1.7 PARTIAL DERIVATIVES

Let $z = f(x, y)$ be function of two independent variables x and y. If we keep y constant and x varies then z becomes a function of x only. The derivative of z with respect to x, keeping y as constant is called partial derivative of 'z', w.r.t. 'x' and is denoted by symbols.

$$\frac{\partial z}{\partial x}, \frac{\partial f}{\partial x}, f_x(x, y) \text{ etc.}$$

Then
$$\frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \frac{f(x + \delta x, y) - f(x, y)}{\delta x}$$

The process of finding the partial differential coefficient of z w.r.t. 'x' is that of ordinary differentiation, but with the only difference that we treat y as constant.

Similarly, the partial derivative of 'z' w.r.t. 'y' keeping x as constant is denoted by

$$\frac{\partial z}{\partial y}, \frac{\partial f}{\partial y}, f_y(x, y) \text{ etc.}$$

$$\frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y}$$

Notation. $\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x \, \partial y} = s, \quad \frac{\partial^2 z}{\partial y^2} = t$

Example 7. *If* $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$, *then find the value of* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

Solution. $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1 - \left(\frac{x}{y}\right)^2}} \cdot \frac{1}{y} + \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right) = \frac{1}{\sqrt{y^2 - x^2}} - \frac{y}{x^2 + y^2}$$

$$x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2 - x^2}} - \frac{xy}{x^2 + y^2} \qquad ...(1)$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1 - \left(\frac{x}{y}\right)^2}}\left(-\frac{x}{y^2}\right) + \frac{1}{1 + \left(\frac{y}{x}\right)^2}\frac{1}{x} = -\frac{x}{y\sqrt{y^2 - x^2}} + \frac{x}{x^2 + y^2}$$

$$y.\frac{\partial u}{\partial y} = -\frac{x}{\sqrt{y^2 - x^2}} + \frac{xy}{x^2 + y^2} \qquad ...(2)$$

On adding (1) and (2), we have $x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = 0$ **Ans.**

Example 8. *Find* $\frac{\partial u}{\partial r}$ *and* $\frac{\partial u}{\partial \theta}$ *if* $u = e^{r\cos\theta}.\cos(r\sin\theta)$

Solution. $u = e^{r\cos\theta}.\cos(r\sin\theta)$

$$\frac{\partial u}{\partial r} = e^{r\cos\theta}.[-\sin(r\sin\theta).\sin\theta] + [\cos\theta.e^{r\cos\theta}].\cos(r\sin\theta)$$

(keeping θ as constant)

$$= e^{r\cos\theta}.[-\sin(r\sin\theta).\sin\theta + \cos(r\sin\theta).\cos\theta]$$

$$= e^{r\cos\theta}.\cos(r\sin\theta + \theta)$$ **Ans.**

$$\frac{\partial u}{\partial \theta} = e^{r\cos\theta}.[-\sin(r\sin\theta).r\cos\theta] + [-r\sin\theta.e^{r\cos\theta}].\cos(r\sin\theta)$$

(keeping *r* as constant)

$$= -r\,e^{r\cos\theta}.[\sin(r\sin\theta).\cos\theta + \sin\theta\cos(r\sin\theta)]$$

$$= -r\,e^{r\cos\theta}.\sin(r\sin\theta + \theta)$$ **Ans.**

Example 9. *If* $u = (1 - 2xy + y^2)^{-1/2}$ *prove that,* $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = y^2 u^3$.

Solution. $u = (1 - 2xy + y^2)^{-1/2}$ ...(1)

Differentiating (1) partially w.r.t. 'x', we get

$$\frac{\partial u}{\partial x} = -\frac{1}{2}(1-2xy+y^2)^{-3/2}(-2y)$$

$$x\frac{\partial u}{\partial x} = xy(1-2xy+y^2)^{-3/2} \quad ...(2)$$

Differentiating (1) partially w.r.t. 'y', we get

$$\frac{\partial u}{\partial y} = -\frac{1}{2}(1-2xy+y^2)^{-3/2}(-2x+2y)$$

$$y\frac{\partial u}{\partial y} = (xy - y^2)(1-2xy+y^2)^{-3/2} \quad ...(3)$$

Subtracting (3) from (2), we get

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = xy(1-2xy+y^2)^{-3/2} - (xy-y^2)(1-2xy+y^2)^{-3/2}$$

$$= y^2(1-2xy+y^2)^{-3/2} = y^2u^3.$$ **Proved.**

1.10 HOMOGENEOUS FUNCTION

A function $f(x, y)$ is said to be homogeneous function in which the power of each term is the same.

