Chapter 2

Lecture 4: Sets and Functions

12/3/2022

Book: Section 2.1

- Sets
- Set Operations
- Functions
- Sequences and Summations
- Cardinality of Sets
- Matrices

2.1 Sets

- **A *set* is an unordered collection of objects.**
- **The *objects* in a set are called the elements, or members, of the set.**
  - Capital letters (A, B, S…) used for sets.
  - Italic lower-case letters (a, x, y…) used to denote elements of sets.

Example

S = {*a*, *b*, *c*, *d*}

We write $a \in S$ to denote that a is an element of the set S.

The notation $e \notin S$ denotes that e is not an element of the set S.

2.1 Sets

Examples

The set **O** of odd positive integers less than 10, then

- **O** = {1, 3, 5, 7, 9}

The set **O** of positive integers less than 100, then

- **O** ={1, 2, 3, … , 99}.

ellipses (...)

2.1 Sets

Equal Sets

- If A and B are sets, then A and B are equal if and only if :

- $A = B$, if A and B are equal sets.

- $\forall x(x \in A \leftrightarrow x \in B)$.

  - The sets {1, 3 , 5} and {3, 5 , 1} are equal, because they have the same elements.
  - {1 , 3 , 3 , 5 , 5 , 5} is the same as the set {1, 3 , 5} because they have the same elements.

2.1 Sets

Examples

- $\{1, 2, 3\}$ is the set containing "1" and "2" and "3."
- $\{1, 1, 2, 3, 3\} = \{1, 2, 3\}$ since repetition is irrelevant.
- $\{1, 2, 3\} = \{3, 2, 1\}$ since sets are unordered.
- $\{1, 2, 3, \ldots\}$ is a way we denote an infinite set.
- $\{1, 2, 3, 4, 5\} = \{5, 4, 3, 2, 1\}$ but $\{1, 2, 3, 4, 5\} \neq \{1, 2, 3, 4\}$

2.1 Sets

Empty Set

- There is a special set that has no elements.
- This set is called the empty set, or null set, and is denoted by $\emptyset$.
- The empty set can also be denoted by $\{\ \}$
- $\varnothing = \{\ \}$ is the empty set, or the set containing no elements.
  - Note that $\varnothing \neq \{\varnothing\}$

2.1 Sets

Builder Notation

- Another way to describe a set is to use set builder notation.
- The set **O** of odd positive integers less than 10 can be expressed by:
  - **O = {x | x odd positive integers less than 10}**
  - **$O = \{x \in Z^+ \mid x \text{ is odd and } x < 10\}$**

Z^+ *Positive integers*

- The set **E** of odd integers greater than 2 can be expressed by:
  - **$E = \{x \mid x \text{ is odd and } x > 2\}$**

