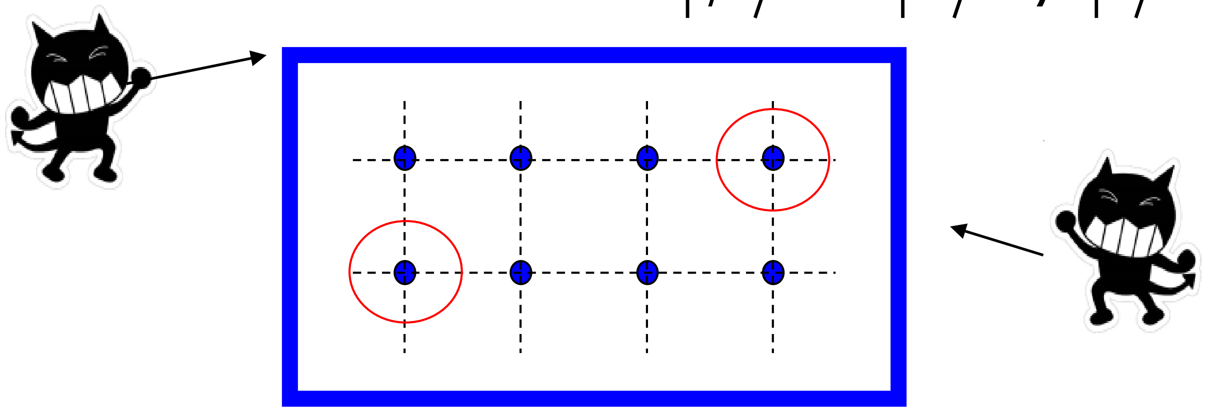
21. The Density Matrix

$$\rho$$

Noise in quantum systems

Quantum systems are typically very fragile and are highly susceptible to noise.

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$



$$|\psi'\rangle = \alpha'|0\rangle + \beta'|1\rangle$$

Suppose we ask our quantum to prepare the same state for the two circled qubits. They might not be exactly the same because:

- decoherence (noise) can affect the qubits
- the gates to prepare the states may not be perfect

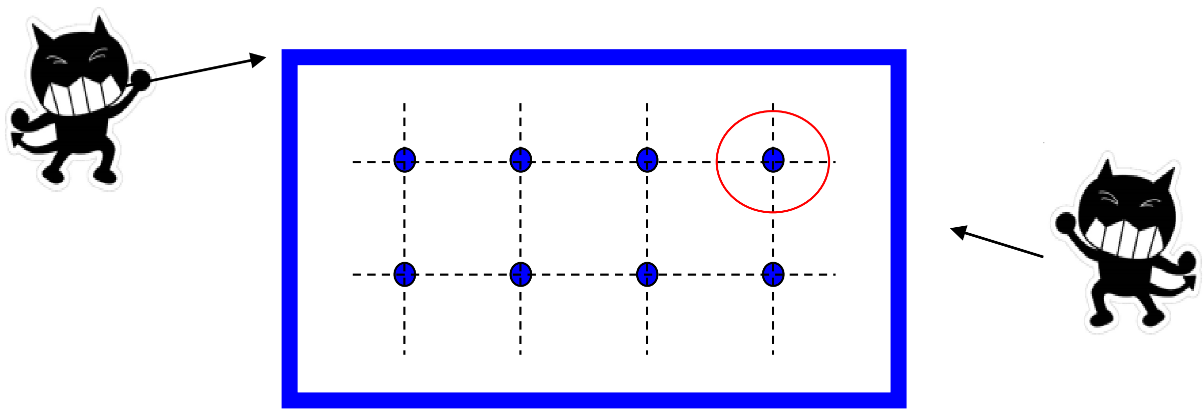
What is decoherence?

Coherence is the effect of having a superposition between states $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

Any state that is not $|0\rangle$ or $|1\rangle$ would have coherence.

e.g. Due to the action of uncontrolled interactions with the environment, the original state randomly changes to the $|0\rangle$ or $|1\rangle$ states

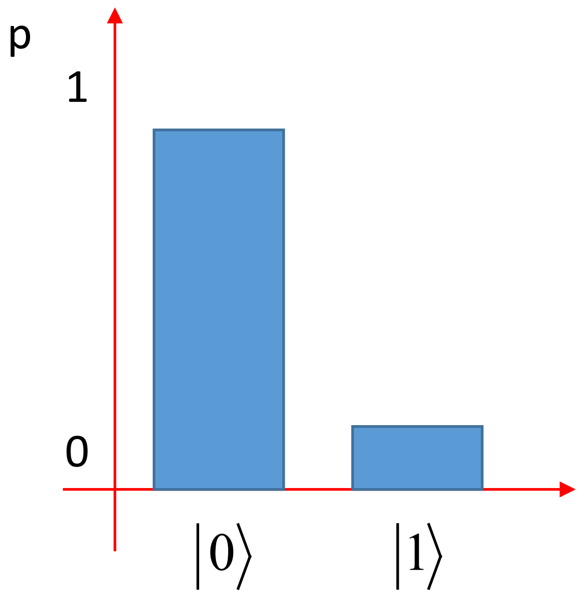
$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \rightarrow \begin{cases} |0\rangle & p = p_0 \\ |1\rangle & p = p_1 \end{cases}$$

