Linear Algebra

Chapter 3: Determinants

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Summary

1. Introduction
2. Properties of Determinants
3. Cramer's Rule
4. Matrix Equations

Introduction

In many applications of linear algebra to geometry and analysis the concept of a determinant plays an important part. Determinants of order two and three were introduced in previous courses. We recall that a determinant of order two was defined by the formula

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Note that the determinant $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, with vertical bars | |, is distinct from the matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, with square brackets [ ]. The determinant is a number assigned to the matrix according to Formula. We write

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Introduction

In other words, if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

Determinants of order three is defined in terms of second-order determinants by the formula

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

For brevity, we write

$$\det(A) = a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + a_{13} \det(A_{13}),$$

where A_{11}, $_{12}$ and A_{13} are obtained from A by deleting the first row and one of the three columns. Generally, for any square matrix A, A_{ij} denote the **submarine** formed by deleting the ith row and jth column of A.

Introduction

For example, if

$$A = \begin{bmatrix} 1 & -2 & 7 & 4 \\ 5 & 8 & 5 & 2 \\ 6 & 3 & 9 & -3 \\ 10 & -1 & 0 & -2 \end{bmatrix}$$

then A_{32} is obtained by crossing out row 3 and column 2,

$$\begin{pmatrix} 1 & -2 & 7 & 4 \\ 5 & 8 & 5 & 2 \\ 6 & 3 & 9 & -3 \\ 10 & -1 & 0 & -2 \end{pmatrix}$$

So that

$$A_{32} = \begin{bmatrix} 1 & 7 & 4 \\ 5 & 5 & 2 \\ 10 & 0 & -2 \end{bmatrix}$$

Introduction

Now, we can now give a general definition of a determinant $\det(A)$ of an $n \times n$ matrix A is using the determinants of $(n-1) \times (n-1)$ its submatrices.

Definition (Determents)

For $n \geq 2$, the **determinant** of a square matrix $A = [a_{ij}]$ of order n is given by

$$\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + \cdots + (-1)^{n+1}a_{1n}\det(A_{1n}) = \sum_{j=1}^{n}(-1)^{j+1}\det(A_{1j}).$$

For a 1×1 matrix (**singleton matrix**) $A = [a_{11}]$, $\det(A) = |a_{11}| = a_{11}$; that is not the absolute value. We aim to define the determinant of a square matrix of order n for any integer n.

Introduction

Example

Compute the determinant of

$$A = \begin{bmatrix} 1 & -5 & -3 \\ 2 & 4 & -1 \\ 0 & -2 & 6 \end{bmatrix}$$

Solution

Computing the determinant of

$$\begin{aligned}
\det(A) &= = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13}) \\
&= (1)\det\begin{bmatrix} 4 & -1 \\ -2 & 6 \end{bmatrix} - (-5)\det\begin{bmatrix} 2 & -1 \\ 0 & 6 \end{bmatrix} + (-3)\det\begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix} \\
&= (1)(24-2) - (-5)(12-0) + (-3)(-4-0) \\
&= 22 + 60 + 12 = 94
\end{aligned}$$

Introduction

In fact the determinant of a matrix can be obtained using the any submatrices according to some row or column. So that

$$\det(A) = a_{i1}\det(A_{i1}) - a_{i2}\det(A_{i2}) + \cdots + (-1)^{i+n} a_{in}\det(A_{in}), \quad \text{for every } i = 1, \cdots, n$$

or

$$\det(A) = a_{1j}\det(A_{1j}) - a_{2j}\det(A_{2j}) + \cdots + (-1)^{n+j} a_{nj}\det(A_{nj}), \quad \text{for every } j = 1, \cdots, n.$$

In the previous example, we can say that

$$\begin{aligned}
\det(A) &= -(2)\det\begin{bmatrix} -5 & -3 \\ -2 & 6 \end{bmatrix} + (4)\det\begin{bmatrix} 1 & -3 \\ 0 & 6 \end{bmatrix} - (-1)\det\begin{bmatrix} 1 & -5 \\ 0 & -2 \end{bmatrix} \\
&= -2(-30-6) + 4(6-0) + 1(-2-0) \\
&= 72 + 24 - 2 = 92
\end{aligned}$$

Properties of Determinants

The determinants has the following properties:

1: If one row of A is multiplied by k to produce B, then $\det(B) = k\det(A)$.

$$\begin{vmatrix} 2 & 5 & 6 \\ 3 & -12 & 9 \\ 1 & 0 & 2 \end{vmatrix} = 3\begin{vmatrix} 2 & 5 & 6 \\ 1 & -4 & 3 \\ 1 & 0 & 2 \end{vmatrix}$$

2: If a multiple of one row of A is added to another row to produce a matrix B, then $\det(B) = \det(A)$.

$$\begin{vmatrix} 2 & 5 & 6 \\ 3 & -12 & 9 \\ 1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 5 & 6 \\ 7 & -2 & 21 \\ 1 & 0 & 2 \end{vmatrix}$$

Properties of Determinants

3: A can be expressed as a sum of two matrices by expressing one of its rows.

$$\begin{vmatrix} 2 & 5 & 6 \\ 6 & 7 & 10 \\ 1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 5 & 6 \\ 5 & -1 & 5 \\ 1 & 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 5 & 6 \\ 1 & 8 & 5 \\ 1 & 0 & 2 \end{vmatrix}$$

4: The determinant vanishes if two adjacent rows are identical.

$$\begin{vmatrix} 2 & 5 & 6 \\ 6 & 7 & 10 \\ 2 & 5 & 6 \end{vmatrix} = 0$$

Properties of Determinants

5: The determinant of the identity matrix is one.

