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Probability and Statistics II

DR. AHMED TAYEL

Department of Engineering Mathematics and Physics, Faculty of Engineering, Alexandria University

ahmed.tayel@alexu.edu.eg

What is a random variable

2.1 Introduction

What is a random variable (RV)

Random + Variable

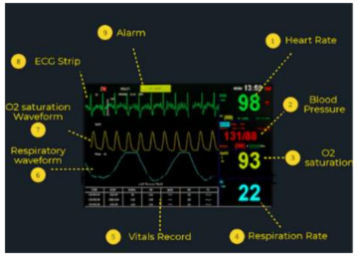
Random:

- NOT based on a **physical law**
- Depends on the **outcome of a random experiment**

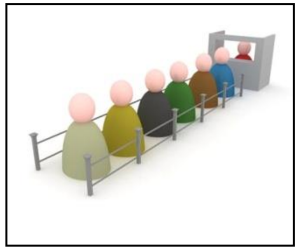
Variable:

- $a \le X \le b$, **or** $X = x_1, x_2, \ldots$
- It takes different <u>**numerical**</u> values

<u>Examples</u>



Heart rate



Queue length

Call duration

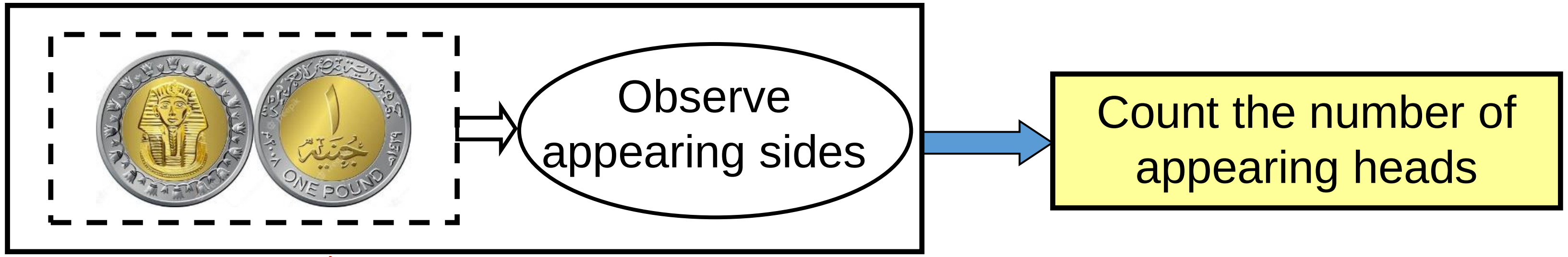
Life time

From random experiment to RV's

Example

A coin is tossed twice and the appearing sides are observed. **Let X be the number of appearing heads.**

Find the possible values of X and the probability of each value.