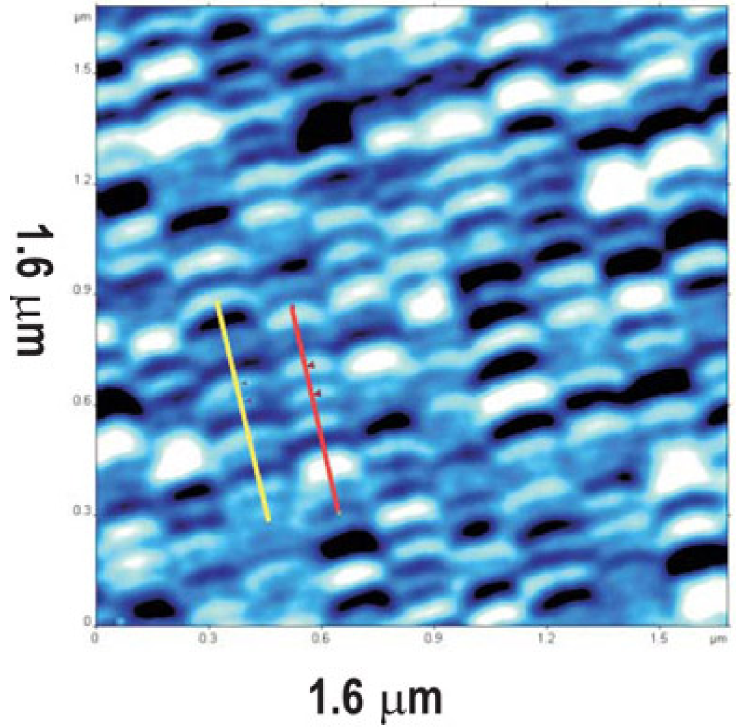


The removal of coherence = decoherence

How to handle noise classically

Taking the example of a classical bit, we know how to handle noise: use probabilities.





The probability of the atom is each state is p_0, p_1

This works for classical bits, but for quantum mechanics there is also a separate probability due to the wavefunction.
How to handle both?

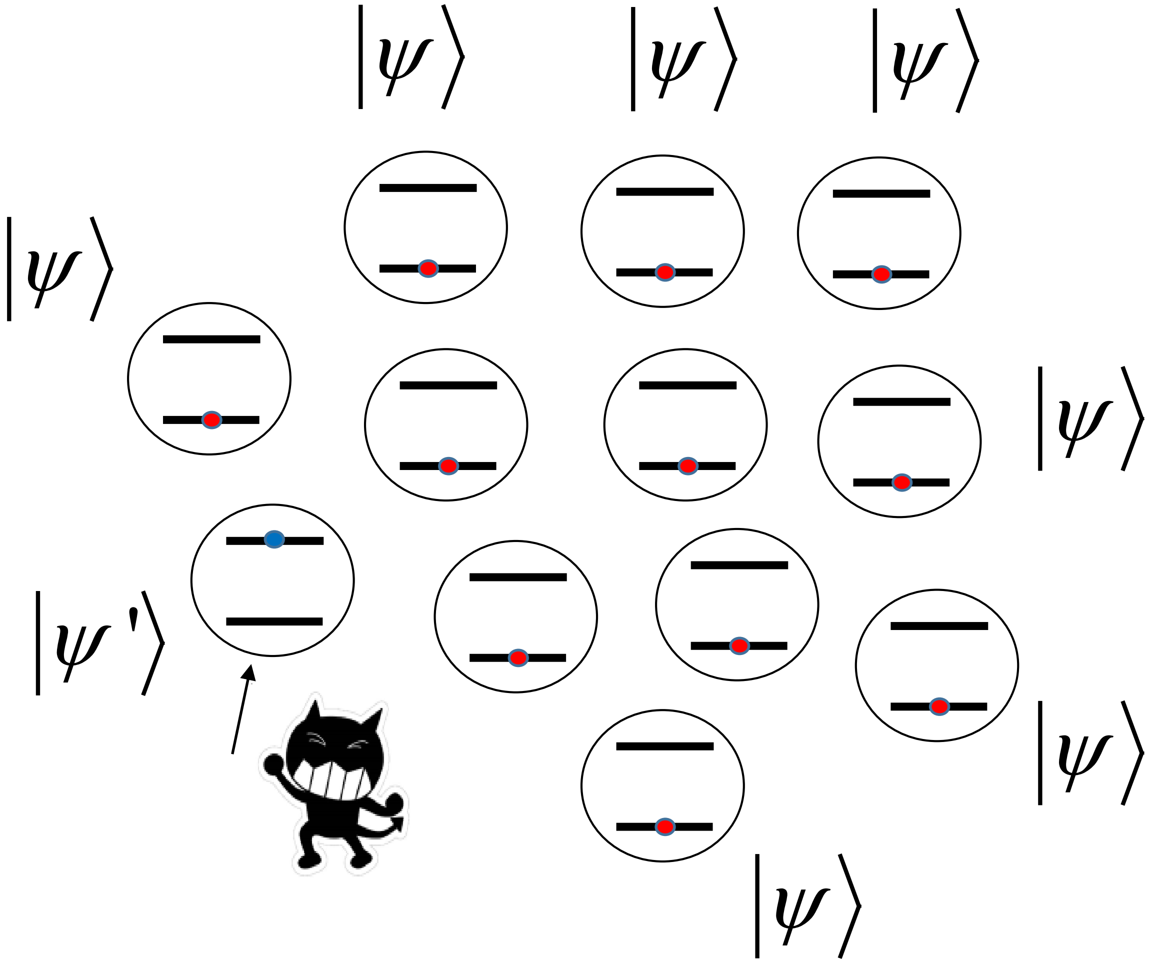
Quantum version of noisy qubits

Suppose we have a bunch of qubits where most of them are supposed to be in the same state but there are a few that are in the wrong state, because of an error.

