Some Applications of Differentiation

I L'Hôpital's Rule

This rule determines the limit of rational functions if the direct substitution gives the indeterminate values $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$.

* Indeterminate values

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty - \infty, \quad 0 \times \infty, \quad 0^0, \quad 1^\infty, \quad \infty^0$$

L'Hôpital's Rule

suppose that we have one of the following Cases,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \quad \text{oR} \quad \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$$

where a Can be any real number, infinity or negative infinity.

In this Cases we have

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Examples:

Evaluate:

① $$\lim_{x \to 0} \frac{x\cos(x) + \tan(2x)}{x\sec(x) + \sin(4x)}$$

by dirct substitution

$$\lim_{x \to 0} \frac{x\cos(x) + \tan(2x)}{x\sec(x) + \sin(4x)} = \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{x(-\sin(x)) + \cos(x) + 2\sec^2(2x)}{x(\sec(x)\tan(x)) + \sec(x) + 4\cos(4x)}$$

$$= \frac{1+2}{1+4} = \frac{3}{5}.$$

$\sin(0)=0$ $\quad \cos(0)=1$

$\tan(0)=0$ $\quad \sec(0)=\frac{1}{\cos(0)}=1$

② $$\lim_{x \to 0} \frac{\sqrt{4+\tan(3x)} - 2\sqrt{1-\sin^{-1}(5x)}}{x}$$

by direct substitution

$$\lim_{x \to 0} \frac{\sqrt{4+\tan(3x)} - 2\sqrt{1-\sin^{-1}(5x)}}{x} = \frac{2-2}{0} = \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\frac{3\sec^2(3x)}{2\sqrt{4+\tan(3x)}} - \frac{\cancel{2}}{\cancel{2}\sqrt{1-\sin^{-1}(5x)}} \cdot \frac{-5}{\sqrt{1-(5x)^2}}}{1}$$

$\sin^{-1}(0)=0$

$$= \frac{3}{2\sqrt{4}} - \frac{-5}{1} = \frac{3}{4} + 5 = \frac{23}{4}.$$

③ $\lim_{x \to 0} \left(\operatorname{Cosec}(x) - \operatorname{Cot}(x) \right)$

by direct substitution

$\lim_{x \to 0} \left(\operatorname{Cosec}(x) - \operatorname{Cot}(x) \right) = \infty - \infty$

$\operatorname{Cosec}(0) = \frac{1}{\sin(0)} = \frac{1}{0} = \infty$

$\operatorname{Cot}(0) = \frac{1}{\tan 0} = \frac{1}{0} = \infty$

$= \lim_{x \to 0} \left(\frac{1}{\sin(x)} - \frac{\cos(x)}{\sin(x)} \right)$

$= \lim_{x \to 0} \frac{1 - \cos(x)}{\sin(x)} = \frac{1-1}{0} = \frac{0}{0}$

$= \lim_{x \to 0} \frac{\sin(x)}{\cos(x)} = \frac{0}{1} = 0$

---

④ $\lim_{x \to 1} (1-x) \tan\left(\frac{\pi}{2} x\right)$

by direct substitution

$\lim_{x \to 1} (1-x) \tan\left(\frac{\pi}{2} x\right) = 0 \times \infty$

$= \lim_{x \to 1} \frac{1-x}{\operatorname{Cot}\left(\frac{\pi}{2} x\right)} = \frac{1-1}{\operatorname{Cot}\left(\frac{\pi}{2}\right)} = \frac{0}{0}$

$= \lim_{x \to 1} \frac{-1}{-\operatorname{Cosec}^2\left(\frac{\pi}{2} x\right) \cdot \left(\frac{\pi}{2}\right)} = \frac{-1}{-\left(\frac{\pi}{2}\right)} = \frac{2}{\pi}$

---

⑤ $\lim_{x \to \frac{\pi}{2}} \cot(x) \cdot \ln(\sec x)$

by direct substitution

$$\lim_{x \to \frac{\pi}{2}} \cot(x) \cdot \ln(\sec x) = 0 \times \infty$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\ln(\sec x)}{\tan(x)} = \frac{\infty}{\infty}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\frac{\cancel{\sec(x)} \tan(x)}{\cancel{\sec(x)}}}{\sec^2(x)} = \lim_{x \to \frac{\pi}{2}} \frac{\tan x}{\sec^2 x}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\sin x}{\cancel{\cos x}} \cdot \cos^{\cancel{2}} x = 0$$

---

⑥ $\lim_{x \to 0} (\cos x)^{\frac{1}{x^2}}$

by direct substitution

$$\lim_{x \to 0} (\cos x)^{\frac{1}{x^2}} = 1^{\infty}$$

when the power is variable, we will use the Logarithm, as follows:

