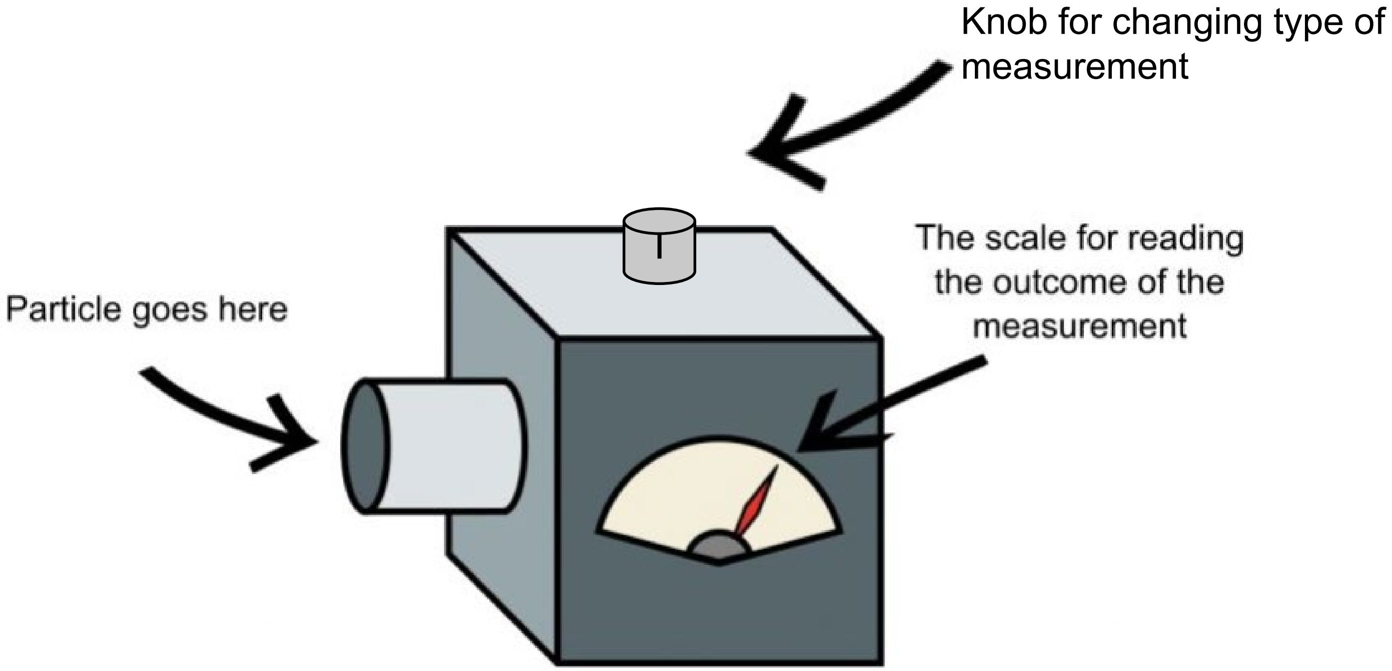
9. Measurements in other bases



Recap: Measurement of a qubit

Measuring a qubit gives a random result, where the probabilities are the absolute value squared.

