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Probability and Statistics II

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Outline

1. Introduction.
2. The method of moments.

1. Introduction

Definition

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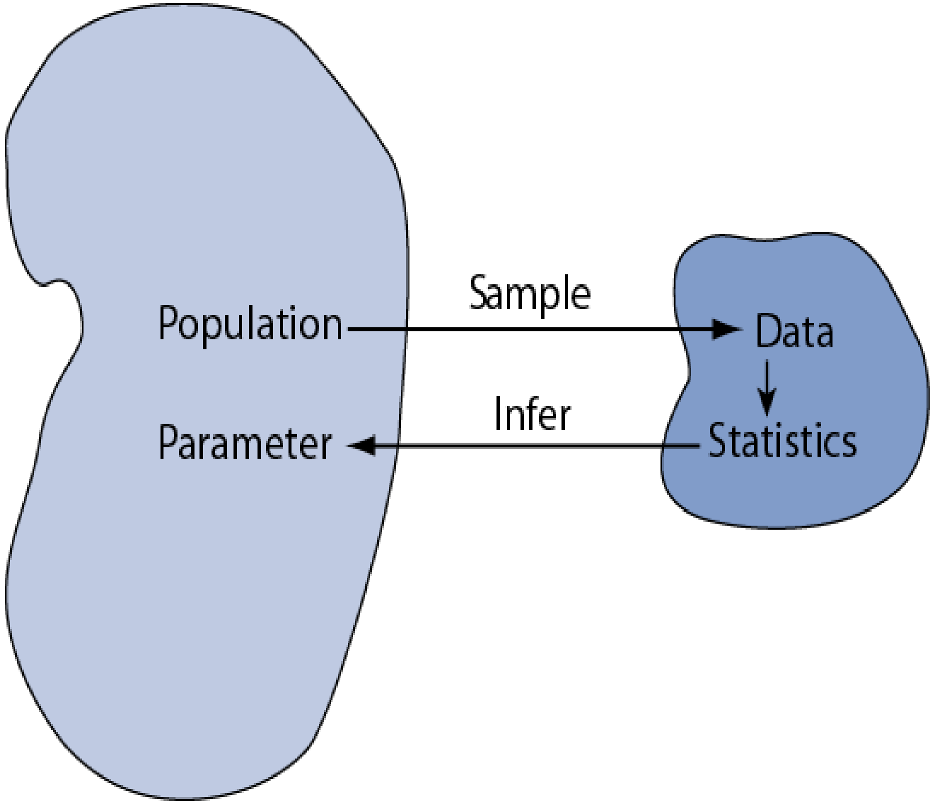
Statistical inference is the process by which information from **sample** data is used to draw conclusions about the **population** from which the sample was selected.

We want to learn about population *parameters* …



…but we can only calculate *sample statistics*

Statistical Inference

Parameter Estimation

Hypothesis testing (Chapter 4)

Point Estimation (Chapter 2)

A point estimate is a single value estimate of a parameter.

Ex: $$\hat{\mu} = \bar{X} = \frac{\sum_i^n X_i}{n}$$

Interval Estimation (Chapter 3)

An interval estimate gives you a range of values where the parameter is expected to lie. **Ex:** $L < \mu < U$ with confidence level **95%**

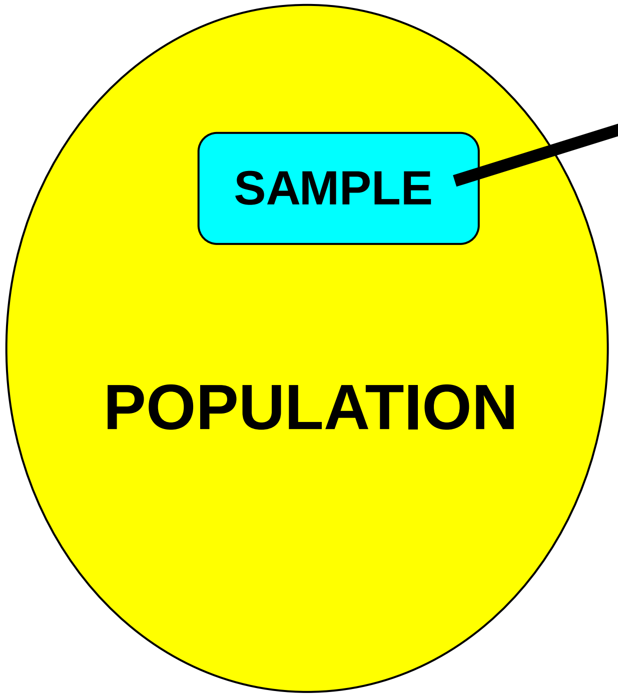
Statistical Inference → Parameter Estimation → Point Estimation

