said to be $m \times n$. Moreover, the matrix of size $1 \times n$, it is said to be a **row matrix** or a **row vector** and the matrix of size $m \times 1$, it is said to be a **column matrix** or a **column vector**. The set of all $m \times n$ matrices with real entries is denoted by $\mathbb{R}^{m \times n}$ and the set of all $m \times n$ matrices with complex entries is denoted by $\mathbb{C}^{m \times n}$.

Example

Consider the matrix

$$Q = \begin{bmatrix} 2 & 3 \\ -1 & 0 \\ 4 & \frac{1}{2} \end{bmatrix}.$$

the order of Q is 3×2 and $q_{11} = 2, q_{12} = 3, q_{21} = -1, q_{22} = 0, q_{31} = 4$ and $q_{32} = \frac{1}{2}$.

Matrix Theory

A matrix in which the number of rows and the number of columns are equal is called a **square matrix**. E.g.

$$\begin{bmatrix} 1 & t \\ 3 & 5 \end{bmatrix}, \qquad \begin{bmatrix} 2 & 0 & 1+i \\ 4 & 5 & -1 \\ 6 & 1 & 2 \end{bmatrix}.$$

A square matrix of order $n \times n$ can be called simply a square matrix of order n. In a square matrix $[a_{ij}]$, the line of entries for which $i = j$, i.e., $a_{11}, a_{22}, a_{33}, \cdots, a_{nn}$ is called the **principal** or **main diagonal** of the matrix and these elements are called **diagonal elements** and their sum is called the **trace** of A, denoted by $tr\,(A)$. For example, in the square matrix

$$A = \begin{bmatrix} 5 & 2 & 3 & -1 \\ 10 & 6 & 6 & 7 \\ 0 & 1 & -10 & 2 \\ 8 & 8 & 8 & 2 \end{bmatrix},$$

the line of elements $[5\ 6\ -10\ 2]$ is the main diagonal A and $tr\,(A) = 3$.

Matrix Theory

A matrix in which every entry is zero is called a **zero matrix** or **null matrix** and is denoted by 0. For example

$$0_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad 0_{4\times 1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

A square matrix $A = [a_{ij}]$ in which $a_{ij} = 0$ for $i > j$ is said to be an **upper triangular** matrix. E.g.

$$\begin{bmatrix} 2 & 5 & 5 \\ 0 & 7 & 3 \\ 0 & 0 & 2 \end{bmatrix}, \qquad \begin{bmatrix} 4 & 5 & 7 & 7 \\ 0 & 0.5 & -1 & 2 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

Similarly, A square matrix A is called an **lower triangular** if $a_{ij} = 0$ for $j > i$. A matrix which is either upper triangular or lower triangular is called simply **triangular**.

Matrix Theory

A square matrix $A = [a_{ij}]$ with $a_{ij} = 0$ for $i \neq j$, in other words all non-diagonal elements are zeros, is called a **diagonal matrix**. Then we can write $A = diag\,(a_{11}, a_{22}, \cdots, a_{nn})$. For example

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{bmatrix} = diag\,(5, 2, 7), \quad \begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = diag\,(7.2, 0, -1).$$

Clearly, every diagonal matrix is both upper and lower triangular matrix. The diagonal matrix whose diagonal elements are all identical is called a **scalar matrix**. For example

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

Matrix Theory

A square matrix $[\delta_{ij}]$ of order n, that is the matrix in which all diagonal elements are ones and all non-diagonal elements are zeros, is called a **unit matrix** or **identity matrix**, notated by I_n or simply I.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

1-Equality of Matrices

Two matrices A and B are equal, denoted by $A = B$, if A and B are of the same type and order such that each entry of A is equal to the corresponding entry of B. Thus, Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same order, say $m \times n$, are equal if $a_{ij} = b_{ij}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

Example

Let $A = \begin{bmatrix} 1 & a \\ b & 0 \end{bmatrix}$ and $B = \begin{bmatrix} c & -5 \\ 4 & d \end{bmatrix}$. If $A = B$, then $a = -5, b = 4, c = 1$ and $d = 0$.

