1.3. Propositional equivalences

Exercises

- Prove that $\neg(p \lor (\neg p \land q))$ and $\neg p \land \neg q$ are logically equivalent using propositional equivalences laws.

Solution

1. Using propositional equivalences laws.
2. $\neg(p \lor (\neg p \land q)) = \neg p \land \neg(\neg p \land q)$

$$= \neg p \land \neg(\neg p) \lor \neg q$$

$$= \neg p \land (p \lor \neg q)$$

$$= (\neg p \land p) \lor (\neg p \land \neg q)$$

$$= \mathbf{F} \lor (\neg p \land \neg q)$$

$$= (\neg p \land \neg q) \lor \mathbf{F}$$

$$= (\neg p \land \neg q)$$

$\neg(p \lor q) \equiv \neg p \land \neg q$ $\neg(p \land q) \equiv \neg p \lor \neg q$	De Morgan's
$\neg(\neg p) \equiv p$	Double Negation
$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$ $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$	Distributive
$p \lor \neg p \equiv \mathbf{T}$ $p \land \neg p \equiv \mathbf{F}$	Negation
$p \lor q \equiv q \lor p$ $p \land q \equiv q \land p$	Commutative
$p \land \mathbf{T} \equiv p$ $p \lor \mathbf{F} \equiv p$	Identity

1.3. Propositional equivalences

Exercises

- Prove that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology using propositional equivalences laws.

Homework

Solution

1. Using propositional equivalences laws.
2. $(p \wedge q) \rightarrow (p \vee q) = \neg (p \wedge q) \vee (p \vee q)$

$$= (\neg p \vee \neg q) \vee (p \vee q)$$

$$= (\neg p \vee p) \vee (\neg q \vee q)$$

$$= T \vee T$$

$$= T$$

$(p \wedge q) \rightarrow (p \vee q) = \neg (p \wedge q) \vee (p \vee q)$ Implication

$\neg(p \vee q) \equiv \neg p \wedge \neg q$

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

De Morgan's

$(p \vee q) \vee r \equiv p \vee (q \vee r)$

$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

Associative

$p \vee q \equiv q \vee p$

$p \wedge q \equiv q \wedge p$

Commutative

$p \vee \neg p \equiv \mathbf{T}$

$p \wedge \neg p \equiv \mathbf{F}$

Negation

1.3. Propositional equivalences

Exercises

- Prove that $(p \to r) \vee (q \to r)$ and $(p \wedge q) \to r$ are logically equivalent using propositional equivalences laws.

Solution

Homework

$((p \to r) \vee (q \to r) = (\neg p \vee r) \vee (\neg q \vee r)$ Implication

1. Using propositional equivalences laws.
2. $(p \to r) \vee (q \to r) = (\neg p \vee r) \vee (\neg q \vee r)$

$= (\neg p \vee \neg q) \vee (r \vee r)$

$= \neg(p \wedge q) \vee r$

$= (p \wedge q) \to r$

$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative
$\neg(p \vee q) \equiv \neg p \wedge \neg q$ $\neg(p \wedge q) \equiv \neg p \vee \neg q$	De Morgan's
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent

Homework

$\neg (p \wedge q) \vee r \equiv \mathbf{(p \wedge q) \to r}$ Definition of implication

Chapter 1

Lecture 3: Predicates and Quantifiers

5/3/2022

Book: Sections 1.4

1.4 Predicates and Quantifiers

Predicate

- Propositional logic cannot adequately express the meaning of all statements in mathematics and in natural language.
- Statements involving variables are called **Predicate**

1.4 Predicates and Quantifiers

Examples of Predicate

- Statement involving variables, such as “$x > 3$”, “$x = y + 3$” and “$x + y = z$”, are often found in mathematical assertions and in computer programs.
- These statements are neither true nor false when the values of the variables are not specified.