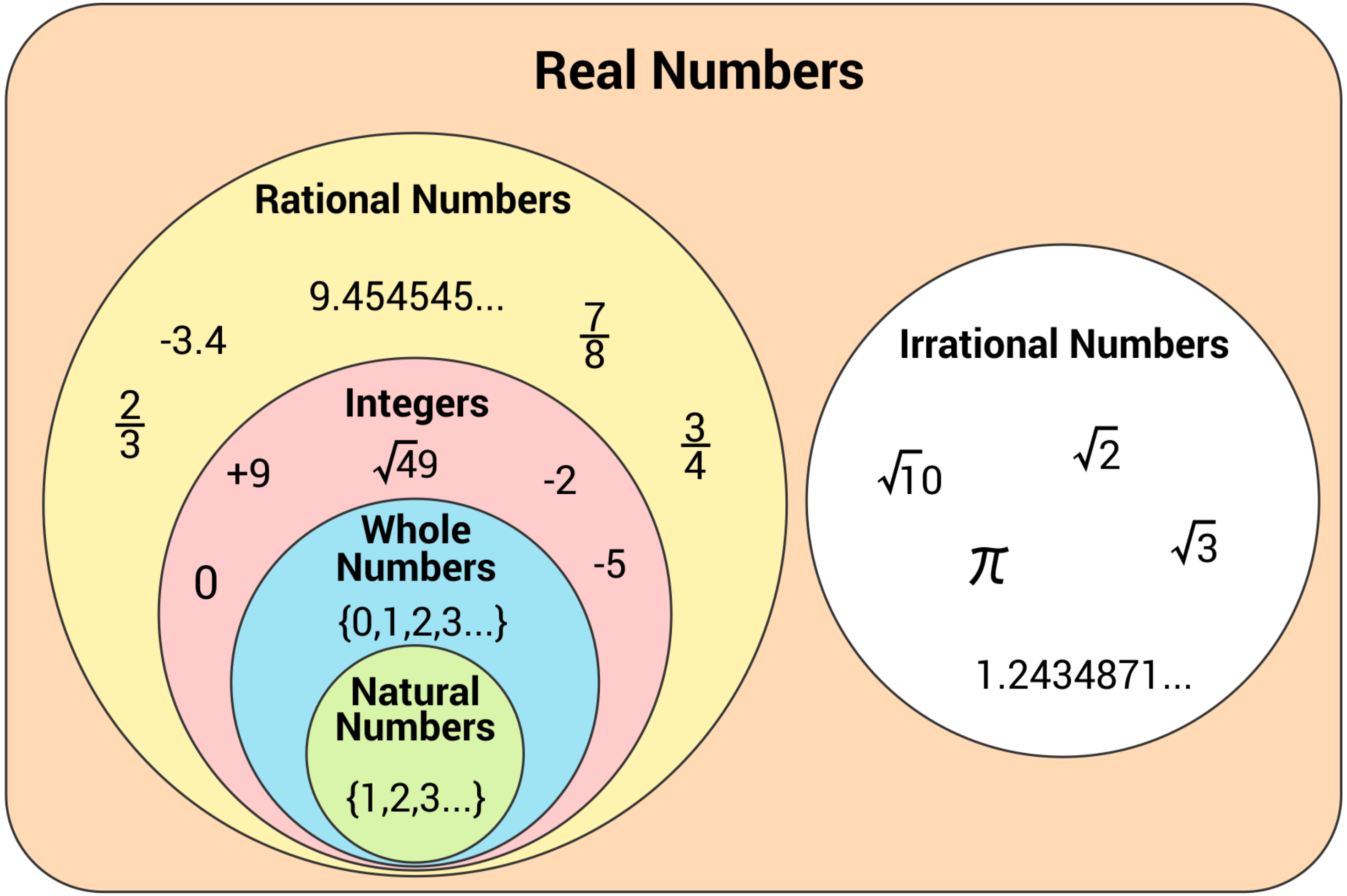
❖ **The vertical bar means “such that” or “where”.**

2.1 Sets

- $N = \{0, 1, 2, 3, \ldots\}$ is the set of natural numbers
- $Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is the set of integers
- $Z^+ = \{1, 2, 3, \ldots\}$ is the set of positive integers
- $Z^- = \{-1, -2, -3, \ldots\}$ is the set of negative integers
- $Q = \{p/q : p, q \in Z, q \neq 0\}$ is the set of rational numbers
  - Any number that can be expressed as a fraction of two integers (where the bottom one is not zero)
- R is the set of real numbers
- R^+ is the set of positive real numbers
- C is the set of complex numbers.

2.1 Sets

- N = natural numbers
- Z = integers
- Z^+ = positive integers
- Z^- = negative integers
- Q = rational numbers
- R real numbers
- R^+ positive real numbers
- C complex numbers.



2.1 Sets

Interval Notation

- Closed interval $[a, b]$
- Open interval (a, b)

$$[a, b] = \{x \mid a \leq x \leq b\}$$

$$[a, b) = \{x \mid a \leq x < b\}$$

$$(a, b] = \{x \mid a < x \leq b\}$$

$$(a, b) = \{x \mid a < x < b\}$$

2.1 Sets

- $x \in S$ means "x is an element of set S."
- $x \notin S$ means "x is not an element of set S."

- **Example:**

  - $4 \in \{1, 2, 3, 4\}$
  - $7 \notin \{1, 2, 3, 4\}$

2.1 Sets

Universal set

- **Universal set "*U*" is:**
  - **The set of all of elements (or the "universe") from which given any set is drawn.**

➢ For the set {-2, 0.4, 2}, ***U*** would be the real numbers

➢ For the set {0, 1, 2}, ***U*** could be the **N**, **Z**, **Q**, **R**

➢ For the set of the vowels of the alphabet, ***U*** would be all the letters of the alphabet.

