NUMERICAL COMPUTATIONS

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INTERPOLATION AND POLYNOMIAL APPROXIMATION

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Interpolation and Polynomial Approximation

Introduction

A census of the population of the United States is taken every 10 years. The following table lists the population, in thousands of people, from 1950 to 2000, and the data are also represented in the figure.

Year	1950	1960	1970	1980	1990	2000
Population (in thousands)	151,326	179,323	203,302	226,542	249,633	281,422

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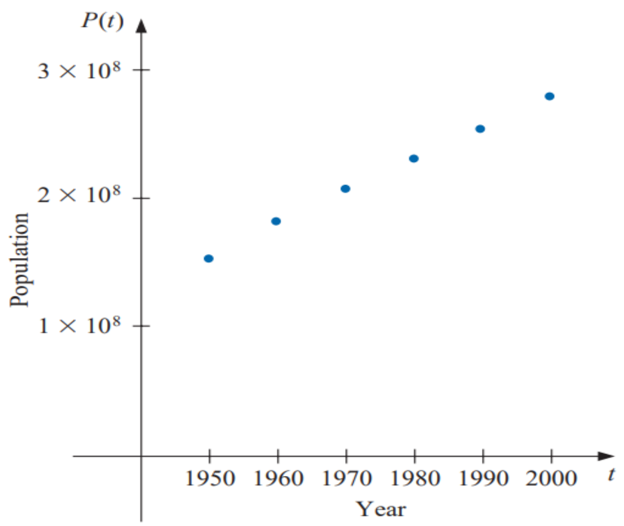
Interpolation and Polynomial Approximation



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Interpolation and Polynomial Approximation

In many problems in engineering and science, the data being considered are known only at a set of discrete points, not as a continuous function.

For example, the continuous function

$$y = f(x)$$

may be known only at n discrete values of x:

$$y_i = y(x_i) \qquad (i = 1, 2, ..., n)$$

Discrete data, or tabular data, may consist of small sets of smooth data, large sets of smooth data, small sets of rough data, or large sets of rough data.

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- Figure illustrates a set of tabular data in the form of a set of $[x, f(x)]$ pairs. The function $f(x)$ is known at a finite set (actually eight) of discrete values of x.
- The value of the function can be determined at any of the eight values of x simply by a table lookup.
- However, a problem arises when the value of the function is needed at any value of x between the discrete values in the table.
- The actual function is not known and can not be determined from the tabular values.
- However, the actual function can be approximated by some known function, and the value of the approximating function can be determined at any desired value of x.
- This process, which is called *interpolation*, is the subject of this chapter.
- The discrete data in Figure are actually values of the function $f(x)=1/x$, which is used as the example problem in this chapter.

x	f(x)
3.20	0.312500
3.30	0.303030
3.35	0.298507
3.40	0.294118
3.50	0.285714
3.60	0.277778
3.65	0.273973
3.70	0.270270

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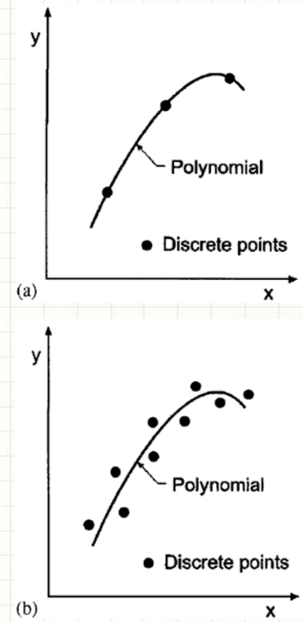
There are two fundamentally different ways to fit a polynomial to a set of discrete data:
a) Exact fits
b) Approximate fits

a) *An exact fit* yields a polynomial that passes exactly through all of the discrete points, as illustrated

b) *An approximate fit* yields a polynomial that passes through the set of data in the best manner possible, without being required to pass exactly through any of the data points, as illustrated



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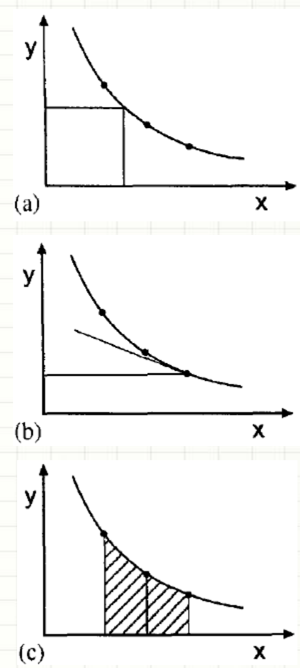
Applications of approximating functions.

