Linear Algebra

Chapter 4: Vector Spaces

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Summary

1. Definition of a Vector Space
2. Subspaces
3. Linear Independence
4. Bases and Dimension

Definition of a Vector Space

Definition (Vector Spaces)

Let V be a nonempty set of objects, called **vectors**, on which are defined two operations, called **addition** and **multiplication by scalars**. The set V is called a **vector space** or **linear space** if it satisfies the following ten axioms which we list in three groups. The axioms must hold for all vectors u, v and w in w and for all scalars c and d.

- **Closure Axioms**

  A 1: The sum of u and v, denoted by $\mathsf{u}+\mathsf{v}$, is in V. (Closure under addition)

  A 2: The scalar multiple of u by c, denoted by $c\mathsf{u}$, is in V. (Closure under scalar multiplication)

- **Axioms of addition**

  A 3: $\mathsf{u}+\mathsf{v}=\mathsf{v}+\mathsf{u}$. (Commutative law of addition)

  A 4: $(\mathsf{u}+\mathsf{v})+\mathsf{w}=\mathsf{u}+(\mathsf{v}+\mathsf{w})$. (Associative law of addition)

  A 5: There is a zero vector $\mathsf{0}$ in V such that $\mathsf{u}+\mathsf{0}=\mathsf{u}$. (Existence of zero element (additive identity))

Definition of a Vector Space

Definition

- **A 6:** For each u in V, there is a vector $-$u in V such that $\text{u} + (-\text{u}) = 0$. (Existence of (negative) additive inverse)
- **Axioms for Multiplication by Scalars**

A 7: $c(\text{u} + \text{v}) = c\text{u} + c\text{v}$. (Distributive law for addition)
A 8: $(c + d)\text{u} = c\text{u} + d\text{u}$. (Distributive law for scalar multiplication)
A 9: $c(d\text{u}) = (cd)\text{u}$. (Associative law of scalar multiplication)
A 10: $1\text{u} = \text{u}$. (Existence of identity)

Definition of a Vector Space

Example

Let $V = \mathbb{R}^n$, with addition and multiplication by scalars defined in the usual way in terms of components, is a vector space the vector space and called real n-dimensional space.

Example

The following example are called **function spaces**. Let the elements of V be the real-valued functions (vectors), with addition of two functions f and g defined in the usual way

$$(f+g)(x) = f(x) + g(x)$$

and scalar multiplication of a function f by a real scalar c is defined as

$$(af)(x) = a\ f(x)$$

The zero element is the function whose values are everywhere zero defined as $O(x) = 0$.

Definition of a Vector Space

Example

The set of all polynomials of degree less or equal n for some fixed positive integer n, with usual addition and scalar multiplications of polynomials, is a vector space. Clearly, we consider that the zero polynomial is zero vector of the space. This linear space is called polynomial linear space and denoted by $\mathbb{P}_n(x)$.

Note that the set of all polynomials of degree equal to n is not a linear space because the closure axioms of addition are not satisfied. For example,

$$P(x) = x^2 + 3x - 5 \quad \text{and} \quad Q(x) = -x^2 + 4x + 9$$

are polynomials of degree 2 while $P + Q$ is not of degree 2.

Definition of a Vector Space

Example

The set of all matrices of the same order, say $m \times n$, forms a vector space under the usual addition and scalar multiplication of matrices where the zero matrix of order $m \times n$ is the zero vector of the space (vector as an element of a vector space not a usual vector).

Example

The solutions set of a homogeneous system $Ax = 0$ forms a vector space under the usual addition and scalar multiplication. While the solutions set of nonhomogeneous system can not be a vector space because it we violate the closure axioms.

Definition of a Vector Space

Theorem

In a given linear space, let u and v be arbitrary vectors and c and d be arbitrary scalars. Then we have the following properties:

1. $0\mathrm{u} = 0$.
2. $c0 = 0$.
3. $(-a)\mathrm{u} = -(a\mathrm{u}) = a(-\mathrm{u})$.
4. *If* $a\mathrm{u} = 0$, *then either* $a = 0$ *or* $u = 0$.
5. *If* $a\mathrm{u} = a\mathrm{v}$ *and* $a \neq 0$, *then* $u = v$.
6. *If* $a\mathrm{u} = b\mathrm{u}$ *and* $\mathrm{u} \neq 0$, *then* $a = b$.
7. $-(\mathrm{u} + \mathrm{v}) = (-\mathrm{u}) + (-\mathrm{v}) = -\mathrm{u} - \mathrm{v}$.
8. $\mathrm{u} + \mathrm{u} = 2\mathrm{u}$, $\mathrm{u} + \mathrm{u} + \mathrm{u} = 3\mathrm{u}$, *and* $\underbrace{\mathrm{u} + \mathrm{u} + \cdots + \mathrm{u}}_{n\text{-}times} = n\mathrm{u}$.

