Linear Algebra

Chapter 2: Linear Equations

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2021-2022

Summary

1. Systems of Linear Equations
2. Solving a Linear System
3. Homogeneous Systems
4. Applications of Linear Systems

Systems of Linear Equations

A linear equation in the variables $x_1, x_2, \cdots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where b and the **coefficients** $a_1; a_2, \cdots, a_n$ are real or complex numbers.
The equations

$$4x_1 - 5x_2 + 2 = x_1 \quad \text{and} \quad x_2 = 2(\sqrt{6} - x_1) + x_3$$

are both linear because they can be rearranged algebraically:

$$3x_1 - 5x_2 = -2 \quad \text{and} \quad 2x_1 + x_2 - x_3 = 2\sqrt{6}.$$

The equations

$$4x_1 + 5x_2 = x_1x_2 \qquad \text{and } x_2 = 2\sqrt{x_1} - 6.$$

are not linear because of the presence of x_1x_2 in the first equation and $\sqrt{x_1}$ in the second one.

Systems of Linear Equations

A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same variables, say, $x_1; x_2, \cdots, x_n$. An example is

$$\begin{aligned} 2x_1 - x_2 + 1.5x_3 &= 8 \\ x_1 \qquad 4x_3 &= -7 \end{aligned}$$

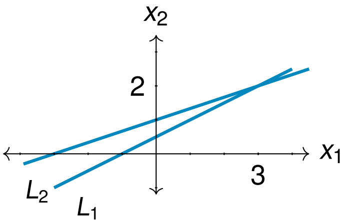
A **solution** of the system of n of variables is a list $(s_1, s_2, \cdots, s_n)$ of numbers that satisfies each equation; that when the values $s_1, s_2, \cdots, s_n$ are substituted for $x_1, x_2, \cdots, x_n$ respectively, all the equations of the system hold.
Finding the solution set of a system of two linear equations in two variables is easy because it amounts to finding the intersection of two lines. Consider the linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

whose equations of straight lines, say L_1 and L_2. A pair of numbers (x_1, x_2) satisfies both equations in the system if and only if the point (x_1, x_2) lies on both L_1 and L_2.

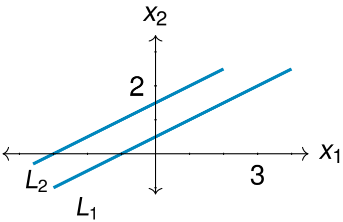
Systems of Linear Equations

Of course, two lines need not intersect in a single point. They could be parallel and therefore there is no a common point (solution), or they could coincide and hence intersect at every point on the line which means that every point on the coincided two lines are a solution of the system.



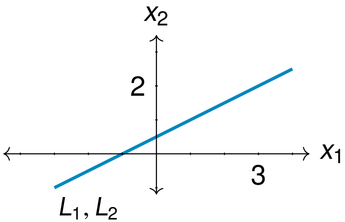
$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

Exactly one solution $(3, 2)$



$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned}$$

No solution



$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned}$$

Infinitely many solutions

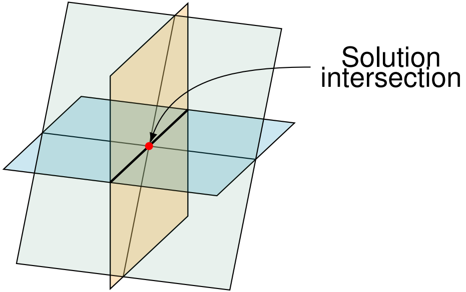
Systems of Linear Equations

Same cases occurs also in the system in three unknowns. The solution set of a system of three linear equations in three variables

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

whose equations of planes, say P_1, P_2 and P_3. A tuple of numbers (x_1, x_2, x_3) satisfies all three equations in the system if and only if the point (x_1, x_2, x_3) lies on the planes P_1, P_2 and P_3. Again we have the three cases:

