VECTOR FUNCTION

Vector Differential Operator Del *i.e.* ∇

The vector differential operator Del is denoted by ∇. It is defined as

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

(*ii*) **Directional derivative.** The component of $\nabla\phi$ in the direction of a vector $\vec{d}$ is equal to $\nabla\phi.\hat{d}$ and is called the directional derivative of ϕ in the direction of $\vec{d}$.

Example 16. *If* $\phi = 3x^2y - y^3z^2$; *find grad* ϕ *at the point (1, –2, –1).*

(*AMIETE, June 2009, U.P., I Semester, Dec. 2006*)

Solution.

$$\text{grad } \phi = \nabla\phi$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(3x^2y - y^3z^2)$$

$$= \hat{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \hat{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \hat{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= \hat{i}\,(6xy) + \hat{j}\,(3x^2 - 3y^2z^2) + \hat{k}\,(-2y^3z)$$

$$\text{grad } \phi \text{ at } (1, -2, -1) = \hat{i}\,(6)\,(1)\,(-2) + \hat{j}\,[(3)\,(1) - 3(4)\,(1)] + \hat{k}\,(-2)(-8)\,(-1)$$

$$= -12\hat{i} - 9\hat{j} - 16\hat{k}$$ **Ans.**

$-12\hat{i} - 9\hat{j} - 16\hat{k}$ **Ans.**

Example 17. *If* $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = yz + zx + xy$ *prove that grad u, grad v and grad w are coplanar vectors.* [*U.P., I Semester, 2001*]

Solution. We have,

$$\text{grad } u = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x + y + z) = \hat{i} + \hat{j} + \hat{k}$$

$$\text{grad } v = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2) = 2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}$$

$$\text{grad } w = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(yz + zx + xy) = \hat{i}(z + y) + \hat{j}(z + x) + \hat{k}(y + x)$$

[For vectors to be coplanar, their scalar triple product is 0]

Example 19. *Find the unit normal to the surface $xy^3z^2 = 4$ at $(-1, -1, 2)$.* (*M.U. 2008*)

Solution. Let $\phi(x, y, z) = xy^3z^2 = 4$

We know that $\nabla\phi$ is the vector normal to the surface $\phi(x, y, z) = c$.

$$\text{Normal vector} = \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Now

$$= \hat{i}\frac{\partial}{\partial x}(xy^3z^2) + \hat{j}\frac{\partial}{\partial y}(xy^3z^2) + \hat{k}\frac{\partial}{\partial z}(xy^3z^2)$$

$$\Rightarrow \quad \text{Normal vector} = y^3z^2\,\hat{i} + 3xy^2z^2\,\hat{j} + 2xy^3z\,\hat{k}$$

Normal vector at $(-1, -1, 2) = -4\hat{i} - 12\hat{j} + 4\hat{k}$

Unit vector normal to the surface at $(-1, -1, 2)$.

$$= \frac{\nabla\phi}{|\nabla\phi|} = \frac{-4\hat{i} - 12\hat{j} + 4\hat{k}}{\sqrt{16+144+16}} = -\frac{1}{\sqrt{11}}(\hat{i} + 3\hat{j} - \hat{k})$$ **Ans.**

Example 20. *Find the rate of change of $\phi = xyz$ in the direction normal to the surface $x^2y + y^2x + yz^2 = 3$ at the point $(1, 1, 1)$.* (*Nagpur University, Summer 2001*)

Solution. Rate of change of $\phi = \Delta\,\phi$

Rate of change of ϕ at $(1, 1, 1) = (\hat{i} + \hat{j} + \hat{k})$

Normal to the surface $\Psi = x^2y + y^2x + yz^2 - 3$ is given as -

$$\nabla\psi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2y + y^2x + yz^2 - 3)$$

$$= \hat{i}(2xy + y^2) + \hat{j}(x^2 + 2xy + z^2) + \hat{k}\,2yz$$

$$(\nabla\psi)_{(1,\,1,\,1)} = 3\hat{i} + 4\hat{j} + 2\hat{k}$$

$$\text{Unit normal} = \frac{3\hat{i} + 4\hat{j} + 2\hat{k}}{\sqrt{9+16+4}}$$

$$\text{Required rate of change of } \phi = (\hat{i} + \hat{j} + \hat{k}).\frac{(3\hat{i} + 4\hat{j} + 2\hat{k})}{\sqrt{9+16+4}} = \frac{3+4+2}{\sqrt{29}} = \frac{9}{\sqrt{29}}$$ **Ans.**

