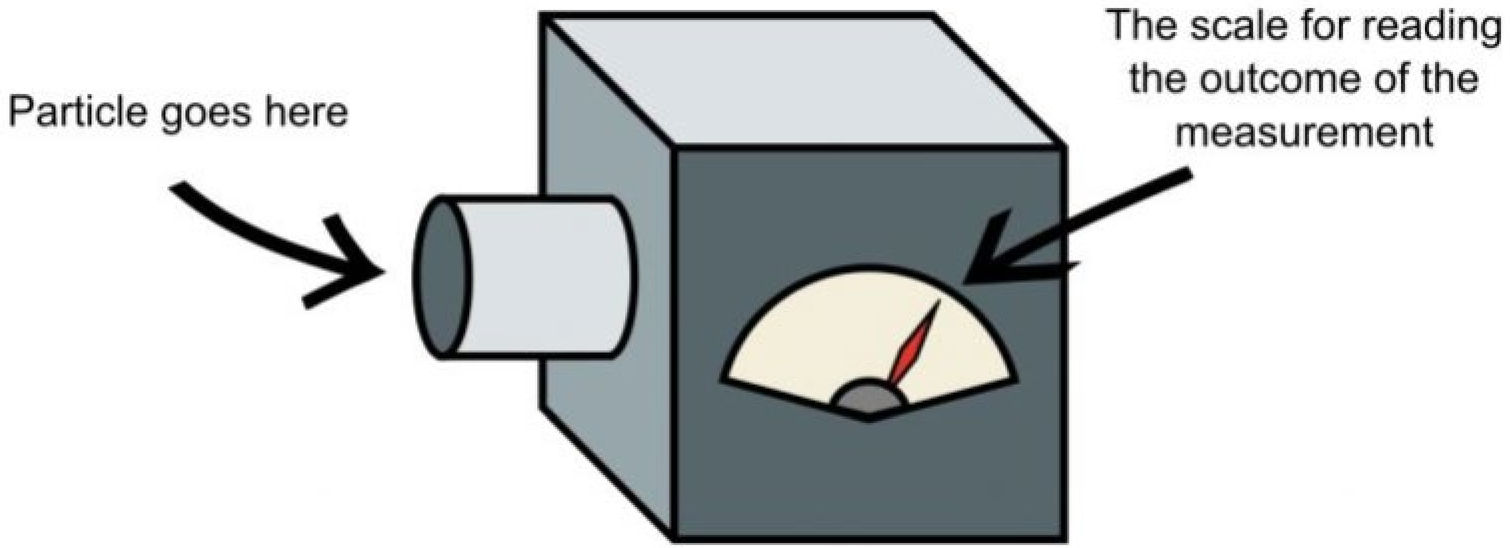
7. Measurements



Measurements

Why do we need the vector to be normalized, or equivalently, why is $\sum_{n=0}^{N} |a_n|^2 = 1$?