First some terminology:

Quantum states without noise = “**pure states**”

Quantum states with noise = “**mixed states**”

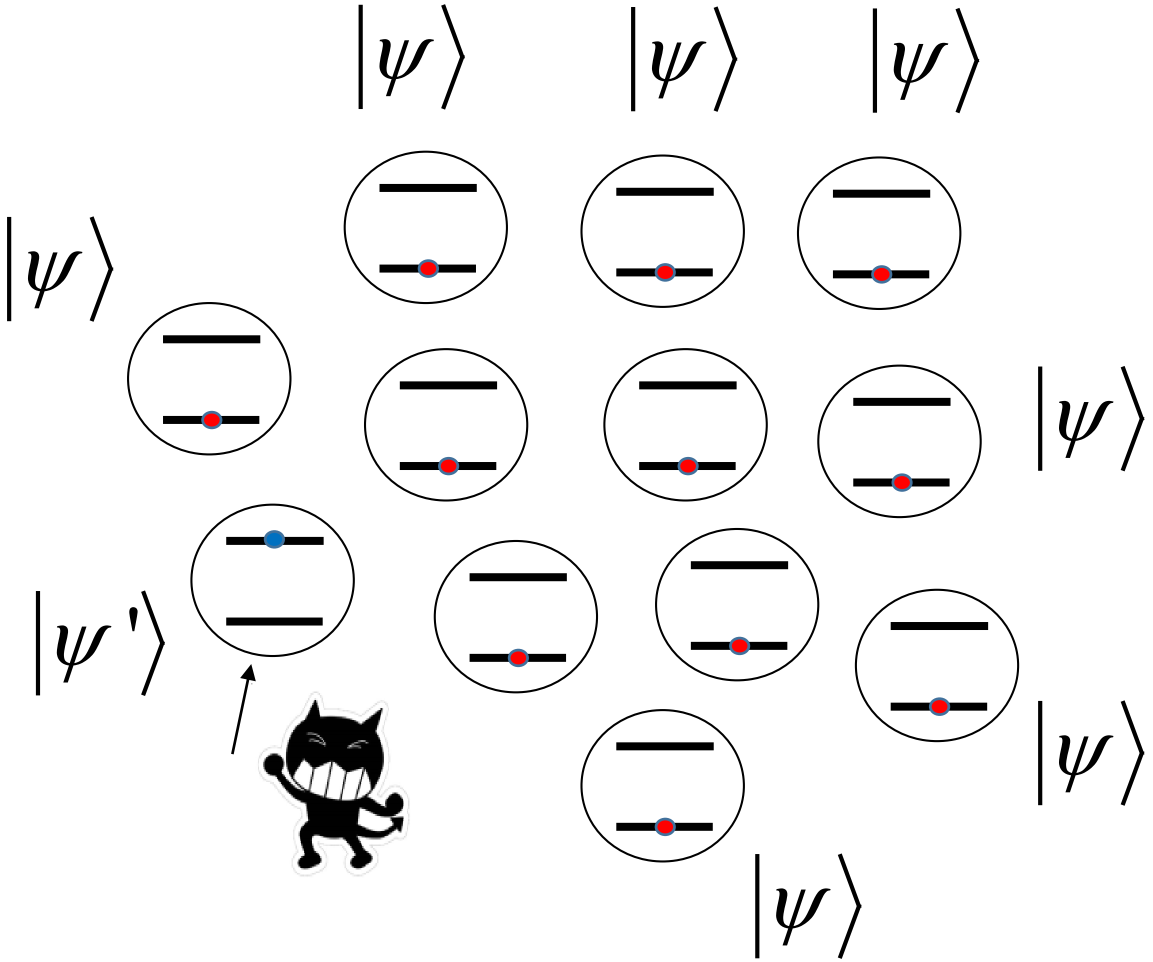


Representing mixed states

How to represent this type of noisy quantum state (mixed state)?

Try:

$$|\Psi\rangle = p_{\text{noerror}}|\psi\rangle + p_{\text{error}}|\psi'\rangle$$



$$(p_{\text{noerror}} + p_{\text{error}} = 1)$$

Representing mixed states

How to represent this type of noisy quantum state (mixed state)?

Try:

