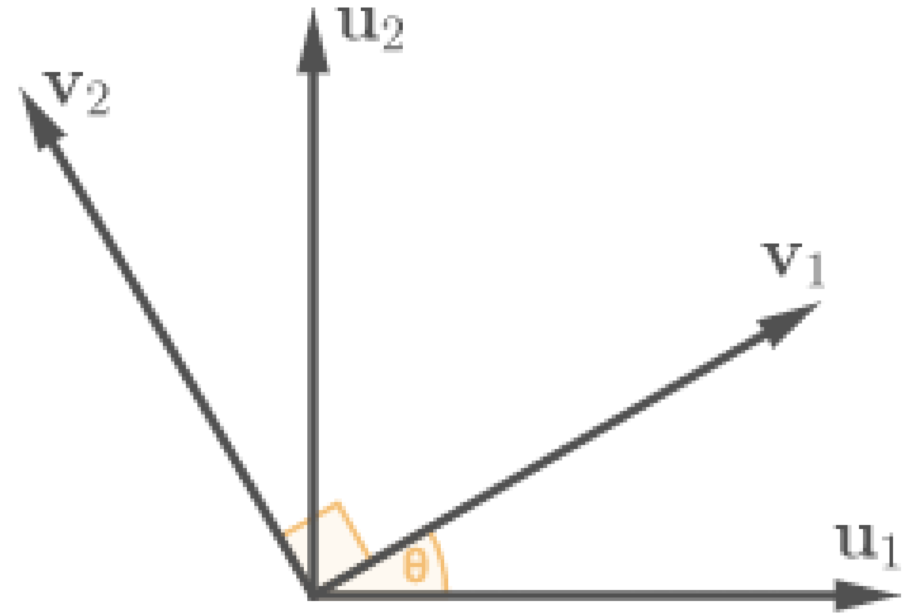
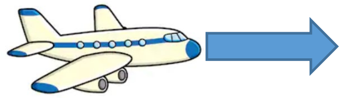
8. Basis rotations



Coordinate systems

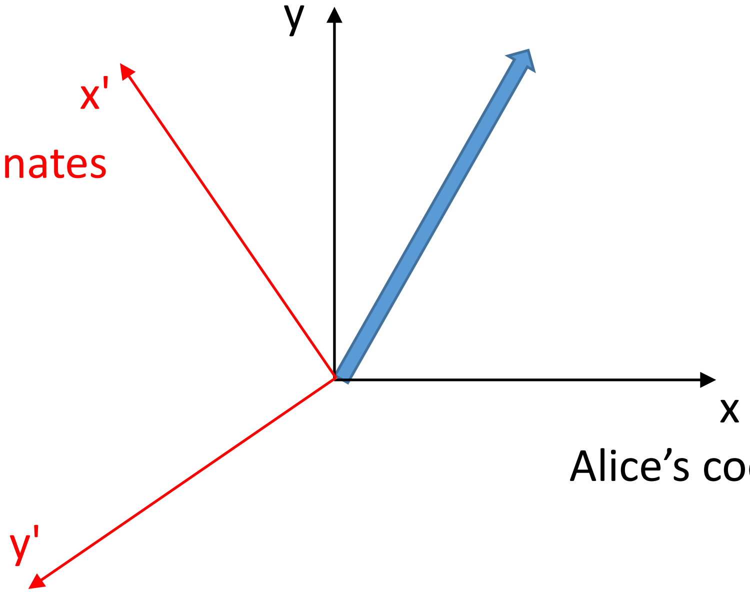
Let’s forget quantum mechanics for a minute and think about another vector space: velocity.

Alice and Bob are looking at a plane flying in the air

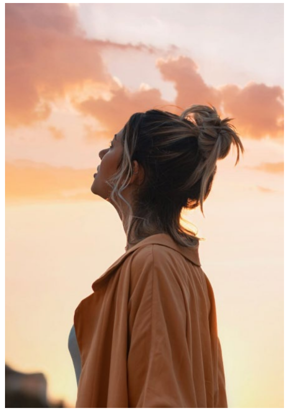


Depending upon the direction that they are facing,
they see the plane moving in different directions

Bob’s coordinates



Alice’s coordinates



Alice and Bob sees the plane move in different directions relative to them, since they are facing different directions (DUH)

Coordinate systems in quantum mechanics

Before we said that a general quantum state can be written

$$|\psi\rangle = a_0|0\rangle + a_1|1\rangle + \ldots + a_N|N\rangle$$

$$= \sum_{n=0}^{N} a_n |n\rangle$$

The $|n\rangle$ are distinct states of the physical system.

We can view the $|n\rangle$ as a particular choice of basis vectors, and the a_n as just the coordinates for that choice.