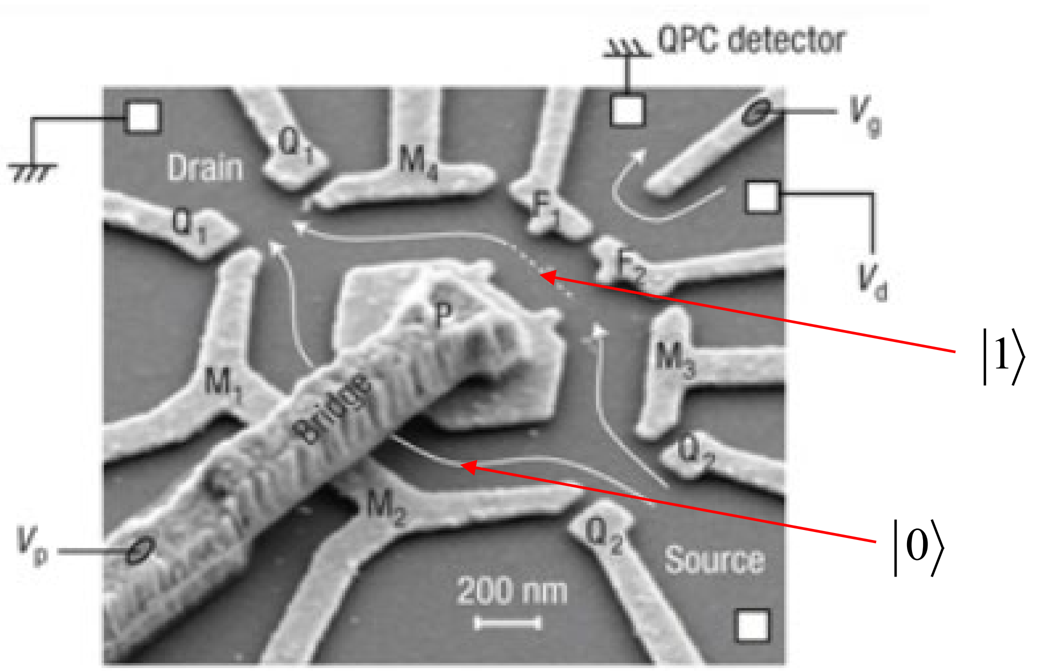
It is because if we try and measure a quantum state to determine which of the states $|n\rangle$ it is in, we end up getting a RANDOM result with probability

$$p_n = |a_n|^2$$

In fact it is not possible to directly measure the coefficients a_n from a single measurement. Since it is intrinsically random, we have to repeat the same measurement many times, and then this will give us the coefficients $|a_n|$

Question

I have a qubit that is in the state $|\psi\rangle = \frac{\sqrt{3}i}{2}|0\rangle + \frac{1}{2}|1\rangle$



which is the state of an electron in a channel, for example.

Find the probability that it is in the left and right channels.

Randomness of probability

The calculated probabilities are the ratio of outcomes when the experiment is repeated many times. What you would really do in the lab is:

1) Prepare the system in an identical way, so that you make the state $|\psi\rangle = \frac{\sqrt{3}i}{2}|0\rangle + \frac{1}{2}|1\rangle$ each time

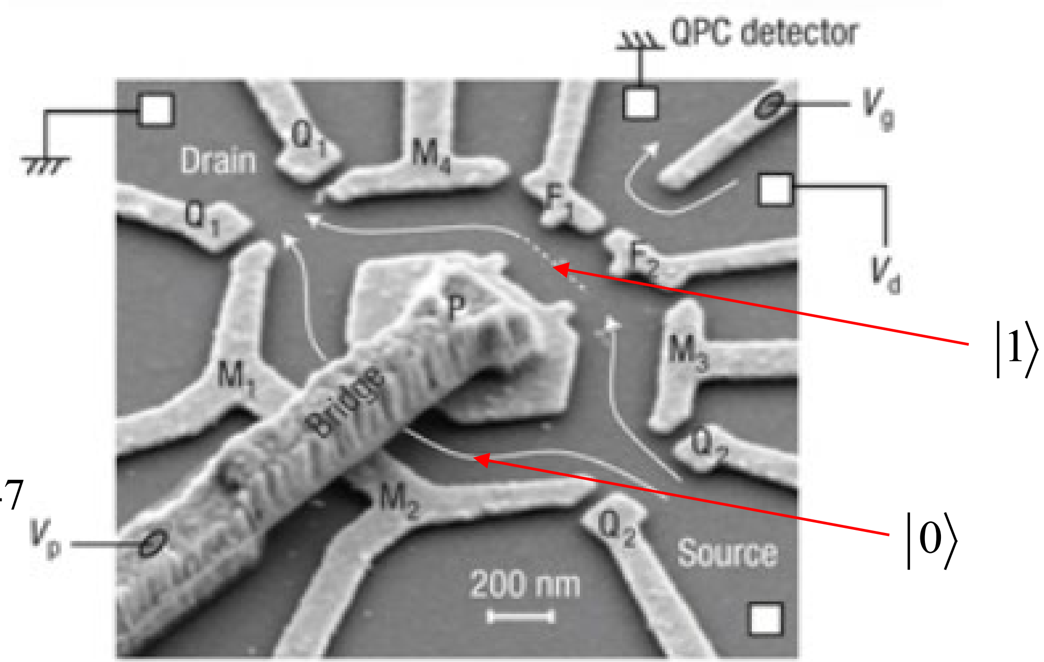
2) Repeat the experiment many times, say NEXP=1000 times.