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

6: The determinant vanishes if some row is zero.

$$\begin{vmatrix} 2 & 5 & 6 \\ 0 & 0 & 0 \\ 4 & -5 & 3 \end{vmatrix} = 0$$

Properties of Determinants

7: The determinant changes sign if two adjacent rows are interchanged.

$$\begin{vmatrix} 2 & 5 & 6 \\ 6 & 7 & 10 \\ 2 & 5 & 6 \end{vmatrix} = -\begin{vmatrix} 2 & 5 & 6 \\ 6 & 7 & 10 \\ 2 & 5 & 6 \end{vmatrix}$$

8: The determinant vanishes it has parallel rows as vectors (in ratio).

$$\begin{vmatrix} 2 & 5 & 6 \\ 8 & 4 & -20 \\ 2 & 1 & -5 \end{vmatrix} = 0$$

Properties of Determinants

Theorem (Triangular Determinate)

If A is a triangular matrix, then $\det(A)$ is the product of the entries on the main diagonal (diagonal elements) of A.

Example

Compute $\det(A)$, where $A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$.

Properties of Determinants

Solution

Applying the properties of determinate to simplify the arithmetic by getting 3 zeros in one column or one row.

$$\det(A) = \begin{vmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix}, \qquad \textit{we factor out 2 from the top row,}$$

$$= 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix}, \qquad \textit{then proceed with row replacements in the first column}$$

Properties of Determinants

Solution

$$= \quad 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{vmatrix},$$

we can use the 3 *in the second column as a pivot to eliminate* -12

$$= \quad 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{vmatrix},$$

interchanging the last two rows

Solution

$$= 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -6 & 2 \end{vmatrix},$$

we can use the -3 in the third column as a pivot to eliminate -6

$$= -2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & -2 \end{vmatrix}.$$

Finally, computing the triangular determinant

$$= -2(1)(3)(-3)(-2) = -36.$$

Properties of Determinants

Theorem (Invertibility Condition)

A square matrix A is invertible if ad only if $\det(A) \neq 0$.

Theorem

For square matrices A and B of the same order n and a scalar k, we have

(a) $\det(A^T) = \det(A)$.

(b) $\det(AB) = \det(A)\det(B)$.

(c) $\det(kA) = k^n \det(A)$.

(d) *If A is invertible, then* $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

From (a) in the previous theorem, we can perform operations on the columns of a matrix in a way that is analogous to the row operations we have considered.

Cramer's Rule

Cramer's rule is needed in a variety of theoretical calculations. For instance, it can be used to study how the solution of $Ax = b$ is affected by changes in the entries of b. However, the formula is inefficient for hand calculations, except for 2×2 or perhaps 3×3 matrices.
For any $n \times n$ matrix A and any b in $\mathbb{R}^n$, let $A_i(b)$ be the matrix obtained from A by replacing ith column by the vector b.

$$A_i(b) = [a_1 \cdots \underbrace{b}_{\text{Column } i} \cdots a_n].$$

Theorem (Cramer's Rule)

Let A be an invertible $n \times n$ matrix. For any b in $\mathbb{R}^n$, the unique solution x of $Ax = b$ has entries given by

$$x_i = \frac{\det(A_i(b))}{\det(A)}.$$

Cramer's Rule

Example

Use Cramer's method to solve the linear system

$$\begin{array}{ccccccc} x_1 & + & 2x_2 & + & x_3 & = & 1 \\ 2x_1 & - & x_2 & + & x_3 & = & -4 \\ x_1 & + & x_2 & + & 2x_3 & = & -5 \end{array}$$

Solution

$$Indeed\ A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix} and\ b = \begin{bmatrix} 1 \\ -4 \\ -5 \end{bmatrix},$$

Cramer's Rule

Solution

$$
\begin{aligned}
\det(A) &= \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix} = -6 \\
\det(A_1(b)) &= \begin{bmatrix} 1 & 2 & 1 \\ -4 & -1 & 1 \\ -5 & 1 & 2 \end{bmatrix} = -6 \\
\det(A_1(b)) &= \begin{bmatrix} 1 & 1 & 1 \\ 2 & -4 & 1 \\ 1 & -5 & 2 \end{bmatrix} = -12
\end{aligned}
$$

Cramer's Rule

Solution

$$\det(A_1(b)) = \begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & -4 \\ 1 & 1 & -5 \end{vmatrix} = 24$$

Therefore, $x_1 = 1, x_2 = 2$ *and* $x_3 = -4$.