<u>Random sample:</u>

Subset of the population selected randomly

It is referred to

- The i.i.d. random variables
- or the observed data



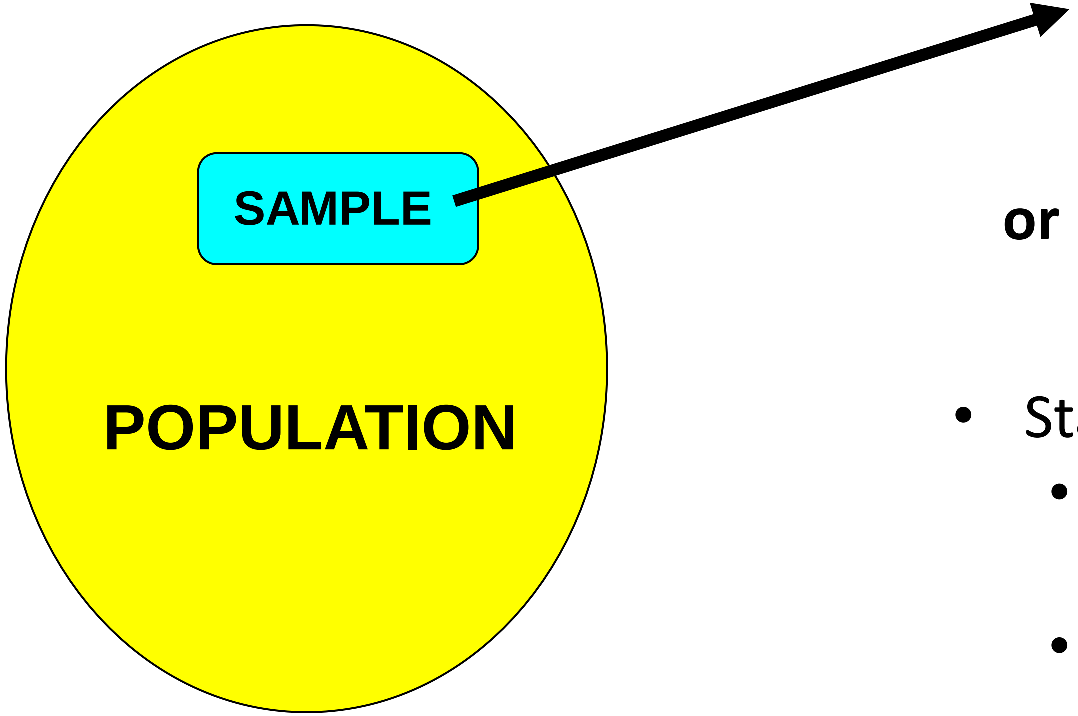
$\{X_1, X_2, \ldots, X_n\}$ **i.i.d. RVs**

or $\{x_1, x_2, \ldots, x_n\}$ **Observed data**

Statistical Inference → Parameter Estimation → Point Estimation

<u>Statistic:</u>

- Any function of the random sample $X_1, X_2, \dots, X_n$ is called a statistic.
- Does not depend on any unknown parameters.



$\{X_1, X_2, \dots, X_n\}$ **i.i.d. RVs**

or $\{x_1, x_2, \dots, x_n\}$ **Observed data**

- Statistic
  - Sample mean $\overline{X} = \dfrac{X_1 + X_2 + \cdots + X_n}{n} = \dfrac{\sum_{i=1}^{n} X_i}{n}$
  - Sample variance $S^2 = \dfrac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{n-1}$

Statistical Inference → Parameter Estimation → Point Estimation

Estimator vs Estimate:

- Assume a **population** with **unknown parameter $\boldsymbol{\theta}$**
- Let $X_1, X_2, \dots, X_n$ be a random sample → X_i has PDF $f(x\,;\,\theta)$.
- The statistic $\hat{\theta} = t(X_1, X_2, \dots, X_n)$ corresponding to θ fn.

