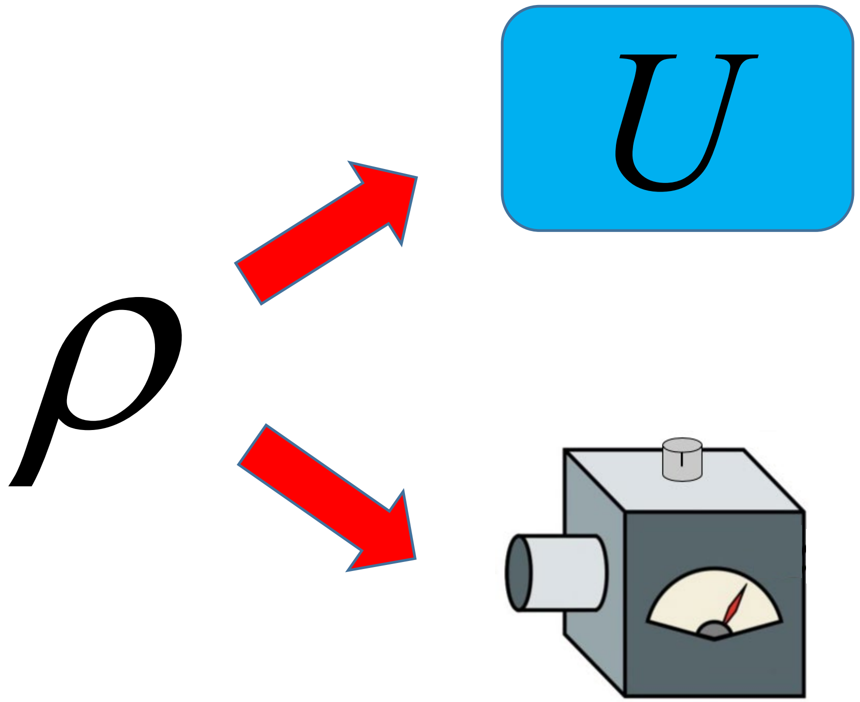
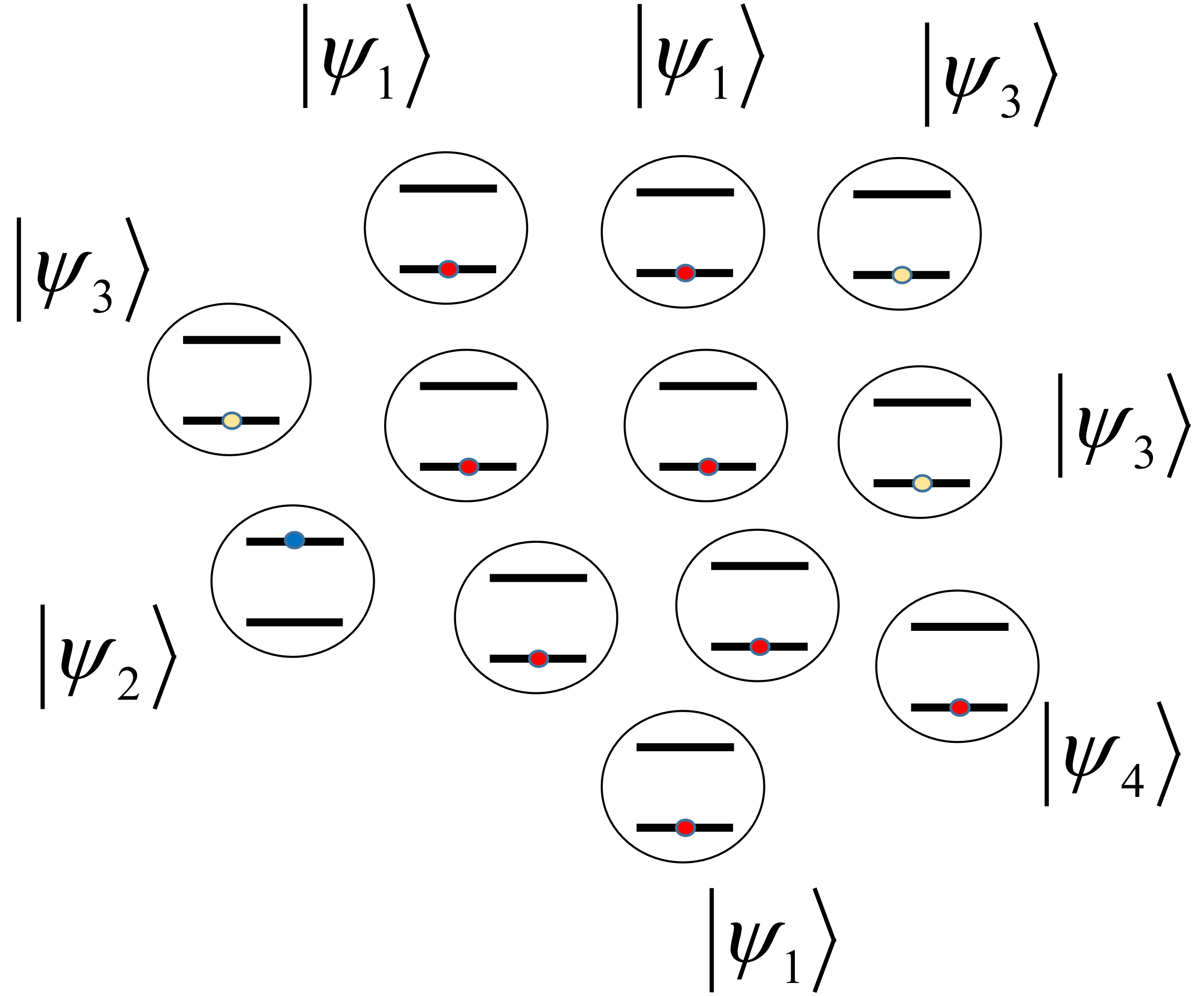
22. Applications of the Density Matrix



Recap: Density matrix definition

$$\rho = \sum_n p_n |\psi_n\rangle\langle\psi_n|$$

$p_n =$ Probability of occurrence of state $|\psi_n\rangle$



What to do with the density matrix?

The density matrix ρ describes the quantum state of a noisy system.