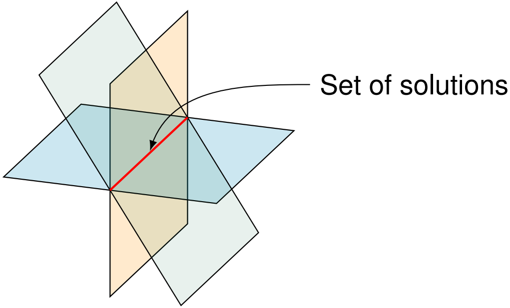
Systems of Linear Equations



The planes are intersected in a point

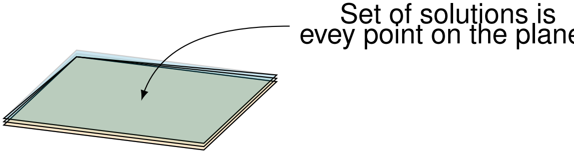
Exactly one solution

Systems of Linear Equations



The planes are intersected in a line

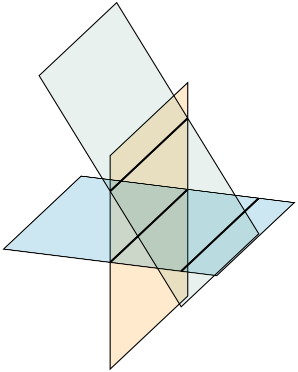
Infinitely many solutions



The planes concoid

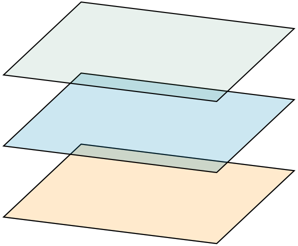
Infinitely many solutions

Systems of Linear Equations



Each pair of planes intersected in a line

No solution



The planes are parallel

No solution

Systems of Linear Equations

Thus any system of linear equations has

- no solution, or
- exactly one solution (unique solution), or
- infinitely many solutions.

Moreover, it is said to be **consistent** if it has either one solution or infinitely many solutions and **inconsistent** if it has no solution.

Systems of Linear Equations

Furthermore, every linear system can be represented as an equation of matrices. Indeed the linear system of m equations in n variables

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

can be written as $Ax = b$, where

$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_2 & \cdots & a_{2n} \\ \vdots & \vdots & ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, is called the **coefficient matrix** (or **matrix of coefficients**),

Systems of Linear Equations

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$ is called the **vector of unknowns** and

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$ is called the **vector of constants**.

$$[A \mid b] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_2 & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right],$$ is called the **augmented matrix** of the system.

Solving a Linear System

The basic method to solve a linear system depends on replacing the system with one of its equivalent systems that is easier to solve. The method use the x_1 term in the first equation of a system to eliminate the x_1 terms in the other equations. Then use the x_2 term in the second equation to eliminate the x_2 terms in the other equations, and so on, until you get a simple equivalent system, called a **triangular system**. For examples, the **lower triangular matrix**

$$\begin{array}{ccccccccc} l_{11}x_1 & & & & & & & = & b_1 \\ l_{21}x_1 & + & l_{22}x_2 & & & & & = & b_2 \\ l_{31}x_1 & + & l_{32}x_2 & + & l_{33}x_3 & & & = & b_4 \\ l_{41}x_1 & + & l_{42}x_2 & + & l_{43}x_3 & + & l_{44}x_3 & = & b_4 \end{array};$$

and the **upper triangular matrix**

$$\begin{array}{ccccccccc} u_{11}x_1 & + & u_{12}x_2 & + & u_{13}x_3 & + & u_{14}x_4 & = & b_1 \\ & & u_{22}x_2 & + & u_{23}x_3 & + & u_{24}x_4 & = & b_2 \\ & & & & u_{33}x_3 & + & u_{34}x_4 & = & b_3 \\ & & & & & & u_{44}x_4 & = & b_4 \end{array}.$$

Solving a Linear System

In fact, the elimination of variables depends only on the coefficients and constants. So to make the process easier, we will apply the eliminations on the augmented matrix.
To solve a given linear system $Ax = b$ in n unknowns, we have follow the following steps:

1: Form the augmented matrix of the system $[A \mid b]$.