  → called **estimator**. $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ $X_1, X_2, \dots$

  one sample
- $t(x_1, x_2, \dots, x_n)$ "The function of the observed data" observed data $190, 180, 170, 200$ cm

  → called **estimate**. $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

Statistical Inference → Parameter Estimation → Point Estimation

<u>Estimation problems occur frequently in real life:</u>

Unknown population parameter

- the mean μ of a single population
- the variance σ^2 (or standard deviation σ) of a single population
- the difference in means of two populations, $\mu_1 - \mu_2$

Reasonable point estimate (By intuition)

- for μ, the estimate is $\hat{\mu} = \overline{X}$, the sample mean
- for σ^2, the estimate is $\hat{\sigma}^2 = S^2$, the sample variance
- for $\mu_1 - \mu_2$, the estimate is $\hat{\mu}_1 - \hat{\mu}_2 = \overline{X}_1 - \overline{X}_2$ the difference between the sample means of two independent random samples

Statistical Inference → Parameter Estimation → Point Estimation

General methods to derive the estimators

The method of moments

(This lecture)

The Method of Maximum Likelihood

(Next lecture)

2. The method of moments

The method of moments

Definition:
Population and sample moments

The k^{th} **population moment** of a RV (about the origin) is

$$\mu_k = E[X^k]$$

$$= \int_{-\infty}^{\infty} x^k \, f(x;\theta) \; dx \quad \text{"Continuous RV"}$$

$$= \sum_{x=-\infty}^{\infty} x^k \, f(x;\theta) \quad \text{"Discrete RV"}$$

depend on the unknown parameter

The k^{th} **sample moment** is

$$m_k = \frac{1}{n} \sum_{i=1}^{n} X_i^k$$

sample

doesn't depend on any unknown parameter

The method of moments

The **Method of Moments** (MoM) consists of equating sample moments and population moments. If a population has t parameters, the MOM consists of solving the system of equations

$$m_k = \mu_k, \; k = 1, 2, \ldots, t$$

for the t parameters.

If one parameter is unknown

$$\mu_1 = m_1$$

One equation in one unknown

If two parameters are unknown

$$\mu_1 = m_1$$
$$\mu_2 = m_2$$

Two equations in two unknown

…

The method of moments

Example

Let **X** be uniformly distributed on the interval (α, 1). Given a random sample of size n, use the method of moments to obtain a formula for estimating the parameter α.

Solution

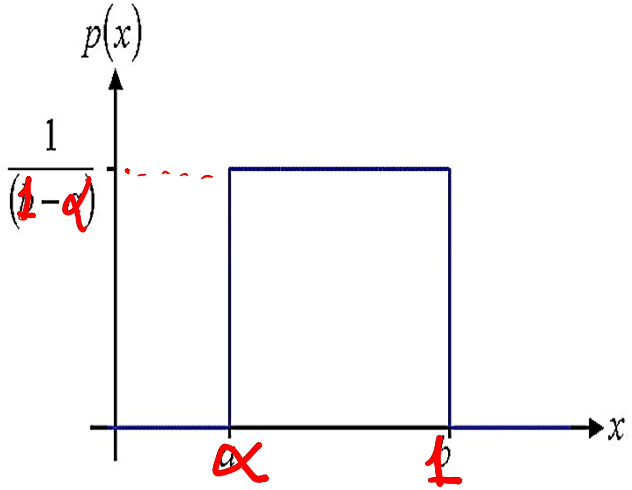
1 unknown → 1 eqn.