A function $f(x, y)$ is a homogeneous function of order n, if the degree of each of its terms in x and y is equal to n. Thus

$$a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \dots + a_{n-1} x y^{n-1} + a_n y^n \qquad \text{...(1)}$$

is a homogeneous function of order n.

The polynomial function (1) which can be written as

$$x^n \left[a_0 + a_1 \left(\frac{y}{x}\right) + a_2 \left(\frac{y}{x}\right)^2 + \dots + a_{n-1} \left(\frac{y}{x}\right)^{n-1} + a_n \left(\frac{y}{x}\right)^n \right] = x^n \varphi\left(\frac{y}{x}\right) \qquad \text{...(2)}$$

(*i*) The function $x^3 \left[1 + \frac{y}{x} + 3\left(\frac{y}{x}\right)^2 + 5\left(\frac{y}{x}\right)^3 \right]$ is a homogeneous function of order 3.

(*ii*) $\dfrac{\sqrt{x} + \sqrt{y}}{x^2 + y^2} = \dfrac{\sqrt{x}\left[1 + \sqrt{\dfrac{y}{x}}\right]}{x^2\left[1 + \left(\dfrac{y}{x}\right)^2\right]} = x^{-3/2} . \dfrac{1 + \sqrt{\dfrac{y}{x}}}{1 + \left(\dfrac{y}{x}\right)^2}$ is a homogeneous function of order – 3/2.

(*iii*) $\sin^{-1} \dfrac{\sqrt{x} + \sqrt{y}}{x^2 + y^2}$ is not a homogeneous function as it cannot be written in the form of $x^n f\left(\dfrac{y}{x}\right)$ so that its degree may be pronounced. It is a function of homogeneous expression.

1.11 EULER'S THEOREM ON HOMOGENEOUS FUNCTION

[redacted]

Statement. If z is a homogeneous function of x, y of order n, then

$$x.\frac{\partial z}{\partial x} + y.\frac{\partial z}{\partial y} = n\, z$$

$$x^2.\frac{\partial^2 z}{\partial x^2} + 2xy.\frac{\partial^2 z}{\partial x \partial y} + y^2.\frac{\partial^2 z}{\partial y^2} = n(n-1)z.$$

I. Deduction from Euler's theorem

If z is a homogeneous function of x, y of degree n and $z = f(u)$, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$$ [redacted]

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)\,[g'(u) - 1]$$ [redacted]

where, $$g(u) = n\frac{f(u)}{f'(u)}$$

Example 21. *If* $u = \cos^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, *show that*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0.$$

Solution. Here, we have, $u = \cos^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

u is not a homogeneous function but if $z = \cos u$, then

$$u = \cos^{-1} z = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{x\left(1+\dfrac{y}{x}\right)}{\sqrt{x}\left(1+\sqrt{\dfrac{y}{x}}\right)} = x^{\frac{1}{2}}\frac{\left(1+\dfrac{y}{x}\right)}{\left(1+\sqrt{\dfrac{y}{x}}\right)} = x^{\frac{1}{2}}\,\phi\left(\frac{y}{x}\right).$$

z is a homogeneous function in x, y of degree $\dfrac{1}{2}$.

By Euler's theorem, we have $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = \dfrac{1}{2}z$

$$x\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + y\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} = \frac{1}{2}z$$

$$x\frac{\partial u}{\partial x}(-\sin u) + y\frac{\partial u}{\partial y}(-\sin u) = \frac{1}{2}\cos u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u. \qquad \Rightarrow \qquad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$$

Example 22. *If* $u = \sin^{-1}\left[\dfrac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right]$, *show that*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + 3\tan u = 0.$$

Solution. We have, $u = \sin^{-1}\left[\dfrac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right]$

Here, u is not a homogeneous function but if $v = \sin u = \dfrac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}$ then v is a homogeneous function in x, y, z of degree -3.

