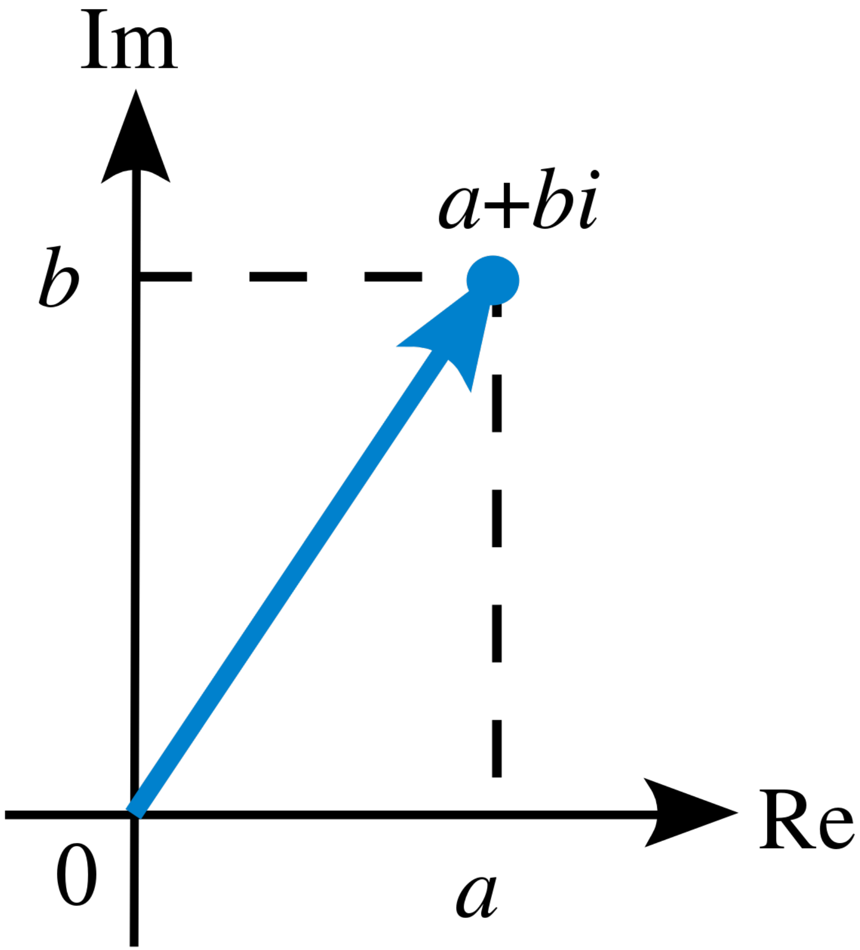
3. Complex numbers



Definition of i

Normally square root of a number does not have a solution: $$x^2 + 4 = 0$$

By defining $$i = \sqrt{-1}$$

We can write the solution to this equation $$x = \pm 2i$$

What properties does i have?

Adding and multiplying imaginary numbers

Addition

$$i + i = 2i \qquad\qquad i - i = 0$$

Treat like an algebraic variable

Multiplication

From the definition we can deduce that

$$i^2 = -1 \qquad\qquad i^3 = -i \qquad\qquad i^4 = 1$$

Complex numbers

Adding and multiplying imaginary numbers only produces either another imaginary number or a real number.

