NUMERICAL COMPUTATIONS

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INTERPOLATION AND POLYNOMIAL APPROXIMATION

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Discrete Data

Revision

The Weierstrass approximation theorem

To fit a polynomial to a set of discrete data:

a) Exact fits

b) Approximate fits

There are alternate algebraic representations that are useful in certain situations.

Direct Fit Polynomials

Lagrange polynomials

A uniqueness theorem

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DIVIDED DIFFERENCE TABLES AND DIVIDED DIFFERENCE POLYNOMIALS

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Suppose that $P_n(x)$ is the nth Lagrange polynomial that agrees with the function f at the distinct numbers $x_0, x_1, \ldots, x_n$. Although this polynomial is unique, there are alternate algebraic representations that are useful in certain situations. The divided differences of f with respect to $x_0, x_1, \ldots, x_n$ are used to express $P_n(x)$ in the form

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)\cdots(x - x_{n-1}), \quad (3.5)$$

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$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)\cdots(x - x_{n-1}),$$

$$a_0 = P_n(x_0) = f(x_0).$$

$$f(x_0) + a_1(x_1 - x_0) = P_n(x_1) = f(x_1);$$

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

$$f[x_i] = f(x_i).$$

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}.$$

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}.$$

$$P_n(x) = f_0^{(0)} + f_0^{(1)}(x - x_0) + f_0^{(2)}(x - x_0)(x - x_1) + f_0^{(3)}(x - x_0)(x - x_1)(x - x_2) + \ldots$$

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$$P_n(x) = f_0^{(0)} + f_0^{(1)}(x - x_0) + f_0^{(2)}(x - x_0)(x - x_1) + f_0^{(3)}(x - x_0)(x - x_1)(x - x_2) + \dots$$
$$P_1(x) = f_0^{(0)} + f_0^{(1)}(x - x_0)$$
$$P_2(x) = f_0^{(0)} + f_0^{(1)}(x - x_0) + f_0^{(2)}(x - x_0)(x - x_1)$$
$$P_3(x) = f_0^{(0)} + f_0^{(1)}(x - x_0) + f_0^{(2)}(x - x_0)(x - x_1) + f_0^{(3)}(x - x_0)(x - x_1)(x - x_2)$$

$$P_2(x) = P_1(x) + f_0^{(2)}(x - x_0)(x - x_1)$$
$$P_3(x) = P_2(x) + f_0^{(3)}(x - x_0)(x - x_1)(x - x_2)$$
$$P_4(x) = P_3(x) + f_0^{(4)}(x - x_0)(x - x_1)(x - x_2)(x - x_3)$$
$$P_5(x) = P_4(x) + f_0^{(5)}(x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)$$

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DIVIDED DIFFERENCE TABLES AND DIVIDED DIFFERENCE POLYNOMIALS

A divided difference **is defined as the ratio of the difference in the function values at two points divided by the difference in the values of the corresponding independent variable.**

Thus, *the first divided difference* at point *i* is defined as

$$f[x_i, x_{i+1}] = \frac{f_{i+1} - f_i}{x_{i+1} - x_i} \qquad f[x_i] = f_i.$$

The second divided difference **is defined as**

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

Similar expressions can be obtained for divided differences of any order.

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x	$f(x)$	First divided differences	Second divided differences	Third divided differences
x_0	$f[x_0]$			
		$f[x_0,x_1]=\dfrac{f[x_1]-f[x_0]}{x_1-x_0}$		
x_1	$f[x_1]$		$f[x_0,x_1,x_2]=\dfrac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$	
		$f[x_1,x_2]=\dfrac{f[x_2]-f[x_1]}{x_2-x_1}$		$f[x_0,x_1,x_2,x_3]=\dfrac{f[x_1,x_2,x_3]-f[x_0,x_1,x_2]}{x_3-x_0}$
x_2	$f[x_2]$		$f[x_1,x_2,x_3]=\dfrac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1}$	
		$f[x_2,x_3]=\dfrac{f[x_3]-f[x_2]}{x_3-x_2}$		$f[x_1,x_2,x_3,x_4]=\dfrac{f[x_2,x_3,x_4]-f[x_1,x_2,x_3]}{x_4-x_1}$
x_3	$f[x_3]$		$f[x_2,x_3,x_4]=\dfrac{f[x_3,x_4]-f[x_2,x_3]}{x_4-x_2}$	
		$f[x_3,x_4]=\dfrac{f[x_4]-f[x_3]}{x_4-x_3}$		$f[x_2,x_3,x_4,x_5]=\dfrac{f[x_3,x_4,x_5]-f[x_2,x_3,x_4]}{x_5-x_2}$
x_4	$f[x_4]$		$f[x_3,x_4,x_5]=\dfrac{f[x_4,x_5]-f[x_3,x_4]}{x_5-x_3}$	
		$f[x_4,x_5]=\dfrac{f[x_5]-f[x_4]}{x_5-x_4}$		
x_5	$f[x_5]$			