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Measurement apparatus

$$|\psi_0^{meas}\rangle = |0\rangle \qquad p_0 = |\alpha|^2$$

$$|\psi_1^{meas}\rangle = |1\rangle \qquad p_1 = |\beta|^2$$

Measurement operators

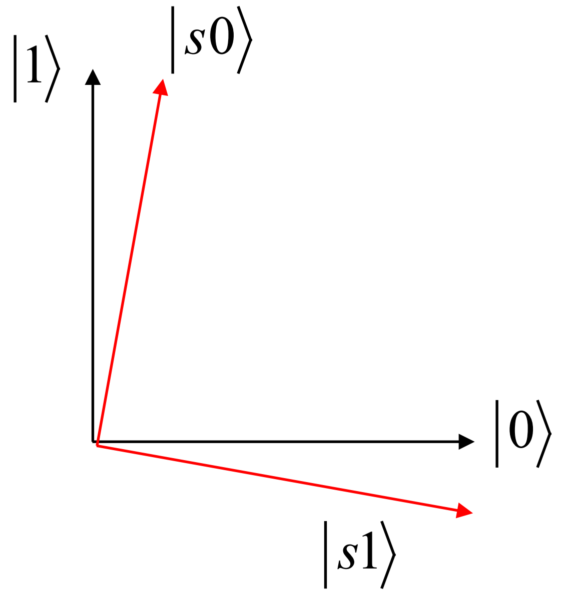
$$M_0 = |0\rangle\langle 0| \qquad M_1 = |1\rangle\langle 1|$$

Probability: $p_n = \langle\psi|M_n^\dagger M_n|\psi\rangle$

New state: $|\psi_n^{meas}\rangle = \dfrac{M_n|\psi\rangle}{\sqrt{p_n}}$

Combining with basis rotations

Actually the $|0\rangle, |1\rangle$ is just one pair of orthogonal states.



We can equally measure in any pair of orthogonal bases

$$|s0\rangle = c|0\rangle + d|1\rangle$$

$$|s1\rangle = d^*|0\rangle - c^*|1\rangle$$

If we have some state $\quad |\psi\rangle = a|s0\rangle + b|s1\rangle$

Then we can also measure in this basis with a similar rule, but for the basis states $|s1\rangle, |s2\rangle$

$$|\psi\rangle = a|s0\rangle + b|s1\rangle \quad \Rightarrow \quad \boxed{\text{Measurement apparatus}}$$

$$|\psi_0^{meas}\rangle = |s0\rangle \qquad p_0 = |a|^2$$

$$|\psi_1^{meas}\rangle = |s1\rangle \qquad p_1 = |b|^2$$

Example: Measuring qubit in $|\pm\rangle$-basis

In Lecture 8 we did the example of writing a qubit in the $|\pm\rangle$-basis.

The state $$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

in the basis of $$|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \qquad |-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

is $$|\psi\rangle = \left(\frac{\alpha+\beta}{\sqrt{2}}\right)|+\rangle + \left(\frac{\alpha-\beta}{\sqrt{2}}\right)|-\rangle$$

So measuring in this basis will give probabilities

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \Rightarrow \boxed{\text{Measurement apparatus}}$$

$$|\psi_0^{meas}\rangle = |+\rangle \quad p_0 = \frac{|\alpha+\beta|^2}{2}$$

$$|\psi_1^{meas}\rangle = |-\rangle \quad p_1 = \frac{|\alpha-\beta|^2}{2}$$

Measurement operators

How to calculate with measurement operators?

We can use the same theory as before: $M_0 = |m_0\rangle\langle m_0|$ $\quad$ $M_1 = |m_1\rangle\langle m_1|$

Probability $$\boxed{p_n = \langle\psi|M_n^\dagger M_n|\psi\rangle}$$

State after the measurement is $$\boxed{|\psi_n^{meas}\rangle = \frac{M_n|\psi\rangle}{\sqrt{p_n}}}$$

Example: Measuring qubit in $|\pm\rangle$-basis

Let's check that this works for the qubit example that we did before. $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

This time use the measurement operator method. Consider just the $|+\rangle$ case

$$M_0 = |+\rangle\langle+|$$

$$p_0 = \left(\langle\psi|M_0^\dagger\right)\left(M_0|\psi\rangle\right)$$

$$M_0|\psi\rangle = |+\rangle\langle+|\left(\alpha|0\rangle + \beta|1\rangle\right) = |+\rangle\left(\alpha\langle+|0\rangle + \beta\langle+|1\rangle\right)$$

Work out the inner products:

$$\langle+|0\rangle = \frac{1}{\sqrt{2}}\left(\langle 0| + \langle 1|\right)|0\rangle = \frac{1}{\sqrt{2}}$$

$$\langle+|1\rangle = \frac{1}{\sqrt{2}}\left(\langle 0| + \langle 1|\right)|1\rangle = \frac{1}{\sqrt{2}}$$

