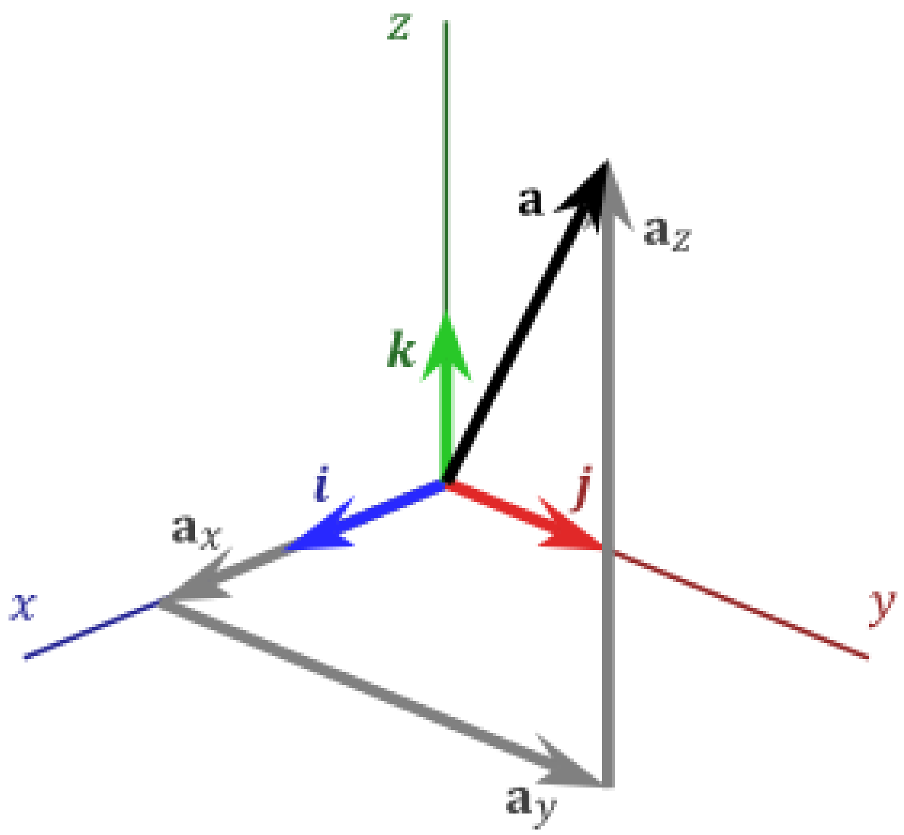
5. Linear algebra II



Matrix terminology

Transpose:
Interchange rows and columns

$$\tilde{\mathbf{T}} = \begin{pmatrix} T_{11} & T_{21} & \dots & T_{n1} \\ T_{12} & T_{22} & \dots & T_{n2} \\ \vdots & \vdots & & \vdots \\ T_{1n} & T_{2n} & \dots & T_{nn} \end{pmatrix} \qquad \mathbf{T} = \begin{pmatrix} T_{11} & T_{12} & \dots & T_{1n} \\ T_{21} & T_{22} & \dots & T_{2n} \\ \vdots & \vdots & & \vdots \\ T_{n1} & T_{n2} & \dots & T_{nn} \end{pmatrix}$$

Symmetric matrices: $\text{symmetric} : \tilde{\mathbf{T}} = \mathbf{T}; \quad \text{antisymmetric} : \tilde{\mathbf{T}} = -\mathbf{T}.$

Conjugate of a matrix:

$$\mathbf{T}^* = \begin{pmatrix} T_{11}^* & T_{12}^* & \dots & T_{1n}^* \\ T_{21}^* & T_{22}^* & \dots & T_{2n}^* \\ \vdots & \vdots & & \vdots \\ T_{n1}^* & T_{n2}^* & \dots & T_{nn}^* \end{pmatrix}$$

Real and imaginary matrices: $\text{real} : \mathbf{T}^* = \mathbf{T}; \quad \text{imaginary} : \mathbf{T}^* = -\mathbf{T}.$

Matrix terminology 2

Hermitian conjugate (adjoint):

$$\mathbf{T}^{\dagger} \equiv \tilde{\mathbf{T}}^{*} = \begin{pmatrix} T_{11}^{*} & T_{21}^{*} & \dots & T_{n1}^{*} \\ T_{12}^{*} & T_{22}^{*} & \dots & T_{n2}^{*} \\ \vdots & \vdots & & \vdots \\ T_{1n}^{*} & T_{2n}^{*} & \dots & T_{nn}^{*} \end{pmatrix}$$

Hermitian matrices:

$$\text{hermitian} : \mathbf{T}^{\dagger} = \mathbf{T};$$

In quantum mechanics physical observables are Hermitian operators

Completeness (identity matrix)

The matrix is called the identity matrix

$$\mathbf{I} \equiv \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

$$\mathbf{I}_{ij} = \delta_{ij}$$

This has the special property that $I|\alpha\rangle = |\alpha\rangle$ for any vector