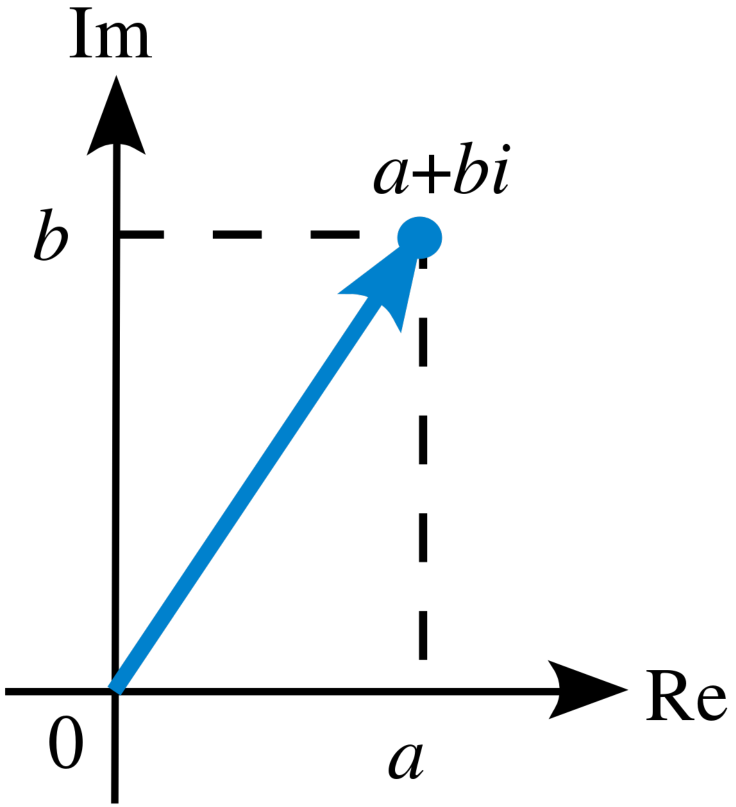
Define a complex number to be

$$c = a + ib$$

$$a, b = \text{real number}$$

Since its just two numbers we can plot it on a graph, or the "complex plane"