The result of the measurement operator is

$$M_0 |\psi\rangle = |+\rangle \left(\alpha \langle + | 0 \rangle + \beta \langle + | 1 \rangle \right) = \left(\frac{\alpha + \beta}{\sqrt{2}} \right) |+\rangle$$

The probability is then

$$p_0 = \left(\langle \psi | M_0^\dagger \right) \left(M_0 |\psi\rangle \right) = \left(\frac{\alpha^* + \beta^*}{\sqrt{2}} \right) \left(\frac{\alpha + \beta}{\sqrt{2}} \right) \langle + | + \rangle = \frac{|\alpha + \beta|^2}{2}$$

Same as other method

Measured state

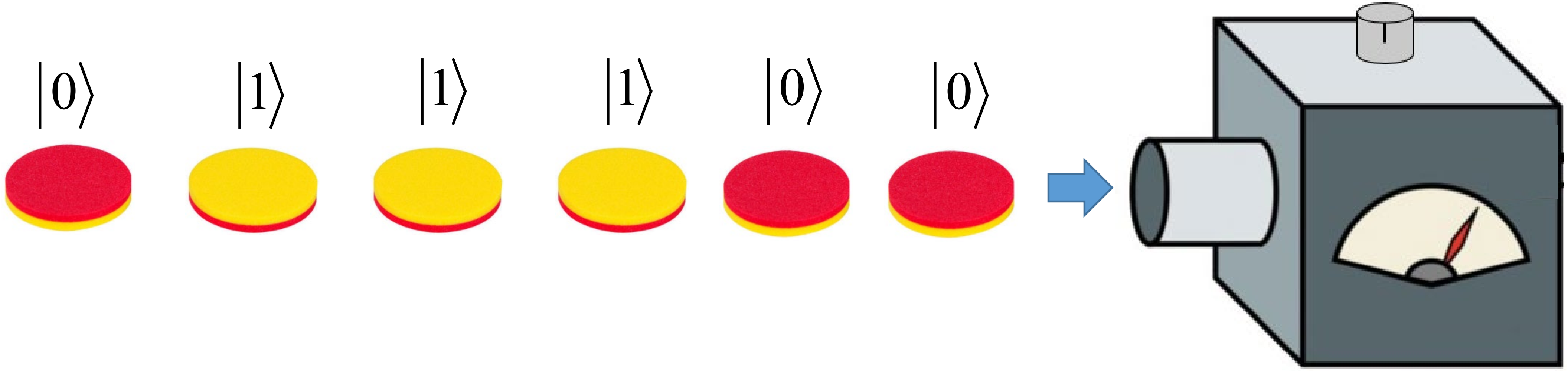
$$|\psi_0^{meas}\rangle = \frac{M_0 |\psi\rangle}{\sqrt{p_0}} = \frac{\left(\frac{\alpha + \beta}{\sqrt{2}} \right) |+\rangle}{\left| \frac{\alpha + \beta}{\sqrt{2}} \right|} \propto |+\rangle$$

The final state is always just the state that you are measuring, up to some phase (unless it is zero).

Distinguishing quantum states: classical case

Say we wish to distinguish between two types of counters, red and yellow, which we call states $|0\rangle, |1\rangle$

Measurement setting to distinguish $|0\rangle, |1\rangle$

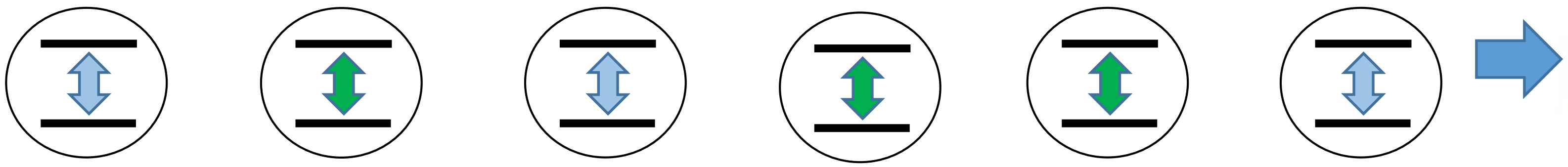


As long as our measurement device is working correctly, we can always distinguish between the two.