H < H, T

T < H, T

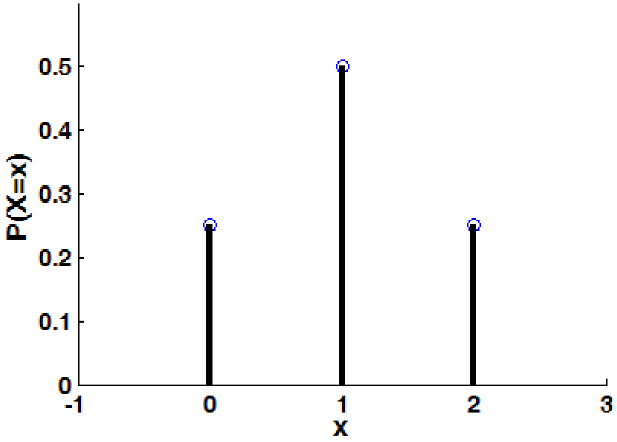
S = {HH, HT, TH, TT}

x	0	1	2
P(X = x)	1/4	1/2	1/4

Σ = 1

Prob. mass fn. (PMF) ← discrete

if cont. PDF



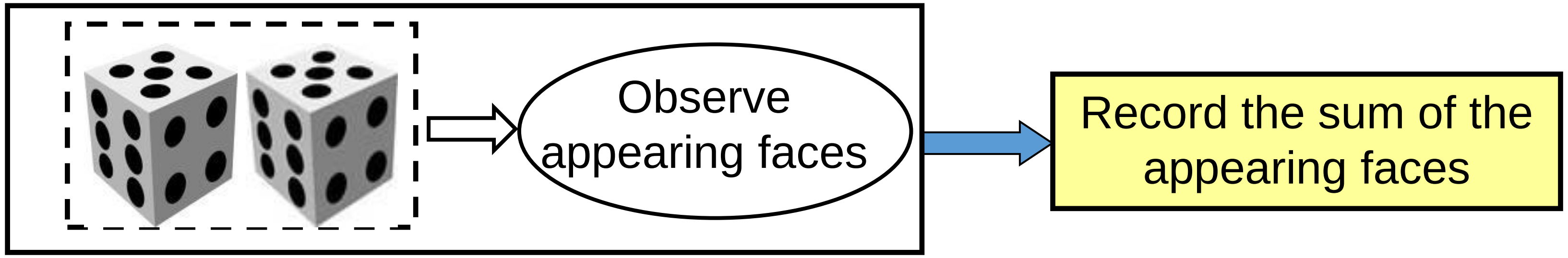
From random experiment to RV's (Cont'd)

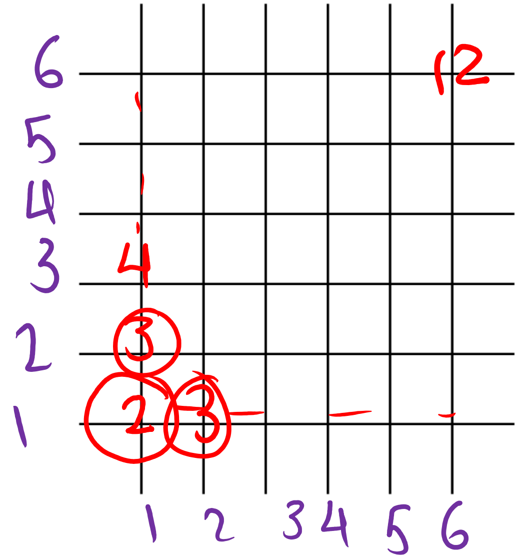
Example

A die is thrown twice.

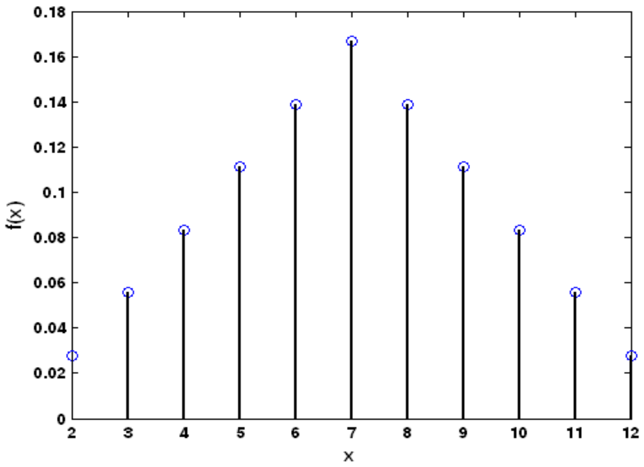
Let X denotes the sum of the appearing numbers.

Find the possible values of X and the probability of each value.





X	2	3	. . .	12
P(X = x)	1/36	2/36	. . .	1/36



Expectation and variance

Introduction to expectation

Example

A class of 100 students with ages ranging from 20 to 24 years old.

A record of the ages of students is introduced in the following table.

Age (x)	20	21	22	23	24
Number of students	5	20	50	15	10
$P_X(x)$	5/100	20/100	50/100	15/100	10/100

$$\frac{20+21}{2} = 20 * \frac{1}{2} + 21 * \frac{1}{2}$$

Q: How to average these numbers?

Motivation

Sum the ages of all students and divide by the number of students

$$\frac{20 * 5 + 21 * 20 + 22 * 50 + 23 * 15 + 24 * 10}{100}$$

The average of X may be computed as follows:

$$20 \times (5/100) + 21 \times (20/100) + 22 \times (50/100) + 23 \times (15/100) + 24 \times (10/100) = 22.05$$

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$$E(X) = \sum x f(x)$$

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prob.

Expectation or mean ($E(X)$, μ_X)

Definition $$E(X) = \sum_x x\, P_X(x)$$

Definition

$$E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

Consts.

$a\, E(X) + \cancel{E(b)}$ b

$$E(aX + b) = a\,E(X) + b$$

What does E(X) represent?