(a) In many applications, the values of the discrete data at the specific points are not all that is needed. Values of the function at points other than the known discrete points may be needed (i.e., interpolation).

(b) The derivative of the function may be required (i.e., differentiation).

(c) The integral of the function may be of interest (i.e., integration).



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These processes are performed by fitting *an approximating function* to the set of discrete data and performing the desired process on *the approximating function*.

Many types of *approximating functions* exist. In fact, *any analytical function* can be used as an approximating function.

Three of the more common approximating functions are:
1. Polynomials
2. Trigonometric functions
3. Exponential functions

Approximating functions should have the following properties:
1. The approximating function should be easy to determine.
2. It should be easy to evaluate.
3. It should be easy to differentiate.
4. It should be easy to integrate.

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Exact Fit Methods

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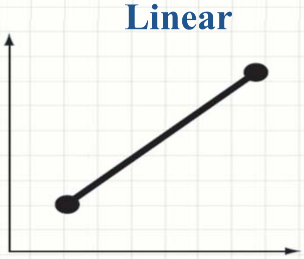
PROPERTIES OF POLYNOMIALS

The general form of an nth-degree polynomial is

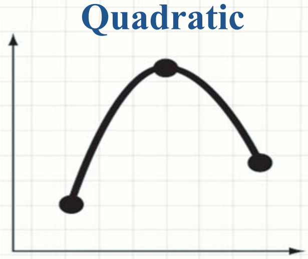
$$\boxed{P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n}$$

- where n denotes the degree of the polynomial and a0 to an are constant coefficients.
- There are n + 1 coefficients,x so n + 1 discrete data points are required to obtain unique values for the coefficients.

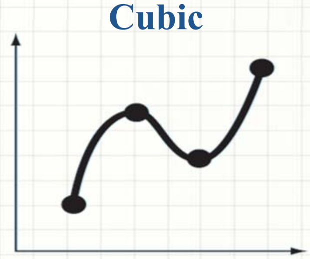
Linear **Quadratic** **Cubic**



$P_1(x) = a_0 + a_1 x$



$P_2(x) = a_0 + a_1 x + a_2 x^2$



$P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

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Differentiation of polynomials

$$\frac{d}{dx}(a_i x^i) = i a_i x^{i-1}$$

The derivatives of the nth-degree polynomial *Pn(x)* are

$$\boxed{P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n}$$

$$\frac{dP_n(x)}{dx} = P_n'(x) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1} = P_{n-1}(x)$$

$$\frac{d^2P_n(x)}{dx^2} = \frac{d}{dx}\left[\frac{dP_n(x)}{dx}\right] = P_n''(x) = 2a_2 + 6a_3 x + \cdots + n(n-1)a_n x^{n-2} = P_{n-2}(x)$$

$$\cdots$$

$$P_n^{(n)}(x) = n!a_n$$

$$P_n^{(n+1)}(x) = 0$$

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Integration of polynomials

$$\int a_i x^i \, dx = \frac{a_i}{i+1} x^{i+1} + \text{constant}$$

The integral of the nth-degree polynomial *Pn(x)* is

$$I = \int P_n(x)\, dx = \int (a_0 + a_1 x + \cdots + a_n x^n)\, dx$$

$$I = a_0 x + \frac{a_1}{2} x^2 + \cdots + \frac{a_n}{n+1} x^{n+1} + \text{constant} = P_{n+1}(x)$$

