Linear Algebra

Chapter 5: Eigenvalues and Eigenvectors

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Summary

1. Eigenvalues and Eigenvectors of Matrices
2. Diagonalization
3. The Cayley-Hamilton theorem

Eigenvalues and Eigenvectors of Matrices

Definition (Eigenvalue and Eigenvector of Matrices)

Let A be as quare matrix of order n. A scalar λ is called an **eigenvalue** of A if there is a nonzero vector x in S such that

$$Ax = \lambda x.$$

The element x is called an **eigenvector** of A corresponding to λ. The scalar λ is called an eigenvalue corresponding to x.

The zero vector is a trivial solution to the eigenvalue equation for any number λ and is not considered as an eigenvector. As an illustration, let

$$A \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

Eigenvalues and Eigenvectors of Matrices

Observe that

$$A \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

So $\mathrm{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 1$. We also have

$$A \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

So $\mathrm{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is another eigenvector of A corresponding to the eigenvalue $\lambda_2 = -1$.

Eigenvalues and Eigenvectors of Matrices

Let's show how to find eigenvalues and eigenvectors for a matrix A. If A has an eigenvalue λ with corresponding eigenvector x, then

$$A\mathrm{x} = \lambda \mathrm{x}$$

which can be written as

$$(\lambda I - A)\mathrm{x} = 0.$$

The previous equation represents a homogeneous equation in x its coefficient matrix $A - \lambda I$. Indeed, we want a nonzero solution for this system. So the system shall by consistent with infinitely many solutions. So that

$$\det(\lambda I - A) = 0$$

So we can state the next definition.

Eigenvalues and Eigenvectors of Matrices

Definition

If A is an $n \times n$ matrix the determinant

$$f(\lambda) = \det(\lambda I - A)$$

is called the **characteristic polynomial** of A.

Example

Calculate the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -7 & 1 \end{bmatrix}$$

Also, compute the dimension of the eigenspace $E(\lambda)$ for each eigenvalue λ.

Solution

Setting $\det(\lambda I - A) = 0$, *we get*

$$0 = \begin{vmatrix} \lambda - 1 & 0 & 0 \\ 3 & \lambda - 1 & 0 \\ -4 & 7 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^3$$

So that, the only eigenvalue is 1. *The corresponding eigenvector of* 1 *is obtained by solving the system:*

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

We get $x_1 = x_2 = 0$ *and* x_3 *is arbitrary. So that, the eigenvector is* $t(0, 0, 1)$ *where* t *is a nonzero scalar.*

Eigenvalues and Eigenvectors of Matrices

Solution

Eigenvalue	*Eigenvector*	$\dim(E(\lambda))$
1	$t(0,0,1)),\ t \neq 0$	1

Eigenvalues and Eigenvectors of Matrices

Example

Calculate the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 37 & 201 \end{bmatrix}$$

Also, compute the dimension of the eigenspace $E(\lambda)$ for each eigenvalue λ.

Solution

Set $\det(\lambda I - A) = 0$, *then*

$$0 = \begin{vmatrix} \lambda - 2 & -1 & -3 \\ -1 & \lambda - 2 & -3 \\ -3 & -3 & \lambda - 20 \end{vmatrix} = \lambda^3 - 24\lambda^2 + 65\lambda - 42 = (\lambda - 1)(\lambda - 2)(\lambda - 21)$$

Eigenvalues and Eigenvectors of Matrices

Solution

So that, the eigenvalues are 1, 2 *and* 21. *The corresponding eigenvector of* 1 *is obtained by solving the system:*

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

We get

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 0 \\ x_1 + x_2 + 3x_3 &= 0 \\ 3x_1 + 3x_2 + 19x_3 &= 0. \end{aligned}$$

By subtracting the last equations from the second after multiplying by 3, *we get* $x_3 = 0$ *and hence* $x_1 + x_2 = 0$. *So that, the eigenvector is* $t(-1, 1, 0)$ *where* t *is a nonzero scalar.*

Eigenvalues and Eigenvectors of Matrices

Solution

The corresponding eigenvector of 2 *is obtained by solving the system:*

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}$$

We get

$$\begin{aligned} x_2 + 3x_3 &= 0 \\ x_1 + 3x_3 &= 0 \\ 3x_1 + 3x_2 + 18x_3 &= 0. \end{aligned}$$

By subtracting the first two equations, we get $x_1 = x_2$ *and hence* $x_1 = -3x_3$*. So that, the eigenvector is* $t(-3, 3, 1)$ *where* t *is a nonzero scalar.*

Eigenvalues and Eigenvectors of Matrices

Solution

The corresponding eigenvector of 2 *is obtained by solving the system:*

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 21x_1 \\ 21x_2 \\ 21x_3 \end{bmatrix}$$

We get

$$\begin{aligned} -19x_1 + x_2 + 3x_3 &= 0 \\ x_1 - 19x_2 + 3x_3 &= 0 \\ 3x_1 + 3x_2 - x_3 &= 0. \end{aligned}$$

By subtracting the first two equations, we get $x_1 = x_2$ *and form the third one we have* $x_3 = 6x_1$. *So that, the eigenvector is* $t(1, 1, 6)$ *where* t *is a nonzero scalar.*