By Euler's Theorem $$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = n\,v$$

$$x\frac{\partial v}{\partial u}\frac{\partial u}{\partial x} + y\frac{\partial v}{\partial u}\frac{\partial u}{\partial y} + z\frac{\partial v}{\partial u}\frac{\partial u}{\partial z} = -3\,v \qquad ...(1)$$

Putting the value of $\dfrac{\partial v}{\partial u}$ in (1), we get

$$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} + z\cos u\frac{\partial u}{\partial z} = -3\sin u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -3\frac{\sin u}{\cos u} = -3\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + 3\tan u = 0$$ **Proved.**

Example 23. *If* $u = \log_e\left(\frac{x^4 + y^4}{x + y}\right)$, *show that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3.$

[redacted]

Solution. We have, $$u = \log_e\left(\frac{x^4 + y^4}{x + y}\right)$$

Here, u is not a homogeneous function but if

$$z = e^u = \frac{x^4 + y^4}{x + y} = \frac{x^4\left[1+\left(\frac{y}{x}\right)^4\right]}{x\left[1+\left(\frac{y}{x}\right)\right]} = x^3\,\varphi\left(\frac{y}{x}\right)$$

Then z is a homogeneous function of degree 3.
By Euler's Deduction formula I

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = 3\frac{e^u}{e^u} = 3$$

Example 24. *If* $f(x, y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2 + y^2}$, *prove that*

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + 2f = 0.$$

[redacted]

Solution. $$f(x, y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2 + y^2}$$

$$= \frac{1}{x^2}\left(\frac{y}{x}\right)^0 + \frac{1}{x^2}\,\frac{1}{\left(\frac{y}{x}\right)} - \frac{1}{x^2}\,\frac{\log\frac{y}{x}}{\left[1+\left(\frac{y}{x}\right)^2\right]}$$

$f(x, y)$ is a homogeneous function of degree -2.
By Euler's Theorem

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = -2f \Rightarrow x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + 2f = 0$$

Example 28. *If* $u = \tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$, *prove that*

(*i*) $x.\dfrac{\partial u}{\partial x} + y.\dfrac{\partial u}{\partial y} = \sin 2u$ [redacted]

(*ii*) $x^2.\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 2\cos 3u \sin u.$ [redacted]

Solution. Here u is not a homogeneous function. We however write

$$z = \tan u = \frac{x^3+y^3}{x-y} = \frac{x^3\left[1+\left(\dfrac{y}{x}\right)^3\right]}{x\left[1-\left(\dfrac{y}{x}\right)\right]} = x^2.\frac{1+\left(\dfrac{y}{x}\right)^3}{1-\left(\dfrac{y}{x}\right)} = x^2\varphi\left(\frac{y}{x}\right)$$

so that z is a homogeneous function of x, y of order 2.

(*i*) By Euler's Theorem [Here $f(u) = \tan u$]

$$\therefore \qquad x\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = \frac{n\,f(u)}{f'(u)} \qquad \text{...(1)}$$

$$= \frac{2\tan u}{\sec^2 u} = \frac{2\sin u\cos^2 u}{\cos u} = 2\sin u\cos u = \sin 2u$$

(*ii*) By deduction II

$$x^2.\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$$

Here $\sin 2u = g(u)$

$$\therefore \qquad x^2.\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 2u\,(2\cos 2u - 1) = 2\sin 2u\cos 2u - \sin 2u$$

$$= \sin 4u - \sin 2u = 2\cos 3u\sin u$$

Proved.

Example 29. *If* $u = \sin^{-1}\left[\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right]$

Prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = \dfrac{-\sin u\cos 2u}{4\cos^3 u}$.

Solution. We have, $u = \sin^{-1}\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$

Let $z = \sin u = \dfrac{x+y}{\sqrt{x}+\sqrt{y}} = \dfrac{x\left[1+\dfrac{y}{x}\right]}{\sqrt{x}\left[1+\sqrt{\dfrac{y}{x}}\right]} = x^{1/2}\ \varphi(x)$

$$z = f(u) = \sin u$$

z is a homogeneous function of degree $\dfrac{1}{2}$.