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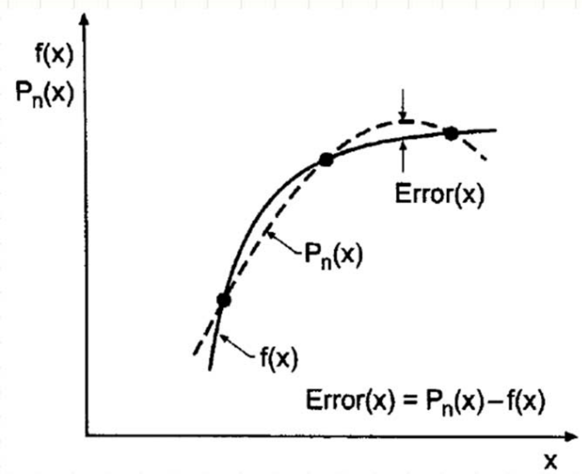
When a polynomial of degree *n, Pn(x),* is fit exactly to a set of *n+1* discrete data points, *(xo,fo), (x1,f1), ... , (xn,fn),* as illustrated in Figure, the polynomial has no error at the data points themselves



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The property of polynomials that makes them suitable as approximating functions is stated by
the Weierstrass approximation theorem:

> If $f(x)$ is a continuous function in the closed interval $a \le x \le b$, then for every $\varepsilon > 0$ there exists a polynomial $P_n(x)$, where the the value of n depends on the value of ε, such that for all x in the closed interval $a \le x \le b$,
>
> $$|P_n(x) - f(x)| < \varepsilon$$

Consequently, any continuous function can be approximated to any accuracy by a polynomial of high enough degree. In practice, low-degree polynomials are employed, so care must be taken to achieve the desired accuracy.

Polynomials satisfy *a uniqueness theorem:*

> A polynomial of degree n passing exactly through $n + 1$ discrete points is *unique*

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Approximation Theorem

- Suppose that f is defined and continuous on $[a, b]$. For each $\varepsilon > 0$, there exists a polynomial $P(x)$, with the property that

$$|f(x) - P_n(x)| < \varepsilon$$

- Polynomials satisfy a uniqueness theorem.

A polynomial of degree n passing exactly through $n + 1$ discrete points is unique.

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DIRECT FIT POLYNOMIALS

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DIRECT FIT POLYNOMIALS

The general procedure for fitting the unique *nth*-degree polynomial *Pn(x)*

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \qquad (4.34)$$

that passes exactly through the *n+1* points (equally spaced or unequally spaced data *(xo,fo), (x1 ,f1), ..., (Xn,fn)*. For simplicity, we write *f(xi) =fi*. Substituting each data point into Eq. (4.34) yields *n+1* equations:

$$f_0 = a_0 + a_1 x_0 + a_2 x_0^2 + \ldots + a_n x_0^n$$

$$f_1 = a_0 + a_1 x_1 + a_2 x_1^2 + \ldots + a_n x_1^n$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$f_n = a_0 + a_1 x_n + a_2 x_n^2 + \ldots + a_n x_n^n$$

There are *n+1* linear equations containing the *n+1* coefficients *a0* to *an*. Equation (4.35) can be solved for *a0* to *an* by Gauss elimination. The resulting polynomial is the unique *nth*-degree (4.34) polynomial that passes exactly through the *n+1* data points.

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Direct Fit Polynomials: Example

- **Find the first-degree direct fit interpolating polynomial that goes through the first two data points, and use it to find the interpolated value at x = 0.35.**

x	$y = 1 + e^{-x}$
0.1	1.9048
0.5	1.6065
0.8	1.4493

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Direct Fit Polynomials: Example

x	$y = 1 + e^{-x}$
0.1	1.9048
0.5	1.6065
0.8	1.4493

$$1.9048 = a_0 + a_1(0.1)$$
$$1.6065 = a_0 + a_1(0.5)$$

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Direct Fit Polynomials: Example

$$1.9048 = a_0 + a_1(0.1)$$

$$1.6065 = a_0 + a_1(0.5)$$

$$\begin{bmatrix} 1 & 0.1 \\ 1 & 0.5 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1.9048 \\ 1.6065 \end{bmatrix}$$

$$a_0 = 1.9794$$

$$a_1 = -0.7458$$

$$P_1(x) = 1.9794 - 0.7458x$$

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Direct Fit Polynomials: Example

$$P_1(x) = 1.9794 - 0.7458x$$

$$P_1(0.35) = P_1(x) = 1.9794 - 0.7458(0.35) = 1.7184$$

$$f(0.35) = 1.7047$$

$$\left| f(0.35) - P_1(0.35) \right| = 0.0137$$

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Direct Fit Polynomials: Example

- Find the second-degree direct fit interpolating polynomial, and use it to find the interpolated value at x = 0.35.