Everything we did with the wavefunction $|\psi\rangle$, we can also do with the density matrix!

- Measurements
- Observables
- Unitary operations (quantum gates)

We will show how these work with the density matrix.

Unitary operations

Recall that for a pure state, under a Hamiltonian H the state evolves like

$$|\psi\rangle \to U|\psi\rangle$$

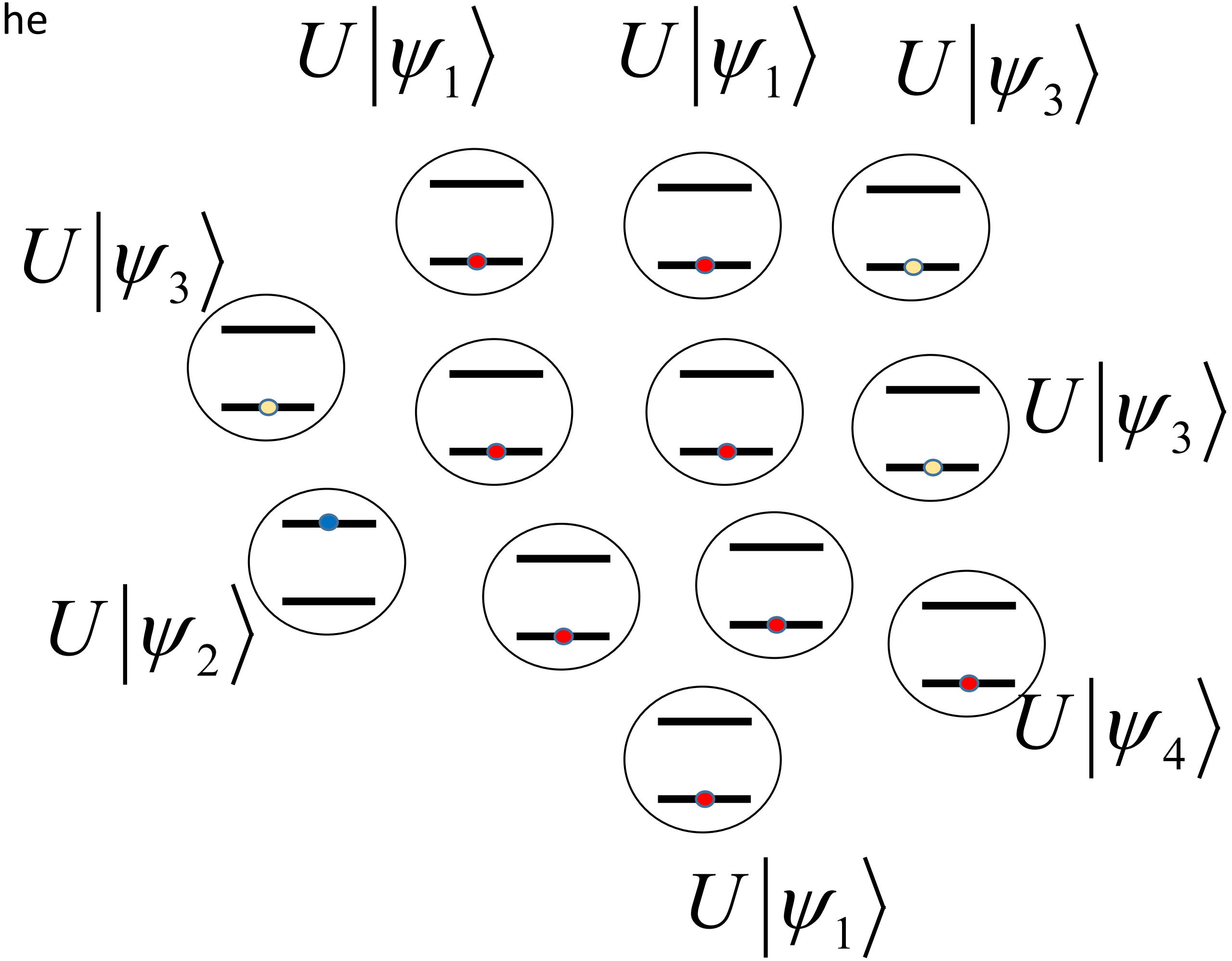
$$U = e^{-iHt/\hbar}$$

So the density matrix evolves as

$$\rho \to \rho' = \sum_n p_n U|\psi_n\rangle\langle\psi_n|U^\dagger$$

Or simply

$$\boxed{\rho \to \rho' = U\rho U^\dagger}$$



Question

The state of a system is such that it is 50% in the state $|+\rangle$ and 50% in the state $|0\rangle$. An X gate is applied to the system. What is the new density matrix of the system?

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Initial state:

$$\rho = \frac{1}{2}|+\rangle\langle+| + \frac{1}{2}|0\rangle\langle 0| = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$$

Final state:

$$\rho' = X\rho X^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{4}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$$