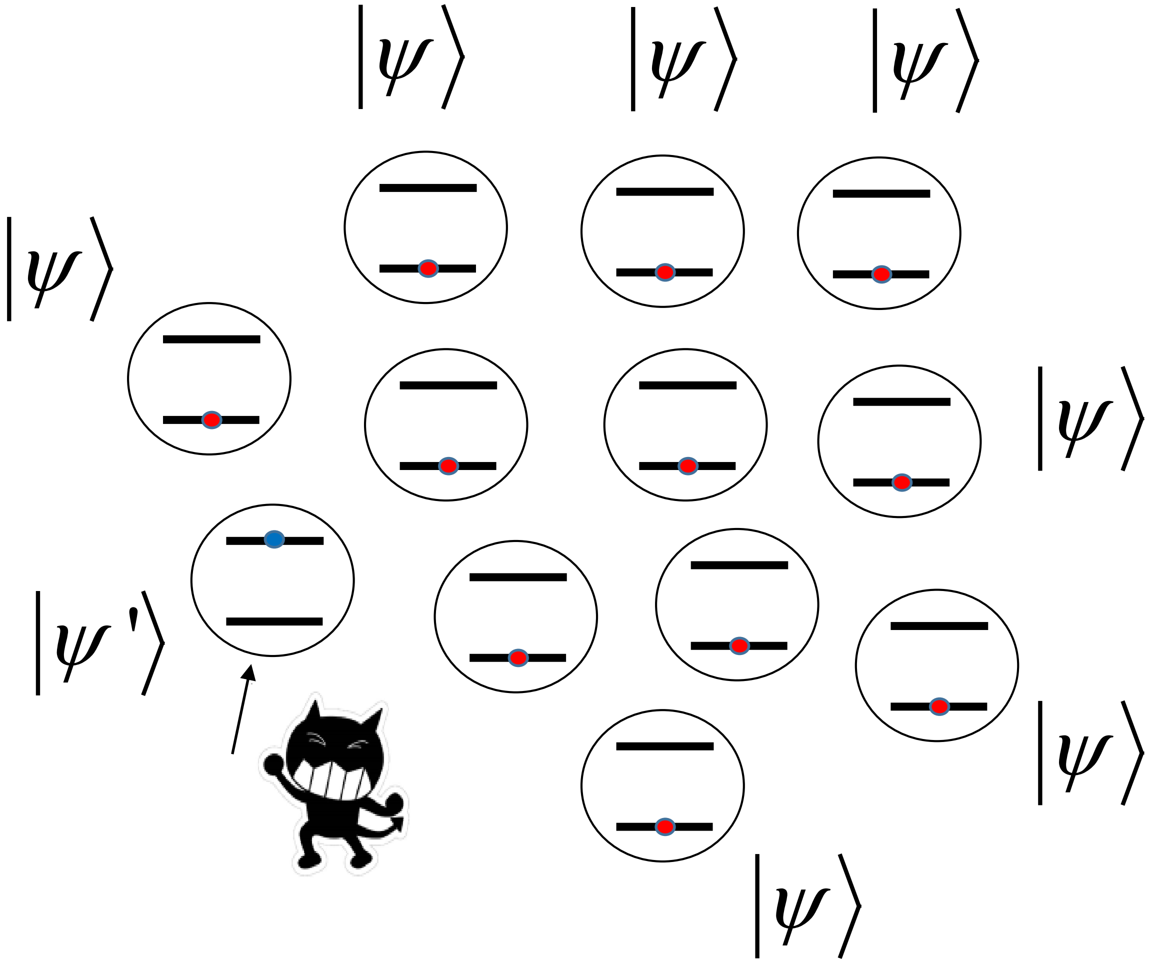
$$\cancel{|\Psi\rangle = p_{\mathrm{noerror}} |\psi\rangle + p_{\mathrm{error}} |\psi'\rangle}$$

NO! This makes a new wavefunction! e.g.

$$p_{\mathrm{noerror}} = p_{\mathrm{error}} = 1/2$$

$$\frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Hmmm..... A $|+\rangle$ state?
Also not normalized



Including probabilities and the wavefunction

In order to have probabilities AND the wavefunction at the same time, we need more degrees of freedom.

For a vector, we are already using most of the degrees of freedom:

$$|\psi\rangle = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{pmatrix}$$

$$\sum_{n=0}^{N-1} |a_n|^2 = 1$$

N complex numbers = 2N degrees of freedom
-1 for normalization

$\Rightarrow$ 2N-1 free variables

The probability distribution for the noise also needs N degrees of freedom p_n, since this is how many orthogonal states there are.

The density matrix (pure states)

Instead of representing a state by a vector, we now represent a state by a matrix. For a pure state:

$$\rho = |\psi\rangle\langle\psi|$$

Example:

Density matrix for the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

Bra-ket notation

$$\rho = \left(\alpha|0\rangle + \beta|1\rangle\right)\left(\alpha^*\langle 0| + \beta^*\langle 1|\right)$$

$$= |\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1| + \alpha\beta^*|0\rangle\langle 1| + \alpha^*\beta|1\rangle\langle 0|$$

Matrix notation

$$\rho = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\begin{pmatrix} \alpha^* & \beta^* \end{pmatrix}$$

$$= \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix}$$

Properties of the density matrix

Although this was for the zero noise case (pure states) we can notice a few things:

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix} \qquad |\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

1. The diagonal elements of the density matrix add to one (normalization). $\mathrm{Tr}(\rho) = 1 \qquad |\alpha|^2 + |\beta|^2 = 1$

