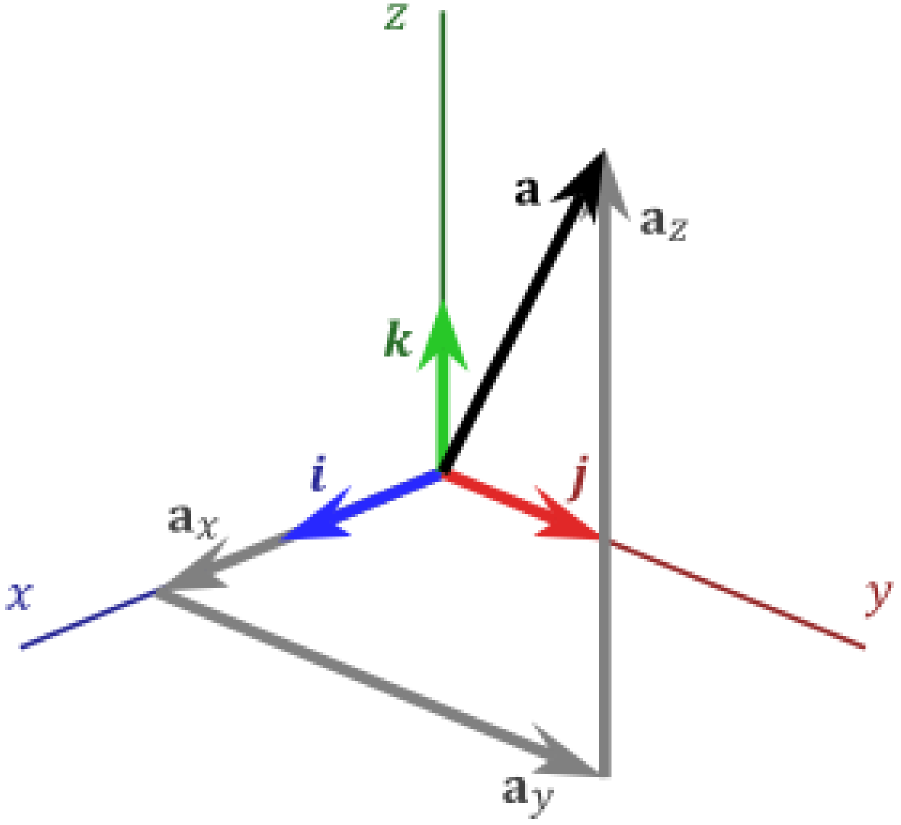
4. Linear algebra I



Linear algebra

(see Appendix of Griffiths "Introduction to quantum mechanics" (2nd edition))