Measurements

Recall that under a measurement the state evolves like

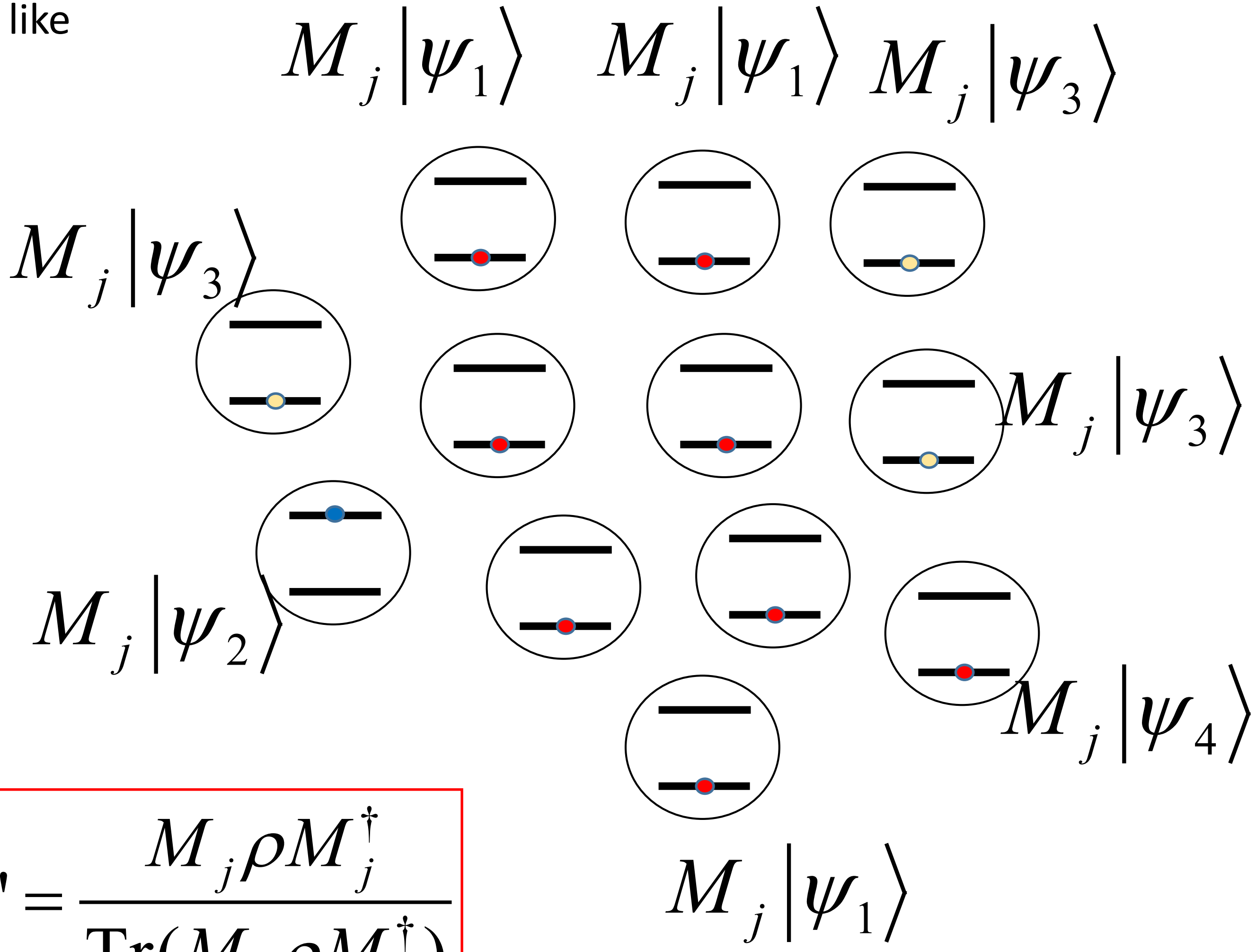
$$|\psi\rangle \to \frac{M_j|\psi\rangle}{\sqrt{\langle\psi|M_j^\dagger M_j|\psi\rangle}}$$

So the density matrix evolves as

$$\rho \to \rho' = \frac{\sum_n p_n M_j|\psi_n\rangle\langle\psi_n|M_j^\dagger}{N}$$

Where N is the normalization factor

$$N = \mathrm{Tr}\left[\sum_n p_n M_j|\psi_n\rangle\langle\psi_n|M_j^\dagger\right]$$
$$= \sum_n p_n \langle\psi_n|M_j^\dagger M_j|\psi_n\rangle$$



$$\boxed{\rho \to \rho' = \frac{M_j\rho M_j^\dagger}{\mathrm{Tr}(M_j\rho M_j^\dagger)}}$$

Question

The state of a system is such that it is 50% in the state $|+\rangle$ and 50% in the state $|0\rangle$, is now measured in the $\{|0\rangle, |1\rangle\}$ basis, and the $|0\rangle$ outcome is obtained. What is the new density matrix?

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Initial state:

$$\rho = \frac{1}{2}|+\rangle\langle+| + \frac{1}{2}|0\rangle\langle 0| = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$$

$$M_0 = |0\rangle\langle 0|$$

Applying the measurement operators

$$M_0 \rho M_0^\dagger = \frac{3}{4}|0\rangle\langle 0|$$

$$Tr\left(M_0 \rho M_0^\dagger\right) = \frac{3}{4}$$

$$\rho' = |0\rangle\langle 0|$$

Observables

Recall that observables are calculated for a pure state by

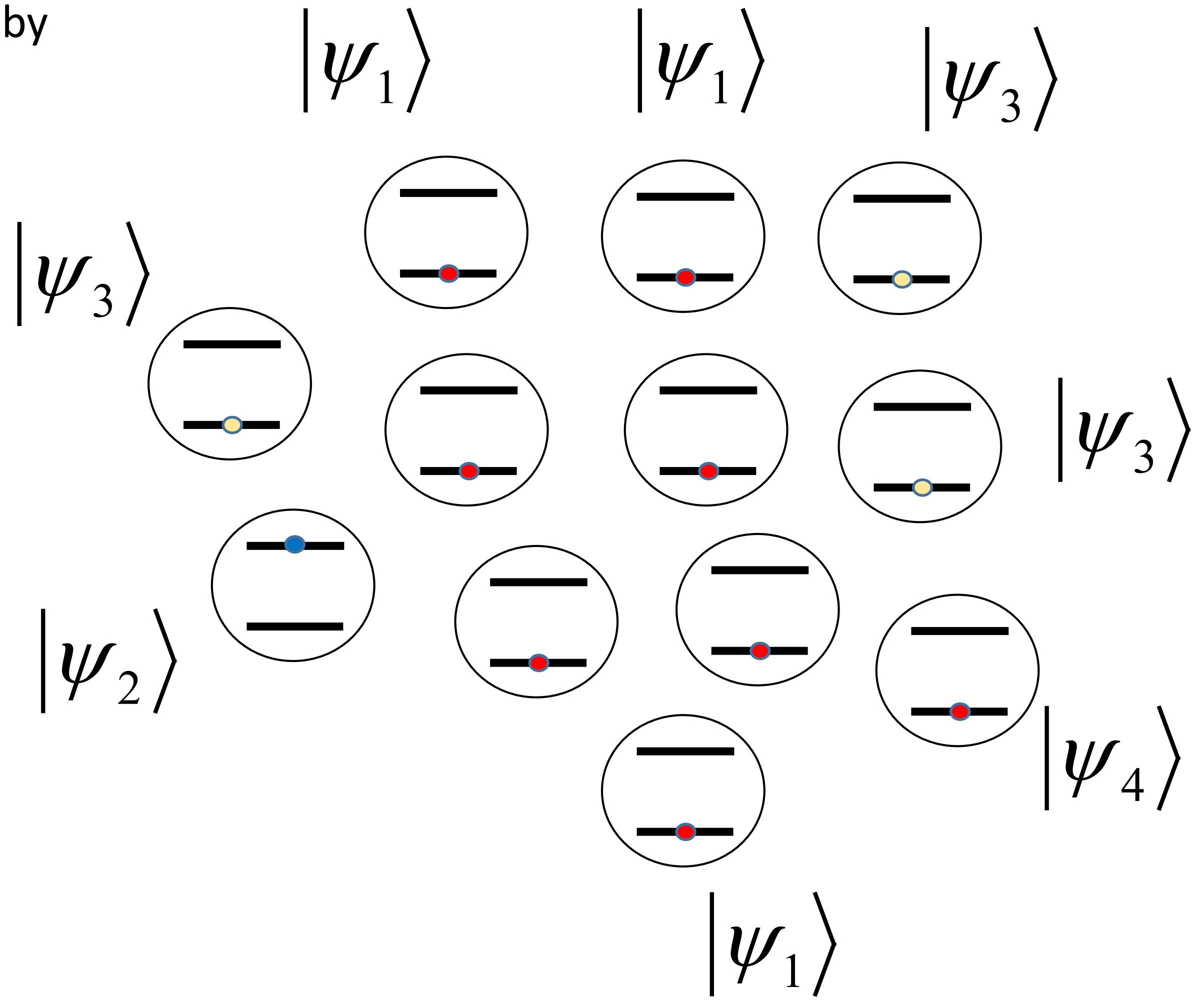
$$\langle A \rangle = \langle \psi | A | \psi \rangle$$