2: Apply elementary row operations to transform the augmented matrix into echelon form.

3: From the obtained echelon form, we can find the ranks the matrices A and $A|b$. Then we have three cases:

Case 1: If $rank\,(A) = rank\,(A|b) = n$, then the system is consistent and has a unique solution.
Case 2: If $rank\,(A) = rank\,(A|b) < n$, then the system is consistent and has infinitely many solutions.
Case 3: If $rank\,(A) \neq rank\,(A|b)$, then the system is inconsistent and has no solution.

Solving a Linear System

Example

Solve the linear system

$$\begin{array}{rcrcrcr} x_1 & + & 7x_2 & + & 7x_3 & = & 6 \\ 2x_1 & - & x_2 & + & x_3 & = & 3 \\ x_1 & - & 2x_2 & + & 3x_3 & = & 6 \end{array}$$

Solution

Forming the augmented matrix of the system

$$A|b = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & 1 & 3 \\ 1 & -2 & 3 & 6 \end{array}\right].$$

Applying the elementary operation.

Solving a Linear System

Solution

$$
\begin{aligned}
A|b \quad &= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & 1 & 3 \\ 1 & -2 & 3 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & -1 & -9 \\ 0 & -3 & 2 & 0 \end{array}\right] \\
&\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & -1 & -9 \\ 0 & 0 & 3 & 9 \end{array}\right].
\end{aligned}
$$

Indeed, $\mathit{rank}\,(A) = \mathit{rank}\,(A|b) = 3$ *and the system is consistent with a unique solution.*

Solving a Linear System

Solution

To get the unique solution, we have to write a simpler linear system equivalent to the given one. This system can be obtained form the echelon form; that is

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 6 \\ & & -3x_2 & - & x_3 & = & -9 \\ & & & & 3x_3 & = & 9 \end{array}.$$

Then we can use the value of x_3 in the second equation to get x_2. Finally, we use the values of both x_2 and x_3 to get the value of x_1 from the first equation.

$$\begin{aligned} 3x_3 = 9 \quad &\Rightarrow \quad \boxed{x_3 = 3} \\ -3x_2 - x_3 = -9 \quad &\Rightarrow \quad -3x_2 - (3) = -9 \Rightarrow \boxed{x_2 = 2} \\ x_1 + x_2 + x_3 = 6 \quad &\Rightarrow \quad x_1 + (2) + (3)6 \Rightarrow \boxed{x_1 = 1} \end{aligned}$$

Solving a Linear System

Example

Discuss the consistence of

$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 1 \\ x & + & 2y & + & z & = & 2 \\ 2x & + & y & + & 2z & = & 2 \end{array}$$

Solution

We have

$$A|b = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

So rank $(A|b) = 3$ *while rank* $(A) = 2$. *Therefore the system is inconsistent and has no solution.*

Solving a Linear System

Example

Solve the following system

$$\begin{aligned} x_1 &+ x_2 &+ x_3 &= 16 \\ x_1 &+ 2x_2 &+ 3x_3 &= 24 \\ x_1 &+ 4x_2 &+ 7x_3 &= 40 \end{aligned}$$

Solution

$$A|b = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 16 \\ 1 & 2 & 3 & 24 \\ 1 & 4 & 7 & 40 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 16 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 16 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solving a Linear System

Solution

Here $\text{rank}(A) = \text{rank}(A|b) = 2 < 3$. *So the system is consistent with infinitely many solutions and has an equivalent system*

$$\begin{array}{ccccccc} x_1 & + & x_2 & + & x_3 & = & 16 \\ & & x_2 & + & 2x_3 & = & 8 \end{array}$$

In this case, the variables x_1 *and* x_2 *corresponding to pivot columns in the matrix are called* ***leading variables*** *and the other variable,* x_3*, is called a* ***free variable****. We can apply the backward substitution by extend the leading variables in terms of the free variable. Hence*

$$\begin{cases} x_1 = 8 + x_3 \\ x_2 = 8 - 2x_3 \\ x_3 \textit{ is free} \end{cases}.$$