x	$y = 1 + e^{-x}$
0.1	1.9048
0.5	1.6065
0.8	1.4493

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Direct Fit Polynomials: Example

x	$y = 1 + e^{-x}$
0.1	1.9048
0.5	1.6065
0.8	1.4493

$$1.9048 = a_0 + a_1(0.1) + a_2(0.1)^2$$

$$1.6065 = a_0 + a_1(0.5) + a_2(0.5)^2$$

$$1.4493 = a_0 + a_1(0.8) + a_2(0.8)^2$$

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Direct Fit Polynomials: Example

$$1.9048 = a_0 + a_1(0.1) + a_2(0.01)$$
$$1.6065 = a_0 + a_1(0.5) + a_2(0.25)$$
$$1.4493 = a_0 + a_1(0.8) + a_2(0.64)$$
$$\begin{bmatrix} 1 & 0.1 & 0.01 \\ 1 & 0.5 & 0.25 \\ 1 & 0.8 & 0.64 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1.9048 \\ 1.6065 \\ 1.4493 \end{bmatrix}$$
$$a_0 = 1.9952$$
$$a_1 = -0.9358$$
$$a_2 = 0.3168$$
$$P_2(x) = 1.9952 - 0.9358x + 0.3168x^2$$

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Direct Fit Polynomials: Example

$$P_2(x) = 1.9952 - 0.9358x + 0.3168x^2$$
$$P_2(0.35) = 1.9952 - 0.9358(0.35) + 0.3168(0.35)^2 = 1.7065$$
$$f(0.35) = 1.7047$$

$$\left| f(0.35) - P_2(0.35) \right| = 0.0018$$

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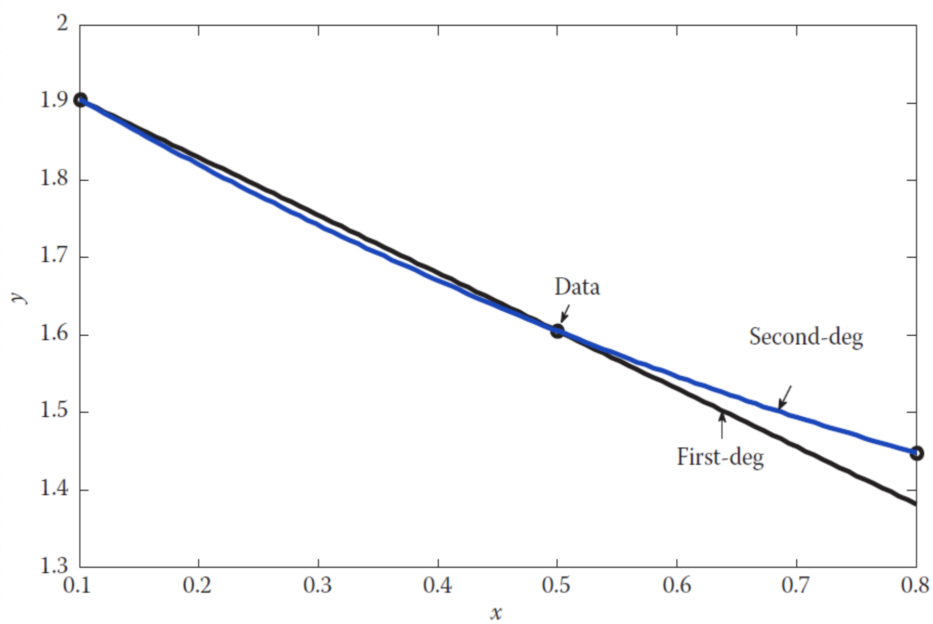
Direct Fit Polynomials: Example



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Example 4.2. Direct fit polynomials.