Distinguishing quantum states: distinguishable quantum case

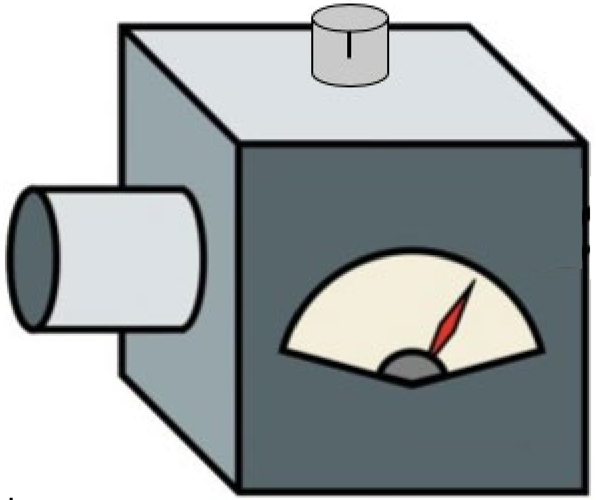
Likewise, if we wish to distinguish between any two orthogonal states, we just have to measure in that basis, and we can distinguish between them.

$|+\rangle \quad |-\rangle \quad |+\rangle \quad |-\rangle \quad |-\rangle \quad |+\rangle$



Measurement setting

$|+\rangle, |-\rangle$



This works because

$$M_0 = |+\rangle\langle+|$$

$$M_0 |+\rangle = |+\rangle\langle+|+\rangle = |+\rangle$$

$$p_0 = \left(\langle\psi| M_0^\dagger\right)\left(M_0|\psi\rangle\right) = \langle+|+\rangle = 1$$

Perfect success probability

$$M_1 = |-\rangle\langle-|$$

$$M_1 |+\rangle = |-\rangle\langle-|+\rangle = 0$$

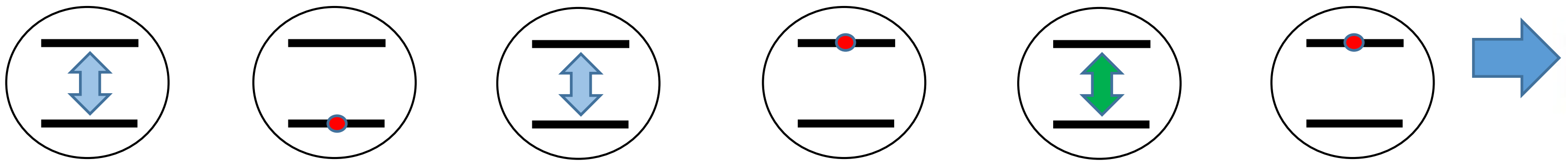
$$p_1 = 0$$

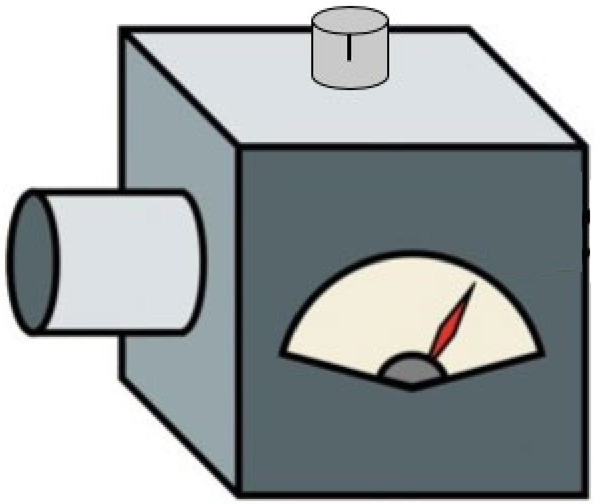
No false positives

Distinguishing quantum states: incomplete distinguishable quantum case

But if the states that we have are non-orthogonal, then there is NO setting that you can make that can distinguish the states.