Solving a Linear System

Example

Solve the following system:

$$
\begin{array}{rcrcrcrcrcl}
x_1 & - & 2x_2 & + & 3x_3 & + & 2x_4 & + & x_5 & = & 10 \\
2x_1 & - & 4x_2 & + & 8x_3 & + & 3x_4 & + & 10x_5 & = & 7 \\
3x_1 & - & 6x_2 & + & 10x_3 & + & 6x_4 & + & 5x_5 & = & 27
\end{array}
$$

Solution

$$
A|b = \left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 2 & -4 & 8 & 3 & 10 & 7 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 2 & -1 & 8 & -13 \\ 0 & 0 & 1 & 0 & 2 & -3 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 2 & -1 & 8 & -13 \\ 0 & 0 & 0 & \frac{1}{2} & -2 & \frac{7}{2} \end{array}\right]
$$

Solving a Linear System

Solution

We get the echelon form and rank $(A) =$ *rank* $(A|b) = 3 < 5$. *Therefore the system is consistent with infinitely many solutions such that* x_2 *and* x_5 *are free variables. We have*

$$\begin{array}{ccccccccccc} x_1 & - & 2x_2 & + & 3x_3 & + & 2x_4 & + & x_5 & = & 10 \\ & & & & 2x_3 & - & x_4 & + & 8x_5 & = & -13 \\ & & & & & & \frac{1}{2}x_4 & - & 2x_5 & = & \frac{7}{2} \end{array}$$

Let $x_2 = t$ *and* $x_3 = s$ *where* s *and* t *are arbitrary real numbers. So that*

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2t - 7s + 33 \\ t \\ -3 - 2s \\ 4s + 7 \\ s \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ -2 \\ 4 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \\ -3 \\ 7 \\ 0 \end{bmatrix}.$$

Homogeneous Systems

A system of linear equations is said to be **homogeneous** if it can be written in the form $Ax = 0$, where A is an $m \times n$ matrix and 0 is the zero vector in $\mathbb{R}^m$. Note that the linear systems in the previous section are called **non-homogeneous**. Obviously, a system $Ax = 0$ always has at least one solution, namely, $x = 0$. This zero solution is usually called the **trivial solution**. The important question is whether there exists a nontrivial solution, that is, a nonzero vector x that satisfies $Ax = 0$. Therefore every homogeneous system is consistent. It has either a unique solution; that is zero solution, or has infinitely many solutions including the zero solution. So we does not need an augmented can solve any homogeneous system and it is enough to work on the matrix A only.

Homogeneous Systems

To solve a given linear system $Ax = 0$ in n unknowns, we have follow the following steps:

1: Apply elementary row operations to transform the matrix A into echelon form.

2: From the obtained echelon form, we can find the ranks the matrix A. Then we have two cases:

Case 1: If $rank\,(A) = n$; then the system is consistent and has only the zero solution.

Case 2: If $rank\,(A) < n$, then the system is consistent and has infinitely many solutions.

In fact, the homogeneous equation $Ax = 0$ has a nontrivial solution if and only if the equation has at least one free variable.

Homogeneous Systems

Example

Discuss the solution of the homogeneous linear system:

$$\begin{array}{rcrcrcl} x & + & 2y & + & 3z & = & 0 \\ 2x & + & 3y & + & 4z & = & 0 \\ 3x & + & 6y & - & z & = & 0 \end{array}$$

Solution

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 6 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & -10 \end{bmatrix}$$

and rank $(A) = 3$. *Therefore the system has only the zero solution. The system describes three planes intersected in the origin.*

Homogeneous Systems

Example

Solve the system $Ax = 0$ where

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 & 1 \\ 2 & 2 & 4 & 4 & 3 \\ 2 & 2 & 4 & 4 & 2 \\ 3 & 5 & 8 & 6 & 5 \end{bmatrix}$$

Solution

We have

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 & 1 \\ 2 & 2 & 4 & 4 & 3 \\ 2 & 2 & 4 & 4 & 2 \\ 3 & 5 & 8 & 6 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 2 & 1 \\ 0 & 2 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Homogeneous Systems