3) Count how many times you get the electron the left channel and right channel

4) Some typical results might be

NLEFT: 753 times
NRIGHT: 247 times

5) Estimate probabilities as $p_0 = \frac{LEFT}{NEXP} = 0.753 \quad p_1 = \frac{NRIGHT}{NEXP} = 0.247$



From this experiment you don't get the i or even any minus signs!

Measurement of a qubit

Say on a particular experimental run, we measure it, and we find it is in the state $|0\rangle$ (or $|1\rangle$).
What is the quantum state now?

Our experiment is saying that it is in $|0\rangle$ (or $|1\rangle$). So it is in the state $|0\rangle$ (or $|1\rangle$)!

The state evolution is then

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Measurement apparatus

$$|\psi_0^{meas}\rangle = |0\rangle \qquad p_0 = |\alpha|^2$$

$$|\psi_1^{meas}\rangle = |1\rangle \qquad p_1 = |\beta|^2$$

Measurement actually irreparably modifies the state!

Wavefunction collapse

Say I make a measurement on the state and I get $|0\rangle$

If I make a measurement again, what do I get?

Since the state is $|\psi\rangle = |0\rangle$ this means $\alpha = 1, \beta = 0$

Then the probability measurement are $p_0 = |\alpha|^2 = 1$ $\quad p_1 = |\beta|^2 = 0$

i.e. once it is in the state $|0\rangle$ we always get $|0\rangle$ thereafter

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \rightarrow \text{Measurement apparatus}$$

$p_0 = |\alpha|^2$: $|\psi_0^{meas}\rangle = |0\rangle \rightarrow$ Measurement apparatus $\xrightarrow{p_0 = 1} |\psi_0^{meas2}\rangle = |0\rangle \ldots\ldots$

$p_1 = |\beta|^2$: $|\psi_1^{meas}\rangle = |1\rangle \rightarrow$ Measurement apparatus $\xrightarrow{p_1 = 1} |\psi_1^{meas2}\rangle = |1\rangle \ldots\ldots$

Since the wavefunction is destroyed, this is cause “collapse of the wavefunction”

Measurement operators

How to frame this mathematically?

Define measurement operators $M_0 = |m_0\rangle\langle m_0|$ $M_1 = |m_1\rangle\langle m_1|$ .....

There is a measurement operator for each measurement outcome, where we are trying to measure the two states $|m_0\rangle, |m_1\rangle, ...$

Then the probability of obtaining each outcome is

$$\boxed{p_n = \langle \psi | M_n^\dagger M_n | \psi \rangle}$$

The state after the measurement is

$$\boxed{|\psi_n^{meas}\rangle = \frac{M_n |\psi\rangle}{\sqrt{p_n}}}$$

Example: qubit (bra-ket notation)

Let's check that this works for the qubit example that we did before.

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

The two cases we are trying to measure are

$\|0\rangle$ case	$\|1\rangle$ case
$M_0 = \|0\rangle\langle 0\|$	$M_1 = \|1\rangle\langle 1\|$
$p_0 = \left(\langle\psi\| M_0^\dagger\right)\left(M_0\|\psi\rangle\right)$	$p_1 = \left(\langle\psi\| M_1^\dagger\right)\left(M_1\|\psi\rangle\right)$
$M_0\|\psi\rangle = \|0\rangle\langle 0\|\left(\alpha\|0\rangle + \beta\|1\rangle\right) = \alpha\|0\rangle$	$M_1\|\psi\rangle = \|1\rangle\langle 1\|\left(\alpha\|0\rangle + \beta\|1\rangle\right) = \beta\|0\rangle$
probability $p_0 = \left(\alpha^*\langle 0\|\right)\left(\alpha\|0\rangle\right) = \|\alpha\|^2$	probability $p_1 = \left(\beta^*\langle 0\|\right)\left(\beta\|0\rangle\right) = \|\beta\|^2$
Measured state $\|\psi_0^{meas}\rangle = \frac{M_0\|\psi\rangle}{\sqrt{p_0}} = \frac{\alpha\|0\rangle}{\|\alpha\|} = e^{i\theta_\alpha}\|0\rangle$	Measured state $\|\psi_1^{meas}\rangle = \frac{M_1\|\psi\rangle}{\sqrt{p_1}} = \frac{\beta\|0\rangle}{\|\beta\|} = e^{i\theta_\beta}\|1\rangle$