Cramer's Rule

Given a square matrix $A = [a_{ij}]$ of order n the **(i, j)-cofactor** of A, denoted by C_{ij}, is the number given by

$$C_{ij} = (-1)^{i+j} \det(A_{ij}),$$

where A_{ij} are the submatrices of A as has been defined in the previous section. These cofactors are the entries of a square matrix of order n, called the **cofactor matrix** of A, $C_A = [C_{ij}]$. The transpose of this matrix is called the **adjugate** (or **classical adjoint**) of A, denoted by $adj\,(A)$.

Theorem

Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det(A)} adj\,(A).$$

Cramer's Rule

Example

Find the inverse of the matrix

$$A = \begin{bmatrix} -3 & 2 & -5 \\ -1 & 0 & -2 \\ 3 & -4 & 1 \end{bmatrix}.$$

Solution

We have

$$\det(A) = \begin{vmatrix} -3 & 2 & -5 \\ -1 & 0 & -2 \\ 3 & -4 & 1 \end{vmatrix} = -6$$

Hence A is invertible. Now

Cramer's Rule

Solution

$$C_A = \begin{bmatrix} +\begin{vmatrix} 0 & -2 \\ -4 & 1 \end{vmatrix} & -\begin{vmatrix} -1 & -2 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} -1 & 0 \\ 3 & -4 \end{vmatrix} \\ \\ -\begin{vmatrix} 2 & -5 \\ -4 & 1 \end{vmatrix} & +\begin{vmatrix} -3 & -5 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} -3 & 2 \\ 3 & -4 \end{vmatrix} \\ \\ +\begin{vmatrix} 2 & -5 \\ 0 & -2 \end{vmatrix} & -\begin{vmatrix} -3 & -5 \\ -1 & -2 \end{vmatrix} & +\begin{vmatrix} -3 & 2 \\ -1 & 0 \end{vmatrix} \end{bmatrix}$$

Cramer's Rule

Solution

So

$$C_A = \begin{bmatrix} -8 & -5 & 4 \\ 18 & 12 & -6 \\ -4 & -1 & 2 \end{bmatrix}$$

and

$$adj\,(A) = \begin{bmatrix} -8 & 18 & -4 \\ -5 & 12 & -1 \\ 4 & -6 & 2 \end{bmatrix}$$

Thus

$$A^{-1} = \frac{1}{\det(A)} adj\,(A) = -\frac{1}{6} \begin{bmatrix} -8 & 18 & -4 \\ -5 & 12 & -1 \\ 4 & -6 & 2 \end{bmatrix}$$

Matrix Equations

We can then write a linear system as a single equation, using a matrix and two vectors, which generalizes the linear equation $ax = b$ for real numbers. As we will see, in some cases the linear system can then be solved using algebraic operations similar to the operations used to solve equations involving real numbers.
To illustrate the process, consider the linear system

$$\begin{array}{rcrcrcr} x & - & 6y & - & 4z & = & -5 \\ 2x & - & 10y & - & 9z & = & -4 \\ -x & + & 6y & + & 5z & = & 3 \end{array}$$

The matrix of coefficients is given by

$$A = \begin{bmatrix} 1 & -6 & -4 \\ 2 & -10 & -9 \\ -1 & 6 & 5 \end{bmatrix}$$

Cramer's Rule

Now let x and b be the vectors

$$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix}$$

Then the original linear system can be rewritten as

$$Ax = b$$

We refer to this equation as the **matrix form** of the linear system and x as the vector form of the solution.

Matrix Equations

f A is invertible, we can multiply both sides of the previous equation on the left by A^{-1}, so that

$$A^{-1}(Ax) = A^{-1}b.$$

Since matrix multiplication is associative, we have

$$(A^{-1}A)x = A^{-1}b.$$

But

$$(A^{-1}A)x = Ix = x.$$

Therefore

$$x = A^{-1}b$$

Cramer's Rule

For the example above, the inverse of the matrix

$$A = \begin{bmatrix} 1 & -6 & -4 \\ 2 & -10 & -9 \\ -1 & 6 & 5 \end{bmatrix} \quad \text{is} \quad A^{-1} = \begin{bmatrix} 2 & 3 & 7 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 1 \end{bmatrix}$$

Therefore, the solution of the linear system in vector form is given by

$$x = A^{-1}b = \begin{bmatrix} 2 & 3 & 7 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$$

That is, $x = -1, \qquad y = 2 \qquad \text{and} \qquad z = -2.$

Matrix Equations

Theorem

If the $n \times n$ matrix A is invertible, then for every vector b, with n components, the linear system $Ax = b$ has the unique solution $x = A^{-1}b$.

Obviously, if A is invertible, then the only solution to the homogeneous equation $Ax = 0$ is the trivial solution $x = 0$.
On the other hand if A is non-invertible, then the homogenous system $Ax = 0$ has infinitely many solutions and the non-homogeneous system $Ax = b$ is consistent with infinitely many solutions or inconsistent.