For a mixed state, we simply average over all the expectation values

$$\langle A \rangle = \sum_n p_n \langle \psi_n | A | \psi_n \rangle$$

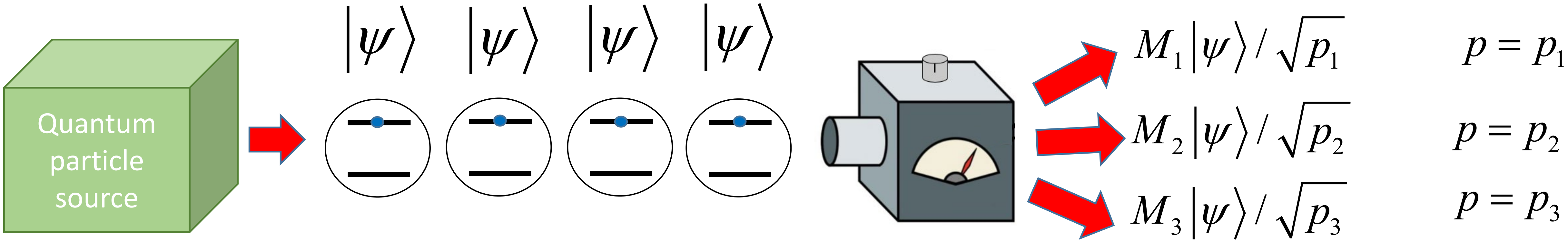
Or another way to write this is

$$\boxed{\langle A \rangle = \mathrm{Tr}(A\rho)}$$



Mixed state after measurement

How do we get a mixed state in the first place? One way is through measurement:



The density matrix of the ensemble after the measurement is

$$\rho' = \sum_n p_n |\psi_n\rangle\langle\psi_n|$$

In this case $|\psi_n\rangle = \frac{M_n|\psi\rangle}{\sqrt{p_n}}$

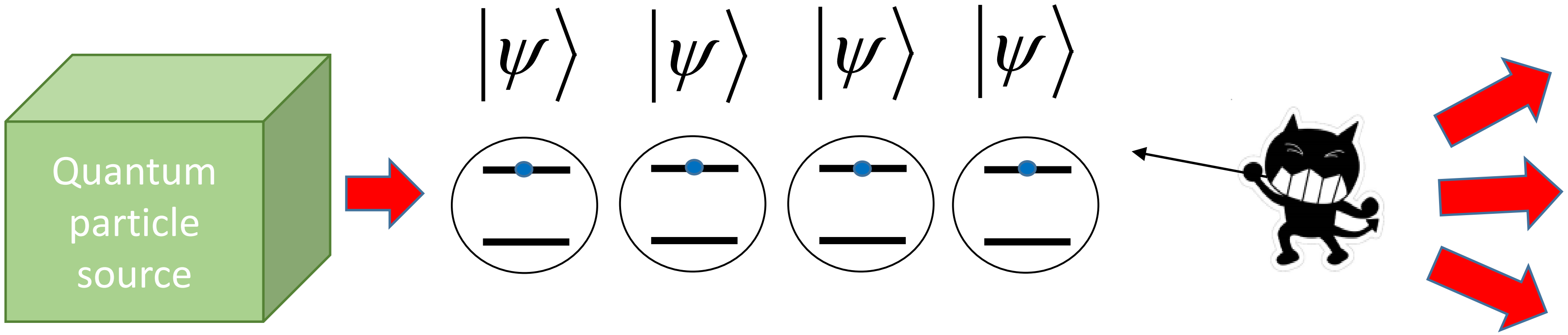
$$\rho' = \sum_n M_n |\psi\rangle\langle\psi| M_n^\dagger$$

Probability cancels!

$$\boxed{\rho' = \sum_n M_n \rho M_n^\dagger}$$

Error channels: decoherence

Actually having a physical error is very similar to a measurement



$E_1|\psi\rangle/\sqrt{p_1}$ $\quad p = p_1$

$E_2|\psi\rangle/\sqrt{p_2}$ $\quad p = p_2$

$E_3|\psi\rangle/\sqrt{p_3}$ $\quad p = p_3$

Using the same formula as for the measurement

$$p_n = \mathrm{Tr}(E_n \rho E_n^\dagger)$$

$$\boxed{\rho' = \sum_n E_n \rho E_n^\dagger}$$

We can view decoherence as being a type of uncontrolled measurement

Trace preserving operation

Even when an error occurs on the quantum state, we must still preserve normalization of the state, since it is still a quantum state