1.4 Predicates and Quantifiers

Predicate

The statement: x is greater than 3

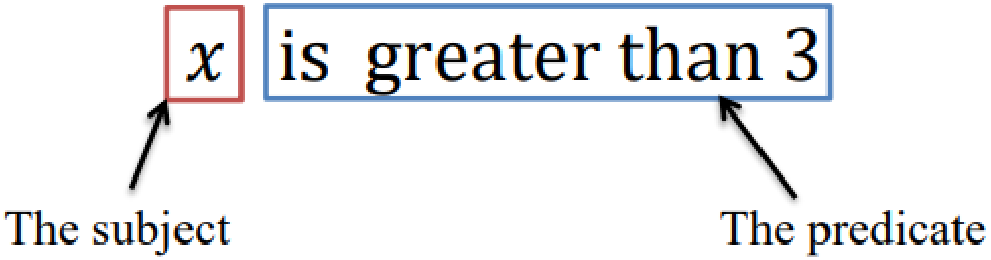
has two parts:

1. The first part:

    x: subject

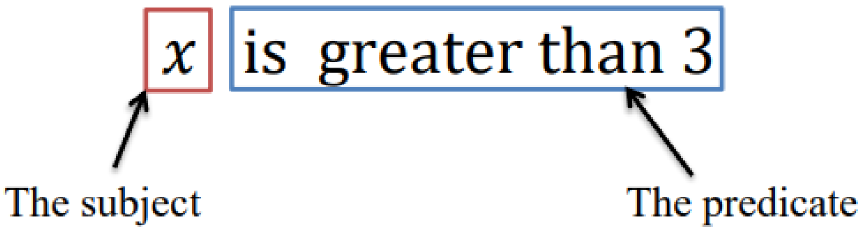
2. The second part:

    "is greater than 3": predicate



1.4 Predicates and Quantifiers

Predicate



- The statement "$x \text{ is greater than } 3$":
  - Can be denoted by $\boldsymbol{P(x)}$, where:
    - $\boldsymbol{P}$ denotes the predicate "$\text{is greater than } 3$"
    - $\boldsymbol{x}$ is the variable.
- The statement $\boldsymbol{P(x)}$ is also said to be the value of the *propositional function* $\boldsymbol{P}$ *at* $\boldsymbol{x}$.
- Once a value has been assigned to the variable $\boldsymbol{x}$, the statement $\boldsymbol{P(x)}$ becomes a proposition and has a truth value.
- P(x) will create a proposition at given values of x.

1.4 Predicates and Quantifiers

Predicate

Example (1)

- ***P(x)* = x > 3, what are the truth values of P(4) and P(2)**

Solution

1. P(x) has no truth values (x is not given a value)
2. P(4): x = 4 in the statement “x > 3.”
   - P(4): “4 > 3,” is **true**.
3. P(2): x = 2 in the statement “x > 3.”
   - “2 > 3,” is **false**.

1.4 Predicates and Quantifiers

Predicate

Example (2)

- ***P(x)* = x < 5, what are the truth values of P(1) and P(10)**

Solution

1. P(x) has no truth values (x is not given a value)
2. P(1) is true: The proposition 1<5 is **true**
3. P(10) is false: The proposition 10<5 is **false**

1.4 Predicates and Quantifiers

Predicate

Example (3)

- Let $P(x)$ denote the statement "$x \leq 4$", what are the truth values of :

  1. $P(0)$ 2. $P(4)$ 3. $P(6)$

Solution

1. $P(0)$ True
2. $P(4)$ True
3. $P(6)$ False

1.4 Predicates and Quantifiers

Predicate

Example (3)

➢ Let *P(x)* denote the statement "the word x contains the letter a.", what are the truth values of :

1. *P(orange)* 2. *P(lemon)* 3. *P(true)* 4. *P(false)*

Solution

1. *P(orange)* True
2. *P(lemon)* False
3. *P(true)* False
4. *P(false)* True

1.4 Predicates and Quantifiers

Predicate

Example (4)

➢ **Let *Q(x,y)* denote the statement "x = y+ 3", what are the truth values of *Q(1,2)* and *Q(3,0)***

Solution

1. *Q(1,2)*: ***False***
   - As x=1 , y = 2 and 1 = 2+3 is false
1. *Q(3,0)*: **True**
   - As x=3 , y = 0 and 3 = 0+3 is true

1.4 Predicates and Quantifiers

Quantifiers

- Quantification expresses the extent to which a predicate is true over a range of elements.

1.4 Predicates and Quantifiers

Types of Quantifiers

- **Quantifiers**
  - **Universal**
    - Represented by $\forall\ x\ P(x)$
    - $P(x)$ for all values of x in the domain
  - **Existential**
    - Represented by $\exists\ x\ P(x)$
    - There exists an element x in the (Domain) such that $P(x)$ is true.
  - **Uniqueness**
    - Represented by $\exists!\ x\ P(x)$
    - There exists unique x in the (Domain) such that $P(x)$ is true.