Solution

So $\operatorname{rank}(A) = 3 < 5$ *and the system* $Ax = 0$ *has infinitely many solutions. The equivalent system is*

$$\begin{array}{ccccccccccc} x_1 & + & x_2 & + & 2x_3 & + & 2x_4 & + & x_5 & = & 0 \\ & & 2x_2 & + & 2x_3 & + & & & 2x_5 & = & 0 \\ & & & & & & & & x_5 & = & 0 \end{array}$$

with free variables x_3 *and* x_4*. Therefore*

$$x = \begin{bmatrix} -x_3 - 2x_4 \\ -x_3 \\ x_3 \\ x_4 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1_3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Applications of Linear Systems

Balancing Chemical Equations

Chemical equations describe the quantities of substances consumed and produced by chemical reactions. For instance, when propane gas burns, the propane C_3H_8 combines with oxygen O_2 to form carbon dioxide CO_2 and water H_2O, according to an equation of the form

$$x_1 \; C_3H_8 + x_2 \; O_2 \rightarrow x_3 \; CO_2 + x_4 \; H_2O$$

To balance this equation, a chemist must find whole numbers x_1, x_2, x_3, x_4 such that the total numbers of carbon C, hydrogen H, and oxygen O atoms on the left match the corresponding numbers of atoms on the right. Comparing the both sided of the reaction equation, we have the homogeneous system:

$$\begin{array}{lcrcllllcl} C & : & 3x_1 = x_3 & \Rightarrow & 3x_1 & & -x_3 & & = & 0 \\ H & : & 8x_1 = 2x_4 & \Rightarrow & 8x_1 & & & -2x_4 & = & 0 \\ O & : & 2x_2 = 2x_3 + x_4 & \Rightarrow & & 2x_2 & -2x_3 & -x_4 & = & 0 \end{array}$$

Applications of Linear Systems
Balancing Chemical Equations

Clearly, this homogeneous equations shall have infinitely many solutions since if it has a unique solutions, then the both sides of the equations will be zero which means that there is no reaction occurs. It is logical that the system has infinity many solutions as the chemical equation has infinitely many balances. Solving the obtained system

$$\begin{bmatrix} 3 & 0 & -1 & 0 \\ 8 & 0 & 0 & -2 \\ 0 & 2 & -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & -1 & 0 \\ 0 & 0 & \frac{8}{3} & -2 \\ 0 & 2 & -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 \\ 0 & 0 & \frac{8}{3} & -2 \end{bmatrix}$$

The equivalent system is

$$\begin{array}{ccccc} 3x_1 & & -x_3 & & = 0 \\ & 2x_2 & -2x_3 & -x_4 & = 0 \\ & & \frac{8}{3}x_3 & -2x_4 & = 0 \end{array}$$

where x_4 is a free variable.

So

$$\begin{cases} x_1 = \frac{1}{4}x_4 \\ x_2 = \frac{5}{4}x_4 \\ x_3 = \frac{3}{4}x_4 \\ x_4 \quad \text{is free} \end{cases}.$$

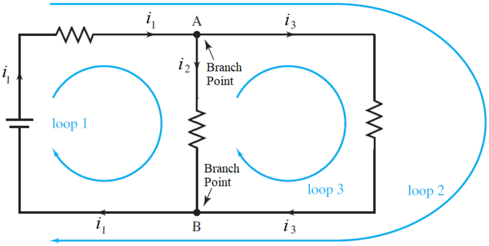
Since the coefficients in a chemical equation must be integers, take $x_4 = 4$, in which case $x_1 = 1$, $x_2 = 5$, and $x_3 = 3$. The balanced equation is

$$C_3H_8 + 5\ O_2 \to 3\ CO_2 + 4\ H_2O.$$

Applications of Linear Systems

Electrical Networks

Current flow in a simple electrical network can be described by a system of linear equations. A voltage source such as a battery forces a current of electrons to flow through the network. When the current passes through a resistor, some of the voltage is "used up"; by Ohm's law, this "voltage drop" across a resistor is given by $V = RI$, where the voltage V is measured in volts (V), the resistance R in ohms (Ω), and the current flow I in amperes (A).