N = natural numbers
Z = integers
Z^+ = positive integers
Z^- = negative integers
Q = rational numbers
R real numbers
R^+ positive real numbers

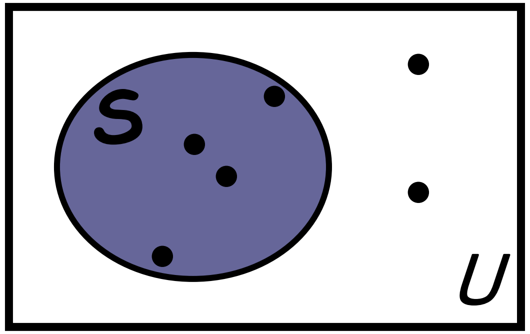
2.1 Sets

- Sets can contain other sets
  - $S = \{ \{1\}, \{2\}, \{3\} \}$
  - $T = \{ \{1\}, \{\{2\}\}, \{\{\{3\}\}\} \}$
  - $V = \{ \{\{1\}, \{\{2\}\}\}, \{\{\{3\}\}\}, \{ \{1\}, \{\{2\}\}, \{\{\{3\}\}\} \} \}$
    - V has only 3 elements!
- Note that $1 \neq \{1\} \neq \{\{1\}\} \neq \{\{\{1\}\}\}$ They are all different
- $\varnothing \neq \{ \varnothing \}$
  - The first is a set of zero elements
  - The second is a set of 1 element
- Replace $\varnothing$ by $\{ \}$, and you get: $\{ \} \neq \{\{ \}\}$

2.1 Sets

Venn diagrams

- Sets can be represented graphically using Venn diagrams.
- **Universal set *U*,** which contains all the objects under consideration, is represented by a ***rectangle***.
- ***Circles*** or other geometrical figures inside this rectangle are used to represent sets.
- ***Points*** represent the particular elements of the set.