Note that

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}.$$

Algebraic Operations of Matrices

2-Addition and Subtraction of Matrices The sum $A + B$ of two matrices $A = [A_{ij}]$ and $B = [b_{ij}]$ having the same order is allowed and is the matrix having their same order and obtained by adding the corresponding element of A and B. That is

$$A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}].$$

Similarly, the subtraction can be defined as

$$A - B = [a_{ij}] - [b_{ij}] = [a_{ij} - b_{ij}].$$

Algebraic Operations of Matrices

Example

If

$$\begin{bmatrix} 3 & 2 & 1 & -3 \\ 2 & -4 & 3 & 0 \\ 6 & 0 & -1 & 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & -3 & 7 & 6 \\ -4 & -5 & 0 & -2 \\ 2 & 4 & -3 & 5 \end{bmatrix},$$

then

$$A + B = \begin{bmatrix} 3+2 & 2+(-3) & 1+7 & -3+6 \\ 2+(-4) & -4+(-5) & 3+0 & 0+(-2) \\ 6+2 & 0+4 & -1+(-3) & 5+5 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 8 & 3 \\ -2 & -4-9 & 3 & -2 \\ 8 & 4 & -4 & 10 \end{bmatrix}$$

and

$$A - B = \begin{bmatrix} 3-2 & 2-(-3) & 1-7 & -3-6 \\ 2-(-4) & -4-(-5) & 3-0 & 0-(-2) \\ 6-2 & 0-4 & -1-(-3) & 5-5 \end{bmatrix} = \begin{bmatrix} 1 & 5 & -6 & -9 \\ 6 & 1 & 3 & 2 \\ 4 & -4 & 2 & 0 \end{bmatrix}.$$

Algebraic Operations of Matrices

The addition and subtraction satisfy the following properties for all matrices A, B and C of the same order $m \times n$:

P1: $A + B$ is an $m \times n$ matrix. (Closure Addition)

P2: $A + B = B + A$. (Commutative Law)

P3: $A + (B + C) = (A + B) + C$. (Associative Law)

P4: $A + 0 = A$. (Additive identity)

3-Scalar Multiplication

Let $A = [a_{ij}]$ be a matrix of order $m \times n$ and k is a scalar. Then kA is a matrix of the same order of A and is defined by $kA = Ak = [ka_{ij}]$. Simply, kA is obtained by multiplying each entry of A by k.

Example

If $A = \begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -3 \end{bmatrix}$, then :

$$2A = \begin{bmatrix} 8 & -10 & 2 \\ 4 & 0 & -6 \end{bmatrix} \quad \text{and} \quad -3A = \begin{bmatrix} -12 & 15 & -3 \\ -6 & 0 & 9 \end{bmatrix}.$$

Note that the matrix $(-1)A$ can be written simply $-A$.

Algebraic Operations of Matrices

The scalar multiplication satisfies the following properties for all matrices A and B of the same order $m \times n$ and scalars k, k_1 and k_2:

P5: $A + (-A) = 0.$ (Additive Inverse)

P6: kA is a matrix of order $m \times n$. (Closure Scalar Multiplication)

P7: $k_1(k_2A) = (k_1k_2)A.$ (Associative Law)

P8: $(k_1 + k_2)A = k_1A + k_2A.$ (Distributive Law 1)

P9: $k(A + B) = kA + kB.$ (Distributive Law 2)

P10: $kA = 0$ if and only if $k = 0$ or $A = 0$. (Cancelation Law)

P11: $1A = A.$ (Scalar Multiplicative Identity)

Algebraic Operations of Matrices

4-Matrix Multiplication

First, the multiplication of two matrices is not always defined. The multiplication AB of two matrices A and B is allowed only if the number of columns of A and the number of rows of B are equal. For example A is of order $l \times m$ and B is of order $m \times n$.
Secondly, the existence of the multiplication AB does not mean that BA is necessarily exist. Moreover, if both AB and BA are defined, it is not necessary $AB = BA$ generally. Therefore the multiplication of matrices is noncommutative; that is $AB \neq BA$ necessarily.
Thirdly, the product AB of the matrices $A = [a_{ij}]$ of order $l \times m$ and $B = [b_{ij}]$ of order $m \times n$ is defined to be the matrix $C[c_{ij}]$ of order $l \times n$ whose elements are given by

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}, \qquad 1 \leq i \leq l, 1 \leq j \leq n.$$

Algebraic Operations of Matrices

Example

Let $A = \begin{bmatrix} 1 & 4 & -5 \\ 1 & 3 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & 5 \\ 8 & 9 & 0 \\ 1 & 3 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 3 \\ 0 & -1 \\ 1 & 2 \end{bmatrix}$. Therefore AB and BC are not defined while AB, BC, AC and CA are all defined. Although both AC and CA are exist, $AC \neq CA$ since AC is of order 2×2 and $C \times A$ is of order 3×3. Now