Electrical Networks

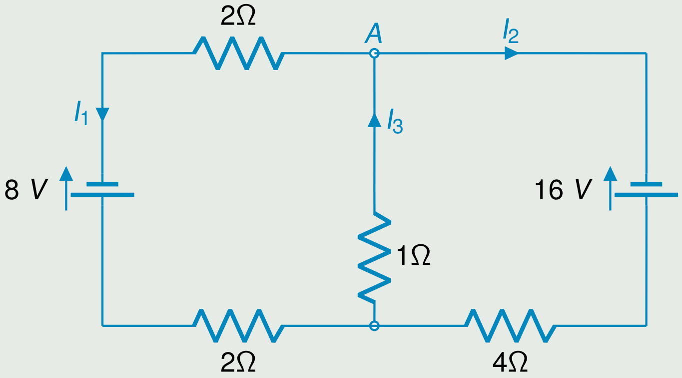
Every electric network in contains some closed loops. The designated directions of such loop currents are arbitrary. If the current direction shown is away from the positive (longer) side of a battery (|⊢) around to the negative (shorter) side, the voltage is positive; otherwise, the voltage is negative. Current flow in a loop is governed by the following rule.

Kirchhoff's Voltage Law

- The algebraic sum of the voltage drops in one direction around a loop equals the algebraic sum of the voltage sources in the same direction around the loop.
- The algebraic sum of the currents flowing into any junction point (branch point) must be zero.

Example

Determine the currents in the shown network.



Solution

The shown network has two junctions in the circuit and three closed loops. Applying Kirchhoff's Law to the junction A and paths results in:

$$I_1 + I_2 = I_3 \quad \Rightarrow \quad \boxed{I_1 + I_2 - I_3 = 0}.$$

We does not need to work on the junction B since it gives the same equation. Applying Kirchhoff's Law to the paths (Loops), it is enough to work on only two loops because we need only three equations to get three unknowns I_1, I_2 and I_2. So

$$\boxed{2I_1 + 2I_2 + 1I_3 = 8}$$

and

$$\boxed{4I_2 + 1I_3 = 8}.$$

Applications of Linear Systems

Electrical Networks

Solution

$$\begin{array}{rcrcrcl} I_1 & + & I_2 & - & I_3 & = & 0 \\ 4I_1 & & & + & I_3 & = & 8 \\ & & 4I_2 & + & I_3 & = & 16 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 4 & 0 & 1 & 8 \\ 0 & 4 & 1 & 16 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -4 & 5 & 8 \\ 0 & 4 & 1 & 16 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -4 & 5 & 8 \\ 0 & 0 & 6 & 24 \end{array}\right].$$

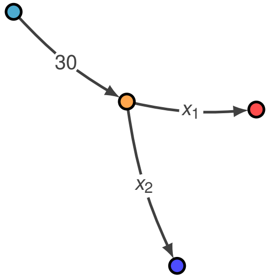
$$\begin{array}{rcrcrcl} I_1 & + & I_2 & - & I_3 & = & 0 \\ & & -4I_2 & + & 5I_3 & = & 8 \\ & & & & 6I_3 & = & 24 \end{array},$$

and applying the backward substitution, we get $I_1 = 1\ A$, $I_2 = 3\ A$, *and* $I_3 = 4\ A$.

Applications of Linear Systems

Network Flow

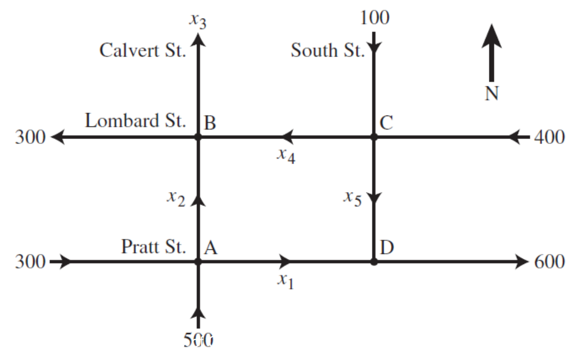
The basic assumption of network flow is that the total flow into the network equals the total flow out of the network and that the total flow into a junction equals the total flow out of the junction. As shown below, 30 units flowing into a junction through one branch, with x_1 and x_2 denoting the flows out of the junction through other branches. Since the flow is "conserved" at each junction, we must have $x_1 + x_2 = 30$.