By Euler's deduction I

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} \quad\Rightarrow\quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\frac{\sin u}{\cos u}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$$

Let $g(u) = \dfrac{1}{2}\tan u$

By Euler's deduction II

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)\ [g'(u) - 1] = \frac{1}{2}\tan u\left(\frac{1}{2}\sec^2 u - 1\right)$$

$$= \frac{1}{4}\frac{\sin u}{\cos u}\left(\frac{1}{\cos^2 u} - 2\right) = \frac{1}{4}\frac{\sin u}{\cos^3 u}(1 - 2\cos^2 u) = \frac{-\sin u\cos 2u}{4\cos^3 u}$$

1.15 CHANGE IN THE INDEPENDENT VARIABLES x AND y BY OTHER TWO VARIABLES u AND v.

Let $z = f(x, y)$

where $x = \varphi(u, v)$

$y = \psi(u, v)$

Then from (5), we obtain

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} \quad \text{...(6)}$$

and

$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v} \quad \text{...(7)}$$

Example 35. *If $w = f(x,y)$, $x = r\cos\theta$, $y = r\sin\theta$, show that*

$$\left(\frac{\partial w}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2$$

Solution. Here, $x = r\cos\theta$, $y = r\sin\theta$

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

Now,

$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}.(\cos\theta)+\frac{\partial f}{\partial y}.(\sin\theta)$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \theta}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \frac{\partial f}{\partial x}.(-r\sin\theta)+\frac{\partial f}{\partial y}.(r\cos\theta)$$

$$\Rightarrow \qquad \frac{1}{r}\frac{\partial w}{\partial \theta} = -\frac{\partial f}{\partial x}\sin\theta+\frac{\partial f}{\partial y}\cos\theta$$

Squaring (1) and (2) and adding, we obtain

$$\left(\frac{\partial w}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2$$

Example 34. *If* $\varphi\,(cx-az,\ cy-bz)=0$ *show that* $ap+bq=c$ *:*

$$\text{where } p=\frac{\partial z}{\partial x} \text{ and } q\equiv\frac{\partial z}{\partial y}$$

Solution. Here, we have

[x and y are independent but z is dependent on x and y]

$$\varphi\,(cx-az,\ cy-bz)=0$$

$$\varphi\,(r,\ s)=0$$

where $\quad r=cx-az, \qquad s=cy-bz$

$$\frac{\partial r}{\partial x}=c-a\frac{\partial z}{\partial x}, \qquad \frac{\partial r}{\partial y}=-a\frac{\partial z}{\partial y}$$

$$\frac{\partial s}{\partial x}=-b\frac{\partial z}{\partial x}, \qquad \frac{\partial s}{\partial y}=c-b\frac{\partial z}{\partial y}$$

We know that, $\quad \dfrac{\partial\varphi}{\partial r}=\dfrac{\partial\varphi}{\partial r}\dfrac{\partial r}{\partial x}+\dfrac{\partial\varphi}{\partial s}\dfrac{\partial s}{\partial x}$

$$0=\frac{\partial\varphi}{\partial r}\left(c-a\frac{\partial z}{\partial x}\right)+\frac{\partial\varphi}{\partial s}\left(-b\frac{\partial z}{\partial x}\right)$$

$$\Rightarrow \qquad 0=c\frac{\partial\varphi}{\partial r}+\frac{\partial z}{\partial x}\left(-a\frac{\partial\varphi}{\partial r}-b\frac{\partial\varphi}{\partial s}\right)$$

$$c\frac{\partial\phi}{\partial r}=\frac{\partial z}{\partial x}\left(a\frac{\partial\phi}{\partial r}+b\frac{\partial\phi}{\partial s}\right) \quad\Rightarrow\quad a\frac{\partial z}{\partial x}=\frac{ac\dfrac{\partial\varphi}{\partial r}}{a\dfrac{\partial\varphi}{\partial r}+b\dfrac{\partial\varphi}{\partial s}} \qquad \text{...(1)}$$

Again $\quad \dfrac{\partial\varphi}{\partial y}=\dfrac{\partial\varphi}{\partial r}\dfrac{\partial r}{\partial y}+\dfrac{\partial\varphi}{\partial s}\dfrac{\partial s}{\partial y}$

$$0=\frac{\partial\varphi}{\partial r}\left(-a\frac{\partial z}{\partial y}\right)+\frac{\partial\varphi}{\partial s}\left(c-b\frac{\partial z}{\partial y}\right)$$

$$0=c\frac{\partial\varphi}{\partial s}-\frac{\partial z}{\partial y}\left(a\frac{\partial\varphi}{\partial r}+b\frac{\partial\varphi}{\partial s}\right) \quad\Rightarrow\quad c\frac{\partial\phi}{\partial s}=\frac{\partial z}{\partial y}\left(a\frac{\partial\phi}{\partial r}+b\frac{\partial\phi}{\partial s}\right)$$

$$\Rightarrow \qquad b\frac{\partial z}{\partial y}=\frac{bc\dfrac{\partial\varphi}{\partial s}}{a\dfrac{\partial\varphi}{\partial r}+b\dfrac{\partial\varphi}{\partial s}} \qquad \text{...(2)}$$