Example 21. *Find the constants m and n such that the surface $m\,x^2 - 2nyz = (m+4)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at the point (1, –1, 2).*

(*M.D.U. Dec. 2009, Nagpur University, Summer 2002*)

Solution. The point P (1, –1, 2) lies on both surfaces. As this point lies in

$$mx^2 - 2nyz = (m+4)x, \text{ so we have}$$

$$m - 2n(-2) = (m+4)$$

$$\Rightarrow \quad m + 4n = m + 4 \quad \Rightarrow \quad n = 1$$

$$\therefore \quad \text{Let } \phi_1 = mx^2 - 2yz - (m+4)x \text{ and } \phi_2 = 4x^2y + z^3 - 4$$

$$\text{Normal to } \phi_1 = \nabla\phi_1$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)[mx^2 - 2yz - (m+4)x]$$

$$= \hat{i}(2mx - m - 4) - 2z\,\hat{j} - 2y\,\hat{k}$$

$$\text{Normal to } \phi_1 \text{ at } (1, -1, 2) = \hat{i}(2m - m - 4) - 4\hat{j} + 2\hat{k} = (m-4)\hat{i} - 4\hat{j} + 2\hat{k}$$

$$\text{Normal to } \phi_2 = \nabla\phi_2$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(4x^2y + z^3 - 4) = \hat{i}\,8xy + 4x^2\,\hat{j} + 3z^2\,\hat{k}$$

$$\text{Normal to } \phi_2 \text{ at } (1, -1, 2) = -8\hat{i} + 4\hat{j} + 12\hat{k}$$

Sinec ϕ_1 and ϕ_2 are orthogonal, then normals are perpendicular to each other.

$$\nabla\phi_1 \,.\, \nabla\phi_2 = 0$$

$$\Rightarrow \quad [(m-4)\hat{i} - 4\hat{j} + 2\hat{k}].[-8\hat{i} + 4\hat{j} + 12\hat{k}] = 0$$

$$\Rightarrow \quad -8(m-4) - 16 + 24 = 0$$

$$\Rightarrow \quad m - 4 = -2 + 3 \quad \Rightarrow \quad m = 5$$

Hence $m = 5$, $n = 1$ **Ans.**

Example 23. *Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$.* (*Nagpur University, Summer 2002*)

Solution. Normal on the surface $(x^2 + y^2 + z^2 - 9 = 0)$

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - 9) = (2x\hat{i} + 2y\hat{j} + 2z\hat{k})$$

Normal at the point $(2, -1, 2) = 4\hat{i} - 2\hat{j} + 4\hat{k}$ ...(1)

Normal on the surface $(z = x^2 + y^2 - 3) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 - z - 3)$

$$= 2x\hat{i} + 2y\hat{j} - \hat{k}$$

Normal at the point $(2, -1, 2) = 4\hat{i} - 2\hat{j} - \hat{k}$ ...(2)

Let θ be the angle between normals (1) and (2).

$$(4\hat{i} - 2\hat{j} + 4\hat{k}).(4\hat{i} - 2\hat{j} - \hat{k}) = \sqrt{16+4+16}\sqrt{16+4+1}\cos\theta$$

$$16 + 4 - 4 = 6\sqrt{21}\cos\theta \quad \Rightarrow \quad 16 = 6\sqrt{21}\cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{8}{3\sqrt{21}} \quad \Rightarrow \quad \theta = \cos^{-1}\frac{8}{3\sqrt{21}}$$ **Ans.**

Example 26. *Find the directional derivative of $\phi(x, y, z) = x^2yz + 4xz^2$ at $(1, -2, 1)$ in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$. Find the greatest rate of increase of ϕ.*