$$a = \mathrm{Re}(c) \qquad b = \mathrm{Im}(c)$$

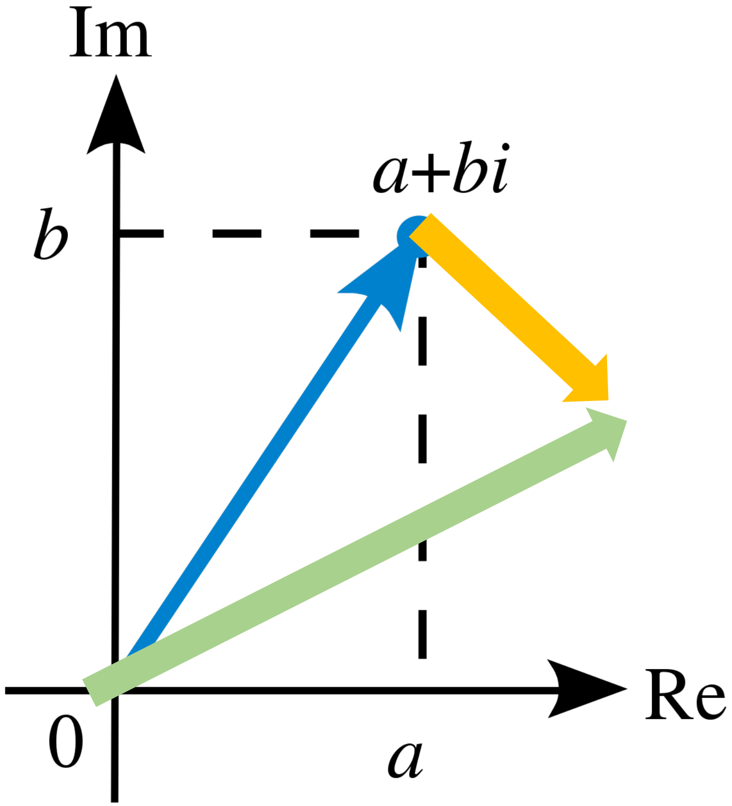


Adding complex numbers

Add the real and imaginary components separately

$$c = a + ib \qquad c' = a' + ib'$$

$$c + c' = a + a' + i(b + b')$$



Question

1) Find $c + c'$
2) Find $c - c'$

$$c = 3 + 3i \qquad c' = 1 - 2i$$

Multiplying complex numbers

Multiply out all terms

$$c = a + ib \qquad c' = a' + ib'$$

$$cc' = aa' - bb' + i(ab' + b'a)$$

Question

1) Find cc'
2) Find c^2

$$c = 3 + 3i \qquad c' = 1 - 2i$$

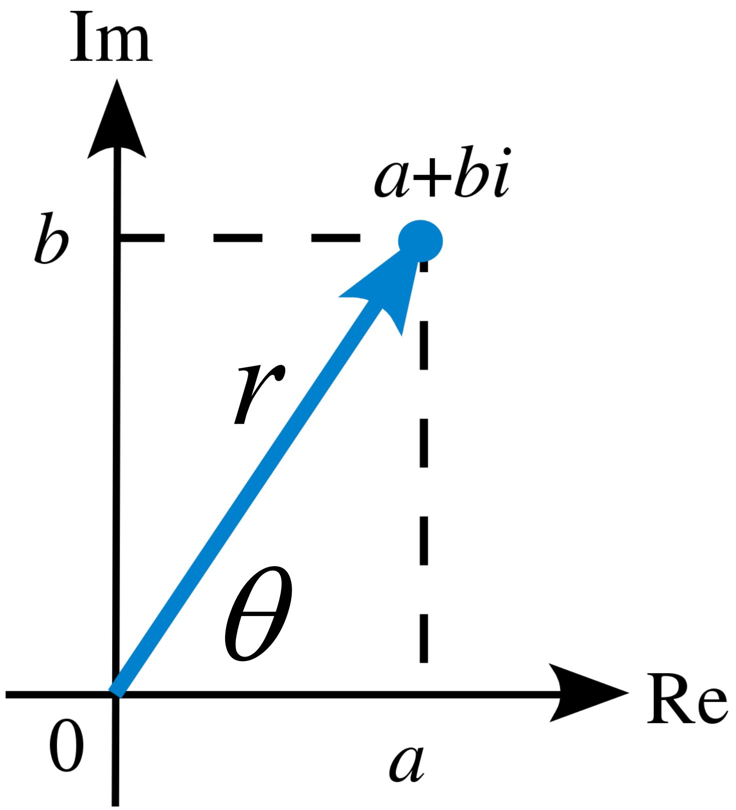
Polar form

Since a vector can be represented by a length and a direction, a complex number can be too

$$|c| = r = \sqrt{a^2 + b^2}$$

$$Arg(c) = \theta = \arctan(b / a)$$

$$c = r\cos\theta + ir\sin\theta$$



Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Proof

$$= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \dots$$

$$= \underbrace{(1 - \frac{\theta^2}{2!} + \dots)}_{\cos\theta} + i\underbrace{(\theta - \frac{\theta^3}{3!} + \dots)}_{\sin\theta}$$

"The most beautiful; formula"

$$e^{i\pi} = -1$$

Polar form again

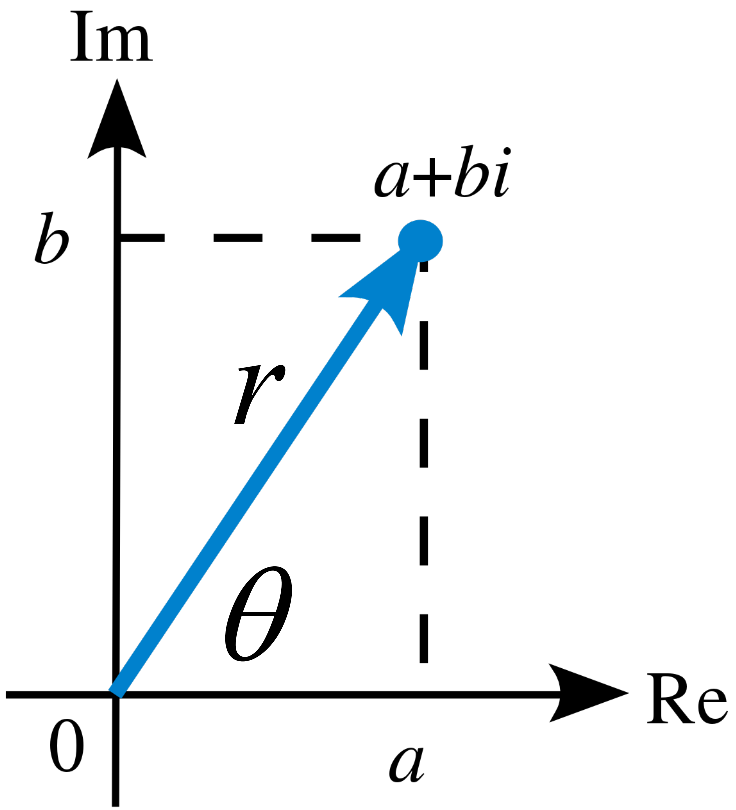
Now we can write the polar form in a nice way