Definition of a Vector Space

Example

Let $V = \mathbb{R}^+$, the set of positive real numbers. Define the "addition" of two elements x and y in V to be their product xy (in the usual sense), and define "scaler multiplication" of an element x in V by a scalar c from the field $\mathbb{R}$ to be x^c. Prove that V is a real linear space with 1 as the zero vector.

Solution

Let $x, y, z \in V = \mathbb{R}^+$ and $c, d \in \mathbb{R}$. Denote the defined addition by $\oplus$ and the defied scalar multiplication by $\odot$. Therefore

A 1: $x \oplus y = xy \in \mathbb{R}^+$ *where the product of every pair o positive numbers is positive.*

A 2: $c \odot x = x^c \in R^+$ *even c is negative.*

A 3: $x \oplus y = xy = yx = y \oplus x.$

Definition of a Vector Space

Solution

A 4: $(x \oplus y) \oplus z = (xy) \oplus z = xyz$ *and* $x \oplus (y \oplus z) = x \oplus (yz) = xyz$. *So* $(x \oplus y) \oplus z = x \oplus (y \oplus z)$.

A 5: *Also* $x \oplus 1 = x1 = x$. *So that* 1 *is the zero of the defined addition.*

A 6: *For every* $x \in V$, *we have* $x \oplus \frac{1}{x} = x\,\frac{1}{x} = 1$ *and* 1 *is our zero vector. Moreover* $\frac{1}{x} \in \mathbb{R}^+$.

A 7: *Also*

$$c \odot (x \oplus y) = c \odot (xy) = (xy)^c = x^x y^c = (x^c) \oplus (y^c) = (c \odot x) \oplus (c \odot y).$$

A 8: *Again*

$$(c + d) \odot x = x^{c+d} = x^c\ x^d = (x^c) \oplus (x^d) = (c \odot x) \oplus (d \cdot x).$$

Definition of a Vector Space

Solution

A 9: *Now*

$$c \odot (d \odot x) = c \odot (x^d) = (x^d)^c = x^{cd} = (cd) \odot x.$$

A 10: *Finally*

$$1 \oplus x = x^1 = x.$$

Since all the axioms hold, hence $\mathbb{R}^+$ *is a real vector space under the defined addition and scalar multiplication.*

Definition of a Vector Space

Example

Let S be the set of all ordered pairs (x_1, x_2) of real numbers. Determine whether or not S is a linear space with the usual addition of vectors and scalar multiplication defined as

$$c(x_1, x_2) = (c\ x_1, 0).$$

Solution

Since the set S includes all the vectors of $\mathbb{R}^2$, hence the closure axioms hold. Also S uses the usual addition which implies that the axioms from 3 to 6 hold all. Now, we have to check the last four axioms. For every $\mathrm{x} = (x_1, x_2)$ and $\mathrm{y} = (y_1, y_2)$ and real numbers a and b, we have:

$$a(\mathrm{x} + \mathrm{y}) = a((x_1, x_2) + (y_1, y_2)) = a(x_1 + y_1, x_2 + y_2) = (a(x_1 + y_1), 0) = (ax_1 + ay_1, 0)$$

and

$$a\mathrm{x} + b\mathrm{y} = a(x_1, x_2) + b(y_1, y_2) = (ax_1, 0) + (ay_1, 0) = (ax_1 + ay_1, 0).$$

Definition of a Vector Space

Solution

Therefore $a(\mathrm{x}+\mathrm{y}) = a\mathrm{x} + b\mathrm{y}$ *and Axiom 7 holds. Also*

$$(a+b)\mathrm{x} = (a+b)(x_1, x_2) = ((a+b)x_1, 0) = (ax_1 + bx_1, 0)$$

$$a\mathrm{x} + b\mathrm{x} = a(x_1, x_2) + b(x_1, x_2) = (ax_1, 0) + (bx_1, 0) = (ax_1 + bx_1, 0).$$

So $(a+b)\mathrm{x} = a\mathrm{x} + b\mathrm{x}$ *and Axiom 8 holds. Again*

$$a(b\mathrm{x}) = a(b(x_1, x_2)) = a(bx_1, 0) = (a(bx_1), 0) = (abx_1, 0)$$

$$(ab)\mathrm{x} = (ab)(x_1, x_2) = ((ab)x_1, 0) = (abx, 0).$$

Hence $a(b\mathrm{x}) = (ab)\mathrm{x}$*; it is Axiom 9. But* $1\mathrm{x} = 1(x_1, x_2) = (1x_1, 0) = (x_1, 0) \neq \mathrm{x}$*, which means that Axiom 10 is not satisfied and* S *under the given operations is not a vector space.*