ρ

ρ'

So we must have

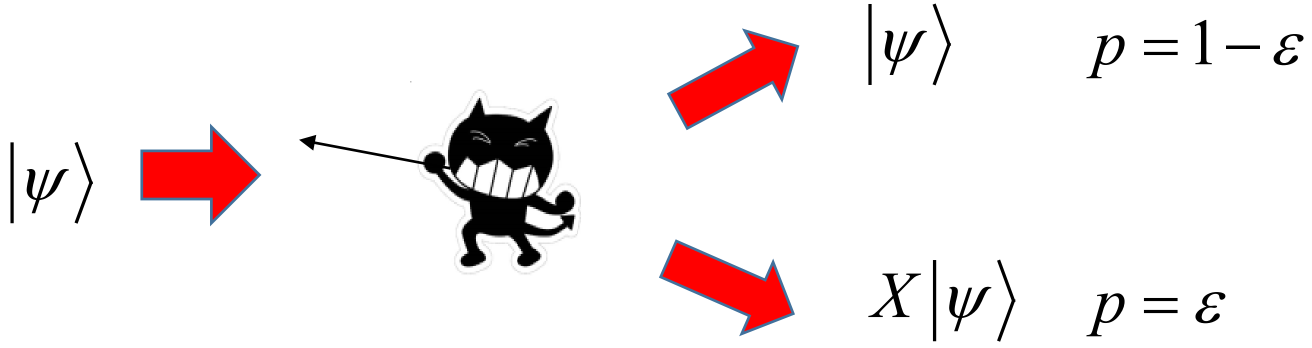
$$\mathrm{Tr}(\rho') = \mathrm{Tr}(\sum_n E_n \rho E_n^\dagger) = \mathrm{Tr}(\underbrace{\sum_n E_n^\dagger E_n}_{=I} \rho) = 1$$

Then the error channel must satisfy

$$\sum_n E_n^\dagger E_n = I$$

Example 1: bit flip channel

Consider an error occurs on a qubit with probability ε where the qubit is flipped by applying X



The error channel for this is written

$$E_0 = \sqrt{1-\varepsilon}\, I, E_1 = \sqrt{\varepsilon}\, X$$

This is trace-preserving

$$\sum_n E_n^\dagger E_n = (1-\varepsilon) I + \varepsilon X^2 = I$$

Example 2: phase flip channel

Consider an error occurs on a qubit with probability ε where the qubit is flipped by applying Z

$|\psi\rangle$

$|\psi\rangle \quad p = 1-\varepsilon$

$Z|\psi\rangle \quad p = \varepsilon$

The error channel for this is written

$$E_0 = \sqrt{1-\varepsilon} I, E_1 = \sqrt{\varepsilon} Z$$

This is trace-preserving

$$\sum_n E_n^\dagger E_n = (1-\varepsilon) I + \varepsilon Z^2 = I$$

Question

Say the initial state is $|+\rangle$ and the phase flip channel acts on the state with probability ε . What is the final state?

$$E_0 = \sqrt{1-\varepsilon} I, E_1 = \sqrt{\varepsilon} Z$$

$$\rho' = \sum_n E_n \rho E_n^\dagger$$

Question

Say the initial state is $|+\rangle$ and the phase flip channel acts on the state with probability ε . What is the final state?

$$E_0 = \sqrt{1-\varepsilon} I, E_1 = \sqrt{\varepsilon} Z \qquad \rho' = \sum_n E_n \rho E_n^\dagger$$

Initial state

$$\rho = |+\rangle\langle+| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Apply

$$E_0 \rho E_0^\dagger = (1-\varepsilon)|+\rangle\langle+| \qquad E_1 \rho E_1^\dagger = \varepsilon Z|+\rangle\langle+|Z = \varepsilon|-\rangle\langle-|$$

$$\rho' = (1-\varepsilon)|+\rangle\langle+| + \varepsilon|-\rangle\langle-| = \frac{1}{2}\begin{pmatrix} 1 & 1-2\varepsilon \\ 1-2\varepsilon & 1 \end{pmatrix}$$

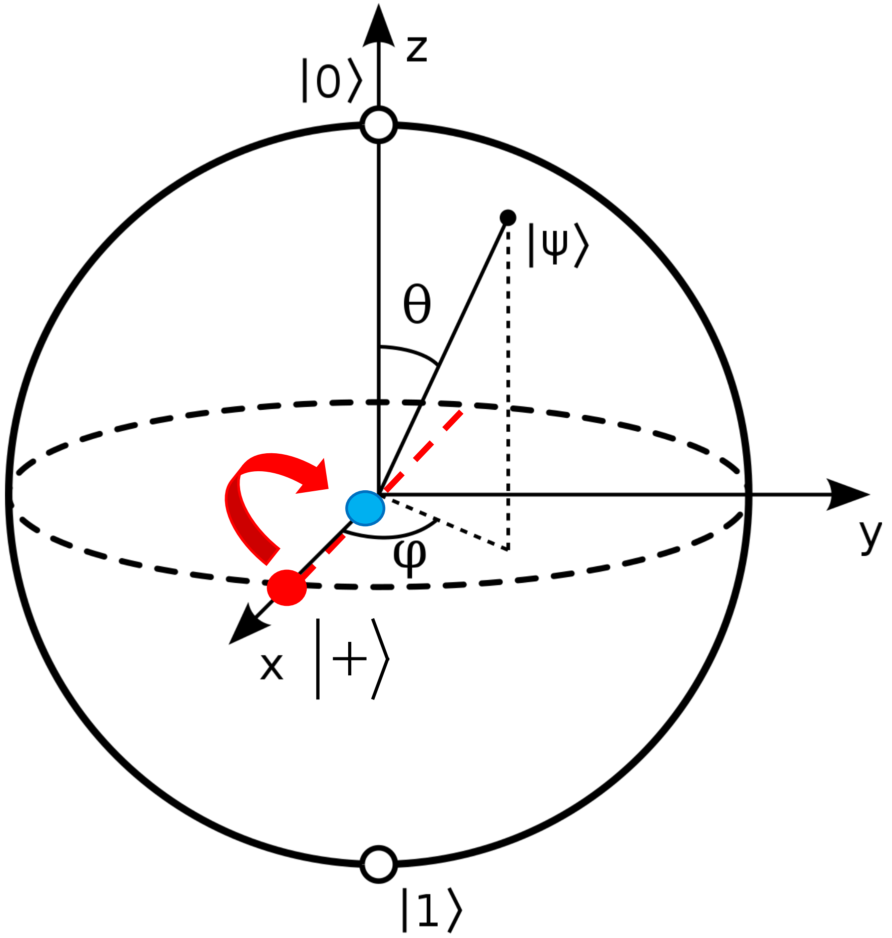
Where is this mixed state on the Bloch sphere?