Put $y = \lim_{x \to 0} (\cos x)^{\frac{1}{x^2}}$

Take ln to both sides

$$\therefore \ln y = \lim_{x\to 0} \ln (\cos x)^{1/x^2}$$

$$= \lim_{x\to 0} \frac{1}{x^2} \ln(\cos x)$$ rational function

$$= \lim_{x\to 0} \frac{\ln(\cos x)}{x^2} = \frac{0}{0}$$

Apply L'Hôpital

$$= \lim_{x\to 0} \frac{\frac{-\sin x}{\cos x}}{2x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{-\tan x}{2x} = \lim_{x\to 0} \frac{-\sec^2 x}{2}$$

$$= \frac{-1}{2}$$

$$\therefore \ln y = \frac{-1}{2} \Rightarrow e^{\ln y} = e^{-1/2}$$

$$y = \lim_{x\to 0} (\cos x)^{1/x^2} = e^{-1/2}$$

---

(7) $$\lim_{x\to 0} \left(1+\tan^{-1}(2x)\right)^{\operatorname{Cosec}(3x)}$$

by direct substitution

$$\lim_{x\to 0} \left(1+\tan^{-1}(2x)\right)^{\operatorname{Cosec}(3x)} = 1^{\infty}$$

Put $y = \lim_{x\to0} \left(1+\tan^{-1}(2x)\right)^{\operatorname{Cosec}(3x)}$

$$\ln y = \lim_{x\to0} \ln\left(1+\tan^{-1}(2x)\right)^{\operatorname{Cosec}(3x)}$$

$$= \lim_{x\to0} \operatorname{Cosec}(3x)\cdot\ln\left(1+\tan^{-1}(2x)\right)$$

$$= \lim_{x\to0} \frac{\ln\left(1+\tan^{-1}(2x)\right)}{\operatorname{Sin}(3x)} = \frac{0}{0}$$

$$= \lim_{x\to0} \frac{\frac{1}{1+\tan^{-1}(2x)}\cdot\frac{2}{1+(2x)^2}}{3\operatorname{Cos}(3x)}$$

$$= \frac{2}{3}$$

$$\therefore \ln y = \frac{2}{3} \rightarrow e^{\ln y} = e^{\frac{2}{3}}$$

$$\therefore y = \lim_{x\to0} \left(1+\tan^{-1}(2x)\right)^{\operatorname{Cosec}(3x)} = e^{\frac{2}{3}}$$

---

(8) $\lim_{x\to\infty} \left(\frac{2x-5}{2x+1}\right)^{3x+4}$

by direct substitution

$$\lim_{x\to\infty} \left(\frac{2x-5}{2x+1}\right)^{3x+4} = 1^{\infty}$$

polyn. of degree 1

$$\lim_{x\to\infty} \frac{2x-5}{2x+1}$$

$$= \frac{\text{Coeff of } x}{\text{Coeff. of } x}$$

poly. of deg 1

$$= \frac{2}{2} = 1$$

Put $y = \lim\limits_{x \to \infty} \left(\frac{2x-5}{2x+1} \right)^{3x+4}$

Take ln to both sides

$$\therefore \ln y = \lim_{x \to \infty} \ln \left(\frac{2x-5}{2x+1} \right)^{3x+4}$$

$$\ln y = \lim_{x \to \infty} (3x+4) \ln \left(\frac{2x-5}{2x+1} \right)$$

$$= \lim_{x \to \infty} \frac{\ln \left(\frac{2x-5}{2x+1} \right)}{\frac{1}{3x+4}} = \frac{0}{0}$$

$$= \lim_{x \to \infty} \frac{\ln(2x-5) - \ln(2x+1)}{\frac{1}{3x+4}}$$

$$\to \frac{d}{dx}(3x+4)^{-1} = -(3x+4)^{-2}(3) = \frac{-3}{(3x+4)^2}$$

$$= \lim_{x \to \infty} \frac{\frac{2}{2x-5} - \frac{2}{2x+1}}{\frac{-3}{(3x+4)^2}}$$

$$= \frac{2}{-3} \lim_{x \to \infty} (3x+4)^2 \left[\frac{1}{2x-5} - \frac{1}{2x+1} \right]$$

$$= \frac{-2}{3} \lim_{x \to \infty} (3x+4)^2 \left[\frac{\cancel{2x}+1 - \overset{-2x+5}{(\cancel{2x}-5)}}{(2x-5)(2x+1)} \right]$$

$\lim\limits_{x \to a} K f(x) = K \lim\limits_{x \to a} f(x)$

$\because$ K is a Const.

$$= \frac{-2}{3} \lim_{x \to \infty} \frac{6(9x^2+24x+16)}{4x^2-8x-5}$$

$$= \frac{-2 * 6}{3} \lim_{x \to \infty} \frac{9x^2+24x+16}{4x^2-8x-5}$$

$$= -4 * \frac{9}{4} = -9$$

$\therefore \ln y = -9$

$\therefore e^{\ln y} = e^{-9}$

$\therefore y = \lim\limits_{x \to \infty} \left(\frac{2x-5}{2x+1}\right)^{3x+4} = e^{-9}$

Scanned with CamScanner