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Divided Difference Tables

Consider a table of data:

x_i	f_i
x_0	f_0
x_1	f_1
x_2	f_2
x_3	f_3

$$f[x_i]=f_i.$$

$$f[x_0,x_1]=\frac{(f_1-f_0)}{(x_1-x_0)} \qquad f[x_1,x_2]=\frac{(f_2-f_1)}{(x_2-x_1)}$$

$$f[x_i,x_{i+1}]=\frac{(f_{i+1}-f_i)}{(x_{i+1}-x_i)}=\frac{(f_i-f_{i+1})}{(x_i-x_{i+1})}=f[x_{i+1},x_i]$$

$$f_i^{(1)}=f[x_i,x_{i+1}]$$

$$f[x_0,x_1,x_2]=\frac{f[x_1,x_2]-f[x_0,x_1]}{(x_2-x_0)} \qquad f_i^{(2)}=f[x_i,x_{i+1},x_{i+2}]$$

$$f[x_0,x_1,\ldots,x_n]=\frac{f[x_1,x_2,\ldots,x_n]-f[x_0,x_1,\ldots,x_{n-1}]}{(x_n-x_0)}$$

$$f_i^{(n)}=f[x_i,x_{i+1},\ldots,x_{i+n}]$$

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Divided Difference Tables

x_i	$f_i^{(0)}$	$f_i^{(1)}$	$f_i^{(2)}$	$f_i^{(3)}$
x_0	$f_0^{(0)}$			
		$f_0^{(1)}$		
x_1	$f_1^{(0)}$		$f_0^{(2)}$	
		$f_1^{(1)}$		$f_0^{(3)}$
x_2	$f_2^{(0)}$		$f_1^{(2)}$	
		$f_2^{(1)}$		
x_3	$f_3^{(0)}$			

$$f_i^{(n)} = f[x_i, x_{i+1}, \ldots, x_{i+n}]$$
$$= \frac{f[x_1, x_2, \ldots, x_n] - f[x_0, x_1, \ldots, x_{n-1}]}{(x_n - x_0)}$$

x	f(x)
3.20	0.312500
3.30	0.303030
3.35	0.298507
3.40	0.294118
3.50	0.285714
3.60	0.277778
3.65	0.273973
3.70	0.270270

x_i	$f_i^{(0)}$	$f_i^{(1)}$	$f_i^{(2)}$	$f_i^{(3)}$
3.20	0.312500			
		−0.094700		
3.30	0.303030		0.028267	
		−0.090460		−0.007335
3.35	0.298507		0.026800	
		−0.087780		−0.009335
3.40	0.294118		0.024933	
		−0.084040		−0.006132
3.50	0.285714		0.023400	
		−0.079360		−0.006668
3.60	0.277778		0.021733	
		−0.076100		−0.006665
3.65	0.273973		0.020400	
		−0.074060		
3.70	0.270270			

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Example: Divided difference polynomials.

x	$f(x)$
3.35	0.298507
3.40	0.294118
3.50	0.285714
3.60	0.277778

Consider the divided difference table presented. Let's interpolate for *f(3.44)* using the divided difference polynomial,

$$P_n(x) = f_i^{(0)} + (x - x_0)f_i^{(1)} + (x - x_0)(x - x_1)f_i^{(2)} + \cdots + (x - x_0)(x - x_1)\cdots(x - x_{n-1})f_i^{(n)}$$

Using *x0=3.35* as the base point. The exact solution is *f(3.44)= 1/3.44 = 0.290698*.

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x_i	$f_i^{(0)}$			
3.35	0.298507			
3.40	0.294118			
3.50	0.285714			
3.60	0.277778			

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x_i	$f_i^{(0)}$	$f_i^{(1)}$		
3.35	0.298507			
		−0.087780		
3.40	0.294118			
		−0.084040		
3.50	0.285714			
		−0.079360		
3.60	0.277778			

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x_i	$f_i^{(0)}$	$f_i^{(1)}$	$f_i^{(2)}$	
3.35	0.298507			
		−0.087780		
3.40	0.294118		0.024933	
		−0.084040		
3.50	0.285714		0.023400	
		−0.079360		
3.60	0.277778			