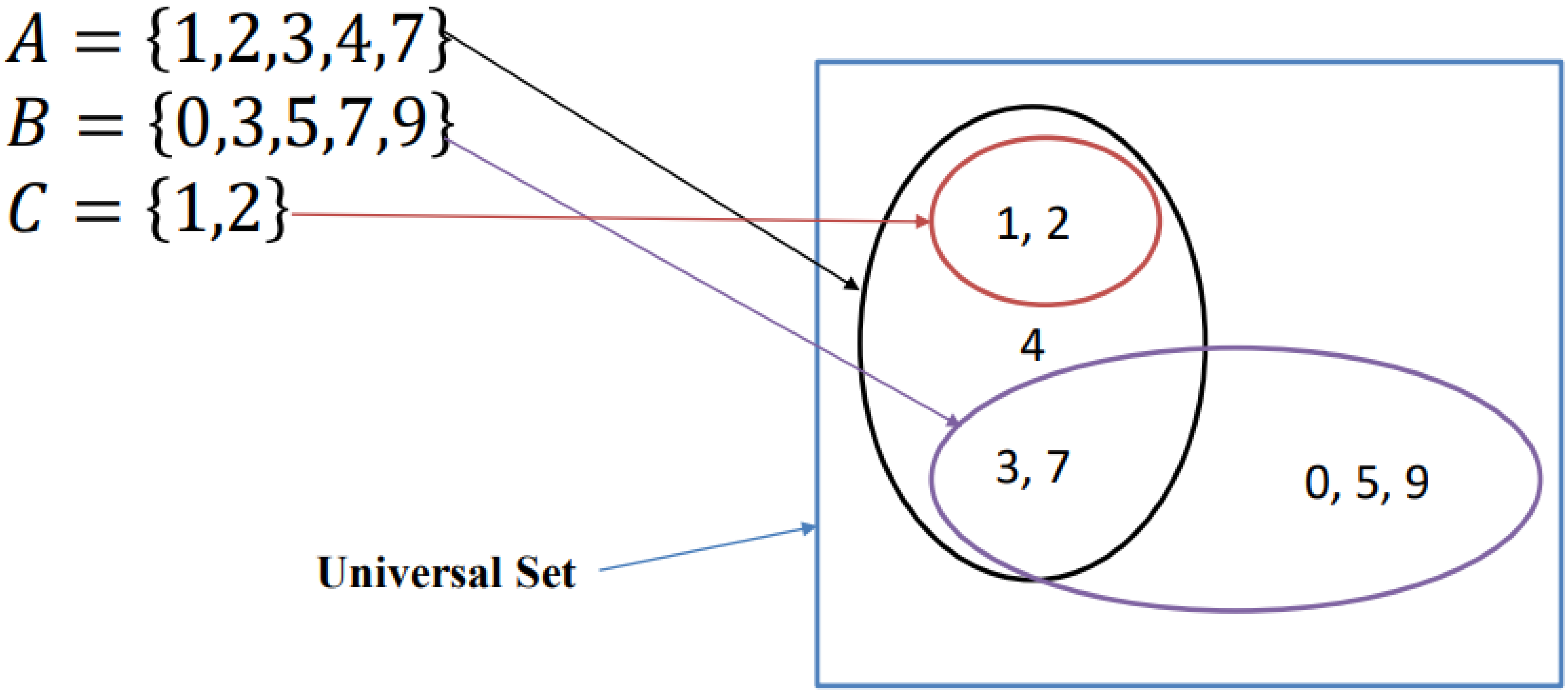


2.1 Sets

Venn diagrams

- Draw the Venn diagrams

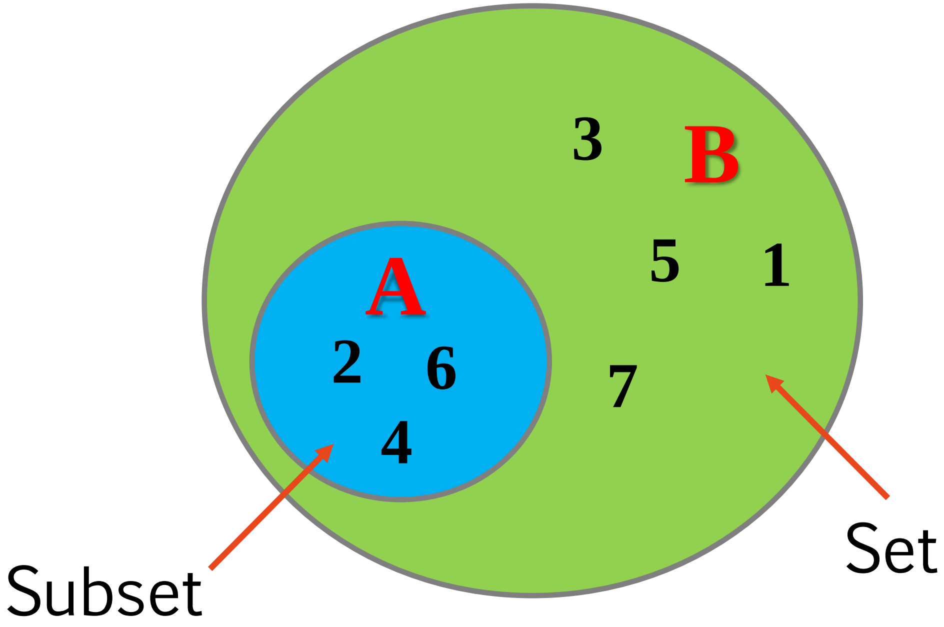
$$A = \{1,2,3,4,7\}$$
$$B = \{0,3,5,7,9\}$$
$$C = \{1,2\}$$



2.1 Sets

Subset set

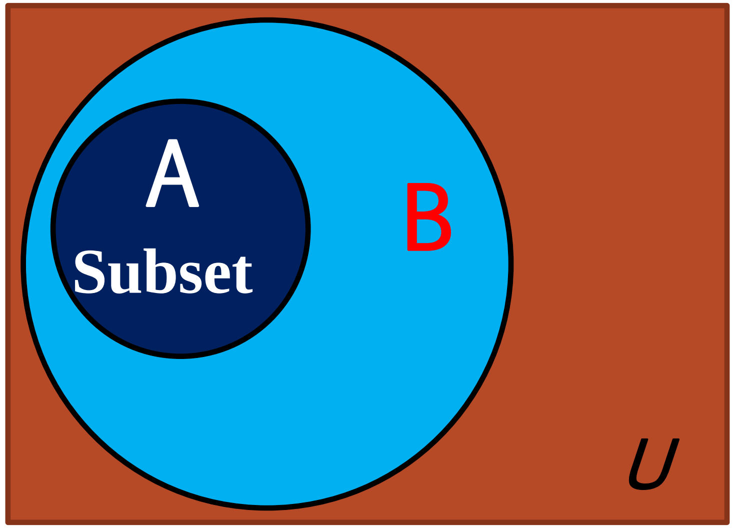
- If A = {2, 4, 6}, B = {1, 2, 3, 4, 5, 6, 7}, A is a subset of B.
- This is specified by $A \subseteq B$ meaning that $\forall\, x\, (x \in A \rightarrow x \in B)$.
- $A \subseteq B$ means: "A is a **subset** of B."
  - **or "B contains A."**
  - **or "every element of A is also in B."**
  - **or $\forall x\, (\underbrace{(x \in A)}_{True} \rightarrow \underbrace{(x \in B)}_{True})$.**