Definition of a Vector Space

Example

Let $V = \mathbb{R}^2$. Determine whether or not V is a vector space with the usual scalar multiplication and addition of vectors defined as

$$(x_1, x_2) + (y_1, y_2) = (x_1 y_1 + x_2 y_2, x_1 y_2 + x_2 y_1).$$

Solution

It is enough to verify the axioms of addition to determiner wether V is a vector space or not. Let $\mathrm{x} = (x_1, x_2)$, $\mathrm{y} = (y_1, y_2)$ *and* $\mathrm{z} = (z_1, z_2)$ *are in* $\mathbb{R}^2$.

A 3: *First*

$$\mathrm{x} + \mathrm{y} = (x_1, x_2) + (y_1, y_2) = (x_1 y_1 + x_2 y_2, x_1 y_2 + x_2 y_1)$$

and

$$\mathrm{y} + \mathrm{x} = (y_1, y_2) + (x_1, x_2) = (y_1 x_1 + y_2 x_2, y_1 x_2 + y_2 x_1).$$

Hence $\mathrm{x} + \mathrm{y} = \mathrm{y} + \mathrm{x}$.

Subspaces

Solution

A 4: *Secondly,*

$$\begin{aligned}
(x+y)+z &= ((x_1,x_2)+(y_1,y_2))+(z_1,z_2) \\
&= (x_1y_1+x_2y_2, x_1y_2+x_2y_1)+(z_1,z_2) \\
&= ((x_1y_1+x_2y_2)z_1+(x_1y_2+x_2y_1)z_2, (x_1y_1+x_2y_2)z_2+(x_1y_2+x_2y_1)z_1) \\
&= (x_1y_1z_1+x_2y_2z_1+x_1y_2z_2+x_2y_1z_2, x_1y_1z_2+x_2y_2z_2+x_1y_2z_1+x_2y_1z_1).
\end{aligned}$$

Similarly,

$$x+(y+z) = (x_1y_1z_1+x_2y_2z_1+x_1y_2z_2+x_2y_1z_2, x_1y_1z_2+x_2y_2z_2+x_1y_2z_1+x_2y_1z_1)$$

and $(x+y)+z = x+(y+z)$.

Subspaces

Solution

A 5: *The defined addition has the zero vector* $\mathbf{0} = (1, 0)$. *Indeed, for every* $\mathbf{x} \in V$ *, we have*

$$\mathbf{x} + \mathbf{0} = (x_1, x_2) + (1, 0) = (x_1(1) + x_2(0), x_1(0) + x_2(1)) = (x_1, x_2) = \mathbf{x}.$$

A 6: *Let* $\mathbf{x} = (1, 1)$ *and assume that* $\mathbf{x} + \mathbf{y} = \mathbf{0}$ *for some* $\mathbf{y} \in V$. *So*

$$(y_1 + y_2, y_1 + y_2) = (1, 0),$$

to get the nonhomogeneous system

$$\begin{aligned} y_1 + y_2 &= 1 \\ y_1 + y_2 &= 0 \end{aligned}$$

is inconsistent and has no solution. It means $\mathbf{x}$ *has no additive inverse which makes Axiom 6 does not hold. Thus* V *is not a vector space.*

Subspaces

Many interesting examples of vector spaces are subsets of a given vector space V that are vector spaces in their own right. For example, the xy-plane in R^3 given by

$$\left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$$

is a subset of $\mathbb{R}^3$. It is also a vector space with the same standard componentwise operations defined on $\mathbb{R}^3$.

Definition (Subspace)

A **subspace** W of a vector space V is a nonempty subset that is itself a vector space with respect to the inherited operations of vector addition and scalar multiplication on V.

Subspaces

The next theorem gives a simple criterion for determining whether or not a subset of a linear space is a subspace.

Theorem

Let S be a nonempty subset of a linear space V. Then S is a subspace if and only if S satisfies the closure axioms.

Example

Let $\mathbb{R}^{2\times 2}$ be the vector space of 2×2 matrices of real entries with the standard operations for addition and scalar multiplication, and let W be the subset of all 2×2 matrices with zero trace. Show that W is a subspace of

Subspaces

Solution

Obviously

$$W = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} | a, b, c \in \mathbb{R} \right\}.$$

Let $w_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{bmatrix}$ *and* $w_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & -a_2 \end{bmatrix}$ *be in* W. *The sum of the two matrices is*

$$w_1 + w_2 = \begin{bmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & -a_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & -(a_2 + a_2) \end{bmatrix} \in W$$

Also, for any scalar k,

$$cw_1 + w_2 = k \begin{bmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{bmatrix} = \begin{bmatrix} ka_1 & kb_1 \\ kc_1 & -ka_1 \end{bmatrix} \in W.$$

Thus, W *is a subspace of* $\mathbb{R}^{2\times 2}$.