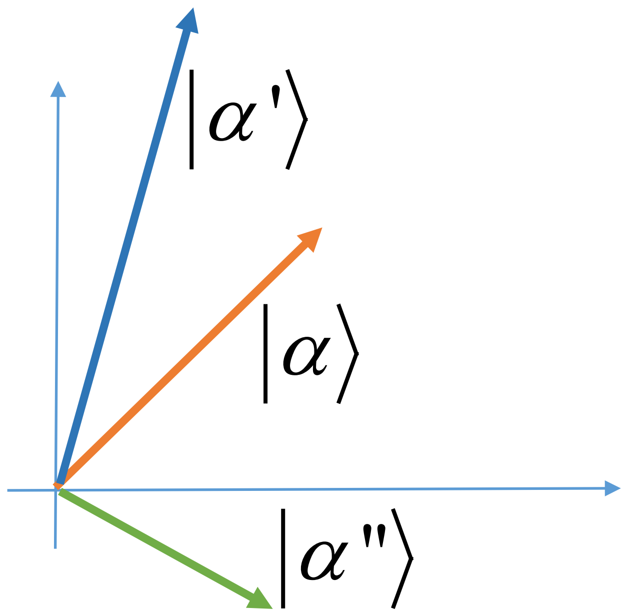
Can be also written as

$$I = |e_1\rangle\langle e_1| + |e_2\rangle\langle e_2| + \dots + |e_n\rangle\langle e_n| = \sum_i |e_i\rangle\langle e_i|$$

Addition and multiplication of operators

Addition of linear transforms: $(\hat{S}+\hat{T})|\alpha\rangle = \hat{S}|\alpha\rangle + \hat{T}|\alpha\rangle$

Multiplication of linear transforms: $|\alpha'\rangle = \hat{T}|\alpha\rangle; \quad |\alpha''\rangle = \hat{S}|\alpha'\rangle = \hat{S}(\hat{T}|\alpha\rangle) = \hat{S}\hat{T}|\alpha\rangle$



$$|\alpha''\rangle = \begin{pmatrix} S_{11} & \cdots & S_{1n} \\ \vdots & & \\ S_{n1} & & S_{nn} \end{pmatrix} \begin{pmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & & \\ T_{n1} & & T_{nn} \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

$$|\alpha\rangle \rightarrow \boxed{\hat{T}} \rightarrow |\alpha'\rangle \rightarrow \boxed{\hat{S}} \rightarrow |\alpha''\rangle$$

The commutator

Operators (and matrices) do NOT obey $ST = TS$

"How much" the operators depart from this is called the commutator

$$[\mathbf{S}, \mathbf{T}] \equiv \mathbf{ST} - \mathbf{TS}.$$

Question

1) Find $X^\dagger$

2) Evaluate $[X,Y]$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Eigenvectors and eigenvalues

Normally, when an operator acts on a vector, it will change the vector $|\alpha\rangle \rightarrow |\alpha'\rangle = \hat{T}|\alpha\rangle$

Some special vectors are unchanged when the operator acts on it (up to a constant)

"Eigenvalue"

"Eigenvector"

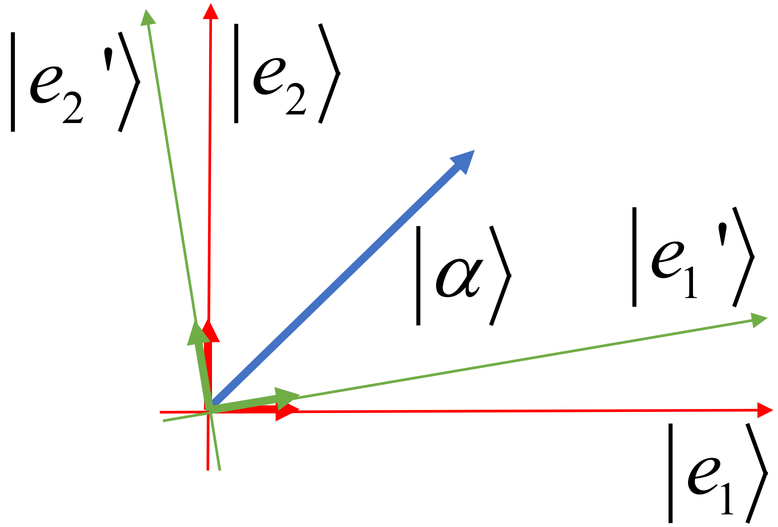
$$\hat{T}|\lambda_1\rangle = E_1|\lambda_1\rangle$$

$$\hat{T}|\lambda_2\rangle = E_2|\lambda_2\rangle$$

$$\cdots$$

Basis transformation

A special type of transformation corresponds to a change of basis



In the original basis $$|\alpha\rangle = a_1|e_1\rangle + a_2|e_2\rangle$$

In the dashed basis $$|\alpha\rangle = a_1'|e_1'\rangle + a_2'|e_2'\rangle$$

The operator that makes such a basis transformation is called a unitary operator $$|e_i'\rangle = U|e_i\rangle$$

Unitary operators have the property that $$U^{-1} = U^{+}$$

Question

1) Verify that $|\lambda_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of X.

What is the eigenvalue?

2) What are the eigenvalues of Z?

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$