Eigenvalues and Eigenvectors of Matrices

Solution

By the way, we can use the matrices to solve the homogenous linear system as:

$$\begin{bmatrix} -19 & 1 & 3 \\ 1 & -19 & 3 \\ 3 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -19 & 3 \\ -19 & 1 & 3 \\ 3 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -19 & 3 \\ 0 & -360 & 60 \\ 0 & 60 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & -19 & 3 \\ 0 & -60 & 10 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigenvalue	*Eigenvector*	$\dim(E(\lambda))$
1	$t(-1,1,0), t \neq 0$	1
2	$t(-3,-3,1), t \neq 0$	1
21	$t(1,1,6), t \neq 0$	1

Eigenvalues and Eigenvectors of Matrices

Example

Calculate the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & 4 \end{bmatrix}$$

Also, compute the dimension of the eigenspace $E(\lambda)$ for each eigenvalue λ.

Solution

Set $\det(\lambda I - A) = 0$, *then*

$$0 = \begin{vmatrix} \lambda - 5 & 6 & 6 \\ 1 & \lambda - 4 & -2 \\ -3 & 6 & \lambda 4 \end{vmatrix} = \lambda^3 - 5\lambda^2 + 8\lambda - 4 = (\lambda - 1)(\lambda - 2)(\lambda - 2)$$

Eigenvalues and Eigenvectors of Matrices

Theorem

The eigenvalues of an $n \times n$ triangular matrix are the diagonal elements.

Theorem

Let A be an $n \times n$ matrix, and let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be distinct eigenvalues with corresponding eigenvectors $\mathrm{v}_1, \mathrm{v}_2, \cdots, \mathrm{v}_n$. Then the set $\{\mathrm{v}_1, \mathrm{v}_2, \cdots, \mathrm{v}_n\}$ is linearly independent.

Diagonalization

In this section, we determine if a matrix A has a factorization of the form

$$A = PDP^{-1}$$

where P is an invertible matrix and D is a diagonal matrix.

Definition (Similar Matrices)

Let A and B be $n \times n$ matrices. We say that A is similar to B if there is an invertible matrix P such that $B = P^{-1}AP$.

This relation is symmetric; that is, if the matrix A is similar to the matrix B, then B is similar to A. Indeed if A is similar to B, then there is an invertible matrix P such that

$$B = P^{-1}AP.$$

Now if $Q = P^{-1}$, so that A can be written as

$$A = Q^{-1}AQ$$

Definition (Diagonalization)

Diagonalization

Example

Let

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

The inverse of P is given by

$$P^1 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Indeed,

$$\begin{bmatrix} -1 & 0 & 0 \end{bmatrix}$$

Diagonalization

Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. Moreover, if $D = P^{-1}AP$, with D a diagonal matrix, then the diagonal entries of D are the eigenvalues of A and the column vectors of P are the corresponding eigenvectors.

Example

Diagonalizae the matrix

$$\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & 1 & -2 \end{bmatrix}$$

Hence calculate A^4.

Example

Determine wether the matrix

$$A = b \begin{array}{ccc} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{array}$$

s diagonalizable or not. If it is so, find the diagonalization.

Solution

First we have to get the eigenvalues and eigenvectors of A. So

Eigenvalue	*Eigenvector*	$\dim(E(\lambda))$
1	$t(1,-1,0), t \neq 0$	1
-1	$t(0,1,-1), t \neq 0$	1
3	$t(2,3,-1), t \neq 0$	1

Therefore A has the linearly independent eigenvectors

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \right\}$$

Thus A is diagonalizable and $P^{-1}AP = D$ *where*

$$P = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 3 \\ 0 & -1 & -1 \end{bmatrix} \qquad and\ D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Solution

The matrix A has the following eigenvalues and eigenvectors

Eigenvalue	Eigenvector	$\dim(E(\lambda))$
7	$t(1,2,3),\ t \neq 0$	1
$1, 1$	$t(1,0,-1) + s(0,1,-1)),\ t,s \neq 0$	2

Therefore A has the linearly independent eigenvectors

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

Thus A is diagonalizable and $P^{-1}AP = D$ where

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad and\ D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Diagonalization

Solution

First we have to get the eigenvalues and eigenvectors of A. So

Eigenvalue	*Eigenvector*	$\dim(E(\lambda))$
$2, 2$	$t(-1, 1, 1)), t \neq 0$	1
4	$t(1, -1, 1)), t \neq 0$	1

which means that A has no three linearly independent eigenvectors and therefore it is non-diagonalizable.

The Cayley-Hamilton theorem

Theorem

Let A be an $n \times n$ matrix and let

$$f(\lambda) = \det(\lambda I - A) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0,$$

be its characteristic polynomial. Then $f(A) = 0$. In other words, every square matrix A satisfies its caharacterstic eaution $f(\lambda) = 0$.

Example

Verify the truth of Cayley-Hamilton theorem for the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix},$$

hence use it to evaluate A^{-1} and $\operatorname{adj}(A)$.

The Cayley-Hamilton theorem

Solution

The characteristic equation is given by

$$\det(\lambda I - A) = 0$$

and

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

Cayley-Hamilton states that

$$A^3 - 6A^2 + 9A - 4I = 0$$

Indeed

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -9 & 5 \\ -9 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 6 & -9 & 5 \end{bmatrix} = \begin{bmatrix} 26 & -29 & 21 \end{bmatrix}$$