Subspaces

Example

Let W be a subset of the vector space $\mathbb{R}^2$ defined as $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} | x, y > 0 \right\}$; that is the set of first quadrant in the plane. Determine wether or not W is a subspace of $\mathbb{R}^3$.

Solution

Let $\mathrm{u} = (x_1, y_1)$ *and* $\mathrm{v} = (x_2, y_2)$ *be in* W. *So that* $x_1, x_2, y_1, y_2 > 0$. *So that*

$$\mathrm{u} + \mathrm{v} = (x_1 + x_2, y_1, y_2) \in W,$$

where both $x_1 + x_2$ *and* $y_+ y_2$ *are positive, and Axiom 1 holds. But*

$$-2\mathrm{u}(x_1, y_1) = (-2x_1, -2y_1) \notin W.$$

Therefore Axiom 2 does not hold and W *is not a subspace of* $\mathbb{R}^2$.

Subspaces

Example

Show that the subset W of the vector space $\mathbb{R}^3$, defined as

$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} | x^3 + y^3 + z^3 \geq 0 \right\}$, is not a subspace of $\mathbb{R}^3$.

Solution

Indeed, Axiom 2 holds since for every vector $\mathrm{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ *and scalar* c, $c\mathrm{u} = \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}$ *which satisfies* $(cx)^3 + (cy)^3 + (cz)^3 = c^3(x^3 + y^3 + z^3) = 0$ *and* $c\mathrm{u} \in W$. *Even so,* W *is not a subspace of* $\mathbb{R}^3$ *where Axiom 1 does not hold. For example,*

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \notin W.$$

Subspaces

Theorem

A nonempty subset W of a vector space V is a subspace of V if and only if for each pair of vectors u and v in W and each scalar c, the vector $\mathrm{u} + c\mathrm{v}$ is in W.

Example

Let

$$S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| 3x - 2y + z = 0 \right\}.$$

Show that S is a subspace of $\mathbb{R}^3$ under the standard componentwise operations.

Subspaces

Solution

Let

$$\mathrm{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \quad \textit{and} \quad \mathrm{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$

in S*. Hence* $3x_1 - 2y_1 + z_1 = 0$ *and* $3x_2 - 2y_2 + z_2 = 0$*. Therefore*

$$\mathrm{u} + c\mathrm{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + c \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + cx_2 \\ y_1 + cy_2 \\ z_1 + cz_2 \end{bmatrix}$$

and

$$3(x_1 + cx_2) - 2(y_1 + cy_2) + (z_1 + cz_2) = (3x_1 - 2y_1 + z_1) + c(3x_2 - 2y_2 + z_2) = 0 + 0 = 0.$$

Therefore $\mathrm{u} + c\mathrm{v} \in W$ *and* W *is a subspace of* $\mathbb{R}^3$.

Subspaces

Theorem

The intersection of any collection of subspaces of a vector space is a subspace of the vector space.

The following example gives an example for the previous theorem and shows that the union of two subspaces does not need to be a subspace.

Example

Let W_1 and W_2 be the subspaces of xy and xz-planes of $\mathbb{R}^3$. So that

$$W_1 = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} | x, y \in \mathbb{R} \right\} \quad \text{and} \quad W_2 = \left\{ \begin{bmatrix} x \\ 0 \\ z \end{bmatrix} | x, z \in \mathbb{R} \right\}.$$

Subspaces

Example

Obviously

$$W_1 \cap W_2 = \left\{ \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} | x \in \mathbb{R} \right\},$$

which represents x-axis and is subspace o $\mathbb{R}^3$. While

$$W_1 \cup W_2 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} | x = 0 \text{ or } z = 0 \text{ and } x \in \mathbb{R} \right\}.$$

This set is not closed under addition since

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix} \notin W_1 \cup W_2.$$

Subspaces

Definition (Linear Combination)

Let $S = \{v_1, v_2, \cdots, v_k\}$ be a nonempty subset of a linear space V. An element x in V of the form

$$\sum_{i=1}^{k} c_i v_i = c_1 v_i + c_2 v_2 + \cdots + c_k v_k.$$

where $c_l, \cdot, c_k$ are scalars, is called a **finite linear combination** of the vectors of S.