1.4 Predicates and Quantifiers

Types of Quantifiers

Quantifiers

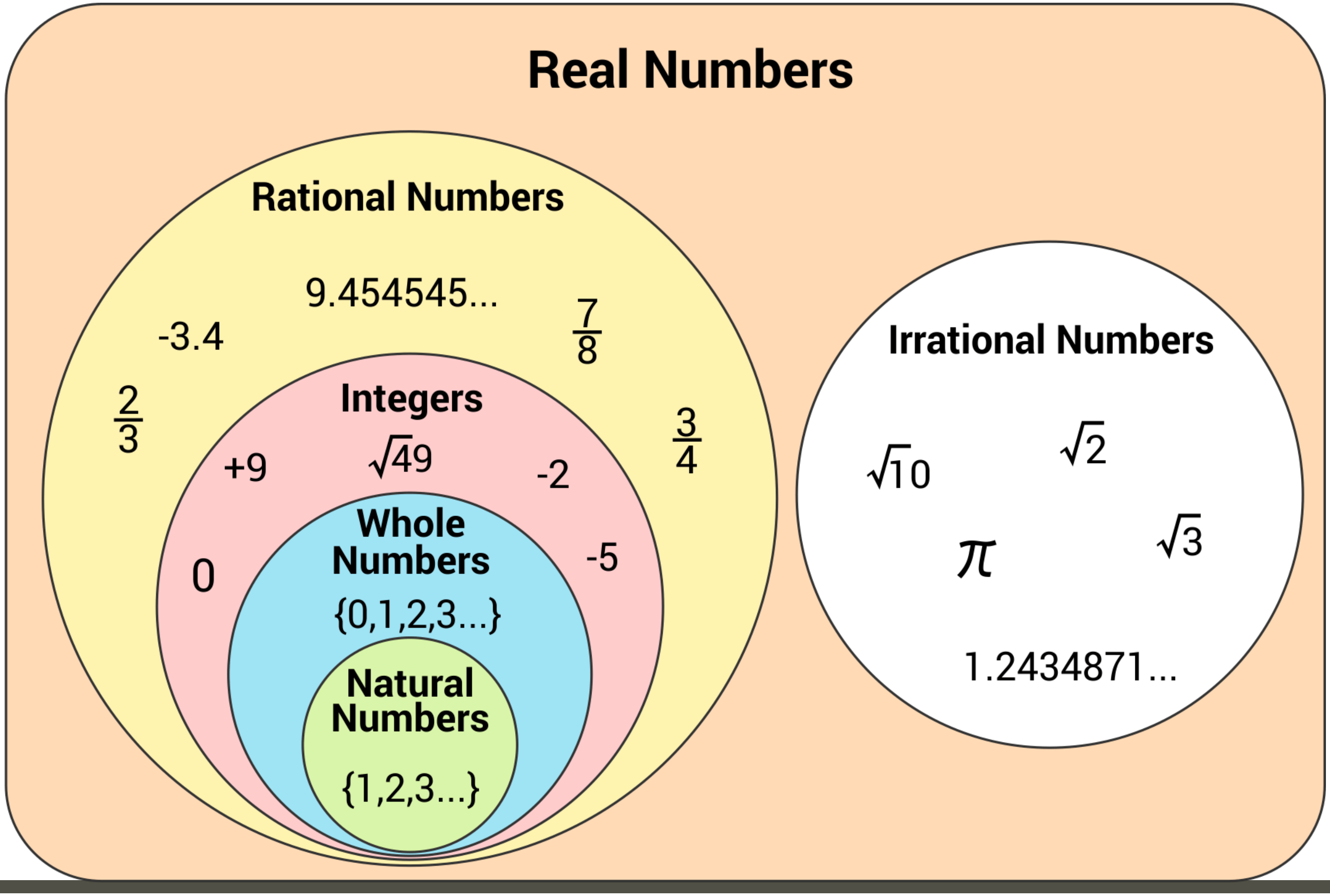
Universal

- Represented by $\forall\ x\ P(x)$
- $\forall$ is called the universal quantifier.
- We read $\forall$x P(x) as "for all x P(x)" or "for every x P(x)"
- It means “for all”
- $P(x)$ for all values of x in the domain
- An element for which *P(x)* is false is called a **counterexample** of $x\ P(x)$.