- A *weighted sum* of the RV values.
- The *average value* of the RV over the long run.
- The *balancing point* of the PMF.

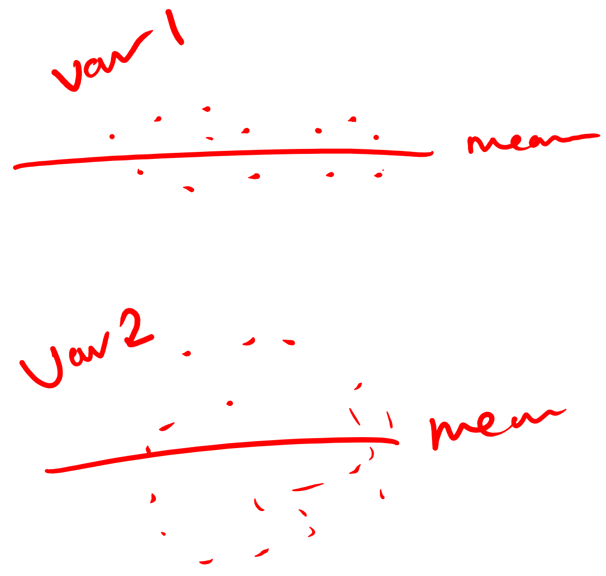
measure to the central tendency

Variance

mean

Definition $\mathbf{Var(X) = E(X - \mu_X)^2}$

Variance is a measure of dispersion



General property

$\text{Var}(aX + b) = a^2\,\text{Var}(X)$ + $\cancel{\text{Var}(b)} \rightarrow 0$

Var 1 < Var 2

Standard deviation $\sigma_X = \sqrt{\text{Var}(X)}$

Moment generating function

Introduction

For a random variable X (discrete or continuous)

- The mean: E(X)
- The variance: $\mathrm{E}(\mathrm{X}^2) - \mathrm{E}(\mathrm{X})^2$

- We need to calculate $\mathrm{E}(\mathrm{X})$, $\mathrm{E}(\mathrm{X}^2)$, ….., $\mathrm{E}(\mathrm{X}^n)$, ….

1st moment **2nd moment** **nth moment**

- Moment generating function (MGF): $\mathrm{M}_\mathrm{X}(t)$

Definition

Definition

$$M_X(t) = E(e^{tX})$$

Discrete **Continuous**

$$E(X) = \left[\frac{d}{dt} M_X(t)\right]_{t=0}$$

$$E(X^2) = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0}$$

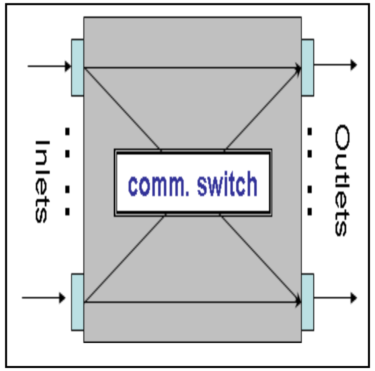
$$E(X^n) = \left[\frac{d^n}{dt^n} M_X(t)\right]_{t=0}$$

MGF for important RV's

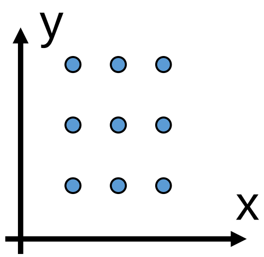
$Binomial(n,p)$	$(1-p+pe^t)^n$
$Geometric(p)$	$\dfrac{pe^t}{1-(1-p)e^t}$
$Poisson(\lambda)$	$e^{\lambda(e^t-1)}$
$Uniform(a,b)$	$\dfrac{e^{bt}-e^{at}}{t(b-a)}$
$Exponential(\lambda)$	$\dfrac{\lambda}{\lambda-t}$
$N(\mu,\sigma^2)$	$e^{t\mu+\frac{1}{2}t^2\sigma^2}$