$|+\rangle \quad |0\rangle \quad |+\rangle \quad |1\rangle \quad |-\rangle \quad |1\rangle$





To see this, suppose we choose the measurement setting $|0\rangle, |1\rangle$

$$M_0 = |0\rangle\langle 0| \qquad M_1 = |1\rangle\langle 1|$$

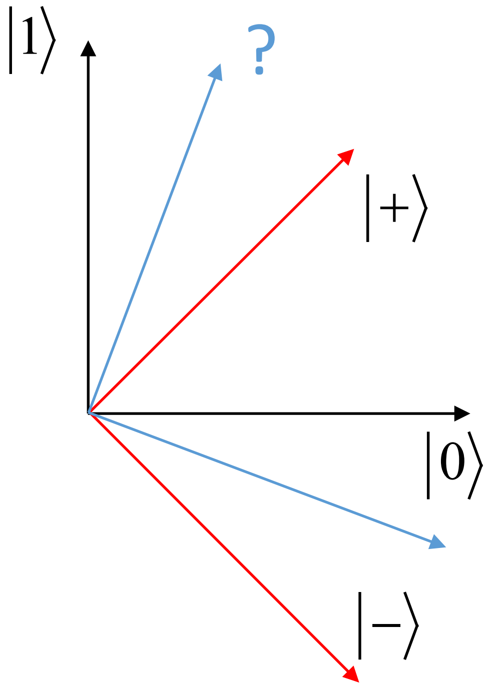
This will get the states $|0\rangle, |1\rangle$ with 100% probability, but the states $|+\rangle, |-\rangle$ would randomly be detected according to

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|\psi_0^{meas}\rangle = |0\rangle \qquad p_0 = 1/2$$

$$|\psi_1^{meas}\rangle = |1\rangle \qquad p_1 = 1/2$$

Are there any other basis choices that we could do to distinguish the states?



Since the basis states need to be orthogonal to each other, we cannot choose any basis to distinguish between the states perfectly.

Exercise:

Choosing halfway between the two bases corresponds to

$$|s0\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$$
$$|s1\rangle = \sin\theta|0\rangle - \cos\theta|1\rangle$$

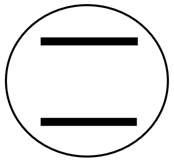
$$\theta = (45+90)/2 = 67.5$$

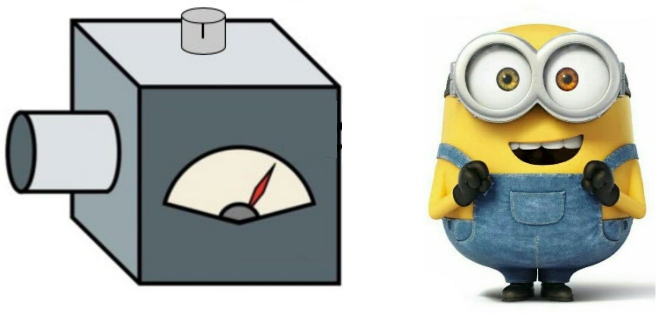
Work out the probabilities of getting $|s0\rangle, |s1\rangle$ and show that it is not possible to distinguish the states.

This is how one of the most famous quantum cryptographic protocols BB84 (Bennett Brassard 1984) works.

Quantum tomography

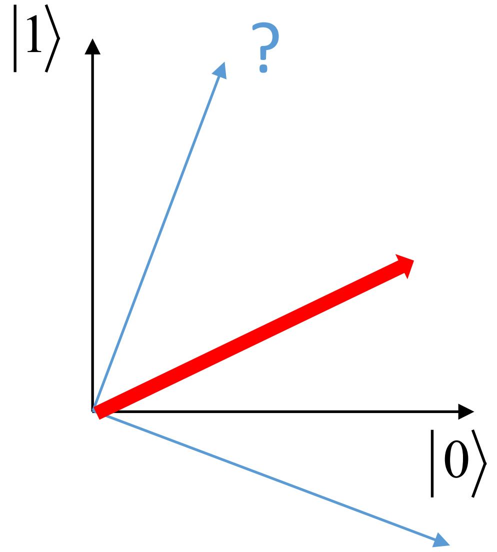
Say Alice prepares a state and sends it to Bob, but Bob doesn't know what it is.