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 4 & -5 \\ 1 & 3 & 6 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 8 & 9 & 0 \\ 1 & 3 & 4 \end{bmatrix} \\ &= \begin{bmatrix} (1)(1)+(4)(8)+(-5)(1) & (1)(3)+(4)(9)+(-5)(3) & (1)(5)+(4)(0)+(-5)(4) \\ (1)(1)+(3)(8)+(6)(1) & (1)(3)+(3)(9)+(6)(3) & (1)(5)+(3)(0)+(6)(4) \end{bmatrix} \\ &= \begin{bmatrix} 28 & 24 & -15 \\ 31 & 48 & 29 \end{bmatrix}. \end{aligned}$$

Algebraic Operations of Matrices

Note that the cancelation low is not varified in matrix multiplication i.e. $AB = AC$ does not mean that $B = C$. Indeed the matricies

$$A = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} -3 & -5 \\ 2 & 1 \end{bmatrix}$$

satisfy $AB = AC = \begin{bmatrix} -21 & -21 \\ 7 & 7 \end{bmatrix}$ and yet $B \neq C$.

Also If $AB = 0$ for some matrices A and B, it is not necessary that A or b is zero. For example

$$\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Algebraic Operations of Matrices

The multiplication of matrices satisfies the following properties for all matrices A and B of the same order and scars k, k_1 and k_2:

P12: $A(BC) = (AB)C$. (Associative Law)

P13: $A(B + C) = AB + AC$. (Distributive Law 1)

P14: $(A + B)C = AC + BC$. (Distributive Law 2)

P15: $(k_1 A)(k_2 B) = (k_1 k_2)AB$. (Homogeneity)

P16: $AI = IA = A$. (Multiplicative identity)

Note that the squaring A^2 of a matrix A is considered but it does mean to square each entry of A. It means that $A^2 = AA$ as a multiplication of matrices. So that the power of matrices is defined only for square matrices where the matrix can be multiplied by itself. Generally, for every positive integer n and square matrix A, we have

$$A^n = \underbrace{AA \cdots A}_{n \text{ times}}.$$

Algebraic Operations of Matrices

The powers of matrices satisfy the following properties:

P17: $A^1 = A$.

P18: $A^m A^n = A^{m+n}$, for all positive integers m and n.

P19: $[A^m]^n = A^{mn}$, for all positive integers m and n.

P20: $A^0 = I$.

P21: $I^n = I$, for every positive integer$n$.

5-The Transpose of a Matrix

The matrix obtained from a given matrix A by interchanging rows into columns and columns into rows is called the **transpose** of A and is denoted by A^T. Therefore, if $A = [a_{ij}]$ is a matrix of order $m \times n$, then its transpose $A^T = [b_{ij}]$ is a matrix of order $n \times m$ such that $b_{ij} = a_{ji}$. For example

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 3 \\ -1 & 0 \\ 2 & 5 \end{bmatrix}.$$

Moreover, a matrix A is said to be **symmetric** if $A^T = A$ and **skew-symmetric** if $A^T = -A$.

For example $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & 4 \\ -3 & 4 & 0 \end{bmatrix}$ is symmetric and $A = \begin{bmatrix} 0 & 3 & -2 \\ -3 & 0 & 4 \\ 2 & -4 & 0 \end{bmatrix}$ is skew-symetric.

Moreover, a matrix A is said to be **orthogonal** if $AA^T = A^TA = I$.

Algebraic Operations of Matrices

According to the definition of matrix transposition, we get the following properties for every matrices A and B and scalar k:

P22: $(A \pm B)^T = A^T \pm B^T$. (Linearity 1)

P23: $(kB)^T = kA^T$. (Linearity 2)

P24: $(A^T)^T = A$. (Idempotent Property)

P25: $(AB)^T = B^T A^T$. (Anti-morphic Property)

P26: $(A^T)^n = (A^n)^T$.

P27: $I^T = I$.

P28: $0^T = 0$.

Raw Echelon Forms

We introduce the concept elementary row operations. Recall the shorthand we used:

E_{ij} : The elementary operation of switching the ith and jth rows of the matrix.

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 2 \\ 4 & 5 & 1 \end{bmatrix} \xrightarrow{E_{13}} \begin{bmatrix} 4 & 5 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & -1 \end{bmatrix}$$

$E_i(c)$: The elementary operation of multiplying the ith row by the nonzero scalar c.

$$\begin{bmatrix} -1 & 5 & -1 \\ 0 & 6 & 3 \\ 4 & 7 & 1 \end{bmatrix} \xrightarrow{E_2(-3)} \begin{bmatrix} -1 & 5 & -1 \\ 0 & -18 & -9 \\ 4 & 7 & 1 \end{bmatrix}$$

$E_{ij}(d)$: The elementary operation of adding d times the jth row to the ith row.