$$\rho' = \frac{1}{2}\begin{pmatrix} 1 & 1-2\varepsilon \\ 1-2\varepsilon & 1 \end{pmatrix}$$

$$\langle X \rangle = 1-2\varepsilon$$

$$\langle Y \rangle = 0$$

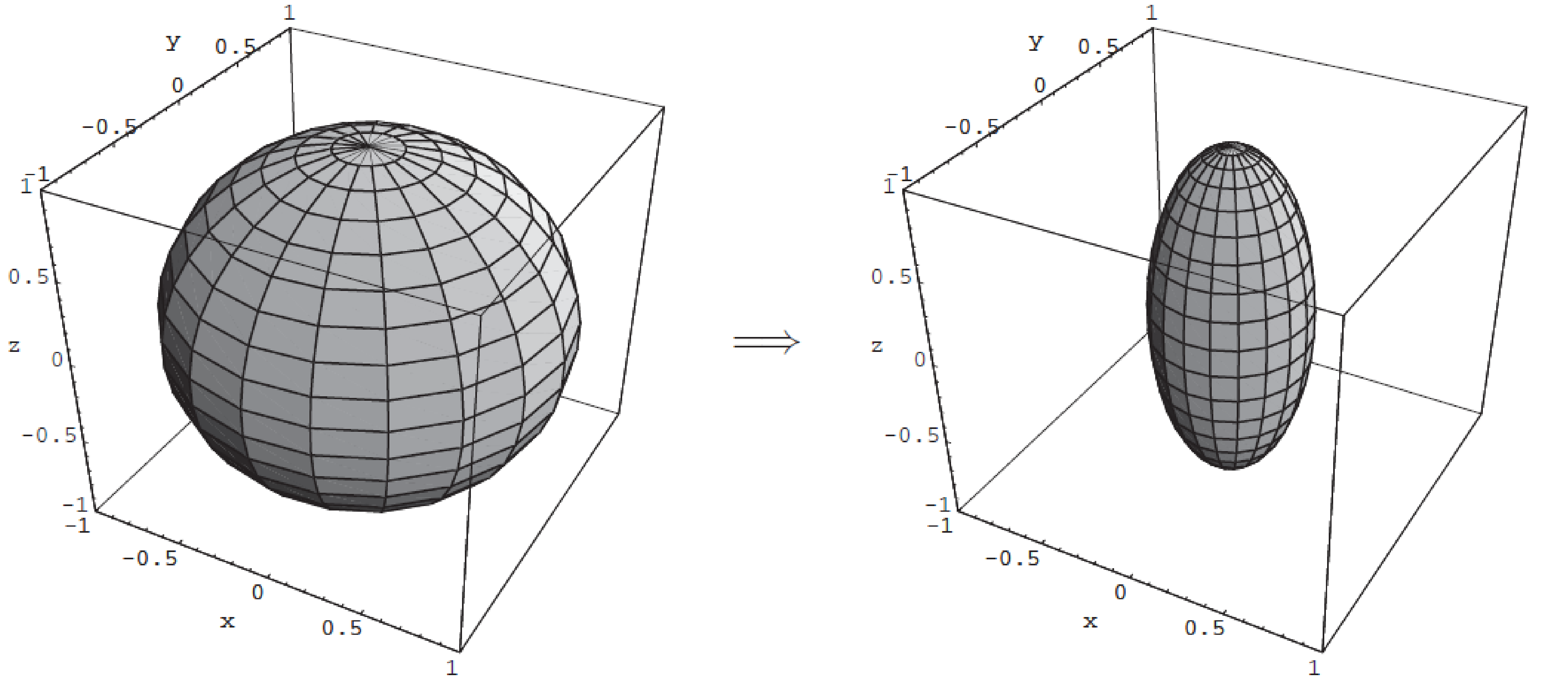
$$\langle Z \rangle = 0$$



States are in the middle of the Bloch sphere. Pure states on the surface.

