Ex:

Find $\frac{dy}{dx}$ if

$$y = \ln\left[\frac{x^4 (1-x^3)^5}{\sin x \cdot \sqrt{1+x^2}}\right]$$

Solution

$$y = \ln x^4 + \ln(1-x^3)^5 - \ln \sin x - \ln(1+x^2)^{1/2}$$

$$y = 4\ln x + 5\ln(1-x^3) - \ln \sin x - \frac{1}{2}\ln(1+x^2)$$

$$\frac{dy}{dx} = \frac{4}{x} + 5\frac{-3x^2}{1-x^3} - \frac{\cos x}{\sin x} - \frac{1}{\cancel{2}}\frac{\cancel{2}x}{1+x^2}$$

$$\frac{dy}{dx} = \frac{4}{x} - \frac{15x^2}{1-x^3} - \cot x - \frac{x}{1+x^2}$$

Logarithmic differentiation:-

Logarithmic differentiation is used to differentiate either ① Complicated function

as: $y = \frac{\sqrt{x+1} \cdot \sin^2 x \cdot (x+1)^3}{\sqrt[3]{(x+3)^2} \cdot \cos^2 x}$, ...

or ② Function with variable power

as: $y = (\sin x)^x$, $y = x^x$, ...

Examples

Find $\frac{dy}{dx}$ if:

① $y = \dfrac{\sqrt[3]{3x+5} \cdot e^{\ln 2x}}{\sqrt{2+\sec x} \cdot \tan(e^{5x})}$

Take ln to both sides

$$\ln y = \ln\left[\frac{(3x+5)^{1/3} \cdot e^{\ln 2x}}{(2+\sec x)^{1/2} \cdot \tan(e^{5x})}\right]$$

$$\ln y = \frac{1}{3}\ln(3x+5) + \ln 2x \, \cancel{\ln e}_{1} - \frac{1}{2}\ln(2+\sec x) - \ln(\tan e^{5x})$$

$$\frac{1}{y}\, y' = \frac{1}{\cancel{3}}\,\frac{\cancel{3}}{3x+5} + \frac{\cancel{2}}{\cancel{2}x} - \frac{1}{2}\,\frac{\sec x \tan x}{2+\sec x} - \frac{5e^{5x}\sec^2(e^{5x})}{\tan(e^{5x})}$$

$\underline{\underline{\times y}}$

$$y' = y\left[\frac{1}{3x+5} + \frac{1}{x} - \frac{1}{2}\,\frac{\sec x \tan x}{2+\sec x} - \frac{5e^{5x}\sec^2(e^{5x})}{\tan(e^{5x})}\right]$$

② $y = (x^2+3)^{x+\cos x}$

Take ln to both sides

$$\ln y = \ln (x^2+3)^{x+\cos x}$$

$$\ln y = (x+\cos x)\ln(x^2+3)$$

$$\frac{1}{y}\, y' = (x+\cos x)\cdot\frac{2x}{x^2+3} + \ln(x^2+3)\cdot(1-\sin x)$$

$\underline{\underline{\times y}}$

$$y' = y\left[(x + \cos x \cdot \frac{2x}{x^2+3} + \ln(x^2+3).(1-\sin x)\right]$$

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③ $y = (1+x)^{\sqrt{\cot x}}$

Take ln to both sides

$$\ln y = \ln (1+x)^{\sqrt{\cot x}}$$

$$\ln y = \sqrt{\cot x}\ \ln(1+x)$$

$$\frac{1}{y} y' = \sqrt{\cot x} \cdot \frac{1}{1+x} + \ln(1+x) \cdot \frac{-\operatorname{cosec}^2 x}{2\sqrt{\cot x}}$$

$\times y$

$$y' = y\left[\frac{\sqrt{\cot x}}{1+x} - \frac{\operatorname{cosec}^2 x\ \ln(1+x)}{2\sqrt{\cot x}}\right]$$

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④ $y = 3^{x^2}$

Take ln to both sides

$$\ln y = \ln 3^{x^2}$$

$$\ln y = x^2 \ln 3$$

$$\frac{1}{y} y' = 2x \ln 3 \rightarrow y' = y \cdot 2x \ln 3$$

$$y' = 3^{x^2} \cdot 2x \ln 3$$

From Ex (4) we Can deduce the following rule:

$$\boxed{\begin{array}{l} \text{If } y = a^{u}, \ a \text{ is a Constant} \\ \qquad a > 0, \ u \equiv u(x) \\ \qquad (u \text{ is a function of } x) \\ \text{Then } y' = a^{u} \cdot u' \cdot \ln a . \end{array}}$$

(5) $y = 5^{\frac{1-x}{1+x}}$

$$y' = 5^{\frac{1-x}{1+x}} \cdot \ln 5 \cdot \frac{(1+x)(-1) - (1-x)(1)}{(1+x)^2}$$

$$-1 - \cancel{x} - 1 + \cancel{x}$$

$$y' = 5^{\frac{1-x}{1+x}} \cdot \ln 5 \cdot \frac{-2}{(1+x)^2}$$

Derivative of inverse Trigonometric functions

If $x = \sin y$, then the inverse sine function is $y = \sin^{-1} x$, which means the angle y that its sine is x.

It's derivative is derived as follows:

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}$$

$$\therefore \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$$

In a similar way, one can deduce the derivatives of the rest inverse trigonometric functions.