(*Uttarakhand, I Semester, Dec. 2006*)

Solution. Here, $\phi(x, y, z) = x^2yz + 4xz^2$

Now, $$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2yz + 4xz^2)$$

$$= (2xyz + 4z^2)\hat{i} + (x^2z)\hat{j} + (x^2y + 8xz)\hat{k}$$

$$\nabla\phi \text{ at } (1, -2, 1) = \{2(1)(-2)(1) + 4(1)^2\}\hat{i} + (1\times 1)\hat{j} + \{1(-2) + 8(1)(1)\}\hat{k}$$

$$= (-4+4)\hat{i} + \hat{j} + (-2+8)\hat{k} = \hat{j} + 6\hat{k}$$

Let $$\hat{a} = \text{unit vector} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{4+1+4}} = \frac{1}{3}(2\hat{i} - \hat{j} - 2\hat{k})$$

So, the required directional derivative at $(1, -2, 1)$

$$= \nabla\phi.\hat{a} = (\hat{j} + 6\hat{k}).\frac{1}{3}(2\hat{i} - \hat{j} - 2\hat{k}) = \frac{1}{3}(-1-12) = \frac{-13}{3}$$

Greatest rate of increase of $\phi = \left|\hat{j} + 6\hat{k}\right| = \sqrt{1+36}$

$$= \sqrt{37}$$ **Ans.**

$= \sqrt{57}$ **Ans.**

Example 27. *Find the directional derivative of the function $\phi = x^2 - y^2 + 2z^2$ at the point P (1, 2, 3) in the direction of the line PQ where Q is the point (5, 0, 4).*

(*AMIETE, Dec. 20010, Nagpur University, Summer 2008, U.P., I Sem., Winter 2000*)

Solution. Directional derivative $= \bar{\nabla}\phi$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 - y^2 + 2z^2) = 2x\hat{i} - 2y\hat{j} + 4z\hat{k}$$

Directional Derivative at the point P (1, 2, 3) $= 2\hat{i} - 4\hat{j} + 12\hat{k}$ ...(1)

$$\overline{PQ} = \bar{Q} - \bar{P} = (5, 0, 4) - (1, 2, 3) = (4, -2, 1) \qquad ...(2)$$

Directional Derivative along $PQ = (2\hat{i} - 4\hat{j} + 12\hat{k}).\dfrac{(4\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{16+4+1}}$ [From (1) and (2)]

$$= \frac{8+8+12}{\sqrt{21}} = \frac{28}{\sqrt{21}}$$ **Ans.**

Example 29. *Find the directional derivative of $\vec{V}^2$, where $\vec{V} = xy^2\,\hat{i} + zy^2\,\hat{j} + xz^2\,\hat{k}$, at the point (2, 0, 3) in the direction of the outward normal to the sphere $x^2 + y^2 + z^2 = 14$ at the point (3, 2, 1).* (*A.M.I.E.T.E., Dec. 2007*)

Solution. $V^2 = \vec{V}.\vec{V}$

$$= (xy^2\,\hat{i} + zy^2\,\hat{j} + xz^2\,\hat{k}).(x\,y^2\,\hat{i} + z\,y^2\,\hat{j} + xz^2\,\hat{k}) = x^2y^4 + z^2y^4 + x^2z^4$$

Directional derivative $= \bar{\nabla}V^2$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2y^4 + z^2y^4 + x^2z^4)$$

$$= (2xy^4 + 2xz^4)\,\hat{i} + (4x^2y^3 + 4y^3z^2)\,\hat{j} + (2y^4z + 4x^2z^3)\,\hat{k}$$

Directional derivative at (2, 0, 3) $= (0 + 2\times 2\times 81)\,\hat{i} + (0+0)\,\hat{j} + (0 + 4\times 4\times 27)\,\hat{k}$

$$= 324\,\hat{i} + 432\,\hat{k} = 108\,(3\,\hat{i} + 4\,\hat{k}) \qquad ...(1)$$

Normal to $x^2 + y^2 + z^2 - 14 = \nabla\phi$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - 14)$$

$$= (2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k})$$

Normal vector at (3, 2, 1) $= 6\,\hat{i} + 4\,\hat{j} + 2\,\hat{k}$ ...(2)