Dephasing channel

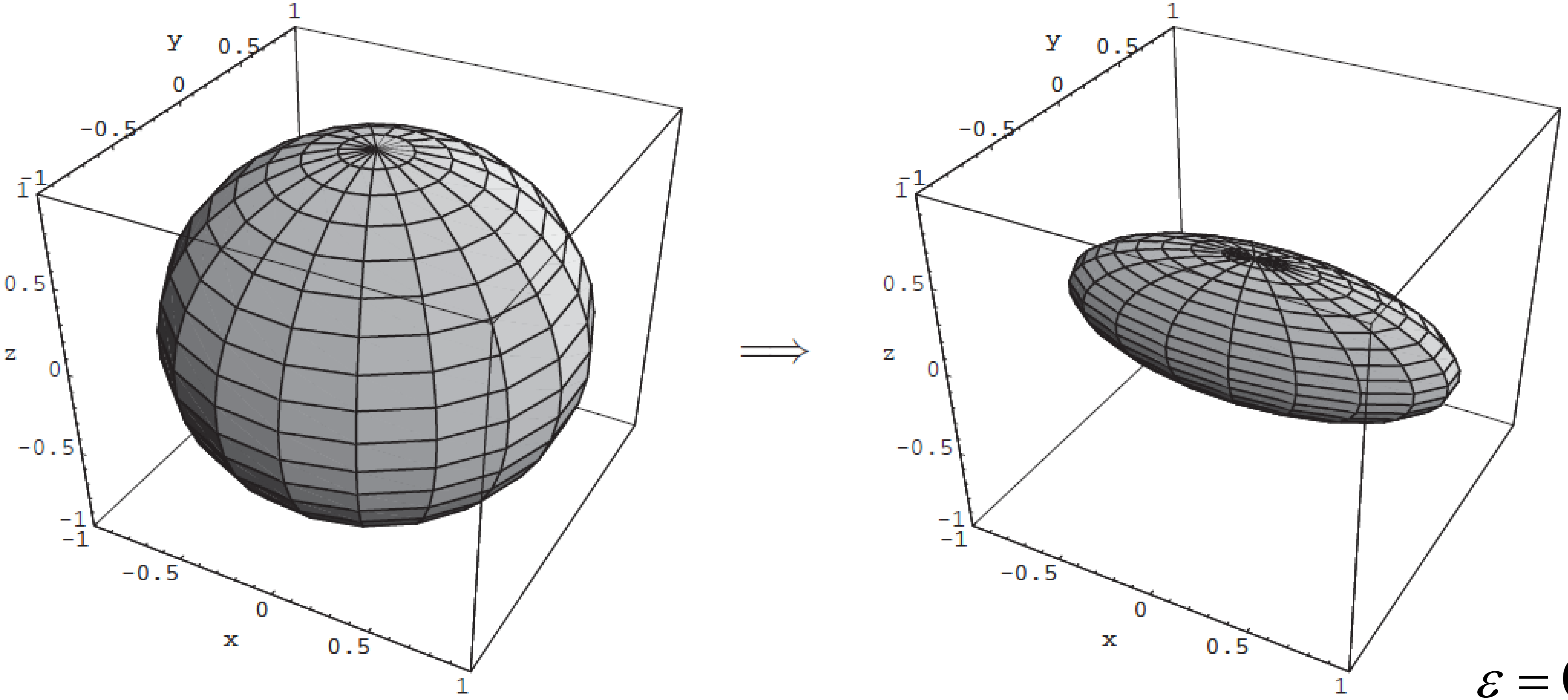
More generally we can visualize the evolution of the dephasing channel on a qubit as



$$\varepsilon = 0.3$$

Bit flip channel

$$E_0 = \sqrt{1-\varepsilon} I, E_1 = \sqrt{\varepsilon} X$$

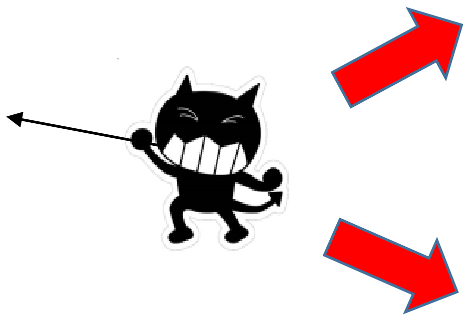


$$\varepsilon = 0.3$$

Completely mixed qubit state

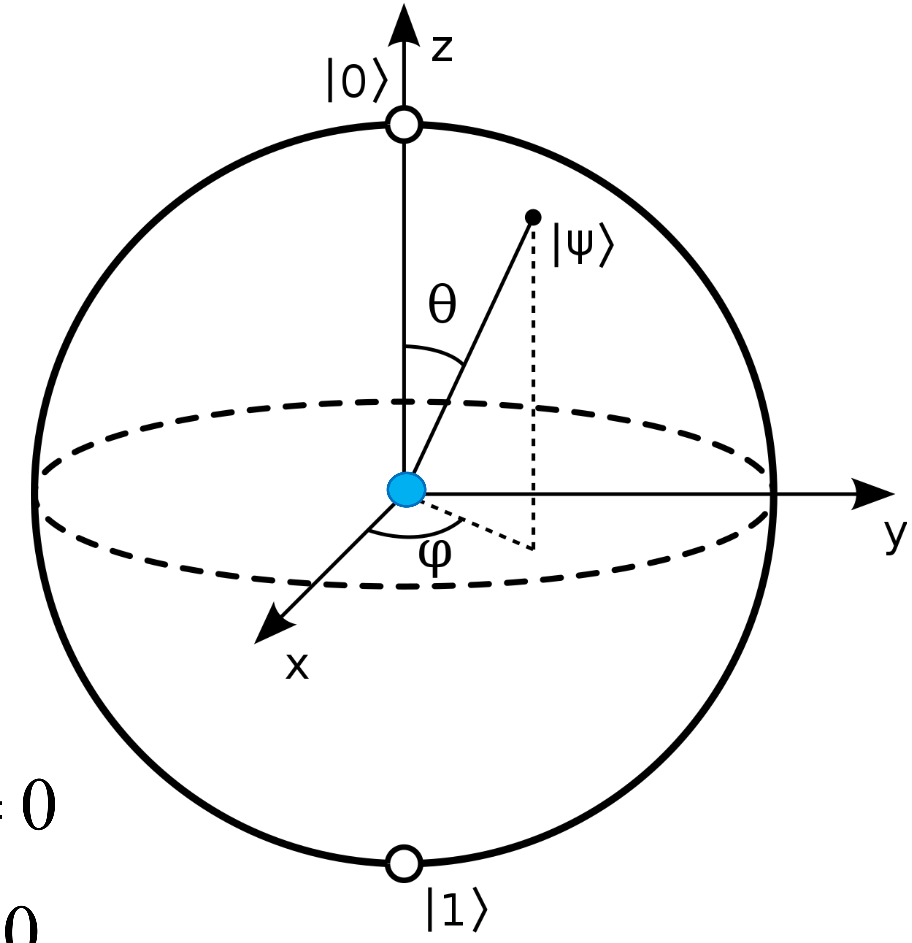
The state at the middle of the Bloch sphere is a completely random state

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



$|0\rangle \quad p = 1/2$

$|1\rangle \quad p = 1/2$



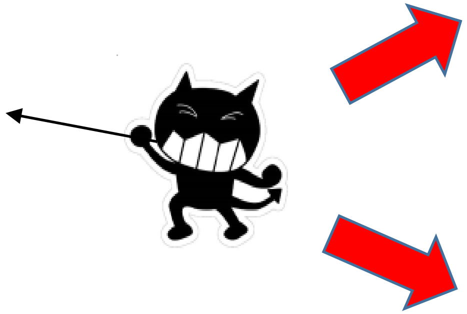
$$\langle X \rangle = 0$$

$$\langle Y \rangle = 0$$

$$\langle Z \rangle = 0$$

Basis invariance of completely mixed state

Actually it doesn't have to be a mixture of $|0\rangle, |1\rangle$



$|+\rangle \quad p = 1/2$

$|-\rangle \quad p = 1/2$

$$\rho = \frac{1}{2}|+\rangle\langle+| + \frac{1}{2}|-\rangle\langle-| = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Or ANY 50/50 mixture of orthogonal states will produce this same state