Example

Let $V = \mathbb{R}^3$ and $S = \left\{ v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$. The vector $v = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$ is a linear combination of $S = \{v_1, v_2, v_3\}$ as

$$v = 1v_1 + 2v_2 + (-1)v_3$$

Subspaces

Example

Show that the vector $v = (6,3,6)$ is a linear combination of the vectors $S = \{(1,2,1),(1,-1,-2),(1,1,3)\}$

Solution

Assume that

$$(6,3,6) = c_1(1,2,1) + c_2(1,-1,-2) + c_3(1,1,3).$$

Comparing the components, we get the liner system

$$\begin{array}{rcrcrcl} c_1 & + & c_2 & + & c_3 & = & 6 \\ 2c_1 & - & c_2 & + & c_3 & = & 3 \\ c_1 & - & 2c_2 & + & 3c_3 & = & 6 \end{array}$$

Subspaces

Solution

The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 2 & -1 & 1 & 3 \\ 1 & -2 & 3 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -3 & -1 & -9 \\ 0 & -3 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -3 & -1 & -9 \\ 0 & 0 & 3 & 9. \end{bmatrix}$$

The system is consistent with the unique solution $c_1 = 1, c_2 = 2, c_3 = 3$. *Hence,*

$$\mathrm{v} = 1(1,2,1) + 2(1,-1,-2) + 3(1,1,3)$$

and it is a linear combination of S.

Subspaces

Example

Prove that the vector $v = (16, 24, 40)$ is a linear combination of the vectors $S = \{v_1 = (1,1,1), v_2 = (1,2,4), v_3 = (1,3,7)\}$

Solution

Let

$$(16, 24, 40) = c_1(1,1,1) + c_2(1,2,4) + c_3(1,3,7).$$

$$\begin{aligned} c_1 &+ c_2 &+ c_3 &= 16 \\ c_1 &+ 2c_2 &+ 3c_3 &= 24 \\ c_1 &+ 4c_2 &+ 7c_3 &= 40 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 16 \\ 1 & 2 & 3 & 24 \\ 1 & 4 & 7 & 40 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 16 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 16 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Subspaces

Solution

The system of the weights is consistent with infinitely many solutions is given by

$$c_1 = 8 + t, \quad c_2 = 8 - 2t \quad c_3 = t$$

for arbitrary parameter t*. So that* v *is a linear combination of* $S = \{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3\}$*; this linear combination can be in infinitely many ways depending on our choosing for* t*. For example*

$$\begin{aligned} t = 1 \quad &: \quad \mathrm{v} = 9\mathrm{v}_1 + 6\mathrm{v}_2 + 1\mathrm{v}_3 \\ t = 0 \quad &: \quad \mathrm{v} = 8\mathrm{v}_1 + 8\mathrm{v}_2 + 0\mathrm{v}_3 \\ t = -2 \quad &: \quad \mathrm{v} = 6\mathrm{v}_1 + 12\mathrm{v}_2 + (-2)\mathrm{v}_3 \end{aligned}$$

Subspaces

Example

Prove that the vector $v = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ can not be written as a linear combination of the vectors

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \right\}.$$

Subspaces

Solution

Setting

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}.$$

Comparing and forming the augemnted matrix of the obtained sysetem, we get

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 1 & 0 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & -1 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

Indeed $\operatorname{rank}(AB) \neq \operatorname{rank}(A)$ *and the system is inconsistent which means that there are no values for* c_1, c_2, c_3 *which means that* v *is not a linear combination of* S.

Subspaces

Definition (Span of a Set of Vectors)

Let V be a vector space and $S = \{\mathrm{v}_1, \mathrm{v}_2, \cdots, \mathrm{v}_n\}$ be a (finite) set of vectors in V. The **span** of S, denoted by **span** (S), is the set

$$\textbf{span}\,(S) = \{c_1\mathrm{v}_1 + c_2\mathrm{v}_2 + \cdots + c_n\mathrm{v}_n | c_1, c_2, \cdots, c_n \text{ are scalars}\}.$$

Theorem

*If $S = \{\mathrm{v}_1, \mathrm{v}_2, \cdots, \mathrm{v}_n\}$ is a set of vectors in a vector space V, then **span** (S) is a subspace of V.*

Subspaces

Example

Let $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. Show that the span of S is the subspace of $\mathbb{R}^{2\times 2}$ of all symmetric matrices.

Solution

*Recall that a 2×2 matrix is symmetric provided that it has the form $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$. Since any matrix in **span** (S) has the form*

$$a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

*Thus **span** (S) is the collection of all 2×2 symmetric matrices.*

Example

Show that the set of matrices

$$S = \left\{ \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

does not span $\mathbb{R}^{2\times 2}$. Describe **span** (S).

