## 2 Maclaurin Expansion :

If $f(x)$ Can be differentiable n times at o, then $f(x)$ Can be expressed in form of an infinite series in the form :

$$\boxed{f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots}$$

This expansion is Called Maclaurin series or Maclaurin polynomial.

Maclaurin expansion for some functions:-

1. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
2. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$
3. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$
4. $(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$
5. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

Proof → ① $f(x) = e^{x}$

$f(x) = e^{x}$ $\quad f(0) = 1$

$f'(x) = e^{x}$ $\quad f'(0) = 1$

$f''(x) = e^{x}$ $\quad f''(0) = 1$

$\vdots \qquad \vdots$

$$\therefore e^{x} = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Since

$$\sinh x = \frac{e^{x} - e^{-x}}{2}, \quad e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$\therefore \sinh x = \frac{1}{2}\left[\cancel{1} + x + \cancel{\frac{x^2}{2!}} + \frac{x^3}{3!} + \cdots \cancel{-1} + x - \cancel{\frac{x^2}{2}} + \frac{x^3}{3!} + \cdots\right]$$

$$\sinh x = \frac{1}{2}\left[2x + 2\frac{x^3}{3!} + 2\frac{x^5}{5!} + \cdots\right]$$

$$\boxed{\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots}$$

Similarly:

$$2 + 2\frac{x^2}{2!} + 2\frac{x^4}{4!} + \cdots$$

$$\cosh x = \frac{e^{x} + e^{-x}}{2} = \frac{1}{2}\left[1 + \cancel{x} + \frac{x^2}{2!} + \cdots + 1 - \cancel{x} + \frac{x^2}{2!} + \cdots\right]$$

$$\boxed{\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots}$$

② $f(x) = \cos x$

$f(x) = \cos x$ $\qquad f(0) = 1$

$f'(x) = -\sin x$ $\qquad f'(0) = 0$

$f''(x) = -\cos x$ $\qquad f''(0) = -1$

$f'''(x) = \sin x$ $\qquad f'''(0) = 0$

$f^{(4)}(x) = \cos x$ $\qquad f^{(4)}(0) = 1$

$$\therefore \cos x = f(0) + \cancel{\frac{f'(0)}{1!}x}^{\,0} + \frac{f''(0)}{2!}x^2 + \cancel{\frac{f'''(0)}{3!}x^3}^{\,0} + \cdots$$

$$\therefore \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\cdots$$

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③ $f(x) = \sin x$

$f(x) = \sin x$ $\qquad f(0) = 0$

$f'(x) = \cos x$ $\qquad f'(0) = 1$

$f''(x) = -\sin x$ $\qquad f''(0) = 0$

$f'''(x) = -\cos x$ $\qquad f'''(0) = -1$

$$\therefore \sin x = \cancel{f(0)}^{\,0} + \frac{f'(0)}{1!}x + \cancel{\frac{f''(0)}{2!}x^2}^{\,0} + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$\therefore \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\cdots$$

④ $f(x) = (1+x)^n$

$f(x) = (1+x)^n \qquad f(0) = 1$

$f'(x) = n(1+x)^{n-1} \qquad f'(0) = n$

$f''(x) = n(n-1)(1+x)^{n-2} \qquad f''(0) = n(n-1)$

$f'''(x) = n(n-1)(n-2)(1+x)^{n-3} \qquad f'''(0) = n(n-1)(n-2)$

$\vdots \qquad \vdots$

$$\therefore (1+x)^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$\boxed{\therefore (1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots}$$

Special Cases :-

at $n = -1$

$$\therefore (1+x)^{-1} = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

at $n = -1$, Replace $(x)$ by $(-x)$

$$\therefore (1-x)^{-1} = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

at $n = \frac{1}{2}$

$$\therefore (1+x)^{1/2} = \sqrt{1+x} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}x^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!}x^3 + \cdots$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots$$

⑤ $f(x) = \ln(1+x)$

$f(x) = \ln(1+x)$ $\qquad$ $f(0) = \ln 1 = 0$

$f'(x) = \frac{1}{1+x} = (1+x)^{-1}$ $\qquad$ $f'(0) = 1$

$f''(x) = -(1+x)^{-2}$ $\qquad$ $f''(0) = -1$

$f'''(x) = 2(1+x)^{-3}$ $\qquad$ $f'''(0) = 2$

$f^{(4)}(x) = -6(1+x)^{-4}$ $\qquad$ $f^{(4)}(0) = -6$

$\vdots$ $\qquad$ $\vdots$

$$\therefore \ln(1+x) = \cancel{f(0)}^{\,0} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$\therefore \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

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## Examples :-

Find Maclaurin's expansion for each of the following functions:

① $\sin(2x)$

solution

$$\because \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Replace $x$ by $2x$

$$\therefore \sin(2x) = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \cdots$$

$$\therefore \sin(2x) = 2x - \frac{\cancel{8}^{4} x^3}{\cancel{6}_{3}} + \frac{\cancel{32}^{4} x^5}{5(\cancel{4})(3)(\cancel{2})} - \cdots$$

$$\sin(2x) = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \cdots$$

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(2) $\cos(3x)$

$$\because \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Replace $x$ by $(3x)$

$$\therefore \cos(3x) = 1 - \frac{(3x)^2}{2} + \frac{(3x)^4}{4(3)(2)} - \cdots$$

$$\cos(3x) = 1 - \frac{9}{2}x^2 + \frac{27}{8}x^4 - \cdots$$

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(3) $e^{-3x}$

$$\because e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Replace $x$ by $-3x$

$$\therefore e^{-3x} = 1 + (-3x) + \frac{(-3x)^2}{2} + \frac{(-3x)^3}{6} + \cdots$$

$$e^{-3x} = 1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3 + \cdots$$

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(4) $\ln(2+3x)$

$$= \ln\left(2\left(1 + \frac{3}{2}x\right)\right)$$

$$= \ln 2 + \ln\left(1 + \frac{3}{2}x\right)$$

$$\because \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Replace $x$ by $\left(\frac{3}{2}x\right)$

$$\therefore \ln(2+3x) = \ln 2 + \frac{3}{2}x - \frac{1}{2}\left(\frac{3}{2}x\right)^2 + \frac{1}{3}\left(\frac{3}{2}x\right)^3 - \cdots$$

$$= \ln 2 + \frac{3}{2}x - \frac{9}{8}x^2 + \frac{9}{8}x^3 - \cdots$$

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(5) $\cos^2(2x)$

$$= \frac{1}{2}(1+\cos 4x)$$

$$\because \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Replace $x$ by $(4x)$

$$\therefore \cos^2(2x) = \frac{1}{2}\left[1 + 1 - \frac{(4x)^2}{2} + \frac{(4x)^4}{4(3)(2)} - \cdots\right]$$

$$\cos^2(2x) = \frac{1}{2}\left[2 - \frac{\cancel{16}^{8}x^2}{\cancel{2}_{1}} + \frac{32}{3}x^4 - \cdots\right]$$

$$\cos^2(2x) = 1 - 4x^2 + \frac{16}{3}x^4 - \cdots$$

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Prove the following relations:

(1) $e^{-x^2}\sin x = x - \frac{7}{6}x^3 + \frac{27}{40}x^5 + \cdots$

Solution

$$\because e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Replace $x$ by $(-x^2)$

$$\therefore e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$$

$$\therefore \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\therefore e^{-x^2} \sin x = \left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots\right) \cdot \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right)$$

$$= x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots$$
$$- x^3 + \frac{x^5}{6} + \cdots$$
$$+ \frac{x^5}{2} - \cdots$$

$$= x - \frac{7}{6}x^3 + \frac{\cancel{81}^{27}}{\cancel{120}_{40}} x^5 + \cdots$$

$$\therefore e^{-x^2} \sin x = x - \frac{7}{6}x^3 + \frac{27}{40}x^5 + \cdots$$

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(2) $$\frac{e^x}{1-x} = 1 + 2x + \frac{5}{2}x^2 + \frac{8}{3}x^3 + \cdots$$

solution

$$\therefore e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$

$$\therefore \frac{e^x}{1-x} = e^x (1-x)^{-1}$$

$$\frac{e^x}{1-x} = \left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots\right)$$
$$\cdot\left(1+x+x^2+x^3+\cdots\right)$$

$$= 1+x+x^2+x^3+\cdots$$
$$+x+x^2+x^3+\cdots$$
$$+\frac{x^2}{2}+\frac{x^3}{2}+\cdots$$
$$+\frac{x^3}{6}+\cdots$$

$$\therefore \frac{e^x}{1-x} = 1+2x+\frac{5}{2}x^2+\frac{\cancel{16}^{\,8}}{\cancel{6}_{\,3}}x^3+\cdots$$
$$= 1+2x+\frac{5}{2}x^2+\frac{8}{3}x^3+\cdots$$

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③ $$\frac{\cos x}{\sqrt{1+x}} = 1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3-\cdots$$

Solution

$$\therefore \cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots$$

$$\therefore (1+x)^n = 1+\frac{nx}{1!}+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots$$

$$\therefore \frac{1}{\sqrt{1+x}} = (1+x)^{-1/2} = 1+\frac{-\frac{1}{2}x}{1}+\frac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{2}x^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)x^3}{6}+\cdots$$

$$\frac{1}{\sqrt{1+x}} = 1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{5}{16}x^3+\cdots$$

$$\therefore \frac{\cos x}{\sqrt{1+x}} = \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right)$$
$$\cdot \left(1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots\right)$$

$$= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots$$
$$- \frac{x^2}{2} + \frac{1}{4}x^3 - \cdots$$

$$\therefore \frac{\cos x}{\sqrt{1+x}} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 + \cdots$$