Multiple random variables

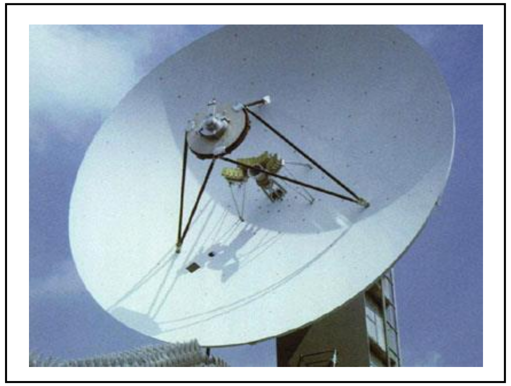
3.1 Introduction



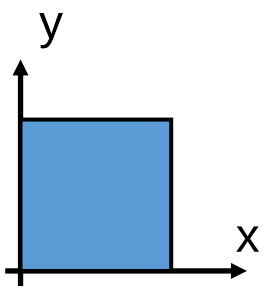
- No of packets
- Destination



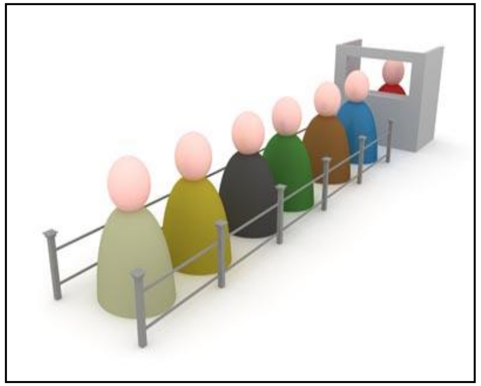
Discrete



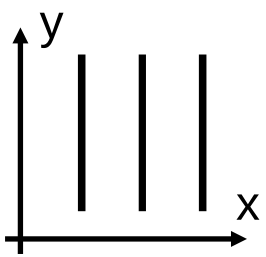
- Transmitted signal
- Received signal



Continuous



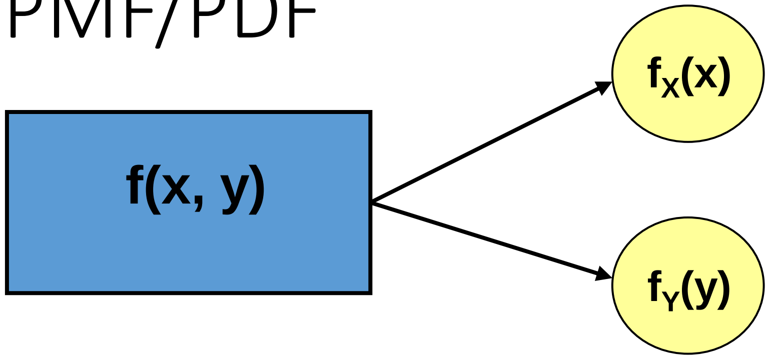
- Waiting time
- Queue length



Mixed

Joint PMF/PDF

Marginal PMF's/PDF's



Independence

X and Y are *independent* if

$$\mathbf{f(x, y) = f_X(x)\, f_Y(y)},$$

for all x and y.

Definition: i.i.d. random variables.

Independent and identically distributed

Expectations

Properties

(1) $E(X + Y) = E(X) + E(Y)$,
$E(X - Y) = E(X) - E(Y)$.

indep. or not

(2) If X and Y are *independent*,

$$E(X\,Y) = E(X)\,E(Y).$$

Variance

For independent random variables *X* and *Y*, the variance of their sum or difference is the sum of their variances

$$\sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y$$
$$\sigma^2_{X-Y} = \sigma^2_X + \sigma^2_Y$$

Variances are added for both the sum *and* difference of two independent random variables because the variation in each variable contributes to the variation in each case