Solution

The equation

$$c_1 \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is equivalent to the system with augmented matrix

Solution

$$\left[\begin{array}{cc|c} -1 & 1 & a \\ 0 & 1 & b \\ 2 & 1 & c \\ 1 & 0 & d \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & 1 & a \\ 0 & 1 & b \\ 0 & 0 & b+c-2a \\ 0 & 0 & a+d-b \end{array}\right].$$

This system is inconsistent if $b+c-2a$ *or* $a+d-b$ *are nonzero. Which means that* S *does not span all matrices of* $\mathbb{R}^{2\times 2}$*. Also*

$$\textbf{span}\,(S) = \left\{ \begin{bmatrix} -a+b & b \\ 2a+b & a \end{bmatrix} | a, b \in \mathbb{R} \right\}$$

Linear Independence

Definition (Linearly Independent and Linearly Dependent)

The set of vectors $S = \{v_1, v_2, \cdots, v_m\}$ in a vector space V is **linearly independent** provided that the only solution to the equation

$$c_1 v_1 + c_2 v_2 + \cdots + c_m v_m = 0$$

is only the trivial solution $c_1 = c_2 = \cdots = c_m = 0$. If the above linear combination has a nontrivial solution, then the set S is called **linearly dependent**.

Note that:

- If a subset T of a set S is dependent, then S itself is dependent. This is logically equivalent to the statement that every subset of an independent set is independent.
- If one vector in S is a scalar multiple of another, then S is dependent.
- If $0 \in S$, then S is dependent.
- The empty set Φ is independent.

Example

Linear Independence

Solution

Assume that

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

to get a homogenous linear system with corresponding coefficient matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Since $\operatorname{rank}(A) = 3$ *and the system has the unique trivial solution. So* $c_1 = c_2 = c_3 = 0$. *Therefore the set* $\{v_1, v_2, v_3\}$ *is linearly independent.*

Linear Independence

Theorem

A set of nonzero vectors is linearly dependent if and only if at least one of the vectors is a linear combination of other vectors in the set.

Theorem

*Let $S = \{\mathrm{x}_1, \mathrm{x}_2, \cdots, \mathrm{x}_k\}$ be an independent set consisting of k vectors of a linear space V. Then every set of $k+1$ vectors in **span** (S) is dependent.*

Theorem

Let $A\mathrm{x} = \mathrm{b}$ be a consistent $m \times n$ linear system. The solution is unique if and only if the column vectors of A are linearly independent.

Bases and Dimension

Definition

A subset S of vectors in a linear space V is called a **basis** for V if

1. S is independent set in V
2. S spans V; $\textbf{span}\,(S) = V$.

The space V is called **finite-dimensional** if it has a finite basis, or if V consists of 0 alone. Otherwise, V is called **infinite-dimensional**.

As an example, the set of coordinate vectors

$$S = \{\mathrm{e}_1, \mathrm{e}_2, \cdots, \mathrm{e}_n\}$$

spans $\mathbb{R}^n$ and is linearly independent, so that S is a basis for $\mathbb{R}^n$. This particular basis is called the **standard basis** for $\mathbb{R}^n$ but it is the only basis as shown in the next example.

Bases and Dimension

Example

Show that the set

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

is a basis for $\mathbb{R}^3$.

Solution

First, to show that B spans $\mathbb{R}^3$, we must show that the equation

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Bases and Dimension

Solution

has a solution for every choice of a, b and c in $\mathbb{R}$. *The corresponding augmented matrix is*

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 1 & 1 & 1 & b \\ 0 & 1 & -1 & c \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & -1 & c \\ 0 & 0 & 1 & b-a \end{array}\right]$$

which is consistent with a unique solution for every a, b and c. Therefore B span $\mathbb{R}^3$. *Also the equation*

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = 0$$

has the unique trivial solution $c_1 = c_2 = c_3 = 0$ *which means that B is linearly independent. So* b *is a basis for* $\mathbb{R}^3$

Bases and Dimension

Example

Is the set $S = \{(1,2,2,3),(1,2,2,5),(2,4,4,8),(2,4,4,6),(1,3,2,5)\}$ a basis of $\mathbb{R}^4$?