$$|c| = r = \sqrt{a^2 + b^2}$$

$$Arg(c) = \theta = \arctan(b/a)$$

$$c = r(\cos\theta + i\sin\theta)$$

$$= re^{i\theta}$$



Question

1) Find $|c|$
2) Find $Arg(c)$

$$c = 3 + 3i$$

Conjugate

The conjugate of a complex number c is defined as

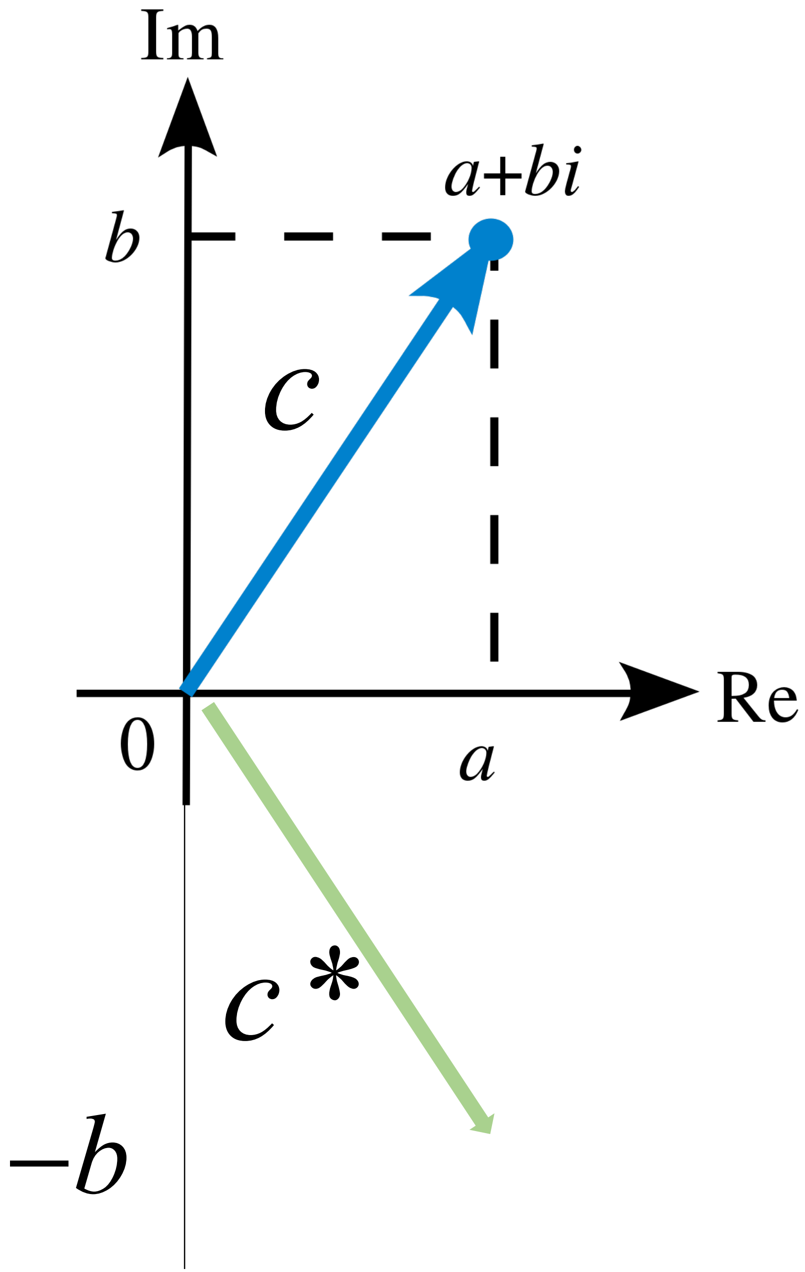
$$c^* = a - ib$$

$$c = a + ib = re^{i\theta}$$

:Polar

$$c^* = re^{-i\theta}$$

$$= r(\cos\theta - i\sin\theta)$$



Adding a conjugate to itself

$$c + c^* = a + ib + a - ib = 2a = 2\,\mathrm{Re}(c)$$

Adding a conjugate to itself gives a real number

$$c - c^* = a + ib - a + ib = 2ib = 2i\,\mathrm{Im}(c)$$

Subtracting a conjugate to itself gives a pure imaginary number

$$\mathrm{Re}(c) = \frac{c + c^*}{2} \qquad \mathrm{Im}(c) = \frac{c - c^*}{2i}$$

Multiplying a conjugate to itself

$$cc^* = (a+ib)(a-ib) = a^2 + b^2$$

$$cc^* = re^{i\theta} re^{-i\theta} = r^2$$

Multiplying a conjugate to itself gives a real number. This is the square of the length (norm) of the polar vector.

Also denoted as

$$|c|^2 \equiv cc^*$$

Question

1) Find $c + c^*$
2) Find $|c|^2$

$$c = 3 + 3i$$

Dividing complex numbers

Dividing is the same as multiplying by its reciprocal

$$\frac{c'}{c} = \frac{a' + ib'}{a + ib}$$

It is the complex number such that when it multiplies c, it gives 1

In polar form

$$\frac{1}{c} = \frac{1}{re^{i\theta}} = \frac{1}{r} e^{-i\theta}$$

So

$$\frac{c'}{c} = c' \frac{1}{c} = r' e^{i\theta'} \frac{1}{r} e^{-i\theta} = \frac{r'}{r} e^{i(\theta' - \theta)}$$

Dividing complex numbers

In Cartesian form

$$\frac{1}{c} = \frac{c^*}{cc^*} = \frac{a - ib}{a^2 + b^2}$$

Now multiplying

$$\frac{c'}{c} = \frac{c'c^*}{cc^*} = \frac{(a' + ib')(a - ib)}{a^2 + b^2}$$

$$= \frac{aa' + bb' + i(ab' - a'b)}{a^2 + b^2}$$

Question

1) Simplify $\dfrac{c}{c^*}$

$$c = 3 + 3i$$