Interpolation and polynomial approximation

Theorem: (Weierstrass approximation theorem)

If f(x) is defined and continuous on [a, b] , then there exists a polynomial p(x) defined on [a, b], with the property that

$$|f(x)-P(x)|<\varepsilon \quad \forall x\in[a,b],\ \varepsilon>0$$

One of the most useful and well-known classes of functions mapping the set of R into R is the class of algebraic polynomials

$$p(x)=a_0+a_1x+a_2x^2+.....+a_nx^n$$

Example: According the following table, Find f(4) by using quadratic and cubic interpolation respectively

X	1	3	5	6
F(x)	0	4	12	19

let $p(x) = a_0 + a_1 x + a_2 x^2$

Then
$$P(1) = f(1) = 0 = a_0 + a_1 + a_2$$
$$P(3) = f(3) = 4 = a_0 + 3a_1 + 9a_2$$
$$P(5) = f(5) = 12 = a_0 + 5a_1 + 25a_2$$

Using Gauss elimination we have $a_0 = -0.5, \quad a_1 = 0, \quad a_2 = 0.5$

Hence $P(x) = -0.5 + 0.5x^2 \Rightarrow f(4) = P(4) = 7.5$

Secondly, let $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

Then

$$P(1) = f(1) = 0 = a_0 + a_1 + a_2 + a_3$$
$$P(3) = f(3) = 4 = a_0 + 3a_1 + 9a_2 + 27a_3$$
$$P(5) = f(5) = 12 = a_0 + 5a_1 + 25a_2 + 125a_3$$
$$P(6) = f(6) = 19 = a_0 + 6a_1 + 36a_2 + 216a_3$$

Using Gauss elimination we have $a_0 = -2, \quad a_1 = 2.3, \quad a_2 = -\frac{2}{5}, \quad a_3 = 1.5$

Hence $P(x) = -2 + 2.3x - 0.4x^2 + 1.5x^3$

Lagrange interpolating polynomial

Theorem: If $x_0, x_1, \dots, x_n$ are (n+1) distinct numbers and f(x) is the function whose values are given at these numbers, then there exists a unique polynomial P(x) of degree at most n with the property that $f(x_k) = P(x_k), \ \forall k = 0, 1, \dots, n$

This polynomial is given by

$$P(x) = L_0(x) f(x_0) + L_1(x) f(x_1) + \dots + L_n(x) f(x_n)$$
$$= \sum_{k=0}^{n} L_k(x) f(x_k)$$

where

$$L_k(x) = \frac{(x - x_0)(x - x_1)\dots(x - x_{k-1})(x - x_{k+1})\dots(x - x_n)}{(x_k - x_0)(x_k - x_1)\dots(x_k - x_{k-1})(x_k - x_{k+1})\dots(x_k - x_n)}$$
$$= \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)}$$

Example: Using the following table and Lagrange polynomial to find f(3)

x	2	2.5	4
F(x)	0.5	0.4	0.25

Solution:

$$L_0(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}=\frac{(x-2.5)(x-4)}{(2-2.5)(2-4)} \Rightarrow L_0(3)=-0.5$$

$$L_1(x)=\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}=\frac{(x-2)(x-4)}{(2.5-2)(2.5-4)} \Rightarrow L_1(3)=\frac{7}{6}$$

$$L_2(x)=\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}=\frac{(x-2)(x-2.5)}{(4-2)(4-2.5)} \Rightarrow L_2(3)=0.1667$$

$$P(x)=L_0(x)f(x_0)+L_1(x)f(x_1)+L_2(x)f(x_2)$$

$$=-0.5(0.5)+\frac{7}{6}(0.4)+0.1667(0.25)=0.325$$

Exercises: Using Lagrange interpolating polynomials to approximate the following

1] Find f(2.5)

x	2	2.2	2.4	2.6	2.8
F(x)	0.52	0.63	0.85	1.1	1.3

2] find cosh (1.1)

x	1	1.2	1.3	1.4	1.5
Cosh(x)	1.543	1.811	1.971	2.151	2.352

3] find $e^{2.2}$

x	1.7	1.9	2	2.1	2.3
e^x	5.474	6.686	7.389	8.166	9.974

Divided Differences Method

Methods for determining the explicit representation of an interpolating polynomial from tabulated data are known as divided-differences method. The divided-differences of f(x) with respect to x_0, x_1,…., x_n are divided by showing P(x) as follows