$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

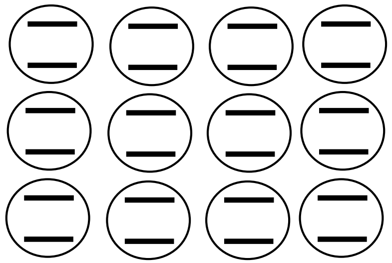
If Alice only sends one copy of the state, it is impossible for Bob to work out what it is, since he doesn't know what basis to measure in.



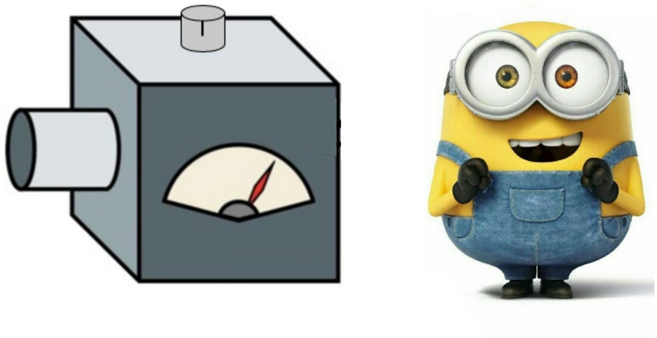
Also, even if he guessed the basis correctly and got a measurement outcome, with only one measurement outcome he could not estimate any probabilities since the sample size is too small

$$p_0 \approx \frac{N_0}{N_0+N_1} \qquad p_1 \approx \frac{N_1}{N_0+N_1}$$

But if Alice sends lots of copies of the same state, then Bob can eventually work out what the state is.



$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$



One simple way would be do the following:

Step 1: Bob first sets his measuring device to measure in the $|0\rangle, |1\rangle$ basis.

After many measurements Bob can get $p_0 = \cos^2\frac{\theta}{2}$ $\quad p_1 = \sin^2\frac{\theta}{2}$

$$\Rightarrow \quad p_0 - p_1 = \cos\theta \quad \Rightarrow \quad \theta = \cos^{-1}(p_0 - p_1)$$

Step 2: Bob now sets him measuring device to the $|+\rangle, |-\rangle$ basis

From Lecture 8 we saw that $$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \left(\frac{\alpha+\beta}{\sqrt{2}}\right)|+\rangle + \left(\frac{\alpha-\beta}{\sqrt{2}}\right)|-\rangle$$

So $$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle = \frac{1}{\sqrt{2}}\left(\cos\frac{\theta}{2} + e^{i\phi}\sin\frac{\theta}{2}\right)|+\rangle + \frac{1}{\sqrt{2}}\left(\cos\frac{\theta}{2} - e^{i\phi}\sin\frac{\theta}{2}\right)|-\rangle$$

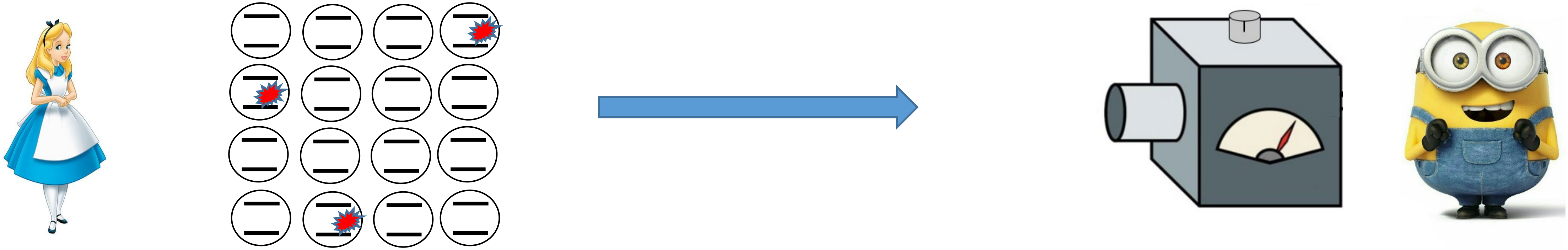
The probabilities of the two outcomes are