The Normal distribution

Normal (Gaussian) random variable

PDF

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty < x < \infty$$

$$-\infty < \mu < \infty, \quad \sigma > 0$$

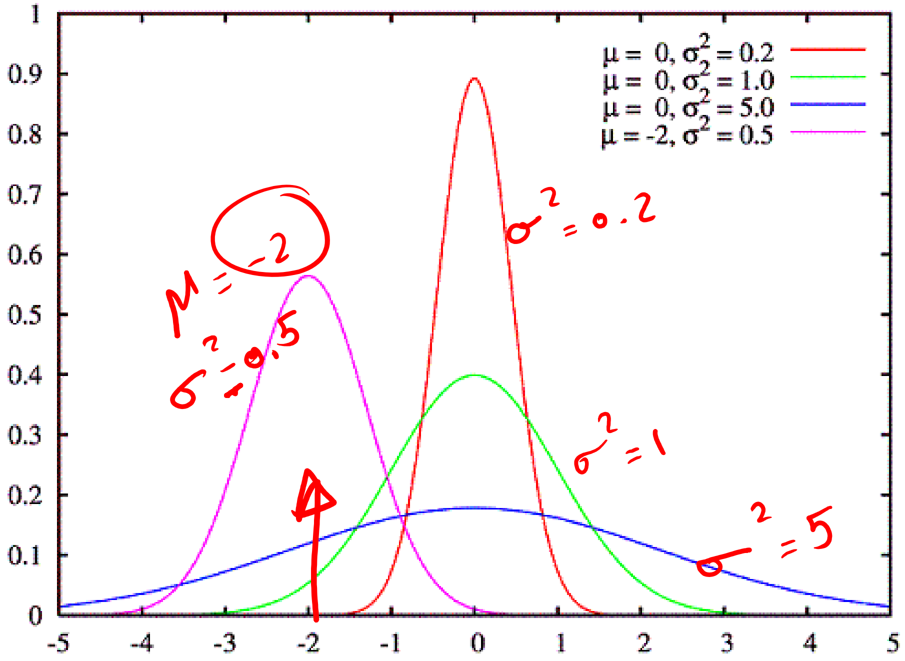
$$X \sim N(\mu, \sigma^2)$$

Mean & variance

$$E(X) = \mu,$$
$$Var(X) = \sigma^2$$

CDF

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

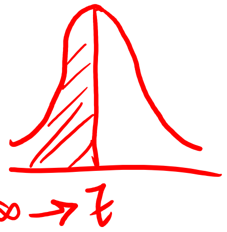


From http://en.wikipedia.org

Standard normal (Gaussian) RV

PDF $Z \sim N(\overset{\mu}{0}, \overset{\sigma^2}{1})$

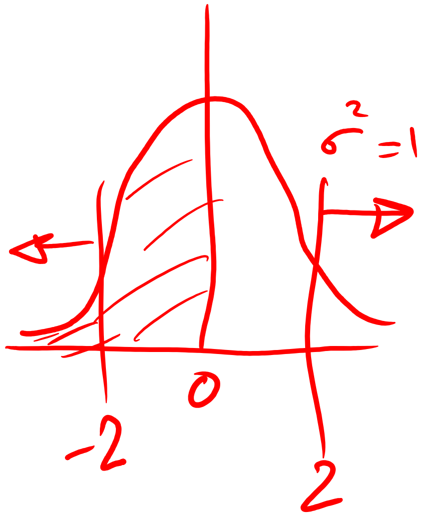
$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, -\infty < z < \infty$$



CDF $\phi(z) = \int_{-\infty}^{z} f_Z(t)\,dt$

+ve z only

z	$\phi(z)$
0.00	0.5000
0.01	0.5040
0.02	0.5080
. . .	. . .
2.99	0.9986



$\phi(0) =$ 0.5

$\phi(-\infty) =$ 0

$\phi(\infty) =$ 1

$\phi(-2) =$ $P(z > 2) = 1 - P(z \leq 2) = 1 - \Phi(2)$

Example

Compute

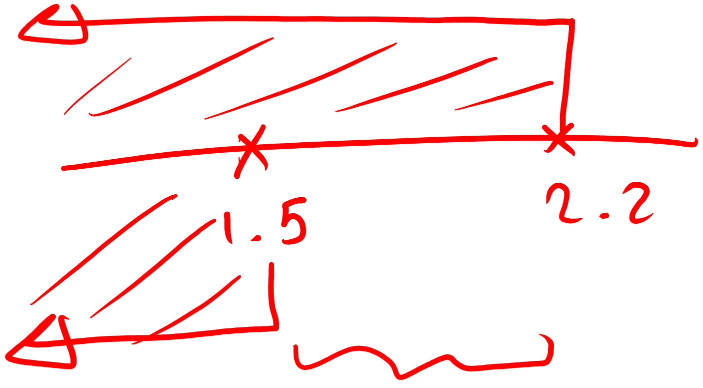
(a) $P(Z < 1)$ $= \Phi(1)$

(b) $P(Z > 2)$ $= 1 - \Phi(2)$

(c) $P(1.5 < Z < 2.2)$ $= \Phi(2.2) - \Phi(1.5)$

(d) $P(Z < -1)$ $= P(Z > 1) = 1 - P(Z \leq 1)$

$= 1 - \Phi(1)$



$$X \sim N(\mu, \sigma^2) \Rightarrow \frac{X-\mu}{\sigma} \sim N(0,1)$$