This is a special property of the completely mixed state

$$\rho = \frac{1}{2}I = \frac{1}{2}UIU^\dagger = \frac{1}{2}I$$

$$\boxed{\rho = \frac{I}{d} \quad \text{For a d-dimensional system}}$$

Question

Verify that a completely mixed state of the states

$$|\theta\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$$

$$\overline{|\theta\rangle} = \sin\theta|0\rangle - \cos\theta|1\rangle$$

gives

$$\rho = \frac{1}{2}I$$

Question

Verify that a completely mixed state of the states $|\theta\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$, $\overline{|\theta\rangle} = \sin\theta|0\rangle - \cos\theta|1\rangle$ gives $\rho = \frac{1}{2} I$

$$\rho = \frac{1}{2}|\theta\rangle\langle\theta| + \frac{1}{2}\overline{|\theta\rangle}\overline{\langle\theta|}$$

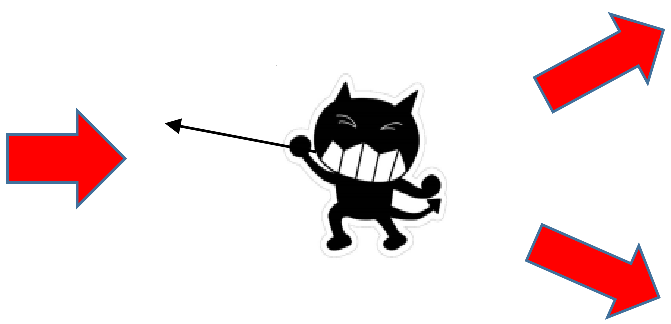
$$= \frac{1}{2}\begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \sin^2\theta & -\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos^2\theta \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Example 3: Depolarizing channel

A more symmetric type of error is the depolarizing channel

ρ



$\rho \qquad p = 1 - \varepsilon$

$\frac{I}{2} \qquad p = \varepsilon$

The final state is

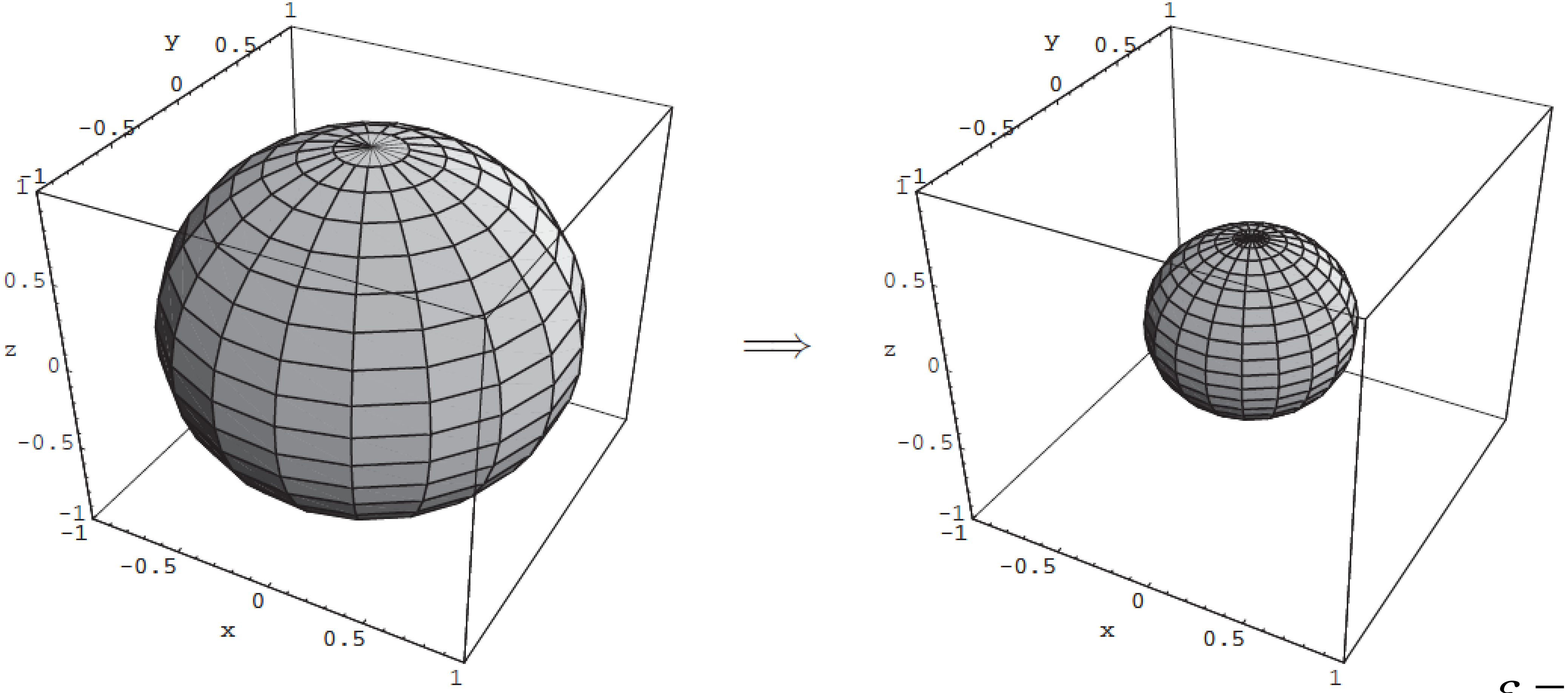
$$\rho \to \varepsilon \frac{I}{2} + (1-\varepsilon)\rho$$

$$\langle X \rangle = (1-\varepsilon)\langle X \rangle_0$$

$$\langle Y \rangle = (1-\varepsilon)\langle Y \rangle_0$$

$$\langle Z \rangle = (1-\varepsilon)\langle Z \rangle_0$$

Depolarizing channel



$$\varepsilon = 0.5$$