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x_i	$f_i^{(0)}$	$f_i^{(1)}$	$f_i^{(2)}$	$f_i^{(3)}$
3.35	0.298507			
		−0.087780		
3.40	0.294118		0.024933	
		−0.084040		−0.006132
3.50	0.285714		0.023400	
		−0.079360		
3.60	0.277778			

$$\boxed{\begin{aligned} P_n(x) = f_i^{(0)} + (x-x_0)f_i^{(1)} + (x-x_0)(x-x_1)f_i^{(2)} + \cdots \\ + (x-x_0)(x-x_1)\cdots(x-x_{n-1})f_i^{(n)} \end{aligned}}$$

$$\begin{aligned} P_n(\boxed{3.44}) = f_0^{(0)} + (\boxed{3.44} - x_0)f_0^{(1)} + (\boxed{3.44} - x_0)(\boxed{3.44} - x_1)f_0^{(2)} \\ + (\boxed{3.44} - x_0)(\boxed{3.44} - x_1)(\boxed{3.44} - x_2)f_0^{(3)} \end{aligned}$$

Substituting the values of x_0 to x_2 and $f_0^{(0)}$ to $f_0^{(3)}$ into Eq.

gives
$$\begin{aligned} P_n(3.44) &= 0.298507 + (3.44 - 3.35)(-0.087780) \\ &+ (3.44 - 3.35)(3.44 - 3.4)(0.024933) \\ &+ (3.44 - 3.35)(3.44 - 3.4)(3.44 - 3.5)(-0.006132) \end{aligned}$$

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$$P_n(3.44) = 0.298507 + (3.44 - 3.35)(-0.087780)$$
$$+ (3.44 - 3.35)(3.44 - 3.4)(0.024933)$$
$$+ (3.44 - 3.35)(3.44 - 3.4)(3.44 - 3.5)(-0.006132)$$

Evaluating Equation term by term gives

$$P_n(3.44) = 0.298507 - 0.007900 + 0.000089 + 0.000001$$

Summing the terms yields the following results and errors:

$P(3.44) = 0.290607$	linear interpolation	$\text{Error}(3.44) = -0.000091$
$= 0.290696$	quadratic interpolation	$= -0.000002$
$= 0.290697$	cubic interpolation	$= -0.000001$

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Newton Divided-Difference Polynomials: Example

Construct the divided differences table and use Newton interpolating polynomials to interpolate at $x = 1$.

x	$y = 2x^3 - 10$
0	-10
1.5	-3.25
2	6
4	118
5	240
6	422

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Newton Divided-Difference Polynomials: Example

x_i	$f_i^{(0)}$	$f_i^{(1)}$	$f_i^{(2)}$	$f_i^{(3)}$	$f_i^{(4)}$	$f_i^{(5)}$
0	**-10**					
		4.5				
1.5	-3.25		**7**			
		18.5		**2**		
2	6		15		**0**	
		56		2		**0**
4	118		22		0	
		122		2		
5	240		30			
		182				
6	422					

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Newton Divided-Difference Polynomials: Example

x_i	$f_i^{(0)}$	$f_i^{(1)}$	$f_i^{(2)}$	$f_i^{(3)}$	$f_i^{(4)}$	$f_i^{(5)}$
0	**-10**					
		4.5				
1.5	-3.25		**7**			
		18.5		**2**		
2	6		15		**0**	
		56		2		**0**
4	118		22		0	
		122		2		
5	240		30			
		182				
6	422					

$$P_1(x) = -10 \ + \ 4.5(x-0)$$

$$P_1(1) = -5.5$$

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Newton Divided-Difference Polynomials: Example

x_i	$f_i^{(0)}$	$f_i^{(1)}$	$f_i^{(2)}$	$f_i^{(3)}$	$f_i^{(4)}$	$f_i^{(5)}$
0	**-10**					
		4.5				
1.5	-3.25		**7**			
		18.5		**2**		
2	6		15		**0**	
		56		2		**0**
4	118		22		0	
		122		2		
5	240		30			
		182				
6	422					

$$P_2(x) = -10 + 4.5(x-0) + 7(x-0)(x-1.5)$$

$$P_2(1) = -9$$

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Newton Divided-Difference Polynomials: Example

x_i	$f_i^{(0)}$	$f_i^{(1)}$	$f_i^{(2)}$	$f_i^{(3)}$	$f_i^{(4)}$	$f_i^{(5)}$
0	**-10**					
		4.5				
1.5	-3.25		**7**			
		18.5		**2**		
2	6		15		**0**	
		56		2		**0**
4	118		22		0	
		122		2		
5	240		30			
		182				
6	422					

$$P_3(x) = -10 + 4.5(x-0) + 7(x-0)(x-1.5) + 2(x-0)(x-1.5)(x-2)$$

$$P_3(1) = -8$$

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Newton Divided-Difference Polynomials: Example