$$\begin{bmatrix} 4 & 5 & -3 \\ -2 & 0 & 2 \\ 5 & 10 & -1 \end{bmatrix} \xrightarrow{E_{32}(2)} \begin{bmatrix} 4 & 5 & -3 \\ -2 & 0 & 2 \\ 1 & 10 & 3 \end{bmatrix}$$

Raw Echelon Forms

Row operations can be applied to any matrix. Two matrices A and B are called **row equivalent** if there is a sequence of elementary row operations that transforms one matrix into the other, notated by $A \sim B$.
Recall that the **leading entry** of a row in some matrix is the first nonzero entry of that row (from left) and the zero row is the row whose all entries are zero. So a zero row has no leading entry. The circled entries of the following matrix are its leading elements

$$\begin{bmatrix} 1 & 5 & 7 & 0 & 0 & 8 & -1 \\ 0 & 0 & 0 & 3 & 5 & 5 & 5 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 \end{bmatrix}$$

Raw Echelon Forms

Definition (Row Echelon Form)

A matrix A is said to be in **row echelon form** or simply **echelon form** if:

1. The all nonzero rows of A precede the all zero rows.
2. The number of zeros to the left of each leading element strictly increases.

For example

$$\begin{bmatrix} 2 & 1 & 7 & 0 & 6 & 8 & -1 & -1 \\ 0 & 6 & 0 & 3 & 5 & 5 & 5 & 4 \\ 0 & 0 & 0 & 0 & 7 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Raw Echelon Forms

Here are some examples of some matrices which is in echelon form and other ones are not.

$$\begin{bmatrix} 1 & 5 & 6 \\ 0 & 8 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Echelon Form

$$\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 5 \end{bmatrix}$$

Echelon Form

$$\begin{bmatrix} 1 & 3 \\ 0 & 5 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Echelon Form

$$\begin{bmatrix} 1 & 3 & 6 \\ 0 & 3 & 2 \\ 0 & 1 & 0 \end{bmatrix}$$

Not Echelon Form

$$\begin{bmatrix} 1 & 3 & 6 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

Not Echelon For

$$\begin{bmatrix} 1 & 5 & 6 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Not Echelon Form

$$\begin{bmatrix} 5 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Echelon Form

$$\begin{bmatrix} 0 & 1 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

Not Echelon Form

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Echelon Form

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Echelon Form

Raw Echelon Forms

Definition (Reduced Echelon Form)

A matrix A is said to be in **row reduced echelon form** or simply **reduced echelon form** if it is in echelon form and additionally

1. The leading entry in each nonzero row is 1.
2. Each leading 1 is the only nonzero entry in its column.

For example

$$\begin{bmatrix} 1 & 0 & 7 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 3 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Raw Echelon Forms

Any nonzero matrix may be **row reduced** (that is, transformed by elementary row operations) into more than one matrix in echelon form, using different sequences of row operations. However, the reduced echelon form obtained from a matrix is unique.

If a matrix A is row equivalent to an echelon matrix U, we call U an **echelon form** (or **row echelon form**) of A; if U is in reduced echelon form, we call U the **reduced echelon form** of A.

Example

Find the echelon form of the matrix

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}.$$

Hence find its reduced echelon form.

Raw Echelon Forms

Solution

Applying elementary operations on A, we get

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \xrightarrow{E_{14}} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$\xrightarrow{E_{21}(1)} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$\xrightarrow{E_{31}(2)} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Raw Echelon Forms

Solution

$$\xrightarrow{E_{32}(-5/2),E_{32}(3/2)} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

$$\xrightarrow{E_{34}} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So that the echelon form of A is $\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Raw Echelon Forms

Solution

The previous operations are called the ***forward phase*** *of the row reduction algorithm. We need some additional elementary operations to get the reduced echelon for of A, which is called* ***backward phase*** *of the row reduction algorithm.*

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{E_2(1/2), E_3(-1/5)} \begin{bmatrix} 1 & 0 & -3 & 3 & 5 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{E_{13}(9), E_{13}(3)} \begin{bmatrix} 1 & 0 & 5 & 0 & -7 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Raw Echelon Forms

Definition (Pivot Positions)

A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A (or simply, leading elements in one of its epsilon form). A pivot column is a column of A that contains a pivot position.

Example

Locate the pivot columns of the matrix

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}.$$

Solution

The echelon form of A is

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So that

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Pivot Positions

Pivot Columns

Raw Echelon Forms

Definition (Matrix Rank)

The **rank** of a matrix A, denoted by $rank\,(A)$, is number of its the pivot columns.

The rank of a matrix can be defined by several ways as the number of its pivot positions, the number of nonzero rows in the echelon form and the number of leading elements in the echelon form. Later some equivalent definitions will be stated.