Solution

Let (x,y,z,u) be an arbitrary vector in $\mathbb{R}^4$. Setting

$$(x,y,z,u) = c_1(1,2,2,3) + c_2(1,2,2,5) + c_3(2,4,4,8) + c_4(2,4,4,6) + c_5(1,3,2,5),$$

we get the system with the augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 1 & 2 & 2 & 1 & x \\ 2 & 2 & 4 & 4 & 3 & y \\ 2 & 2 & 4 & 4 & 2 & z \\ 3 & 5 & 8 & 6 & 5 & u \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 1 & 2 & 2 & 1 & x \\ 0 & 1 & 1 & 0 & 1 & \frac{1}{2} - \frac{3}{2}x \\ 0 & 0 & 0 & 0 & 1 & y - 2x \\ 0 & 0 & 0 & 0 & 0 & z - 2x \end{array}\right]$$

Hence, the system is inconsistent if $z \neq 2x$. Therefore, S does not span $\mathbb{R}^4$ and therefore it can not be a basis of $\mathbb{R}^4$.

Bases and Dimension

Theorem

Let V be a finite-dimensional linear space. Then every basis for V has the same number of elements.

According the previous theorem, we can get the following definition.

Definition (Dimension of a Vector Space)

The **dimension** of the vector space V, denoted by $\dim(V)$, is the number of vectors in any basis of V.

If $V = \{0\}$, which called the **zero vector space**, we say V has dimension 0.

Bases and Dimension

Example

The space $\mathbb{R}^n$ has dimension n. One basis is the set of n unit coordinate vectors.

$$\{\mathrm{e}_1, \mathrm{e}_2, \cdots, \mathrm{e}_n\}.$$

Example

The space $\mathbb{R}^{m\times n}$ has dimension mn. One basis is the set of matrices. $\{\mathrm{e}_{ij} | 1 \leq i \leq m, 1 \leq j \leq n\}$, where e_{ij} is the matrix in which the entry in (i, j)-position is one and zero otherwise.

Example

The space $\mathbb{P}_n(x)$ has dimension $n+1$. One basis is the set of $n+1$ polynomials

$$\{1, x, x^2, \cdots x^n\}.$$

Bases and Dimension

Theorem

Suppose that V is a vector space with $\dim(V) = n$.

1. *If $S = \{v_1, v_2, \cdots, v_n\}$ is linearly independent, then **span** $(S) = V$ and S is a basis.*
2. *If $S = \{v_1, v_2, \cdots, v_n\}$ and **span** $(S) = V$, then S is linearly independent and S is a basis.*

Example

Determine whether

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for $\mathbb{R}^3$.

Bases and Dimension

Solution

Since $\dim(\mathbb{R}^3) = 3$*, it is enough to show that the set* B *is a basis is linearly independent. Let*

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

be the matrix whose column vectors are the vectors of B*. The determinant of this matrix is* 1*, so that the homogeneous system* $A\mathrm{x} = \mathrm{b}$ *for every vector* b *has a unique solution and therefore the set* B *is linearly independent, by Theorem 58. Thus* B *is a basis of* $\mathbb{R}^3$*.*

Given a set $S = \{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3, \cdots, \mathrm{v}_n\}$ to find a basis for **span** (S):

1. Form a matrix A whose column vectors are $\mathrm{v}_1, \mathrm{v}_2, \cdots, \mathrm{v}_n$.
2. Find an row echelon form for A.
3. The pivot columns of A are a basis for **span** (S).

Example

Let

$$S = \left\{ \begin{bmatrix} 2 \\ 4 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 9 \\ -9 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ -5 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 8 \\ 9 \\ 4 \\ -6 \end{bmatrix} \right\}.$$

Find a basis for the span of S.

Solution

Start by constructing the matrix whose column vectors are the vectors in S. We reduce the matrix

$$A = \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix} \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Observe that the pivoting positions are in columns 1, 2, and 4. Therefore, a basis B for $\textbf{span}(S)$ *is given by*

$$B = \left\{ \begin{bmatrix} 2 \\ 4 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 9 \\ -9 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 2 \\ 5 \end{bmatrix} \right\}.$$

Bases and Dimension

Example

Find a basis for $\mathbb{R}^4$ that contains the vectors

$$\mathrm{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathrm{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

Solution

Notice that the set $\{\mathrm{v}_1, \mathrm{v}_2\}$ *shall be linearly independent. However, it cannot span* $\mathbb{R}^4$ *since* $\dim(R^4) = 4$. *To find a basis, form the set* $S = \{\mathrm{v}_1, \mathrm{v}_2, \mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3, \mathrm{e}_4\}$. *Since* ***span*** $\{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3, \mathrm{e}_4\} = \mathbb{R}^4$, *we know that* ***span*** $(S) = \mathbb{R}^4$.
As in previous example, we can get a basis from the pivot columns of the matrix.