Example: qubit (matrix notation)

Let's check that this works for the qubit example that we did before.

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

The two cases we are trying to measure are

$|0\rangle$ case

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$M_0 |\psi\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$$

probability

$$p_0 = \left(\langle\psi|M_0^\dagger\right)\left(M_0|\psi\rangle\right) = \begin{pmatrix} \alpha^* & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ 0 \end{pmatrix} = |\alpha|^2$$

Measured state $\left|\psi_0^{meas}\right\rangle = \dfrac{M_0|\psi\rangle}{\sqrt{p_0}} = \dfrac{1}{|\alpha|}\begin{pmatrix} \alpha \\ 0 \end{pmatrix} = e^{i\theta_\alpha}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|1\rangle$ case

$$M_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$M_1 |\psi\rangle = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ \beta \end{pmatrix}$$

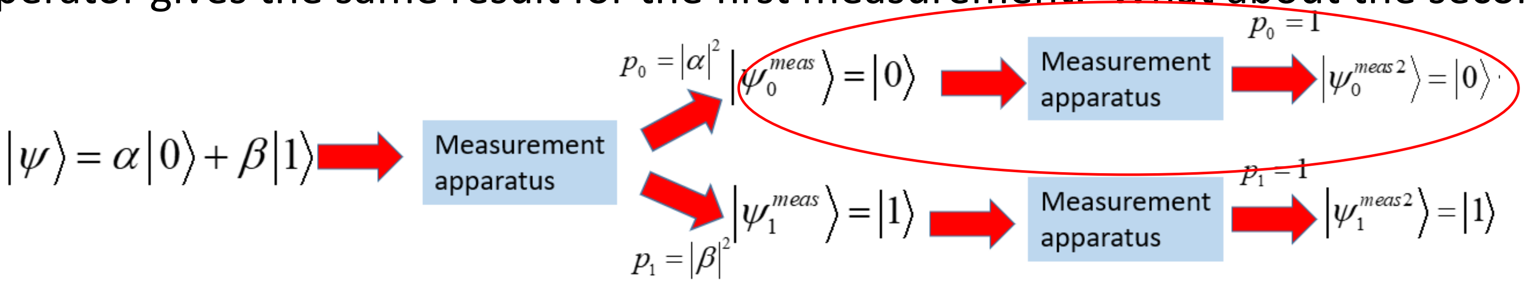
probability

$$p_0 = \left(\langle\psi|M_1^\dagger\right)\left(M_1|\psi\rangle\right) = \begin{pmatrix} 0 & \beta^* \end{pmatrix}\begin{pmatrix} 0 \\ \beta \end{pmatrix} = |\beta|^2$$

Measured state $\left|\psi_1^{meas}\right\rangle = \dfrac{M_1|\psi\rangle}{\sqrt{p_1}} = \dfrac{1}{|\beta|}\begin{pmatrix} 0 \\ \beta \end{pmatrix} = e^{i\theta_\beta}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Example: 2 measurements of a qubit

So the measurement operator gives the same result for the first measurement. What about the second?