pop. moment → $\mu_1 = m_1$ ← sample moment

The first population moment

$$\mu_1 = E(X) = \int_{-\infty}^{\infty} x\, f(x;\alpha)\, dx = \int_{\alpha}^{1} x \cdot \frac{1}{1-\alpha}\, dx = \frac{1}{1-\alpha}\left[\frac{x^2}{2}\right]_{\alpha}^{1} = \frac{1+\alpha}{2}$$

$$x^n \int \rightarrow \frac{x^{n+1}}{n+1}$$



$$\frac{1}{1-\alpha}\left[\frac{1}{2} - \frac{\alpha^2}{2}\right]$$

$$\frac{1}{2}\,\frac{1-\alpha^2}{1-\alpha} = \frac{1}{2}\,\frac{\cancel{(1-\alpha)}(1+\alpha)}{\cancel{(1-\alpha)}}$$

The method of moments

Example

Let **X** be uniformly distributed on the interval (α, 1). Given a random sample of size n, use the method of moments to obtain a formula for estimating the parameter α.

Solution

idea

α better estimator

min $(x_i$'s$)$

The first population moment

$$\mu_1 = \frac{1+\alpha}{2}$$

The first sample moment

$$m_1 = \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

α hat

Using MME

$$m_1 = \mu_1 \;\Rightarrow\; \overline{X} = \frac{1+\hat{\alpha}}{2} \;\Rightarrow\; \hat{\alpha} = 2\overline{X} - 1$$

Estimator for α

The method of moments

Example

Given a random sample of size n from a Poisson population, use the method of moments to obtain a formula for estimating the parameter λ. **Hint:** $M_X(t) = e^{\lambda(e^t - 1)}$

What is this function?

Definition: The moment generating function (MGF)

Definition

$$M_X(t) = E(e^{tX})$$

$$E(X) = \left[\frac{d}{dt} M_X(t)\right]_{t=0}$$

$$E(X^2) = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0}$$

$$E(X^n) = \left[\frac{d^n}{dt^n} M_X(t)\right]_{t=0}$$

MGF for important RV's

$Binomial(n,p)$	$(1-p+pe^t)^n$
$Geometric(p)$	$\frac{pe^t}{1-(1-p)e^t}$
$Poisson(\lambda)$	$e^{\lambda(e^t-1)}$
$Uniform(a,b)$	$\frac{e^{bt}-e^{at}}{t(b-a)}$
$Exponential(\lambda)$	$\frac{\lambda}{\lambda-t}$
$N(\mu,\sigma^2)$	$e^{t\mu+\frac{1}{2}t^2\sigma^2}$

The method of moments

$e^{\square} \xrightarrow{d/dt} e^{\square}\ \square'$

Example

Given a random sample of size n from a Poisson population, use the method of moments to obtain a formula for estimating the parameter λ. **Hint:** $M_X(t) = e^{\lambda(e^t - 1)}$

Solution

1 unk. → 1 eq. → $\mu_1 = m_1$

$E[X]$ $\bar{x}$

$$E(X) = \left[\frac{d}{dt} M_X(t)\right]_{t=0}$$

$\lambda e^t - \cancel{\lambda}^{0}$

$$E(X) = \left[e^{\lambda(e^t-1)}\,\lambda e^t\right]_{t=0}$$

$$E(X) = \lambda$$

The method of moments

Example

Given a random sample of size n from a Poisson population, use the method of moments to obtain a formula for estimating the parameter λ. **Hint:** $M_X(t) = e^{\lambda(e^t - 1)}$

Solution

The first population moment

$$\mu_1 = E(x) = \lambda$$

The first sample moment

$$m_1 = \overline{X}$$

Using MME

$$m_1 = \mu_1 \Rightarrow \quad \overline{X} = \boxed{\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} x_i}$$

The method of moments

$$\frac{d}{dt}(\square\,\Delta) = \square\,\Delta' + \square'\,\Delta$$

Example

Given a random sample of size n from a $\mathbf{N(\mu, \sigma^2)}$ population, use the method of moments to obtain formulas for estimating the parameters $\mathbf{\mu}$ and $\mathbf{\sigma^2}$. 2 unks. → 2 eqs.