Adding (1) and (2), we get

$$a\frac{\partial z}{\partial x}+b\frac{\partial z}{\partial y}=\frac{ac\dfrac{\partial\varphi}{\partial r}+bc\dfrac{\partial\phi}{\partial s}}{a\dfrac{\partial\varphi}{\partial r}+b\dfrac{\partial\varphi}{\partial s}}$$

$$\Rightarrow \qquad a\frac{\partial z}{\partial x}+b\frac{\partial z}{\partial y}=c \qquad\Rightarrow\quad ap+bq=c$$

Proved.

1.20 TANGENT PLANE TO A SURFACE

Let $f(x, y, z) = 0$ be the equation of a surface S. Now we wish to find out the equation of a tangent plane to S at the point $P\ (x_1, y_1, z_1)$.

Hence all these tangent lines will lie in a plane known as tangent plane.

Equation of tangent plane

$$\boxed{(x - x_1)\frac{\partial F}{\partial x} + (y - y_1)\frac{\partial F}{\partial y} + (z - z_1)\frac{\partial F}{\partial z} = 0}$$

Equation of the normal to the plane.

$$\boxed{\frac{x - x_1}{\dfrac{\partial F}{\partial x}} = \frac{y - y_1}{\dfrac{\partial F}{\partial y}} = \frac{z - z_1}{\dfrac{\partial F}{\partial z}}}$$

Example 49. *Find the equation of the tangent plane and normal line to the surface*

$$x^2 + 2y^2 + 3z^2 = 12 \text{ at } (1, 2, -1).$$

Solution. $$F(x, y, z) = x^2 + 2y^2 + 3z^2 - 12$$

$$\frac{\partial F}{\partial x} = 2x, \quad \frac{\partial F}{\partial y} = 4y, \quad \frac{\partial F}{\partial z} = 6z$$

At the point (1, 2, – 1) $\frac{\partial F}{\partial x} = 2, \quad \frac{\partial F}{\partial y} = 8, \quad \frac{\partial F}{\partial z} = -6$

Hence the equation of the tangent plane at (1, 2, – 1) is

$$2(x-1) + 8(y-2) - 6(z+1) = 0$$

$$\Rightarrow \quad 2x + 8y - 6z = 24 \quad \Rightarrow \quad x + 4y - 3z = 12$$

Equation of normal is $\frac{x-1}{2} = \frac{y-2}{8} = \frac{z+1}{-6} \Rightarrow \frac{x-1}{1} = \frac{y-2}{4} = \frac{z+1}{-3}$ **Ans.**

Example 50. *Show that the surface $x^2 - 2yz + y^3 = 4$ is perpendicular to any number of the family of surfaces $x^2 + 1 = (2 - 4a)y^2 + az^2$ at the point of intersection* (1, – 1, 2).

Solution.

$$f(x, y, z) = x^2 - 2yz + y^3 - 4 = 0 \qquad \text{...(1)}$$

$$F(x, y, z) = x^2 + 1 - (2 - 4a)y^2 - az^2 = 0 \qquad \text{...(2)}$$

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = -2z + 3y^2, \quad \frac{\partial f}{\partial z} = -2y$$

Direction ratios to the normal of the tangent plane to (1) are

$$2x,\ -2z + 3y^2,\ -2y$$

DRs at the point (1, – 1, 2) are 2, – 1, 2.

Now differentiating (2), we get

$$\frac{\partial F}{\partial x} = 2x, \quad \frac{\partial F}{\partial y} = -2(2 - 4a)y, \quad \frac{\partial F}{\partial z} = -2az.$$

Direction ratios to the normal of the tangent plane to (2) are

$$2x,\ (-4 + 8a)y,\ -2az.$$

DRs at the point (1, – 1, 2) are 2, 4 – 8a, – 4a

Now

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = (2)(2) + (-1)(4 - 8a) + 2(-4a)$$
$$= 4 - 4 + 8a - 8a = 0.$$

Hence, the given surfaces are perpendicular at (1, – 1, 2). **Ans.**