Let u is a function of x, $u \equiv u(x)$

① $y = \sin^{-1} u \longrightarrow y' = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$

$y = \cos^{-1} u \longrightarrow y' = \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx}$

② $y = \tan^{-1} u \longrightarrow y' = \frac{1}{1+u^2} \frac{du}{dx}$

$y = \cot^{-1} u \longrightarrow y' = \frac{-1}{1+u^2} \frac{du}{dx}$

③ $y = \sec^{-1} u \longrightarrow y' = \frac{1}{u\sqrt{u^2-1}} \frac{du}{dx}$

$y = \csc^{-1} u \longrightarrow y' = \frac{-1}{u\sqrt{u^2-1}} \frac{du}{dx}$

Examples

① If $y = \sin^{-1}(\ln e^{\sin\sqrt{x}})$

Prove that: $y' = \frac{1}{2y}$

Solution

$\because y = \sin^{-1}(\sin\sqrt{x}\,\underbrace{\ln e}_{1}) = \sin^{-1}(\sin\sqrt{x})$

$\therefore y = \sqrt{x}$

$\therefore y' = \frac{1}{2\sqrt{x}} = \frac{1}{2y}$

② Find $\frac{dy}{dx}$ if :

(i) $y = x^3 \sin^{-1}\sqrt{x} + \cot^{-1}(x \sec x)$

$$y' = x^3 \cdot \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{2\sqrt{x}} + 3x^2 \sin^{-1}\sqrt{x}$$

$$+ \frac{-1}{1+(x\sec x)^2} \cdot \left[x \sec x \tan x + \sec x\right]$$

(ii) $y = \sec^{-1}\sqrt{x} + \tan^{-1}(e^{3x})$

$$y' = \frac{1}{\sqrt{x}\sqrt{(\sqrt{x})^2 - 1}} \cdot \frac{1}{2\sqrt{x}} + \frac{1}{1+(e^{3x})^2} \cdot e^{3x}(3)$$

$$y' = \frac{1}{2x\sqrt{x-1}} + \frac{3e^{3x}}{1+e^{6x}}$$

(iii) $y = \tan^{-1}\left(\frac{5-x^2}{5+x^2}\right)$

$$y' = \frac{1}{1+\left(\frac{5-x^2}{5+x^2}\right)^2} \cdot \frac{(5+x^2)(-2x) - (5-x^2)(2x)}{(5+x^2)^2}$$

(iv) $y = \ln(x + \cos^{-1}(\tan x))$

$$y' = \frac{1}{x+\cos^{-1}(\tan x)} \cdot \left(1 - \frac{1}{\sqrt{1-\tan^2 x}} \cdot \sec^2 x\right)$$

Implicit differentiation :

* $y = f(x)$ is Called explicit function
(we Can separate y individually in one side and x in the other side)

as: $y = x^2 + 2x - 6$

* A function which Can't be written in the form $y = f(x)$ is said to be implicit function

as: $x^2 y + \sin^2(x+3y) = y^2$

To get the derivative of the implicit function :

1. we differentiate each term with respect to x,
2. Collect the terms Containing y' in one side,
3. divide by the Coefficient of y'

as in the following examples

Examples

Find y' for each of the following Functions

① $x^3 - 3x^2y^4 + 7y^2 = 10$

$$3x^2 - 3x^2 \cdot 4y^3y' + y^4 \cdot (-6x) + 14yy' = 0$$

$$3x^2 - 12x^2y^3y' - 6xy^4 + 14yy' = 0$$

$$y'(14y - 12x^2y^3) = 6xy^4 - 3x^2$$

$$y' = \frac{6xy^4 - 3x^2}{14y - 12x^2y^3}$$

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② $x + \cos^{-1}y = xy$

$$1 - \frac{1}{\sqrt{1-y^2}}y' = xy' + y$$

$$1 - y = xy' + \frac{1}{\sqrt{1-y^2}}y'$$

$$y'\left(x + \frac{1}{\sqrt{1-y^2}}\right) = 1 - y$$

$$y' = \frac{1-y}{x + \frac{1}{\sqrt{1-y^2}}}$$

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③ $y = e^{-x} + e^{y}$

$y' = -e^{-x} + e^{y} y'$

$y' - e^{y} y' = -e^{-x}$

$y'(1 - e^{y}) = -e^{-x}$

$y' = \dfrac{-e^{-x}}{1 - e^{y}}$

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④ $\sin^{-1} x + \tan 2y = 5$

$\dfrac{1}{\sqrt{1-x^2}} + \sec^2(2y) \cdot 2y' = 0$

$2y' \sec^2(2y) = -\dfrac{1}{\sqrt{1-x^2}}$

$y' = \dfrac{-1}{2\sec^2(2y)\sqrt{1-x^2}}$

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⑤ If $y = e^{\tan^{-1} x}$

prove that $(1+x^2)y'' + (2x-1)y' = 0$

Proof

$y' = e^{\tan^{-1} x} \cdot \dfrac{1}{1+x^2}$ $\quad \times (1+x^2)$

$(1+x^2)y' = e^{\tan^{-1} x}$

$$(1+x^2)y' = y$$
$$(1+x^2)y'' + 2xy' = y'$$
$$(1+x^2)y'' + 2xy' - y' = 0$$
$$(1+x^2)y'' + (2x-1)y' = 0 \longrightarrow \text{?}$$

H.W

① Find $\frac{dy}{dx}$ if:

(i) $y = e^{(x+y)^3}$

(ii) $\ln y = x + e^{y}$

(iii) $\tan^{-1}(y) = x^2 + y^2$

(iv) $y^2 = \sin^3(2x) + \cos^3(2y)$

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② If $y = e^{\sin^{-1}x}$

Prove that:

$$(1-x^2)y'' - xy' - y = 0$$