Unit normal vector $= \dfrac{6\,\hat{i} + 4\,\hat{j} + 2\,\hat{k}}{\sqrt{36+16+4}} = \dfrac{2(3\,\hat{i} + 2\,\hat{j} + \hat{k})}{2\sqrt{14}} = \dfrac{3\,\hat{i} + 2\,\hat{j} + \hat{k}}{\sqrt{14}}$ [From (1), (2)]

Directional derivative along the normal $= 108(3\,\hat{i} + 4\,\hat{k}).\dfrac{3\,\hat{i} + 2\,\hat{j} + \hat{k}}{\sqrt{14}}.$

$$= \frac{108\times(9+4)}{\sqrt{14}} = \frac{1404}{\sqrt{14}}$$ **Ans.**

Example 30. *Find the directional derivative of $\nabla (\nabla f)$ at the point $(1, -2, 1)$ in the direction of the normal to the surface $xy^2z = 3x + z^2$, where $f = 2x^3y^2z^4$.* (*U.P., I Semester, Dec 2008*)

Solution. Here, we have

$$f = 2x^3 y^2 z^4$$

$$\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2x^3y^2z^4) = 6x^2y^2z^4\hat{i} + 4x^3yz^4\hat{j} + 8x^3y^2z^3\hat{k}$$

$$\nabla(\nabla f) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(6x^2y^2z^4\hat{i} + 4x^3yz^4\hat{j} + 8x^3y^2z^3\hat{k})$$

$$= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$

Directional derivative of $\nabla(\nabla f)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2)$$

$$= (12y^2z^4 + 12x^2z^4 + 72x^2y^2z^2)\hat{i} + (24xyz4 + 48x3yz2)\hat{j}$$
$$+ (48xy^2z^3 + 16x^3z^3 + 48x^3y^2z)\hat{k}$$

Directional derivative at $(1, -2, 1) = (48 + 12 + 288)\hat{i} + (-48 - 96)\hat{j} + (192 + 16 + 192)\hat{k}$

$$= 348\hat{i} - 144\hat{j} + 400\hat{k}$$

Normal to$(xy^2z - 3x - z^2) = \nabla(xy^2z - 3x - z^2)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xy^2z - 3x - z^2)$$

$$= (y^2z - 3)\hat{i} + (2xyz)\hat{j} + (xy^2 - 2z)\hat{k}$$

Normal at$(1, -2, 1) = \hat{i} - 4\hat{j} + 2\hat{k}$

Unit Normal Vector $= \dfrac{\hat{i} - 4\hat{j} + 2\hat{k}}{\sqrt{1 + 16 + 4}} = \dfrac{1}{\sqrt{21}}(\hat{i} - 4\hat{j} + 2\hat{k})$

Directional derivative in the direction of normal

$$= (348\hat{i} - 144\hat{j} + 400\hat{k})\frac{1}{\sqrt{21}}(\hat{i} - 4\hat{j} + 2\hat{k})$$

$$= \frac{1}{\sqrt{21}}(348 + 576 + 800) = \frac{1724}{\sqrt{21}}$$ **Ans.**

DIVERGENCE OF A VECTOR FUNCTION

The divergence of a vector point function $\vec{F}$ is denoted by *div F* and is defined as below.

Let $\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$

$$\operatorname{div} \vec{F} = \vec{\nabla} . \vec{F} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (\hat{i} F_1 + \hat{j} F_2 + \hat{k} F_3) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

It is evident that div F is scalar function.

$$\Big(\quad \partial x \qquad \partial y \qquad \partial z \Big)$$

If the fluid is compressible, there can be no gain or loss in the volume element. Hence

$$\operatorname{div} \vec{V} = 0 \qquad \text{...(1)}$$

and V is called a *Solenoidal* vector function.