$$P(x)=a_0+a_1(x-x_0)+a_2(x-x_0)(x-x_1)+\ldots+a_n(x-x_0)\ldots.(x-x_{n-1})$$

For appropriate constants a_0, a,…., a_n

At x=x_0 in P(x), we have

$$a_0=P(x_0)=f(x_0)=f[x_0]$$

Similarly, at x=x_1 in P(x), we have

$$f(x_1)=P(x_1)=a_0+a_1(x_1-x_0)$$

$$f(x_1)=f(x_0)+a_1(x_1-x_0)$$

$$\therefore a_1=\frac{f(x_1)-f(x_0)}{(x_1-x_0)}=f[x_0,x_1]$$

Similarly, at x=x_2 in P(x), we have

$$f(x_2) = P(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$
$$f(x_2) = f(x_0) + f[x_0, x_1](x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$
$$\therefore a_2 = \frac{f[x_1, x_2] - f[x_0, x_1]}{(x_2 - x_0)} = f[x_0, x_1, x_2]$$

In general we have

$$P(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2]\,(x - x_0)(x - x_1) + ... + f[x_0, x_1, x_2, ..., x_n]\,(x - x_0)....(x - x_{n-1}) \quad (1)$$

This equation is Newton's interpolating divided difference formula

x_0	$F(x_0)$				
		$F[x_0 , x_1]$			
x_1	$F(x_1)$		$F[x_0 , x_1, x_2]$		
		$F[x_1 , x_2]$		$F[x_0 , x_1, x_2, , x_3]$	
x_2	$F(x_2)$		$F[x_1 , x_2, x_3]$		
		$F[x_2 , x_3]$			
x_3	$F(x_3)$				

Example: Using the following table to find f(1.5) by using Newton's divided difference formula

x	1	1.3	1.7	1.9
F(x)	2.56	3.42	5.76	6.88

Solution:

1	2.56			
		$\frac{43}{15}$		
1.3	3.42		$\frac{179}{42}$	
		5.85		$\frac{-655}{126}$
1.7	5.76		$\frac{-5}{12}$	
		5.6		
1.9	6.88			

$$P(x)=2.56+\frac{43}{15}(x-1)+\frac{179}{42}(x-1)(x-1.3)-\frac{655}{126}(x-1)(x-1.3)(x-1.7)$$

$$f(1.5)=P(1.5)=4.52349$$

Differences:

Forward-Differences operator Δ

$$\Delta f(x_n)=f(x_{n+1})-f(x_n)$$

$$\therefore \Delta f(x_0)=f(x_1)-f(x_0)$$

$$\therefore \Delta^2 f(x_0)=\Delta f(x_1)-\Delta f(x_0)$$

$$\therefore \Delta^3 f(x_0)=\Delta^2 f(x_1)-\Delta^2 f(x_0)$$

Newton's forward divided difference

When $x_0, x_1, \ldots\ldots, x_n$ are arranged consecutively with equal spacing, that is

$$x_1=x_0+h, \quad x_2=x_1+h,\ldots\ldots,x_{n+1}=x_n+h$$

And so on we have $$\Delta^n f(x_0)=\Delta^{n-1} f(x_1)-\Delta^{n-1} f(x_0)$$

Then
$$f[x_0,x_1]=\frac{f(x_1)-f(x_0)}{x_1-x_0}=\frac{1}{h}\Delta f(x_0)$$

$$f[x_0,x_1,x_2]=\frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}=\frac{1}{2h}[\frac{1}{h}\Delta f(x_1)-\frac{1}{h}\Delta f(x_0)]=\frac{1}{2h^2}\Delta^2 f(x_0)$$

Similarly we can prove
$$f[x_0,x_1,x_2,x_3]=\frac{1}{3!h^3}\Delta^3 f(x_0)$$

$$f[x_0,x_1,x_2,x_3,...,x_n]=\frac{1}{n!h^n}\Delta^n f(x_0)$$

Substituting from above equations in the following Newton's interpolating divided difference formula

$$P(x)=f[x_0]+f[x_0,x_1](x-x_0)+f[x_0,x_1,x_2]\,(x-x_0)(x-x_1)+...+f[x_0,x_1,x_2,...,x_n]\,(x-x_0)....(x-x_{n-1})$$

Then we have

$$P(x)=f[x_0]+\frac{1}{h}\Delta f(x_0)(x-x_0)+\frac{1}{2!h^2}\Delta^2 f(x_0)(x-x_0)(x-x_1)+\ldots+\frac{1}{n!h^n}\Delta^n f(x_0)(x-x_0)\ldots.(x-x_{n-1}) \qquad (2)$$