- **Values x can represent called the “domain” or “universe of discourse”.**

1.4 Predicates and Quantifiers

Universal Quantifiers



1.4 Predicates and Quantifiers

Universal Quantifiers

Example (5)

What is the truth value of $\forall x\,(x^2 \geq x)$ for real and integers numbers

Solution

- If **Domain** is all real numbers, the truth value is **false**

  (take $x = 0.5$, this is called a **counterexample**).

- If **Domain** is the set of integers, the truth value is **true**.

Remark: you need to one counterexample to prove that the universal quantification is false.

1.4 Predicates and Quantifiers

Universal Quantifiers

Example (6)

Let *P(x)* be the statement "x+1 > x."

- What is the truth value for the $\forall x P(x)$, *where the domain consists of all real numbers.*

Solution

- **Domain** is all real numbers, the truth value for the quantification $\forall x\ P(x)$ is **true**

Remark: you need to one counterexample to prove that the universal quantification is false.

1.4 Predicates and Quantifiers

Universal Quantifiers

Example (7)

- Suppose that P(x) is "$x^2 > 0$"
  - What is the truth value of the quantification $\forall x\, P(x)$, where the domain consists of all integer numbers?

Solution

- The statement $\forall x\, P(x)$ is false
- Because the value of $x = 0$ is a *counterexample*

Remark: you need to one counterexample to prove that the universal quantification is false.

1.4 Predicates and Quantifiers

Universal Quantifiers

Example (8)

Let $Q(x)$ be the statement "x < 2."

- What is the truth value for the $\forall x Q(x)$, *where the domain consists of all real numbers.*

Solution

$Q(x)$ is not **true** for every real x ($\forall x P(x)$) as for x = 3 (*counterexample)*

Then $\forall x P(x)$ is false.

Remark: you need to one counterexample to prove that the universal quantification is false.

1.4 Predicates and Quantifiers

Universal Quantifiers

Example (9)

Suppose that P(x) is "$x^2 > 0$"

- What is the truth value of the quantification $\forall x\, P(x)$, where the domain consists of all integer numbers?

Solution

- The statement $\forall x\, P(x)$ is false
- Because the value of $x = 0$ is a *counterexample*

Remark: you need to one counterexample to prove that the universal quantification is false.

1.4 Predicates and Quantifiers

Universal Quantifiers

Example (10)

Let P(x) is "$\frac{X}{2} < x$"

- What is the truth value of the quantification ∀x P(x), where the domain consists of all real numbers?

Solution

- The statement ∀x P(x) is false
- Because all negative values of x are *counterexample*

Remark: you need to one counterexample to prove that the universal quantification is false.

1.4 Predicates and Quantifiers

Types of Quantifiers

Quantifiers

Existential

- Represented by $\exists\, x\, P(x)$
- There exists an element x in the (Domain) such that $P(x)$ is true.
- $\exists\, x\, P(x)$: There exists an element x in the universe of discourse (Domain) such that P(x) is true.

1.4 Predicates and Quantifiers

Types of Quantifiers

Example (11)

Let $P(x)$: $x = x + 1$, Domain is the set of all real numbers:

Solution

- The truth value of $\exists x\, P(x)$ is false (there is no real x such that $x = x + 1$).

1.4 Predicates and Quantifiers

Types of Quantifiers

Example (12)

Example : Let $P(x)$: $x^2 = x$, Domain is the set of all real numbers:

Solution

- The truth value of $\exists x\, P(x)$ is **true** (take $x = 1$).

1.4 Predicates and Quantifiers

Quantifiers Over Finite Domains

Example (12): Find the truth value

Let $P(x)$: $x^2 \geq x$, the domain is the set {0.5, 1, 2, 3}.

- $\forall x\, P(x) \equiv P(0.5) \wedge P(1) \wedge P(2) \wedge P(3)$
  - $\equiv F \wedge T \wedge T \wedge T$
  - $\equiv F$
- $\exists x\, P(x) \equiv P(0.5) \vee P(1) \vee P(2) \vee P(3)$
  - $\equiv F \vee T \vee T \vee T$
  - $\equiv T$

1.4 Predicates and Quantifiers

Remember

Statement	*When True?*	*When False?*
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

1.4 Predicates and Quantifiers

Homework

Let $P(x)$ be the statement "$x = x^2$." If the domain consists of the integers, what are the truth values?

a) $P(0)$ **b)** $P(1)$ **c)** $P(2)$
d) $P(-1)$ **e)** $\exists x P(x)$ **f)** $\forall x P(x)$

Thank you