Linear algebra is a generalization of vector spaces to

(1) Complex numbers
(2) Arbitrary dimensions

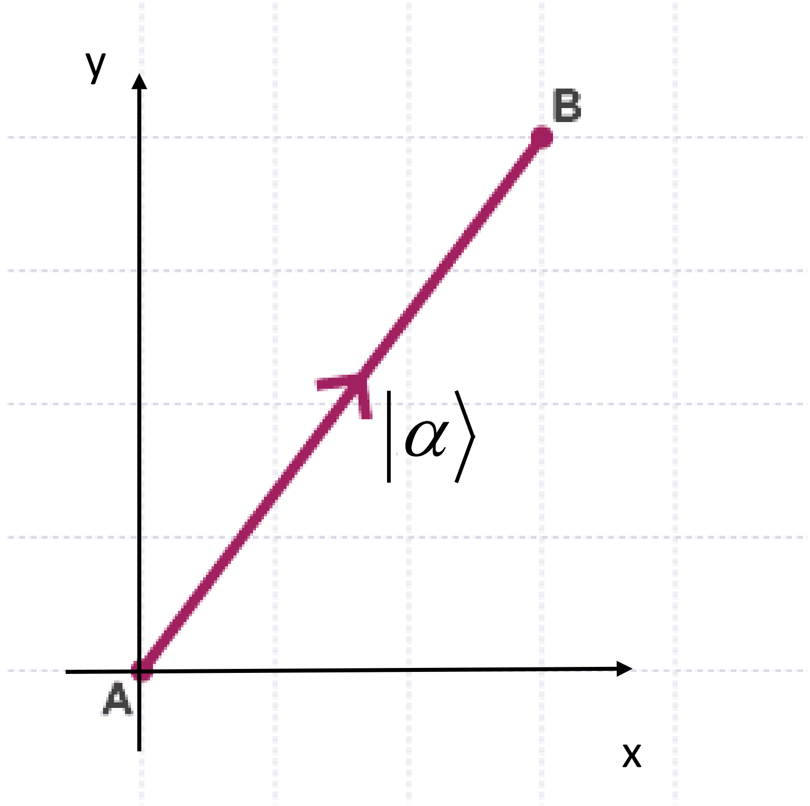
Vectors and scalars

Vectors are quantities that have a magnitude and direction.

Scalars are quantities that only have magnitude.

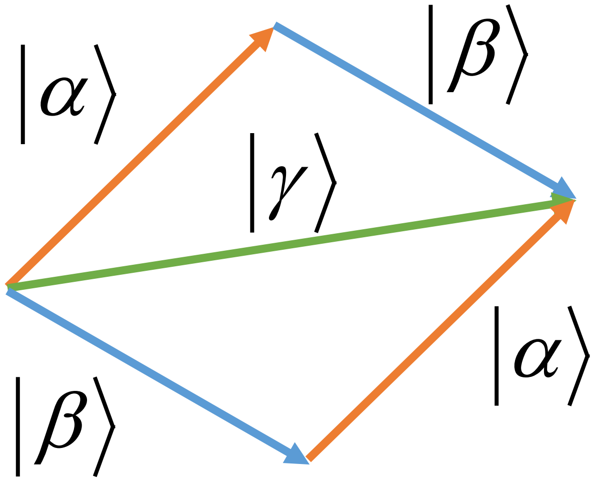
Denote a vector as $|\alpha\rangle$

(Just a different notation to $\vec{\alpha}$)



Vector addition

Define as $|\alpha\rangle + |\beta\rangle = |\gamma\rangle$



Vector addition is commutative $|\alpha\rangle + |\beta\rangle = |\beta\rangle + |\alpha\rangle$

Vector addition is associative $|\alpha\rangle + (|\beta\rangle + |\gamma\rangle) = (|\alpha\rangle + |\beta\rangle) + |\gamma\rangle$

There is a zero vector $|\alpha\rangle + |0\rangle = |\alpha\rangle$

There is an inverse vector $|\alpha\rangle + |-\alpha\rangle = |0\rangle$

Scalar multiplication

Multiplying a vector by a scalar is another vector $a|\alpha\rangle = |\gamma\rangle$

Scalar multiplication is distributive

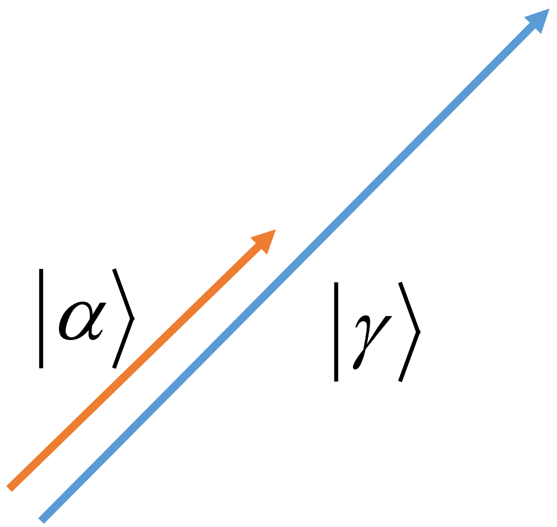
$$a(|\alpha\rangle + |\beta\rangle) = a|\alpha\rangle + a|\beta\rangle$$
$$(a+b)|\alpha\rangle = a|\alpha\rangle + b|\alpha\rangle$$