Assume $x = x_0 +$ s h then $x - x_0 =$ s h. Also we can prove that

$$x-x_1=x_0+sh-x_1=x_0-x_1+sh=-h+sh=h(s-1)$$
$$x-x_2=x_0+sh-x_2=x_0-x_2+sh=-2h+sh=h(s-2)$$
$$x-x_{n-1}=x_0+sh-x_{n-1}=x_0-x_{n-1}+sh=-(n-1)h+sh=h(s-(n-1))$$

Substituting in (2) we have

$$P(x)=f[x_0]+\frac{1}{h}\Delta f(x_0)(sh)+\frac{1}{2!h^2}\Delta^2 f(x_0)s(s-1)h^2+\ldots+\frac{1}{n!h^n}\Delta^n f(x_0)s(s-1)\ldots.(s-(n-1))h^n$$

Hence we have the following polynomial which is known Newton's forward divided difference

$$P(x)=f(x_0)+\Delta f(x_0)s+\frac{1}{2!}\Delta^2 f(x_0)s(s-1)+\ldots+\frac{1}{n!}\Delta^n f(x_0)s(s-1)\ldots.(s-(n-1)) \quad (3)$$

$f(x_0)$				
	$\Delta f(x_0)$			
$f(x_1)$		$\Delta^2 f(x_0)$		
	$\Delta f(x_1)$		$\Delta^3 f(x_0)$	
$f(x_2)$		$\Delta^2 f(x_1)$		$\Delta^4 f(x_0)$
	$\Delta f(x_2)$		$\Delta^3 f(x_1)$	
$f(x_3)$		$\Delta^2 f(x_2)$		
	$\Delta f(x_3)$			
$f(x_4)$				

Example I: Using Newton's forward divided difference interpolation to find f(1.5) according to the following table

x	1	3	5	7	9
F(x)	4	5	8	11	16

Solution: since h=2, x=1.5, x_0=1 , x= x_0 +s h, then 1.5=1+s(2) , hence s=0.25

4				
	1			
5		2		
	3		-2	
8		0		4
	3		2	
11		2		
	5			
16				

Since

$$P(x) = f(x_0) + \Delta f(x_0)s + \frac{1}{2!}\Delta^2 f(x_0)s(s-1) + \frac{1}{3!}\Delta^3 f(x_0)s(s-1)(s-2) + \frac{1}{4!}\Delta^4 f(x_0)s(s-1)(s-2)(s-3)$$

Then

$$P(1.5) = 4 + (1)(0.25) + \frac{1}{2!}(2)(0.25)(0.25-1) + \frac{1}{3!}(-2)(0.25)(0.25-1)(0.25-2) + \frac{1}{4!}(4)(0.25)(0.25-1)(0.25-2)(0.25-3)$$

backward-Differences operator ∇

$$\nabla f(x_n) = f(x_n) - f(x_{n-1})$$
$$\therefore \nabla^2 f(x_n) = \nabla f(x_n) - \nabla f(x_{n-1})$$

Newton's backward divided difference

If the interpolating nodes are reordered as $x_n, x_{n-1}, \ldots\ldots, x_0$ then
Newton's interpolating divided difference formula (1) can be written as follows

$$P(x) = f[x_n] + f[x_{n-1}, x_n](x - x_n) + f[x_n, x_{n-1}, x_{n-2}](x - x_n)(x - x_{n-1}) + \ldots + f[x_0, x_1, \ldots, x_n](x - x_n)\ldots.(x - x_1) \quad (4)$$

Similarly as forward difference we can say

$$f[x_{n-1},x_n]=\frac{f(x_n)-f(x_{n-1})}{x_n-x_{n-1}}=\frac{1}{h}\nabla f(x_n)$$

$$f[x_{n-2},x_{n-1},x_n]=\frac{1}{2!h^2}\nabla^2 f(x_n)$$

$$f[x_0,....,x_{n-1},x_n]=\frac{1}{n!h^n}\nabla^n f(x_n)$$

Substituting from above equations in Newton's interpolating divided difference formula (4) Then we have

$$P(x)=f(x_n)+\frac{1}{h}\nabla f(x_n)(x-x_n)+\frac{1}{2!h^2}\nabla^2 f(x_n)(x-x_n)(x-x_{n-1})+...+\frac{1}{n!h^n}\nabla^2 f(x_n)(x-x_n)....(x-x_1) \quad (5)$$