Solution

Hint: $M_X(t) = e^{t\mu + \frac{1}{2}t^2\sigma^2}$

$\mu + \frac{1}{2}\sigma^2\,\cancel{2}\,t$

The first two population moments

$$E(X) = \left[\frac{d}{dt} M_X(t)\right]_{t=0}$$

$$E(X) = \left[e^{t\mu + \frac{1}{2}t^2\sigma^2}\,(\mu + t\,\sigma^2)\right]_{t=0}$$

$$E(X) = \mu$$

diff. one more time

$$E(X^2) = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0}$$

$$E(X) = \left[e^{t\mu + \frac{1}{2}t^2\sigma^2}\,\sigma^2 + e^{t\mu + \frac{1}{2}t^2\sigma^2}\,(\mu + t\,\sigma^2)^2\right]_{t=0}$$

$$E(X^2) = \sigma^2 + \mu^2$$

The method of moments

Example

Given a random sample of size n from a $\mathbf{N}(\boldsymbol{\mu}, \boldsymbol{\sigma}^2)$ population, use the method of moments to obtain formulas for estimating the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}^2$.

Solution

Hint: $M_X(t) = e^{t\,\mu + \frac{1}{2} t^2 \sigma^2}$

The first two population moments

$$\mu_1 = E(x) = \mu \;, \qquad \mu_2 = E(x^2) = \sigma^2 + \mu^2$$

$\bar{X}^2$

$\hat{\sigma}^2 + \hat{\mu}^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2$

The first two sample moments

$$\frac{1}{n}\sum_{i=1}^{n} X_i = m_1 = \bar{X} \quad \text{and} \quad m_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2$

Using MME

$$m_1 = \mu_1 \;\Rightarrow\; \bar{X} = \hat{\mu} \qquad \hat{\mu} = \bar{X}$$

$$m_2 = \mu_2 \;\Rightarrow\; \frac{1}{n}\sum_{i=1}^{n} X_i^2 = \hat{\sigma}^2 + \hat{\mu}^2 \qquad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2 = \frac{1}{n}\sum (X_i - \bar{X})^2$$

The method of moments

Example

Given a random sample of size n from a $\mathbf{N(\mu, \sigma^2)}$ population, use the method of moments to obtain formulas for estimating the parameters μ and σ^2.

Solution

Hint: $M_X(t) = e^{t\mu + \frac{1}{2}t^2\sigma^2}$

$$\hat{\mu} = \bar{X}$$ Similar to previous estimator

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2 \boxed{=} \frac{1}{n}\sum (X_i - \bar{X})^2 \Rightarrow E(\ ) \neq \sigma^2$$

$$\frac{1}{n-1} \quad \checkmark \quad E(\ ) = \sigma^2$$

$$\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n}\left[\sum_{i=1}^{n} x_i^2 + \underbrace{\sum_{i=1}^{n} \bar{x}^2}_{+n\bar{x}^2} - 2\underbrace{\sum_{i=1}^{n} x_i\bar{x}}_{n\bar{x}}\right]$$

const.

$-2n\bar{x}^2$

$$= \frac{1}{n}\left[\sum x_i^2 - n\bar{x}^2\right] = \frac{1}{n}\sum x_i^2 - \bar{x}^2$$

More Examples

- Example: Find method of moments estimators (MME's) of θ based on a random sample $X_1, \ldots, X_n$ from the following pdf:

1 unk.

continuous

$$f(x;\theta) = (\theta+1)x^{-\theta-2};\ 1 < x,\ \text{zero otherwise};\ \theta > 0.$$

1 eqn.

$$\mu_1 = E(X) = \int_1^\infty x(\theta+1)x^{-\theta-2}\,dx = (\theta+1)\int_1^\infty x^{-\theta-1}\,dx$$

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$$= (\theta+1)\left[\frac{x^{-\theta}}{-\theta}\right]_1^\infty = \frac{\theta+1}{\theta}(1-0) = \frac{\theta+1}{\theta}$$

$$m_1 = \bar{X} \Rightarrow \mu_1 = m_1 \Rightarrow \frac{\hat{\theta}+1}{\hat{\theta}} = \bar{X} \Rightarrow \hat{\theta}+1 = \bar{X}\hat{\theta}$$

$$\hat{\theta}(\bar{X}-1) = 1 \Rightarrow \boxed{\hat{\theta} = \frac{1}{\bar{X}-1}}$$

- Example: Find method of moments estimators (MME's) of θ based on a random sample $X_1, \ldots, X_n$ from the following pdf:

one unk.