Example 35. *If* $u = x^2 + y^2 + z^2$, *and* $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, *then find div* $(u\bar{r})$ *in terms of u.*

(*A.M.I.E.T.E., Summer 2004*)

Solution. $$\operatorname{div}\ (u\vec{r}) = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) . [(x^2 + y^2 + z^2)\ (x\hat{i} + y\hat{j} + z\hat{k})]$$

$$= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) . [(x^2 + y^2 + z^2) x\hat{i} + (x^2 + y^2 + z^2) y\hat{j} + (x^2 + y^2 + z^2) z\hat{k}]$$

$$= \frac{\partial}{\partial x}(x^3 + xy^2 + xz^2) + \frac{\partial}{\partial y}(x^2 y + y^3 + yz^2) + \frac{\partial}{\partial z}(x^2 z + y^2 z + z^3)$$

$$= (3x^2 + y^2 + z^2) + (x^2 + 3y^2 + z^2) + (x^2 + y^2 + 3z^2) = 5\ (x^2 + y^2 + z^2) = 5\ u$$ **Ans.**

Example 39. *Find the directional derivative of div $(\vec{u})$ at the point (1, 2, 2) in the direction of the outer normal of the sphere $x^2 + y^2 + z^2 = 9$ for $\vec{u} = x^4\hat{i} + y^4\hat{j} + z^4\hat{k}$.*

Solution. $\text{div}\,(\vec{u}) = \nabla.\vec{u}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(x^4\hat{i} + y^4\hat{j} + z^4\hat{k}) = 4x^3 + 4y^3 + 4z^3$$

Outer normal of the sphere $= \nabla(x^2 + y^2 + z^2 - 9)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - 9) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

Outer normal of the sphere at (1, 2, 2) $= 2\hat{i} + 4\hat{j} + 4\hat{k}$ ...(1)

Directional derivative $= \vec{\nabla}(4x^3 + 4y^3 + 4z^3)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(4x^3 + 4y^3 + 4z^3) = 12x^2\hat{i} + 12y^2\hat{j} + 12z^2\hat{k}$$

Directional derivative at (1, 2, 2) $= 12\hat{i} + 48\hat{j} + 48\hat{k}$ ...(2)

Directional derivative along the outer normal $= (12\hat{i} + 48\hat{j} + 48\hat{k}).\dfrac{2\hat{i} + 4\hat{j} + 4\hat{k}}{\sqrt{4 + 16 + 16}}$

[From (1), (2)]

$$= \frac{24 + 192 + 192}{6} = 68$$ **Ans.**

Example 36. *Find the value of n for which the vector $r^n\vec{r}$ is solenoidal, where $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$.*

Solution. Divergence $\vec{F} = \vec{\nabla}.\vec{F} = \vec{\nabla}.r^n\vec{r} = \nabla.(x^2 + y^2 + z^2)^{n/2}(x\hat{i} + y\hat{j} + z\hat{k})$

$$= \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right].[(x^2 + y^2 + z^2)^{n/2}x\hat{i} + (x^2 + y^2 + z^2)^{n/2}y\hat{j} + (x^2 + y^2 + z^2)^{n/2}z\hat{k}]$$

$$= \frac{n}{2}(x^2 + y^2 + z^2)^{n/2-1}(2x^2) + (x^2 + y^2 + z^2)^{n/2} + \frac{n}{2}(x^2 + y^2 + z^2)^{n/2-1}(2y^2)$$

$$+ (x^2 + y^2 + z^2)^{n/2} + \frac{n}{2}(x^2 + y^2 + z^2)^{n/2-1}(2z^2) + (x^2 + y^2 + z^2)^{n/2}$$

$$= n(x^2 + y^2 + z^2)^{n/2-1}(x^2 + y^2 + z^2) + 3(x^2 + y^2 + z^2)^{n/2}$$

$$= n(x^2 + y^2 + z^2)^{n/2} + 3(x^2 + y^2 + z^2)^{n/2} = (n + 3)(x^2 + y^2 + z^2)^{n/2}$$

If $r^n\vec{r}$ is solenoidal, then $(n + 3)(x^2 + y^2 + z^2)^{n/2} = 0$ or $n + 3 = 0$ or $n = -3$. **Ans.**

5.31 CURL (*U.P., I semester, Dec. 2006*)

The curl of a vector point function F is defined as below

$$\text{curl } \vec{F} = \vec{\nabla} \times \vec{F} \qquad (\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k})$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times (F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \hat{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

Curl $\vec{F}$ is a vector quantity.