2. The density matrix is Hermitian: $\rho = \rho^\dagger$ (actually it is a positive operator: all eigenvalues are >=0)

3. The off-diagonal elements are there only if there is a superposition. ("coherence")

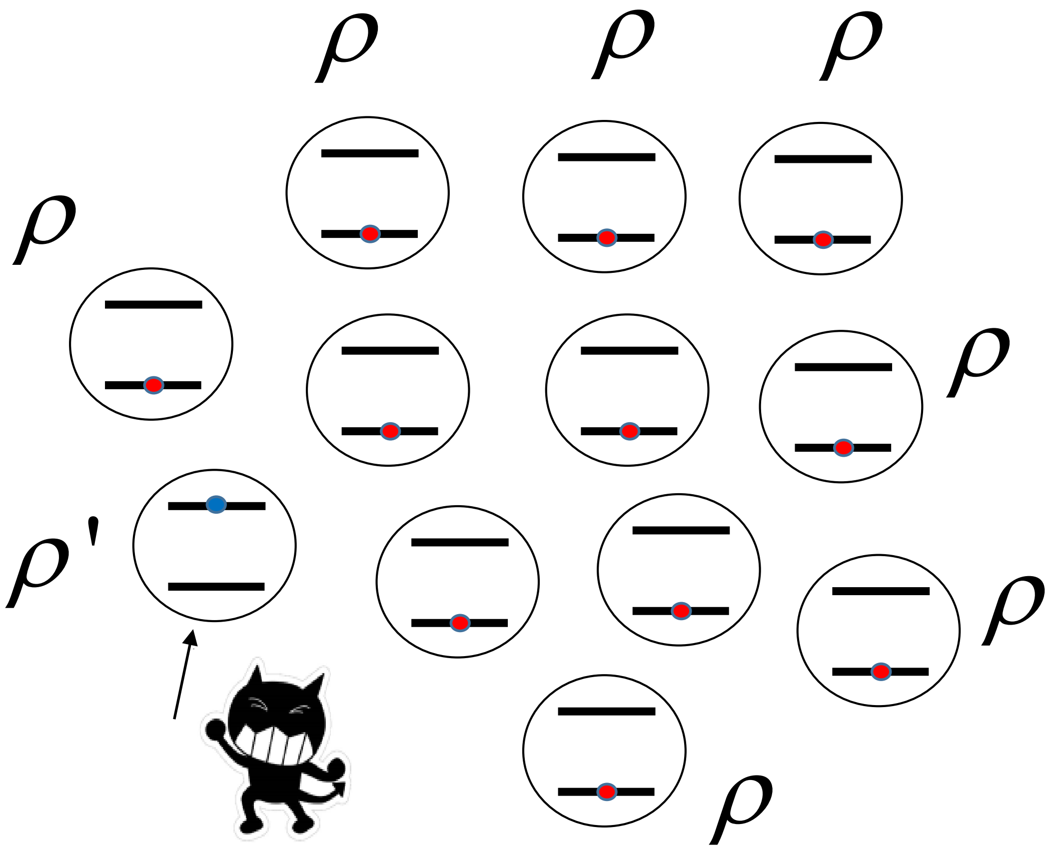
Density matrix (with noise)

Now we can try to do the case we were trying before.

Defining
$$\rho = |\psi\rangle\langle\psi|$$
$$\rho' = |\psi'\rangle\langle\psi'|$$

$$\boxed{\rho_{\text{all}} = p_{\text{noerror}}\rho + p_{\text{error}}\rho'}$$

$$(p_{\text{noerror}} + p_{\text{error}} = 1)$$



Example: Sanity check

For the previous example:

$$p_{\text{noerror}} = p_{\text{error}} = 1/2$$

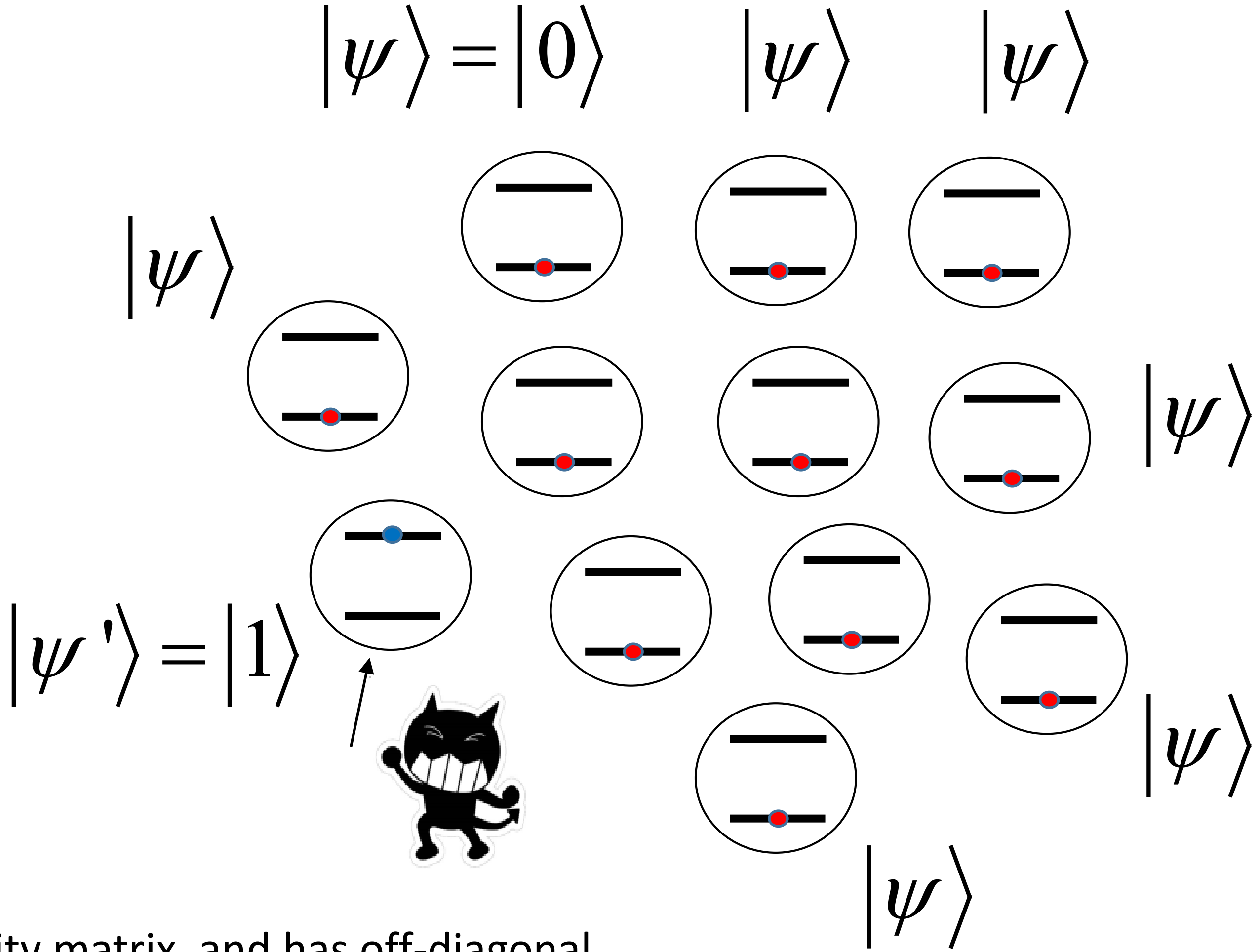
$$\rho_{\text{all}} = \frac{1}{2}\rho + \frac{1}{2}\rho'$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Compare to the pure state $|+\rangle$