Say after the first measurement we got $|\psi_0^{meas}\rangle = |0\rangle$

$\|0\rangle$ case	$\|1\rangle$ case
$M_0\|\psi_0^{meas}\rangle = e^{i\theta_\alpha}\|0\rangle\langle 0\|0\rangle = e^{i\theta_\alpha}\|0\rangle$	$M_1\|\psi_0^{meas}\rangle = e^{i\theta_\beta}\|1\rangle\langle 1\|0\rangle = 0$
probability $p_0 = \left(\langle\psi\|M_0^\dagger\right)\left(M_0\|\psi\rangle\right) = \langle 0\|0\rangle = 1$	probability $p_1 = \left(\langle\psi\|M_1^\dagger\right)\left(M_1\|\psi\rangle\right) = 0$
Measured state $\|\psi_0^{meas2}\rangle = \frac{M_0\|\psi_0^{meas}\rangle}{\sqrt{p_0}} = e^{i\theta_\alpha}\|0\rangle$	Measured state $\|\psi_0^{meas2}\rangle = 0$

The global phase

In the qubit example above, we had after the first measurement $\left|\psi_0^{meas}\right\rangle = e^{i\theta_\alpha}\left|0\right\rangle$

Doesn't the $e^{i\theta_\alpha}$ matter? ⇒ NO.

As we saw in the example after the second measurement, the probability is

$$p_0 = \left\langle \psi_0^{meas} \middle| \psi_0^{meas} \right\rangle = \left\langle 0 \middle| 0 \right\rangle e^{i\theta_\alpha} e^{-i\theta_\alpha} = 1$$

Such "global phases", i.e. a phase that multiplies the whole wavefunction NEVER MATTERS.

So all the following states are physically exactly the same: $\left|\psi\right\rangle \qquad -\left|\psi\right\rangle \qquad i\left|\psi\right\rangle \qquad \frac{1-i}{\sqrt{2}}\left|\psi\right\rangle$

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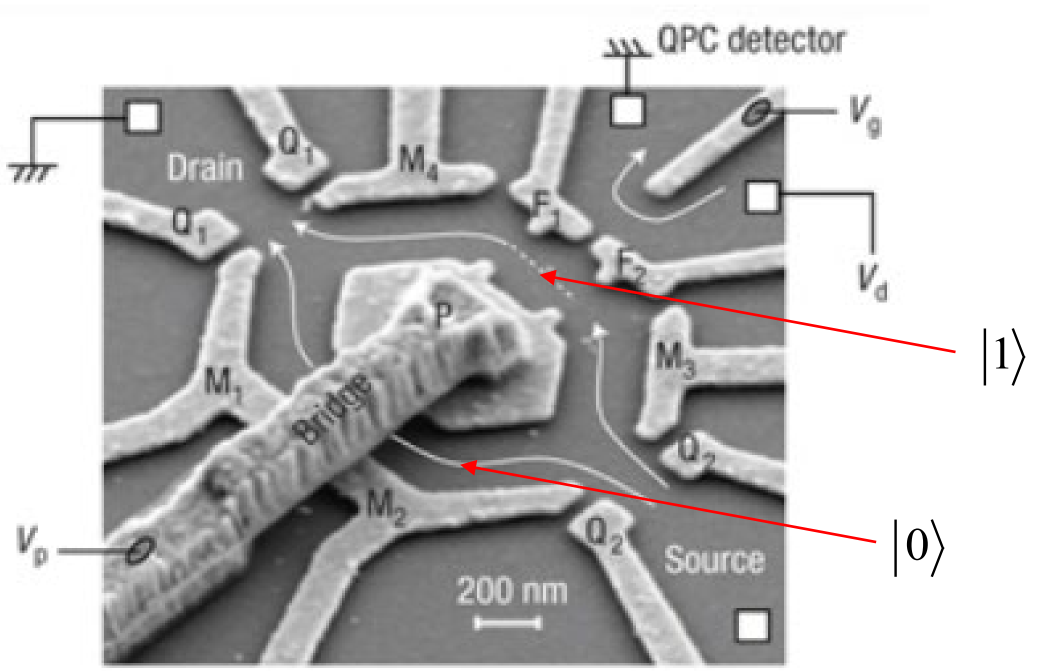

which is the state of an electron in a channel, for example.

1) Find the probability that it is in the left and right channel using measurement operators.

2) Find the wavefunction after the measurement in each case.