Solution

$$\begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Trivially, this matrix has rank 4 to get four pivot columns. We've intentionally set the vectors v_1 *and* v_2 *in the left of the standard unit vectors to get them in the basis.*

$$\begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Observe that the pivot columns are 1, 2, 3, and 6. A basis is therefore given by the set of vectors $\{v_1, v_2, e_1, e_4\}$*.*

Bases and Dimension

Theorem

If W is a strictly subspace of a finite-dimensional linear space V, then $\dim(W) < \dim(V)$.

Definition (Null, Row and Column Spaces)

Let A be an $m \times n$ matrix.

1. The **null space** of A is the set of all vectors in $\mathbb{R}^n$ such that $A\mathrm{x} = 0$, , denoted by $\textbf{null}\,(A)$ and its dimension is called the **nullity**.
2. The **row space** of A, denoted by $\textbf{col}\,(A)$, is the set of all linear combinations (span) of the row vectors of A, denoted by $\textbf{row}\,(A)$ and its dimension is called the **row rank**, denoted by $\text{rank}\,(A)$; that is the same in Chapter 1.
3. The **column space** of A, denoted by $\textbf{col}\,(A)$, is the set of all linear combinations (span) of the column vectors of A, denoted by $\text{col}(A)$ and its dimension is called the **column rank**, denoted by $\textbf{col}\ \text{rank}\,(A)$,.

Bases and Dimension

Example

Find a basis for the column and row spaces of the matrix.

$$A = \begin{bmatrix} 3 & 4 & -1 & -6 \\ 2 & 3 & 2 & -3 \\ 2 & 1 & -14 & -9 \\ 1 & 3 & 13 & 3 \end{bmatrix}$$

Bases and Dimension

Solution

We have

$$
\begin{aligned}
A &= \begin{bmatrix} 3 & 4 & -1 & -6 \\ 2 & 3 & 2 & -3 \\ 2 & 1 & -14 & -9 \\ 1 & 3 & 13 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 13 & 3 \\ 2 & 3 & 2 & -3 \\ 2 & 1 & -14 & -9 \\ 3 & 4 & -1 & -6 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & -3 & -24 & -9 \\ 0 & -5 & -40 & -15 \\ 0 & -5 & -40 & -15 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & -5 & -40 & -15 \\ 0 & -5 & -40 & -15 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

Bases and Dimension

Solution

Therefore,

$$\textbf{\textit{row}}\,(A) = \textbf{\textit{span}} \left\{ \begin{bmatrix} 1 \\ 3 \\ 13 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 8 \\ 3 \end{bmatrix} \right\}$$

and

$$\textbf{\textit{col}}\,(A) = \textbf{\textit{span}} \left\{ \begin{bmatrix} 3 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 1 \\ 3 \end{bmatrix} \right\}$$

Therefore the dimension of the column a row space of A is 2; that is $\textbf{\textit{col}}\ \mathrm{rank}\,(A) = 2$.

Bases and Dimension

Example

Find the null and row spaces of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix}.$$

Solution

We have

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

So The row space is of the basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ *and its dimension is* 2.

Bases and Dimension

Solution

Also the system $A\mathrm{x} = 0$ *can be reduced to*

$$\begin{aligned} x_1 + x_2 + x + 3 &= 0 \\ x_2 + 2x_3 &= 0 \end{aligned}$$

and

$$\mathrm{x} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \qquad \mathit{and}\ t \in \mathbb{R}.$$

So that

$$\textbf{\textit{null}}\,(A) = \textbf{\textit{span}} \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

Thus the nullity of A equals one.

Bases and Dimension

Theorem (Row and Column Ranks)

The row rank and the column rank of a matrix A are equal.

Theorem

For a matrix A of order $m \times n$, we have

$$\text{column rank}(A) + \text{nullity}(A) = n.$$

Theorem (Invertibility of a Matrix)

Let A be an $n \times n$ matrix. Then the following statements are equivalent.

1. *The matrix A is invertible.*
2. *The linear system $Ax = b$ has a unique solution for every vector b.*

Bases and Dimension

Theorem (Invertibility of a Matrix)

3. *The homogeneous linear system $Ax = 0$ has only the trivial solution.*
4. *The matrix A is row equivalent to the identity matrix.*
5. *The determinant of the matrix A is nonzero.*
6. *The column vectors of A are linearly independent.*
7. *The column vectors of A span $\mathbb{R}^n$.*
8. *The column vectors of A are a basis for $\mathbb{R}^n$.*
9. $\text{rank}\,(A) = n$.
10. $\mathrm{row}(A) = \mathrm{col}(A) = \mathbb{R}^n$
11. $\textbf{\textit{null}}\,(A) = \{0\}$.
12. *The number of pivot columns of the reduced row echelon form of A is n.*