2.1 Sets

Subset set

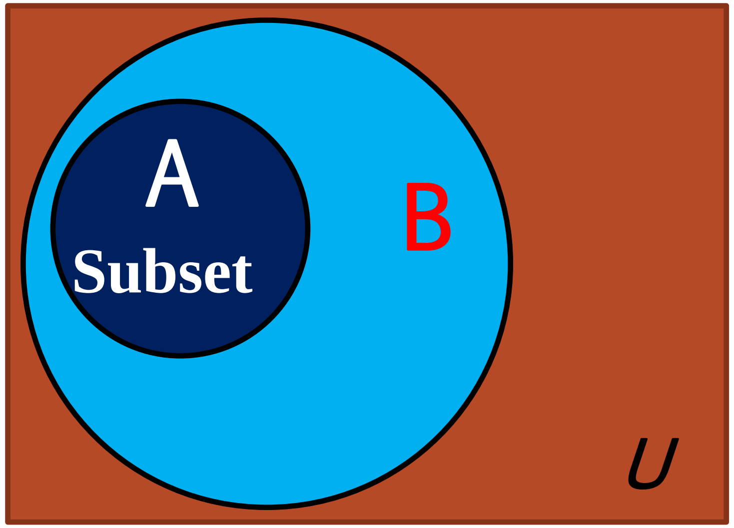
- $A \subseteq B$ means: "A is a **subset** of B."
  - **or "B contains A."**
  - **or "every element of A is also in B."**
  - **or $\forall x\ ((x \in A) \rightarrow (x \in B))$.**



2.1 Sets

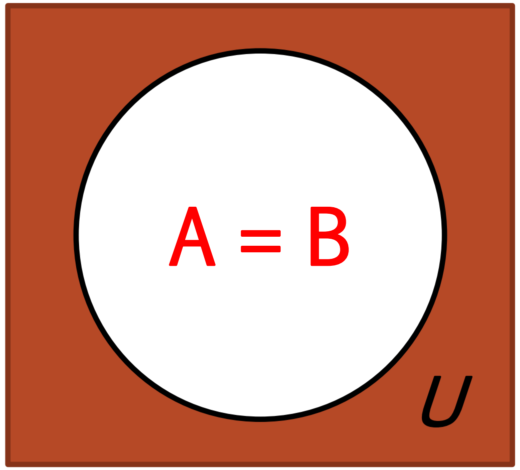
Subset set

- A $\subseteq$ B means "A is a **subset** of B."
- A $\supseteq$ B means "A is a **superset** of B."



- A = B if and only if A and B have exactly the same elements:
  - iff A $\subseteq$ B and B $\subseteq$ A
  - iff A $\subseteq$ B and A $\supseteq$ B
  - iff $\forall x\, ((x \in A) \leftrightarrow (x \in B))$.