If Curl $\overline{F} = 0$, the field F is termed as *irrotational*.

Example 41. *Find the divergence and curl of* $\vec{v} = (xyz)\hat{i} + (3x^2 y)\hat{j} + (xz^2 - y^2 z)\hat{k}$ *at* (*2, –1, 1*) (*Nagpur University, Summer 2003*)

Solution. Here, we have

$$\vec{v} = (x\,y\,z)\hat{i} + (3x^2 y)\hat{j} + (xz^2 - y^2 z)\hat{k}$$

$$\text{Div. } \vec{v} = \nabla\phi$$

$$\text{Div } \vec{v} = \frac{\partial}{\partial x}(x\,y\,z) + \frac{\partial}{\partial y}(3x^2 y) + \frac{\partial}{\partial z}(xz^2 - y^2 z)$$

$$= yz + 3x^2 + 2x\,z - y^2 \qquad = -1 + 12 + 4 - 1 = 14 \text{ at } (2, -1, 1)$$

$$\text{Curl } \overline{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2 y & xz^2 - y^2 z \end{vmatrix} \qquad = -2yz\,\hat{i} - (z^2 - xy)\,\hat{j} + (6xy - xz)\,\hat{k}$$

$$= -2yz\,\hat{i} + (xy - z^2)\,\hat{j} + (6xy - xz)\,\hat{k}$$

Curl at (2, –1, 1)

$$= -2(-1)(1)\,\hat{i} + \{(2)(-1) - 1\}\,\hat{j} + \{6(2)(-1) - 2(1)\}\,\hat{k}$$

$$= 2\hat{i} - 3\hat{j} - 14\hat{k}$$ **Ans.**

Example 43. *Prove that* $(y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$ *is both solenoidal and irrotational.* (*U.P., I Sem, Dec. 2008*)

Solution. Let $\vec{F} = (y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$

For solenoidal, we have to prove $\vec{\nabla}.\vec{F} = 0$.

Now, $$\vec{\nabla}.\vec{F} = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\cdot\left[(y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}\right]$$
$$= -2 + 2x - 2x + 2 = 0$$

Thus, $\vec{F}$ is solenoidal. For irrotational, we have to prove Curl $\bar{F} = 0$.

Now, $$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 + 3yz - 2x & 3xz + 2xy & 3xy - 2xz + 2z \end{vmatrix}$$
$$= (3z + 2y - 2y + 3z)\hat{i} - (-2z + 3y - 3y + 2z)\hat{j} + (3z + 2y - 2y - 3z)\hat{k}$$
$$= 0\hat{i} + 0\hat{j} + 0\hat{k} = 0$$

Thus, $\vec{F}$ is irrotational.

Hence, $\vec{F}$ is both solenoidal and irrotational. **Proved.**

Example 44. *Determine the constants a and b such that the curl of vector* $\bar{A} = (2xy + 3yz)\hat{i} + (x^2 + axz - 4z^2)\hat{j} - (3xy + byz)\hat{k}$ *is zero.* (*U.P. I Semester, Dec 2008*)

Solution. $$\text{Curl } A = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times [(2xy + 3yz)\hat{i} + (x^2 + axz - 4z^2)\hat{j} - (3xy + byz)\hat{k}]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$$

$$= [-3x - bz - ax + 8z]\hat{i} - [-3y - 3y]\hat{j} + [2x + az - 2x - 3z]\hat{k}$$
$$= [-x(3 + a) + z(8 - b)]\hat{i} + 6y\hat{j} + z(-3 + a)\hat{k}$$
$$= 0 \quad \text{(given)}$$

i.e., $3 + a = 0$ and $8 - b = 0$, $\quad -3 + a = 0$

$a = -3, 3 \quad b = 8 \quad \Rightarrow \quad a = 3$ **Ans.**