The plane example suggests we could equally write the SAME state, but just in terms of a different set of basis vectors. So we should be able to write for Bob

$$|\psi\rangle = b_0|0\rangle_B + b_1|1\rangle_B + \ldots + b_N|N\rangle_B$$

$$= \sum_{n=0}^{N} b_n |n\rangle_B$$

As with the original basis, we should have

$$\langle n|m\rangle = \delta_{nm} = \begin{cases} 0 & \text{if } n = m \\ 1 & \text{if } n \neq m \end{cases}$$

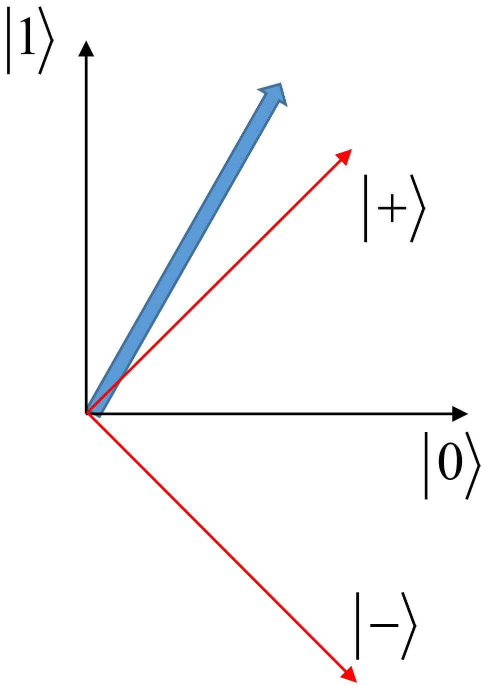
Example: qubit

Say the state in Alice's basis is $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

Bob can represent the exact same state using the basis vectors

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

These are orthogonal vectors and are normalized



$$\langle +|+\rangle = \frac{1}{2}(\langle 0| + \langle 1|)(|0\rangle + |1\rangle) = 1$$
$$\langle +|-\rangle = \frac{1}{2}(\langle 0| + \langle 1|)(|0\rangle - |1\rangle) = 0$$

How to write the vector in terms of the new coordinates? First need to invert the basis equations

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$
$$|-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

Adding the two equations: $$|0\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)$$

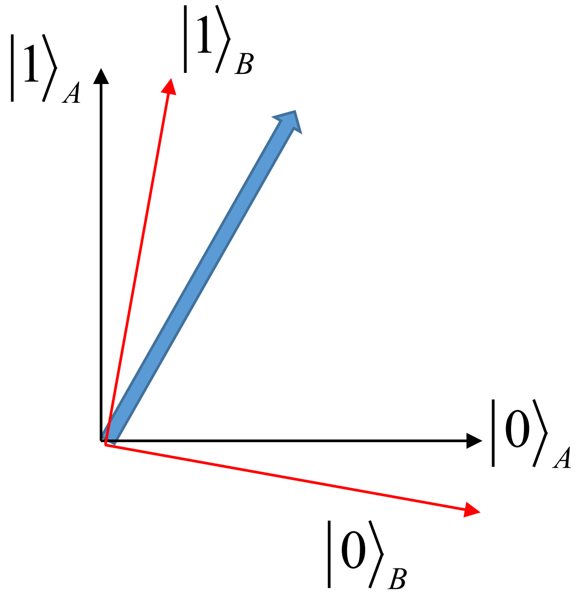
Subtracting the two equations: $$|1\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right)$$

Then substitute into the state

$$|\psi\rangle = \frac{\alpha}{\sqrt{2}}\left(|+\rangle + |-\rangle\right) + \frac{\beta}{\sqrt{2}}\left(|+\rangle - |-\rangle\right)$$
$$= \left(\frac{\alpha+\beta}{\sqrt{2}}\right)|+\rangle + \left(\frac{\alpha-\beta}{\sqrt{2}}\right)|-\rangle$$

The coefficients in the new basis are $$b_0 = \frac{\alpha+\beta}{\sqrt{2}} \qquad b_1 = \frac{\alpha-\beta}{\sqrt{2}}$$

General qubit rotations



The previous rotation was just an example, there are an infinity of different basis states that can be taken.

The most general pair of basis states that can be take are

$$|0\rangle_B = \alpha|0\rangle_A + \beta|1\rangle_A$$
$$|1\rangle_B = \beta^*|0\rangle_A - \alpha^*|1\rangle_A$$

where

$$|\alpha|^2 + |\beta|^2 = 1$$

These are also orthogonal and normalized because

$$\langle 0|0\rangle_B = \left(\alpha^* \langle 0|_A + \beta^* \langle 1|_A\right)\left(\alpha|0\rangle_A + \beta|1\rangle_A\right) = |\alpha|^2 + |\beta|^2 = 1$$

$$\langle 1|0\rangle_B = \left(\beta\langle 0|_A - \alpha\langle 1|_A\right)\left(\alpha|0\rangle_A + \beta|1\rangle_A\right) = 0$$

Question

Previously we saw that a way to parameterize an general qubit state was $$|0\rangle_B = \cos\frac{\theta}{2}|0\rangle_A + e^{i\phi}\sin\frac{\theta}{2}|1\rangle_A$$

1) What is the orthogonal vector $|1\rangle_B$?
2) Show that the vectors are orthogonal, i.e. $\langle 0|1\rangle_B = 0$
3) What are the vectors $|0\rangle_A, |1\rangle_A$ in terms of $|0\rangle_B, |1\rangle_B$?