x_i	$f_i^{(0)}$	$f_i^{(1)}$	$f_i^{(2)}$	$f_i^{(3)}$	$f_i^{(4)}$	$f_i^{(5)}$
0	**-10**					
		4.5				
1.5	-3.25		**7**			
		18.5		**2**		
2	6		15		**0**	
		56		2		**0**
4	118		22		0	
		122		2		
5	240		30			
		182				
6	422					

$$P_4(x) = -10 + 4.5(x-0) + 7(x-0)(x-1.5) + 2(x-0)(x-1.5)(x-2)$$

$$P_5(x) = -10 + 4.5(x-0) + 7(x-0)(x-1.5) + 2(x-0)(x-1.5)(x-2)$$

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Newton Divided-Difference Polynomials: Example

$$P_1(x) = -10 + 4.5(x-0)$$

$$P_2(x) = -10 + 4.5(x-0) + 7(x-0)(x-1.5)$$

$$P_3(x) = -10 + 4.5(x-0) + 7(x-0)(x-1.5) + 2(x-0)(x-1.5)(x-2)$$

$$P_4(x) = -10 + 4.5(x-0) + 7(x-0)(x-1.5) + 2(x-0)(x-1.5)(x-2) + 0$$

$$P_5(x) = -10 + 4.5(x-0) + 7(x-0)(x-1.5) + 2(x-0)(x-1.5)(x-2) + 0 + 0$$

$$P_1(x) = -10 + 4.5(x-0)$$

$$P_2(x) = P_1(x) + 7(x-0)(x-1.5)$$

$$P_3(x) = P_2(x) + 2(x-0)(x-1.5)(x-2)$$

$$P_4(x) = P_3(x) + 0$$

$$P_5(x) = P_4(x) + 0 + 0$$

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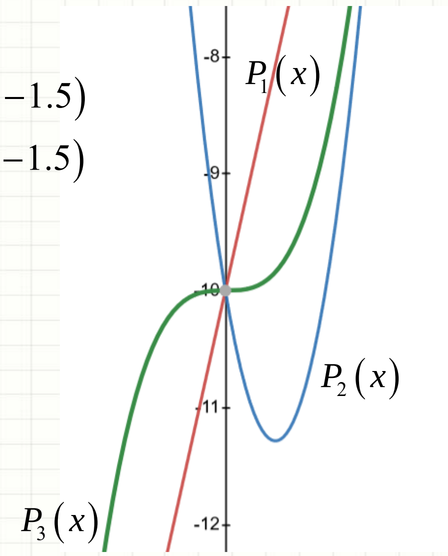
Newton Divided-Difference Polynomials: Example

$$P_1(x) = -10 + 4.5(x-0)$$
$$P_2(x) = -10 + 4.5(x-0) + 7(x-0)(x-1.5)$$
$$P_3(x) = -10 + 4.5(x-0) + 7(x-0)(x-1.5) + 2(x-0)(x-1.5)(x-2)$$



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- The data points **do not have to be in any specific order** to apply the divided difference concept.
- However, just as for the direct fit polynomial, the Lagrange polynomial, more **accurate results are obtained if the data are arranged in order of closeness to the interpolated point**.
- This method works for both **equally spaced data and unequally spaced data**.

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- A major advantage of the Newton forward-difference polynomial, in addition to its simplicity, is that each higher-degree polynomial is obtained from the previous lower degree polynomial by adding the next term.
- The work already performed for the lower-degree polynomial does not have to be repeated.
- his feature is not for the direct fit polynomial and the Lagrange polynomial, where all of the work must be repeated each time the degree of the polynomial is changed.

DIFFERENCE TABLES AND DIFFERENCE POLYNOMIALS

- Fitting approximating polynomials to tabular data is simpler when the values of the independent variable are equally spaced.
- Implementation of polynomial fitting for equally spaced data is best accomplished in terms of differences.

Table 4.6. Table of Differences

x	$f(x)$			
x_0	f_0			
		$(f_1 - f_0)$		
x_1	f_1		$(f_2 - 2f_1 + f_0)$	
		$(f_2 - f_1)$		$(f_3 - 3f_2 + 3f_1 - f_0)$
x_2	f_2		$(f_3 - 2f_2 + f_1)$	
		$(f_3 - f_2)$		
x_3	f_3			

Numerical Methods Simulations

https://alimirlou.github.io/NewtonPolynomial/

https://www.dcode.fr/newton-interpolating-polynomial

https://planetcalc.com/9023/

https://freemagee.github.io/number-difference-table/

https://alteredqualia.com/visualization/hn/sequence/#

https://www.easycalculation.com/algebra/newtons-forward-difference.php

https://www.desmos.com/calculator

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