In a similar fashion, the flow at each junction is described by a linear equation. The problem of

Example

The shown network shows the traffic flow (in vehicles per hour) over several one-way streets in downtown Baltimore during a typical early afternoon. Determine the general flow pattern for the network.



Solution

Write equations that describe the flow, and then find the general solution of the system. Label the street intersections (junctions) and the unknown flows in the branches, as shown in the figure. At each intersection, set the flow in equal to the flow out.

Intersection	Flow in	Flow out
A	$300 + 500$	$x_1 + x_2$
B	$x_2 + x_4$	$300 + x_3$
C	$100 + 400$	$x_4 + x5$
D	$x_1 + x_5$	600

Also, the total flow into the network $(500 + 300 + 100 + 400)$ *equals the total flow out of the network* $300 + x_3 + 600)$, *which simplifies to* $x_3 = 400$. *Combine this equation with a rearrangement of the first four equations to obtain the following system of equations:*

Applications of Linear Systems

Network Flow

Solution

$$
\begin{array}{rcrcrcrcrcl}
x_1 & + & x_2 & & & & & & & = & 800 \\
 & & x_2 & - & x_3 & + & x_4 & & & = & 300 \\
 & & & & & & x_4 & + & x_5 & = & 500 \\
x_1 & & & & & & & + & x_5 & = & 600 \\
 & & & & x_3 & & & & & = & 400
\end{array}
$$

Row reduction of the associated augmented matrix leads to

$$
\begin{array}{rrrrcrcl}
x_1 & & & & + & x_5 & = & 600 \\
 & x_2 & & & - & x_5 & = & 200 \\
 & & x_3 & & & & = & 400 \\
 & & & x_4 & + & x_5 & = & 500
\end{array}
$$

Applications of Linear Systems

Network Flow

Solution

The general flow pattern for the network is described by

$$\begin{cases} x_1 = 600 - x_5 \\ x_2 = 200 + x_5 \\ x_3 = 400 \\ x_4 = 500 - x_5 \\ x_5 \textit{ is free} \end{cases}.$$

A negative flow in a network branch corresponds to flow in the direction opposite to that shown on the model. Since the streets in this problem are one-way, none of the variables here can be negative. This fact leads to certain limitations on the possible values of the variables. For instance, $x_5 \leq 500$ *because* $x - 4$ *cannot be negative.*

Temperature Distribution

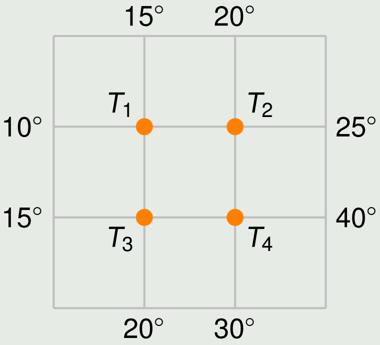
Engineers are interested in knowing the temperature distribution inside the dam in a specific period of time so they can determine the thermal stress to which the dam is subjected. Assuming the boundary temperatures are held constant during that specific period of time, the temperature inside the dam will reach certain equilibrium after some time has passed. Finding this equilibrium temperature distribution at different points on the plate (the dam) is desirable, but extremely difficult. However, one can consider a few points on the plate and approximate the temperature of these points. This approximation is based on a very important physical property called the **Mean-Value Property**: If a plate has reached thermal equilibrium, and P is a point on the plate, and C is a circle centered at P fully contained in the plate, then the temperature at P is the average value of the temperature function over C.