Example

Determine the rank of the matrix

$$A = \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix}$$

Raw Echelon Forms

Solution

Reducing the matrix A to its echelon form

$$A = \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix} \xrightarrow{E_{21}(-2),E_{21}(-3)} \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & -6 & 4 & 14 & -20 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix}$$

$$\xrightarrow{E_{32}(-2),E_{42}(-3)} \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & -10 & 15 \end{bmatrix}$$

$$\xrightarrow{E_{43}(5/2)} \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix Inverse

The concept of division of two matrices or the reciprocal of a matrix is not considered in matrix algebra. But we investigates the matrix analogue of the reciprocal, or multiplicative inverse, of a nonzero number. Recall that the multiplicative inverse of a number such as 5 is $\frac{1}{5}$ or 5^{-1} which satisfies

$$5^{-1} \cdot 5 = 5 \cdot 5^{-1} = 1,$$

where 1 is the multiplicative identity of numbers. The matrices has a multiplicative identity; that is I. Therefore, an $n \times n$ matrix A is said to be **invertible**, if there is an $n \times n$ matrix B such that

$$AB = BA = I_n.$$

Then B is called an **inverse** of A. In fact, B is "the unique inverse of A". This unique inverse is denoted by A^{-1}, so that

$$A^{-1}A = AA^{-1} = I.$$

Note that, $A^{-1} \neq \dfrac{1}{A}$ since the reciprocal of matrices is not considerer but A^{-1} is a notation for the inverse of A. Moreover, a matrix which has no an inverse is called **singular**.

Matrix Inverse

Example

If $A \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ and $B = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$, then

$$\begin{aligned} AB &= \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}\begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \text{and} \\ BA &= \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Thus $B = A^{-1}$. Also $A = B^{-1}$.

Matrix Inverse

The following theorem gives some properties for the inverses of matrices.

Theorem

Let A and B be invertible matrices of same order and nonzero scalar k. Then

1. $(A^{-1})^{-1} = A$.
2. $(AB)^{-1} = B^{1}A^{-1}$.
3. $(kA)^{-1} = \frac{1}{k}A^{-1}$.
4. $(A^{T})^{-1} = (A^{-1})^{T}$.
5. $(A^{n})^{-1} = (A^{-1})^{n}$.

Matrix Inverse

Theorem

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *If* $ad - bc \neq 0$, *then* A *is invertible and*

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, *then* A *is noninvertible.*

Example

Find a 2×2 matrix C such that $A^T CB = I$ where $A = \begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}$.

Matrix Inverse

Solution

Multiplying the equation $A^TCB = I$ *by* B^{-1} *from right, we get*

$$\begin{aligned} A^TC(BB^{-1}) &= IB^{-1} \\ A^TCI &= B^{-1} \\ A^TC &= B^{-1}. \end{aligned}$$

Multiplying the last matrix equation by $(A^T)^{-1}$ *from left, we get*

$$\begin{aligned} C &= (A^T)^{-1}B^{-1} \\ &= (A^{-1})^TB^{-1} \\ &= \frac{-1}{7}\begin{bmatrix} 4 & -1 \\ -3 & -1 \end{bmatrix}^T \frac{1}{4}\begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} \\ &= \frac{-1}{28}\begin{bmatrix} 4 & -3 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} = \frac{-1}{28}\begin{bmatrix} 10 & -2 \\ 1 & 11 \end{bmatrix}. \end{aligned}$$

Matrix Inverse

Theorem

If an $n \times n$ matrix A is a row equivalent to I_n, then it is invertible. Moreover, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

Using the previous theorem, we can get an algorithm for evaluating the inverse of a given matrix which is called **Gauss–Jordan method**.

Row reduce the augmented matrix $[A|I]$. If A is row equivalent to I, then $[A|I]$ is row equivalent to$[I|A^{-1}]$. Otherwise, A is noninvertible.

Matrix Inverse

Example

Find the inverse of the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix},$$

if it exists.

Solution

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right]$$

Matrix Inverse

Solution

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{9}{2} & 7 & -\frac{3}{2} \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{array}\right]$$

Therefore $A \sim I$ implies A is invertible and

$$A^{-1} = \begin{bmatrix} -\frac{9}{2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix}.$$

Matrix Inverse

Example

Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 9 \\ -1 & -1 & 0 \end{bmatrix}$, if it exists.

Solution

$$
\begin{aligned}
[A \mid I] &= \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 9 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 1 \end{array}\right] \\
&\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & -4 & -2 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 0 & 3 & -1 & 1 \end{array}\right]
\end{aligned}
$$

Therefore A and I are not equivalent and A is noninvertible.