To illustrate interpolation by a direct fit polynomial, consider the simple function $y=f(x)=1/x$, and construct the following set of six significant figure data :

x	$f(x)$
3.35	0.298507
3.40	0.294118
3.50	0.285714
3.60	0.277778

Let's interpolate for y at $x = 3.44$ using linear, quadratic, and cubic interpolation. The exact value is

$$y(3.44) = f(3.44) = \frac{1}{3.44} = 0.290698\ldots$$

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Let's illustrate the procedure in detail for a quadratic polynomial:

x	$f(x)$
3.35	0.298507
3.40	0.294118
3.50	0.285714
3.60	0.277778

$$P_2(x) = a + bx + cx^2$$

To center the data around $x = 3.44$, the first three points are used. Applying $P_2(x)$ at each of these data points gives the following three equations:

$$0.298507 = a + b(3.35) + c(3.35)^2 \tag{4.38.1}$$
$$0.294118 = a + b(3.40) + c(3.40)^2 \tag{4.38.2}$$
$$0.285714 = a + b(3.50) + c(3.50)^2 \tag{4.38.3}$$

Solving Eqs. (4.38) for a, b, and c by Gauss elimination without scaling or pivoting yields

$$P_2(x) = 0.876561 - 0.256080x + 0.0249333x^2 \tag{4.39}$$

Substituting $x = 3.44$ into Eq. (4.39) gives

$$P_2(3.44) = 0.876561 - 0.256080(3.44) + 0.0249333(3.44)^2 = 0.290697 \tag{4.40}$$

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The error is Error $(3.44) = P_2(3.44) - f(3.44) = 0.290697 - 0.290698 = -0.000001$.

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For a linear polynomial, use $x = 3.40$ and 3.50 to center that data around $x = 3.44$.

x	$f(x)$
3.35	0.298507
3.40	0.294118
3.50	0.285714
3.60	0.277778

The resulting linear polynomial is

$$P_1(x) = 0.579854 - 0.0840400x \tag{4.41}$$

Substituting $x = 3.44$ into Eq. (4.41) gives $P_1(3.44) = 0.290756$.

For a cubic polynomial, all four points must be used. The resulting cubic polynomial is

$$P_3(x) = 1.121066 - 0.470839x + 0.0878000x^2 - 0.00613333x^3 \tag{4.42}$$

Substituting $x = 3.44$ into Eq. (4.42) gives $P_3(3.44) = 0.290698$.

The results are summarized below, where the results of linear, quadratic, and cubic interpolation, and the errors, Error$(3.44) = P(3.44) - 0.290698$, are tabulated. The advantages of higher-degree interpolation are obvious.

$P(3.44) = 0.290756$	linear interpolation	Error $= 0.000058$
$= 0.290697$	quadratic interpolation	$= -0.000001$
$= 0.290698$	cubic interpolation	$= 0.000000$

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The main advantage of direct fit polynomials:

- Simple
- Direct

The main disadvantage of direct fit polynomials:

- Each time the degree of the polynomial is changed, all of the work required to fit the new polynomial must be redone.
- The results obtained from fitting other degree polynomials is of no help in fitting the next polynomial.
- This procedure is quite laborious using direct fit polynomials.

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The direct fit polynomial which is presented , while quite straightforward in principle, has several disadvantages.

Each time the degree of the polynomial is changed, all of the work required to fit the new polynomial must be redone.

It requires a considerable amount of effort to solve the system of equations for the coefficients.

For a high-degree polynomial (n greater than about 4), the system of equations can be ill-conditioned, which causes large errors in the values of the coefficients.

A simpler, more direct procedure is desired.

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LAGRANGE POLYNOMIALS

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Lagrange Interpolating Polynomials

- **The linear Lagrange interpolating polynomial through (x_0, f_0) and (x_1, f_1) is**

$$P_1(x) = L_0(x) f_0 + L_1(x) f_1$$

$$L_0(x) = \frac{x - x_1}{x_0 - x_1}$$

$$L_1(x) = \frac{x - x_0}{x_1 - x_0}$$

$$P_1(x) = \frac{x - x_1}{x_0 - x_1} f_0 + \frac{x - x_0}{x_1 - x_0} f_1$$

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Lagrange Interpolating Polynomials

- The second-degree Lagrange interpolating polynomial through (x_0, f_0), (x_1, f_1) and (x_2, f_2) is

$$P_2(x) = L_0(x)\, f_0 + L_1(x)\, f_1 + L_2(x)\, f_2$$

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}$$

$$L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

$$P_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f_2$$

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Lagrange Interpolating Polynomials