$$p_+ = \frac{1}{2}\left|\cos\frac{\theta}{2} + e^{i\phi}\sin\frac{\theta}{2}\right|^2 = \frac{1}{2}\left(\cos^2\frac{\theta}{2} + e^{i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} + e^{-i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right) = \frac{1}{2} + \frac{\cos\phi\sin\theta}{2}$$
$$p_- = \frac{1}{2}\left|\cos\frac{\theta}{2} - e^{i\phi}\sin\frac{\theta}{2}\right|^2 = \frac{1}{2}\left(\cos^2\frac{\theta}{2} - e^{i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} - e^{-i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right) = \frac{1}{2} - \frac{\cos\phi\sin\theta}{2}$$

Since we know what θ is from Step 1, then we can work out ϕ

$$p_+ - p_- = \cos\phi\sin\theta \qquad\qquad \phi = \cos^{-1}\left(\frac{p_+ - p_-}{\sin\theta}\right)$$

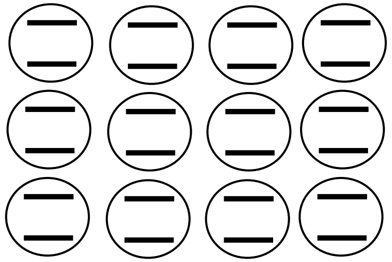
The above procedure works as long as all the qubits sent by Alice are in “pure states ”.



If some of the qubits are different to the others (due to noise etc., called a “mixed state”) then more measurements will be necessary

The name “tomography” comes from the fact that many different measurements need to be made to reproduce the state. This is a bit like CT (computerized tomography) scan to make a 3D model using X-rays from different directions.

Question

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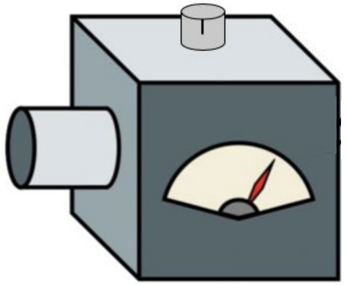


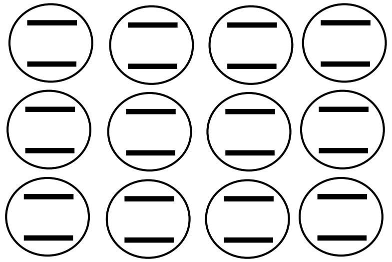

Alice prepares 2000 copies of a particular pure quantum state and sends it to Bob so he can perform tomography.

Bob first measures in the $|0\rangle, |1\rangle$ basis with 1000 copies and gets 352 counts for the state $|0\rangle$ and the remaining in $|1\rangle$.

Bob then measures in the $|+\rangle, |-\rangle$ basis with 1000 copies and gets 853 counts in the state $|+\rangle$ and the remaining in $|-\rangle$.

What was Alice's state?

Question



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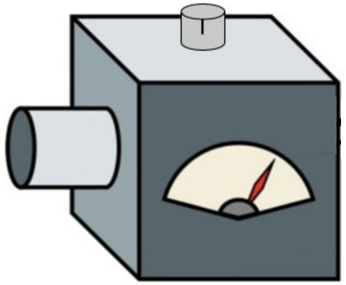

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What was Alice's state?

$$\theta = \cos^{-1}(p_0 - p_1) \approx 1.87(rad)$$

$$\phi = \cos^{-1}\left(\frac{p_+ - p_-}{\sin\theta}\right) \approx 0.74(rad)$$