Assume $x=x_n$+ s h then $x-x_n$ = s h. Also we can prove that

$$x-x_{n-1}=x_n+sh-x_{n-1}=x_n-x_{n-1}+sh=h+sh=h(s+1)$$
$$x-x_{n-2}=x_n+sh-x_{n-2}=x_n-x_{n-2}+sh=2h+sh=h(s+2)$$

Substituting in (5) we have

$$P(x)=f(x_n)+\frac{1}{h}\nabla f(x_n)(sh)+\frac{1}{2!h^2}\nabla^2 f(x_n)(sh)(s+1)h+\ldots+\frac{1}{n!h^n}\nabla^2 f(x_n)(s)(s+1)(s+2)\ldots(s+n-1)h^n$$

Hence we have the following polynomial which is known Newton's backward divided difference

$$P(x)=f(x_n)+\nabla f(x_n)(s)+\frac{1}{2!}\nabla^2 f(x_n)(s)(s+1)+\ldots+\frac{1}{n!}\nabla^2 f(x_n)(s)(s+1)(s+2)\ldots(s+n-1) \qquad (6)$$

$f(x_4)$				
	$\nabla f(x_4)$			
$f(x_3)$		$\nabla^2 f(x_4)$		
	$\nabla f(x_3)$		$\nabla^3 f(x_4)$	
$f(x_2)$		$\nabla^2 f(x_3)$		$\nabla^4 f(x_4)$
	$\nabla f(x_2)$		$\nabla^3 f(x_3)$	
$f(x_1)$		$\nabla^2 f(x_2)$		
	$\nabla f(x_1)$			
$f(x_0)$				

Example II: Using Newton's backward divided difference interpolation to find f(8) according to the following table

x	1	3	5	7	9
F(x)	4	5	8	11	16

Solution: Since h=2, x=8, x_4=1 , x= x_4 +s h, then 8=9+s(2) , hence s= - 0.5

16				
	5			
11		2		
	3		2	
8		0		4
	3		-2	
5		2		
	1			
4				

Since

$$P(x) = f(x_4) + \nabla f(x_4)(s) + \frac{1}{2!}\nabla^2 f(x_4)(s)(s+1) + \frac{1}{3!}\nabla^3 f(x_4)(s)(s+1)(s+2) + \frac{1}{4!}\nabla^4 f(x_4)(s)(s+1)(s+2)(s+3)$$

Then

$$P(8) = 16 + (5)(-0.5) + \frac{1}{2!}(2)(-0.5)(-0.5+1) + \frac{1}{3!}(2)(-0.5)(-0.5+1)(-0.5+2) + \frac{1}{4!}(4)(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)$$

Example: Find f(1.5) and f(8) using the following table

x	1	3	5	7	9
F(x)	4	5	8	11	16

Solution: Using one table only to find f(1.5) and f(8) as follows

4				
	1			
5		2		
	3		-2	
8		0		4
	3		2	
11		2		
	5			
16				

To find f(1.5), we follow the same steps in Example I and to find f(8) we follow the same steps in Example II

Exercises: Using Newton's interpolation method to evaluate the following

1] find $e^{1.8}$

x	1.7	1.9	2	2.1	2.3
e^x	5.474	6.686	7.389	8.166	9.974

2] Find f(1.3) and f(1.95)

x	1.1	1.2	1.5	1.7	1.8	2
f(x)	1.112	1.219	1.636	2.054	2.323	3.011

3] Find f(2.5)

x	-1	0	2	4	7
f(x)	2	1	11	117	666

4] Find f(8.35)

x	8.1	8.3	8.6	8.7
f(x)	16.9441	17.56492	18.50515	18.82091

5] Find f(0.05) and f(0.75)

x	0	0.2	0.4	0.6	0.8
f(x)	1	1.2214	1.4918	1.8221	2.2255

6] Find $\sqrt{1.03}$ & $\sqrt{1.23}$

x	1	1.05	1.1	1.15	1.2
$\sqrt{x}$	1	1.0247	1.0488	1.0724	1.0954

x	2	2.2	2.4	2.6	2.8	3
ln(x+2)	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094

7] Find ln(4.1)& $\ln(4.9)$

x	0.1	0.3	0.5	0.7	0.9
Sin(x+1)	0.8912	0.9636	0.9975	0.9917	0.9463

8] Find sin(1.2)& $\sin(1.85)$