$$(A \subseteq B) \equiv (B \supseteq A)$$

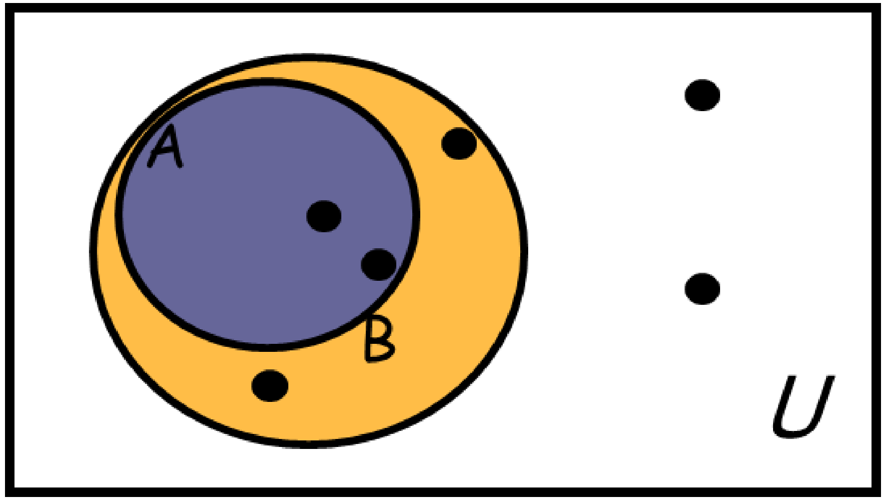


2.1 Sets

$(A \subseteq B) \equiv (B \supseteq A)$

Proper Subsets

- $A \subset B$ means "A is a proper subset of B."
  - $A \subseteq B$, and $A \neq B$.
  - $\forall x\, ((x \in A) \rightarrow (x \in B)) \land \exists x\, ((x \in B) \land (x \notin A))$



2.1 Sets

Cardinality of Set

- The cardinality is the number of distinct elements in S.
- The cardinality of S is denoted by $|S|$.

Example

If S = {1, 2, 3, 4, 5}	Then $|S| = 5$
If S = {3,3,3,3,3}	Then $|S| = 1$
If S = ∅	Then $|S| = 0$
If S = {∅, {a}, {b}, {a, b}}	Then $|S| = 4$
If S be the set of odd positive integers less than 10	Then $|S| = 5$
If S be the set of letters in the English alphabet	Then $|S| = 26$

2.1 Sets

Cardinality of Set

Example

$S = \{a, b, c, d\}$

$|S| = 4$

$A = \{1, 2, 3, 7, 9\}$

$|A| = 5$

$\emptyset = \{\ \}$

$|\emptyset| = 0$

Example

$S = \{a, b, c, \{2\}\}$

$|S| = 4$

$A = \{1, 2, 3, \{2, 3\}, 7, 9\}$

$|A| = 6$

$\{\emptyset\} = \{\{\}\}$

$|\{\emptyset\}| = 1$

2.1 Sets

Power Set

Power Set is **the set of all subsets**.
If the set is S, then the power set of S is denoted by $P(S)$.
The number of elements in the power set is $2^{|S|}$

Example

$S = \{1,2,3\}$

$|S| = 3$

$|P(S)| = 2^{|S|} = 2^3 = 8$ **elements**

$P(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

2.1 Sets

Cartesian Products

- The Cartesian Product of two sets A and B is:
  - **$A \times B = \{ (a, b) \mid a \in A \wedge b \in B\}$**

Example

- Given A = { a, b } and B = { 0, 1 }, what is their Cartesian product?

Solution

A x B = { (a,0), (a,1), (b,0), (b,1) }

$|A \times B| = |A| * |B| = 2 * 2 = 4$ *Cardinality*

2.1 Sets

Cartesian Products

Example

What is the Cartesian product of:
A = { I , 2} and B = {a, b, c}

Solution

The Cartesian product A x B is:
A x B = {(1 , a), (1, b), (1 , c) , (2 , a), (2 , b), (2 , c) }.

$$|A \times B| = |A| * |B| = 2 * 3 = 6$$

Note that: A x B ≠ B x A

B x A = {(a, I), (a, 2), (b, I), (b, 2), (c , 1), (c , 2)}

2.1 Sets

Cartesian Products

Example

What is the Cartesian product A × B × C, where A = {0, l}, B = { I, 2}, and C = {0, 1 , 2}?

Solution

A × B × C = {(0, 1 , 0), (0 , I, I), (0 , 1, 2), (0 , 2 , 0), (0 , 2 , I), (0 , 2 , 2),

(1 , 1 , 0), (1 , 1 , 1), (1 , 1 , 2), (1, 2 , 0), (1 , 2 , 1), (1 , 2 , 2)}.

$|A \text{ x } B \text{ x } C| = |A| * |B| * |C| = 2 * 2 * 3 = 12$

2.1 Sets

Solve

Use set builder notation to give a description of each of these sets.

- a) {0, 3, 6, 9, 12}
- b) {-3, -2, -1,0, 1 , 2, 3}
- c) {m, n, o, p}

Determine whether each of these pairs of sets are equal.

- a) {1, 3, 3, 3, 5, 5, 5, 5, 5}, {5, 3, 1}
- b) {{1}}, {1, {1}}
- c) ϕ, {ϕ}

Suppose that A = {2, 4, 6}, B = {2, 6}, C = {4, 6}, and D = {4, 6, 8}. Determine which of these sets are subsets of which other of these sets.