$$\rho_{+} = |+\rangle\langle +|$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$



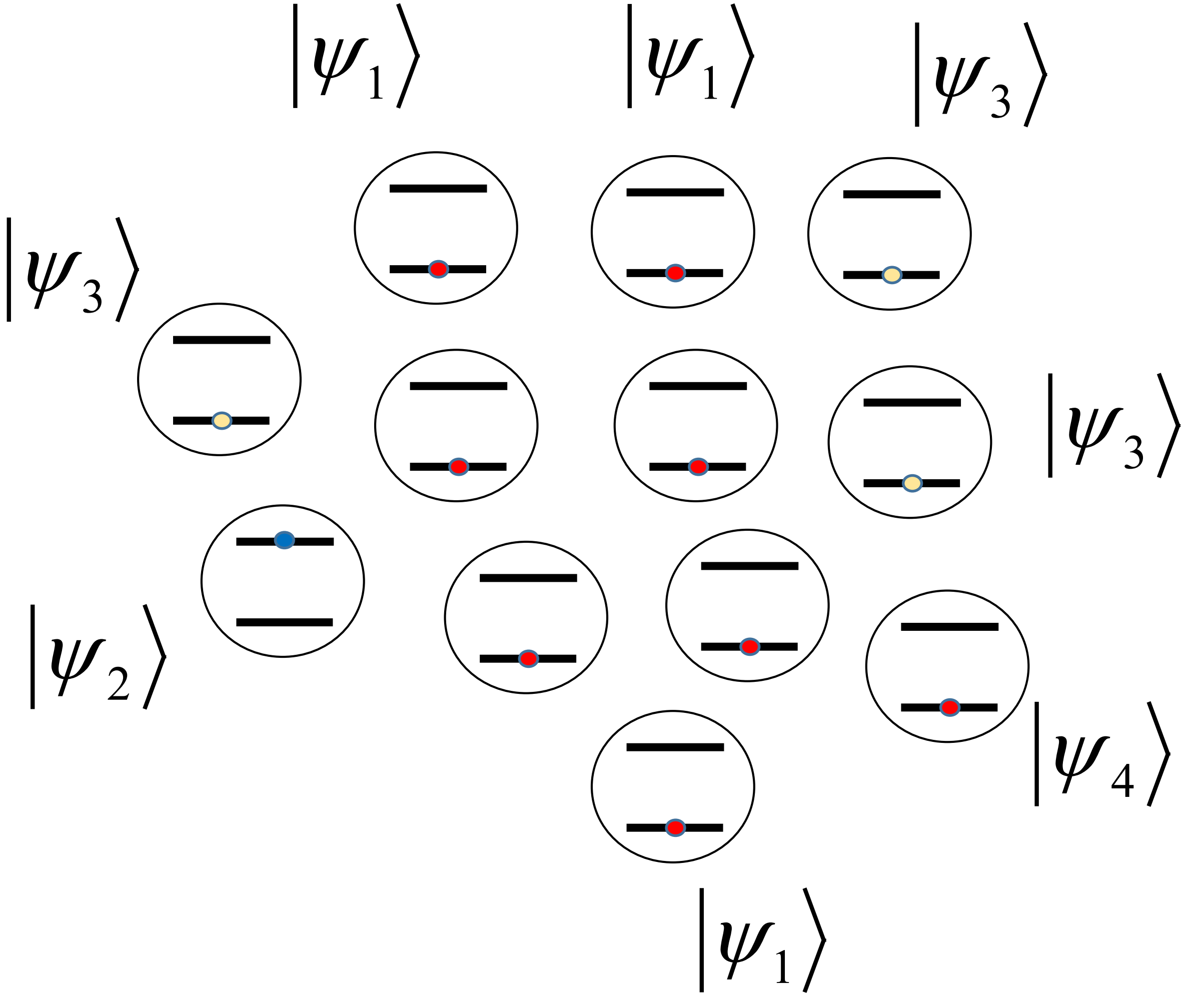
Different density matrix, and has off-diagonal elements, indicating coherence.

Density matrix general definition

If there are different states we can naturally extend the definition of the density matrix to more states.