- The third-degree Lagrange interpolating polynomial through (x_0, f_0), (x_1, f_1), (x_2, f_2) and (x_3, f_3) is

$$P_3(x) = L_0(x)\, f_0 + L_1(x)\, f_1 + L_2(x)\, f_2 + L_3(x)\, f_3$$

$$P_3(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} f_0 + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} f_1 + \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} f_2 + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} f_3 +$$

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Lagrange Interpolating Polynomials

- In general, the nth-degree Lagrange interpolating polynomial that goes through $n + 1$ points (x_0, f_0), …, (x_n, f_n) is formed as

$$L_{n,k}(x) = \frac{(x - x_0)(x - x_1)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_0)(x_k - x_1)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)}$$

$$= \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)}.$$

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Lagrange Interpolating Polynomials: Example

- Find the first-degree Lagrange interpolating polynomial that goes through the first two data points, and use it to find the interpolated value at $x = 0.35$.

x	$y = 1 + e^{-x}$
0.1	1.9048
0.5	1.6065
0.8	1.4493

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Lagrange Interpolating Polynomials: Example

x	$y = 1 + e^{-x}$
0.1	1.9048
0.5	1.6065
0.8	1.4493

$$P_1(x) = \frac{x - x_1}{x_0 - x_1} f_0 + \frac{x - x_0}{x_1 - x_0} f_1$$

$$P_1(x) = \frac{x - 0.5}{0.1 - 0.5}(1.9048) + \frac{x - 0.1}{0.5 - 0.1}(1.6065)$$

$$P_1(0.35) = 1.7184$$

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Lagrange Interpolating Polynomials: Example

- **Find the second-degree Lagrange interpolating polynomial, and use it to find the interpolated value at x = 0.35.**

x	$y = 1 + e^{-x}$
0.1	1.9048
0.5	1.6065
0.8	1.4493

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Lagrange Interpolating Polynomials: Example

$$P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f_2$$

$$P_2(x) = \frac{(x-0.5)(x-0.8)}{(0.1-0.5)(0.1-0.8)}(1.9048) + \frac{(x-0.1)(x-0.8)}{(0.5-0.1)(0.5-0.8)}(1.6065) + \frac{(x-0.1)(x-0.5)}{(0.8-0.1)(0.8-0.5)}(1.4493)$$

x	$y = 1 + e^{-x}$
0.1	1.9048
0.5	1.6065
0.8	1.4493

$$P_2(0.35) = 1.7065$$

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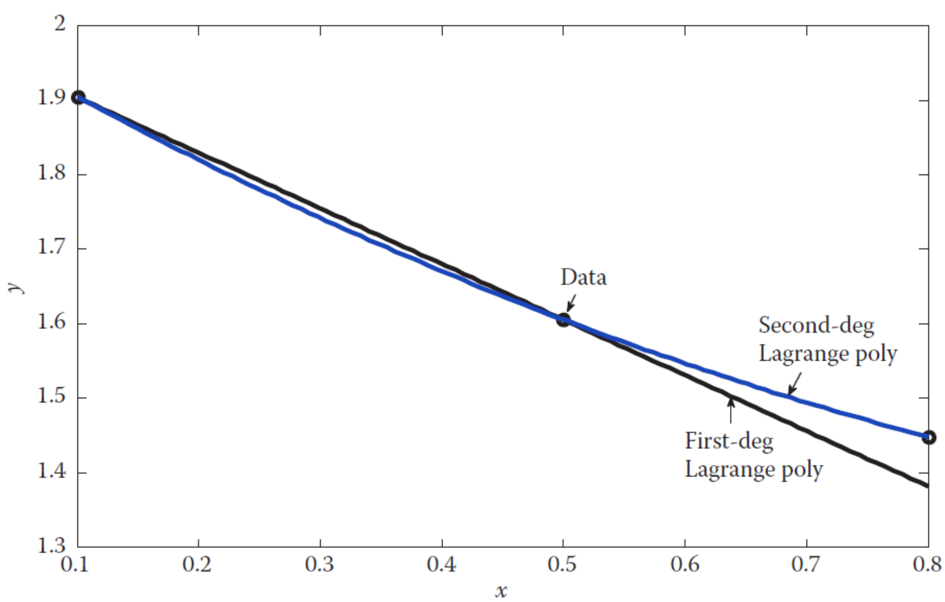
Lagrange Interpolating Polynomials: Example



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Example 4.3. Lagrange polynomials.