$$f(x;\theta) = \theta^2 x e^{-\theta x} \;;\; 0 < x, \text{ zero otherwise}; \; \theta > 0 .$$

$$\boxed{e^{-\infty} = 0}$$

$$\mu_1 = E(X) = \int_0^\infty x\,\theta^2 x e^{-\theta x}\,dx = \theta^2 \int_0^\infty x^2 e^{-\theta x}\,dx$$

$$= \theta^2 \left[\frac{-1}{\theta} x^2 e^{-\theta x} - \frac{2}{\theta^2} x e^{-\theta x} - \frac{2}{\theta^3} e^{-\theta x} \right]_0^\infty$$

$$= \theta^2 \left[0 - \left(\frac{-2}{\theta^3} \right) \right] = \theta^2 \frac{2}{\theta^3} = \frac{2}{\theta}$$

D	I
x^2	$e^{-\theta x}$
$2x$	$\frac{-1}{\theta} e^{-\theta x}$
2	$\frac{1}{\theta^2} e^{-\theta x}$
0	$\frac{-1}{\theta^3} e^{-\theta x}$

$+$, $-$, $+$

$$m_1 = \bar{X} = \frac{2}{\hat{\theta}} \Rightarrow \boxed{\hat{\theta} = \frac{2}{\bar{X}}}$$

- Example: Find method of moments estimators (MME's) of θ based on a random sample $X_1, \dots, X_n$ from Geometric distribution:

↳ one unit.

$$f(x\,;\theta) = \theta\,(1-\theta)^{x-1} \qquad \text{for} \qquad x = 1, 2, 3, \dots$$

Hint: $\sum_{x=1}^{\infty} x z^x = \frac{z}{(1-z)^2}$

$$\mu_1 = E(X) = \sum_{x=1}^{\infty} x\, \underbrace{\theta\,(1-\theta)^{x-1}}_{}$$

$$= \theta \sum_{x=1}^{\infty} x\, (\underbrace{(1-\theta)}_{z})^x\, (1-\theta)^{-1}$$

z^x

const.

$$\mu_1 = \frac{\theta}{1-\theta} \sum_{x=1}^{\infty} x\, z^x \qquad \text{where } z = 1-\theta$$

$$= \frac{\theta}{1-\theta} \frac{z}{(1-z)^2} = \frac{\theta}{1-\theta} \frac{1-\theta}{(\cancel{1}-(1-\theta))^2} = \frac{\cancel{\theta}}{\cancel{1-\theta}} \frac{\cancel{1-\theta}}{\theta^{\cancel{2}}} = \frac{1}{\theta}$$

$$m_1 = \bar{X} = \frac{1}{\hat{\theta}}$$

$$\Rightarrow \hat{\theta} = \frac{1}{\bar{X}}$$

- Example: Let X be a random variable having a gamma distribution

$$f(x) = \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^{\alpha}\Gamma(\alpha)} \quad , \quad x \geq 0, \alpha > 0, \beta > 0$$

with unknown parameters α and β. Find the moment estimators of the unknown parameters. 2 eqs.

(Hint: $\mu_1 = E(X) = \alpha\beta$ and $Var(X) = \alpha\beta^2$)

Note: $Var(X) = E(X^2) - \mu^2$

$$E(X^2) - (\alpha\beta)^2 = \alpha\beta^2 \Rightarrow E(X^2) = \alpha\beta^2 + \alpha^2\beta^2$$

$$m_1 = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad , \quad m_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

$$\alpha\beta = \bar{X} \rightarrow (1)$$

$$\underbrace{\alpha\beta^2}_{(\alpha\beta)\beta} + \underbrace{\alpha^2\beta^2}_{(\alpha\beta)^2} = \frac{1}{n}\sum X_i^2 \rightarrow (2)$$

2 eqs. in 2 unks.

Sub. from (1) in (2)

$$\bar{X}\hat{\beta} + \bar{X}^2 = \frac{1}{n}\sum X_i^2 \Rightarrow \hat{\beta} = \frac{\frac{1}{n}\sum X_i^2 - \bar{X}^2}{\bar{X}}$$

Sub. in (1)

$$\hat{\alpha} = \frac{\bar{X}}{\hat{\beta}}$$