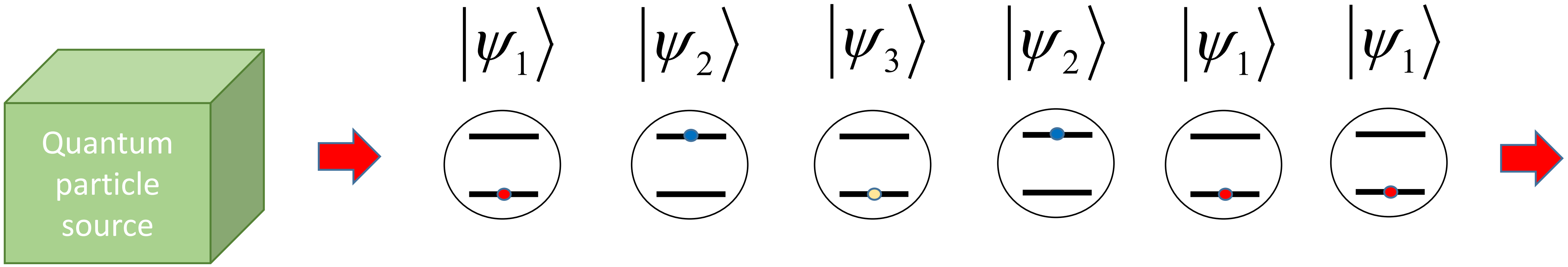
$$\rho = \sum_n p_n |\psi_n\rangle\langle\psi_n|$$

$p_n =$ Probability of occurrence of state $|\psi_n\rangle$



Ensembles or repetitions?

Actually you can still use density matrices even without having an ensemble, it could be equally if they are



$$\boxed{\rho = \sum_n p_n |\psi_n\rangle\langle\psi_n|}$$

The density matrix is like the "average" state that the system is in.

Density matrices of density matrices

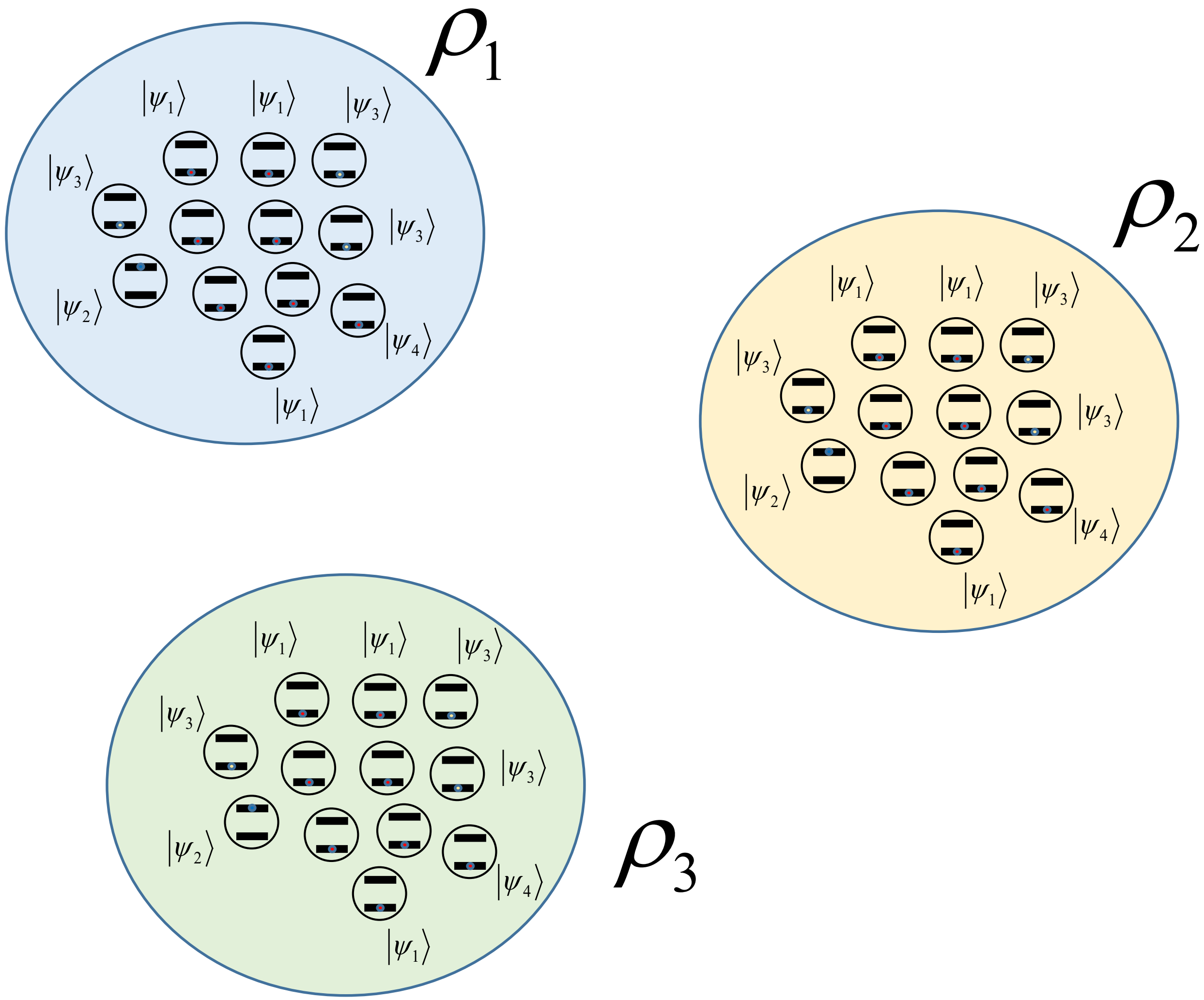
Three systems that are already density matrices?
How to make the global density matrix?