Consider the four points given, in Example 4.2, which satisfy the simple function $y=f(x)=1/x$,:

x	$f(x)$
3.35	0.298507
3.40	0.294118
3.50	0.285714
3.60	0.277778

Let's interpolate for y at $x = 3.44$ using linear, quadratic, and cubic interpolation. The exact value is

$$y(3.44) = f(3.44) = \frac{1}{3.44} = 0.290698\ldots$$

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x	$f(x)$
3.35	0.298507
3.40	0.294118
3.50	0.285714
3.60	0.277778

Linear interpolation using the two closest points, $x = 3.40$ and 3.50, yields

$$P_1(3.44) = \frac{(\boxed{3.44} - 3.50)}{(3.40 - 3.50)}(0.294118) + \frac{(\boxed{3.44} - 3.40)}{(3.50 - 3.40)}(0.285714) = 0.290756$$

44

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x	$f(x)$
3.35	0.298507
3.40	0.294118
3.50	0.285714
3.60	0.277778

Quadratic interpolation using the three closest points, $x = 3.35$, 3.40, and 3.50, gives

$$
\begin{aligned}
P_2(3.44) = &\frac{(\boxed{3.44} - 3.40)(\boxed{3.44} - 3.50)}{(3.35 - 3.40)(3.35 - 3.50)}(0.298507) \\
&+ \frac{(\boxed{3.44} - 3.35)(\boxed{3.44} - 3.50)}{(3.40 - 3.35)(3.40 - 3.50)}(0.294118) \\
&+ \frac{(\boxed{3.44} - 3.35)(\boxed{3.44} - 3.40)}{(3.50 - 3.35)(3.50 - 3.40)}(0.285714) = 0.290697
\end{aligned}
$$

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45

x	$f(x)$
3.35	0.298507
3.40	0.294118
3.50	0.285714
3.60	0.277778

Cubic interpolation using all four points yields

$$
\begin{aligned}
P_3(3.44) = &\frac{(\boxed{3.44} - 3.40)(\boxed{3.44} - 3.50)(\boxed{3.44} - 3.60)}{(3.35 - 3.40)(3.35 - 3.50)(3.35 - 3.60)}(0.298507) \\
&+ \frac{(\boxed{3.44} - 3.35)(\boxed{3.44} - 3.50)(\boxed{3.44} - 3.60)}{(3.40 - 3.35)(3.40 - 3.50)(3.40 - 3.60)}(0.294118) \\
&+ \frac{(\boxed{3.44} - 3.35)(\boxed{3.44} - 3.40)(\boxed{3.44} - 3.60)}{(3.50 - 3.35)(3.50 - 3.40)(3.50 - 3.60)}(0.285714) \\
&+ \frac{(\boxed{3.44} - 3.35)(\boxed{3.44} - 3.40)(\boxed{3.44} - 3.50)}{(3.60 - 3.35)(3.60 - 3.40)(3.60 - 3.50)}(0.277778) = 0.290698
\end{aligned}
$$

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The results of linear, quadratic, and cubic interpolation, and the errors, *Error(3.44)= P(3.44)- 0.290698* , are

$P(3.44) = 0.290756$	linear interpolation	Error $= \ \ 0.000058$
$= 0.290697$	quadratic interpolation	$= -0.000001$
$= 0.290698$	cubic interpolation	$= \ \ 0.000000$

The main advantage of the Lagrange polynomial:
It is easier than Direct fit.

The main disadvantages:
All of the work must be redone for each degree polynomial.
A large amount of computational effort is involved, especially for higher-degree polynomials.

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Lagrange Interpolating Polynomials: Advantages and Disadvantages

- The Lagrange polynomial can be used for both unequally spaced data and equally spaced data.
- No system of equations must be solved to evaluate the polynomial. However, a large amount of computational effort is involved, especially for higher-degree polynomials.
- Lagrange polynomial cannot be stored and used in the construction of a higher-degree polynomial.

As observed in Examples, none of the information gathered in the construction of $P_1(x)$ was saved and used in the construction of $P_2(x)$.

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Numerical Methods Simulations

https://planetcalc.com/8680/

https://elsenaju.eu/Calculator/Lagrange-Interpolation.htm

https://www.dcode.fr/lagrange-interpolating-polynomial

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