Applications of Linear Systems

Temperature Distribution

Example

Consider the square plate divided into a gird of nine smaller squares needs, as shown in the figure. Find the interior temperatures T_1, T_2, T_3 and T_4.



Applications of Linear Systems

Temperature Distribution

Solution

The distribution of temperature and other properties in a continuous material can be approximated by linear equations. Using Mean-Value Property, we get the following system of four equations in nine unknowns:

$$
\begin{aligned}
T_1 &= \frac{1}{4}(T_2 + T_3 + 10 + 15);\\
T_2 &= \frac{1}{4}(20 + 25 + T_4 + T_1);\\
T_3 &= \frac{1}{4}(T_4 + T_1 + 15 + 20);\\
T_4 &= \frac{1}{4}(40 + T_2 + T_3 + 30),
\end{aligned}
$$

Applications of Linear Systems

Temperature Distribution

Solution

$$
\begin{array}{rcrcrcrl}
4T_1 & - & T_2 & - & T_3 & + & & = 25; \\
-T_1 & + & 4T_2 & + & & - & T_4 & = 45; \\
-T_1 & + & & + & 4T_3 & - & T_4 & = 35; \\
& - & T_2 & - & T_3 & + & 4T_4 & = 70.
\end{array}
$$

This system has the augmented matrix

$$
\left[\begin{array}{cccc|c}
4 & -1 & -1 & 0 & 25 \\
-1 & 4 & 0 & -1 & 45 \\
-1 & 0 & 4 & -1 & 35 \\
0 & -1 & -1 & 4 & 70
\end{array}\right] \sim \left[\begin{array}{cccc|c}
1 & -4 & 0 & 1 & -45 \\
0 & 1 & 1 & -4 & -70 \\
0 & 0 & 8 & -16 & -290 \\
0 & 0 & 0 & 24 & 675
\end{array}\right]
$$

and $T_1 = 16.875^\circ, T_2 = 22.5^\circ, T_3 = 20^\circ$ *and* $T_4 = 28.125^\circ$.

Applications of Linear Systems

Nutrition

Designing a healthy diet involves selecting foods from different groups that, when combined in the proper amounts, satisfy certain nutritional requirements. Here, the linear algebra plays an important rule to get the optimal quantities of meals.

Example

The table below gives the amount, in milligrams (mg), of vitamin A, vitamin C, and calcium contained in 1 gram (g) of four different foods. Suppose that a dietician wants to prepare a meal that provides 200 mg of vitamin A, 250 mg of vitamin C, and 300 mg of calcium. How much of each food should be used?

	Food 1	Food 2	Food 3	Food 4
Vitamen A	10	20	30	10
Vitamen C	50	30	25	10
Calsiom	60	20	40	25

Applications of Linear Systems

Nutrition

Solution

Let x_1, x_2, x_3 and x_4 be the amounts of foods 1 through 4, respectively. The amounts for each of the foods needed to satisfy the dietician's requirement can be found by solving the linear system

$$\begin{aligned}
10x_1 + 30x_2 + 20x_3 + 10x_4 &= 200 \\
50x_1 + 30x_2 + 25x_3 + 10x_4 &= 250 \\
60x_1 + 20x_2 + 40x_3 + 25x_4 &= 300
\end{aligned}$$

with the augmented matrix

$$\left[\begin{array}{cccc|c} 10 & 30 & 20 & 10 & 200 \\ 50 & 30 & 25 & 10 & 250 \\ 60 & 20 & 40 & 25 & 300 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 3 & 2 & 1 & 20 \\ 0 & 12 & 75 & 40 & 75 \\ 0 & 0 & 92 & \frac{299}{6} & 10 \end{array}\right]$$

Solution

Rounded to two decimal places, the solution to the linear system is given by $x_1 = 0.63 + 0.11t$, $x_2 = 3.13 + 0.24t$, $x_3 = 5 - 0.92t$ *and* $x_4 = t$. *Observe that each of these values must be nonnegative. Hence, particular solutions can be found by choosing nonnegative values of* t *such that* $t \leq 5.4$.