We can still follow the same rule

$$\rho = \sum_m p_m \rho_m$$

Where each ensemble has its own distribution inside

$$\rho_m = \sum_n p_n^{(m)} \left|\psi_n\right\rangle^{(m)} \left\langle\psi_n\right|^{(m)}$$



Question

A machine produces the state $|0\rangle$ 25% of the time, the state $|1\rangle$ 25% of the time, and the state $|-\rangle$ 50% of the time. Find the density matrix of the qubit.

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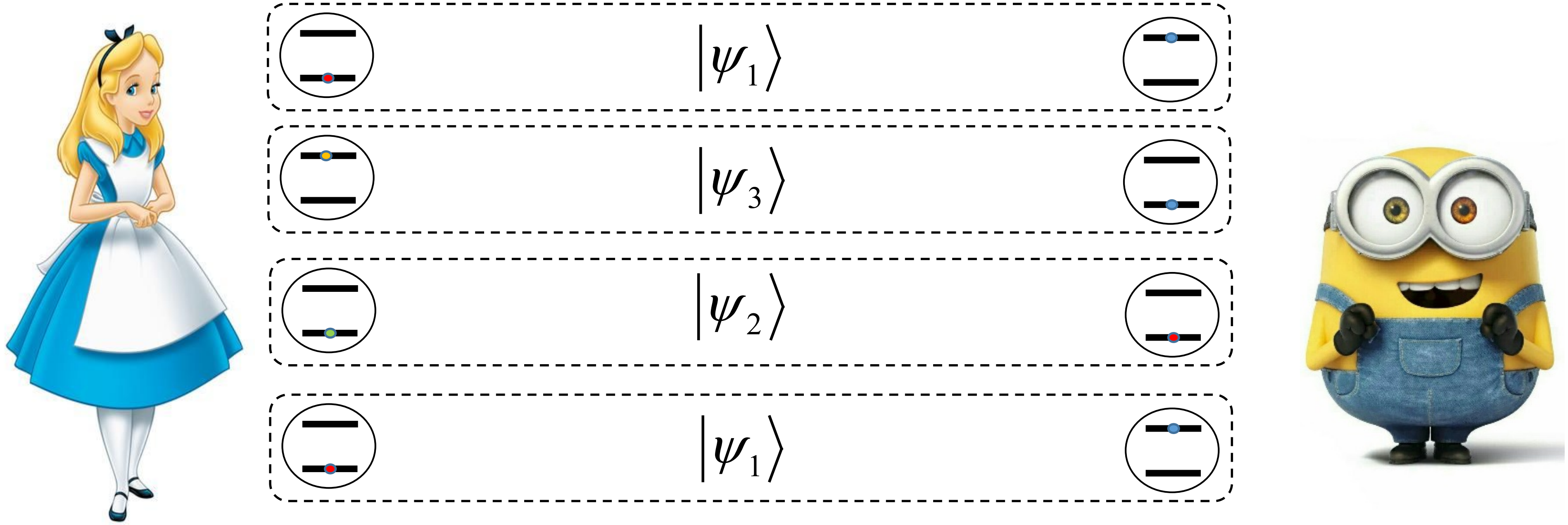
$$\rho_0 = |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \rho_1 = |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \rho_- = |-\rangle\langle -| = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\rho = \frac{1}{4}\rho_0 + \frac{1}{4}\rho_1 + \frac{1}{2}\rho_- = \frac{1}{4}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Composite systems

The wavefunction $|\psi_n\rangle$ in the definition $\rho = \sum_n p_n |\psi_n\rangle\langle\psi_n|$ can be a composite system

e.g.



Here the $|\psi_n\rangle$ could be entangled states, or product states $|\psi_n\rangle = |\psi_n^A\rangle \otimes |\psi_n^B\rangle$, any state.

Independent composite systems

Say Alice and Bob each have a bunch of qubits, and they are very far away and have never been in contact with each other. In this case the density matrix of the composite system is

$$\rho = \rho_A \otimes \rho_B = \sum_n \sum_{n'} p_n^A p_{n'}^B \left(\left| \psi_n^A \right\rangle \otimes \left| \psi_{n'}^B \right\rangle \right) \left(\left\langle \psi_n^A \right| \otimes \left\langle \psi_{n'}^B \right| \right)$$

“Product state”

States are independent AND
The probabilities are independent

$$\rho_A = \sum_n p_n^A \left|\psi_n^A\right\rangle\left\langle\psi_n^A\right| \qquad \rho_B = \sum_n p_n^B \left|\psi_n^B\right\rangle\left\langle\psi_n^B\right|$$

Question

Alice and Bob share pairs of qubits, half of them are in the Bell state $\frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right)$ and the other half are in the state $\rho_A \otimes \rho_B$ where $\rho_A = \frac{1}{4}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $\quad \rho_B = |0\rangle\langle 0|$

Find the density matrix of the composite system.

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Find the density matrix of the composite system.

Bell state:

$$\rho = \frac{1}{2}\left(|00\rangle - |11\rangle\right)\left(\langle 00| - \langle 11|\right)$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

Product state:

$$\rho_A = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| + \frac{1}{4}|0\rangle\langle 1| + \frac{1}{4}|1\rangle\langle 0|$$

$$\rho_A \otimes \rho_B = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|10\rangle\langle 10|$$

$$+\frac{1}{4}|00\rangle\langle 10| + \frac{1}{4}|10\rangle\langle 00|$$

$$= \frac{1}{4}\begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Combined:

$$\rho = \frac{1}{8}\begin{pmatrix} 4 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ -2 & 0 & 0 & 2 \end{pmatrix}$$