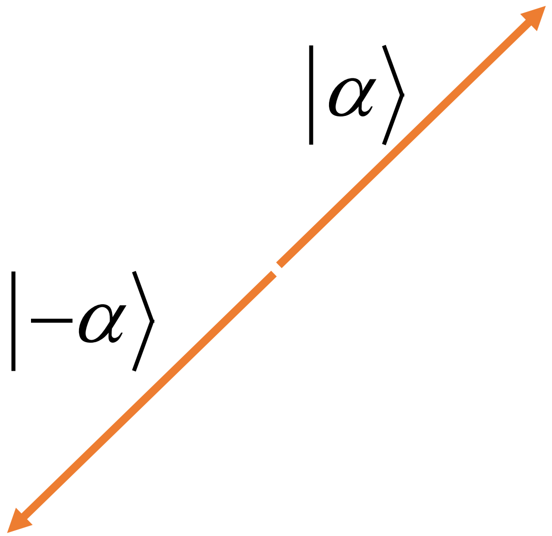
Scalar multiplication is associative $a(b|\alpha\rangle) = (ab)|\alpha\rangle$

Multiplication by zero $0|\alpha\rangle = |0\rangle$

Multiplication by one $1|\alpha\rangle = |\alpha\rangle$

Inverse of a vector $|-\alpha\rangle = (-1)|\alpha\rangle = -|\alpha\rangle$





Components

Components: Given a particular basis $|e_1\rangle,\ |e_2\rangle, \ldots, |e_n\rangle$, the n-tuple $|\alpha\rangle \leftrightarrow (a_1, a_2, \ldots, a_n)$
by writing the vector as

$$|\alpha\rangle = a_1|e_1\rangle + a_2|e_2\rangle + \cdots + a_n|e_n\rangle$$

$$|\alpha\rangle = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

Addition: $|\alpha\rangle + |\beta\rangle \leftrightarrow (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$

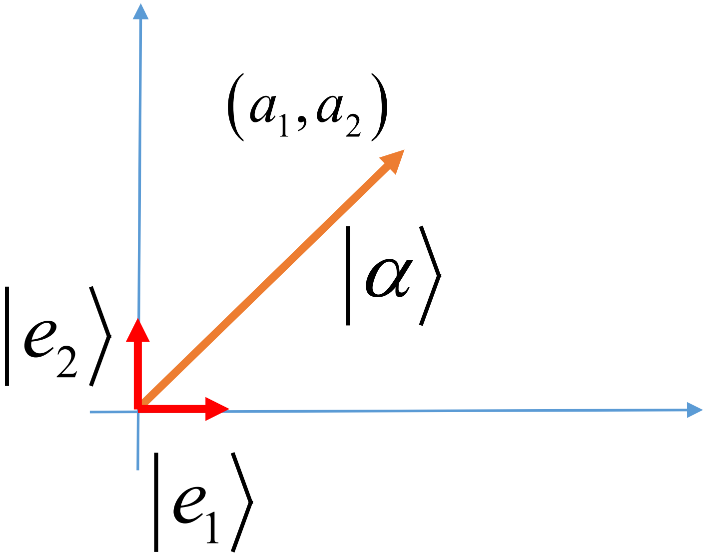
Scalar multiplication: $c|\alpha\rangle \leftrightarrow (ca_1, ca_2, \ldots, ca_n)$

Null vector: $|0\rangle \leftrightarrow (0, 0, \ldots, 0)$

Inverse vector: $|-\alpha\rangle \leftrightarrow (-a_1, -a_2, \ldots, -a_n)$

Basis: A set of vectors that you can construct any vector out of. E.g.

$$|e_1\rangle,\ |e_2\rangle, \ldots, |e_n\rangle$$



Question

1) Find $|\alpha\rangle + 2|\beta\rangle$

2) Find $|\beta\rangle - |\alpha\rangle$

$$|\alpha\rangle = 2|e_1\rangle + i|e_2\rangle$$
$$|\beta\rangle = (1-i)|e_1\rangle + |e_2\rangle$$

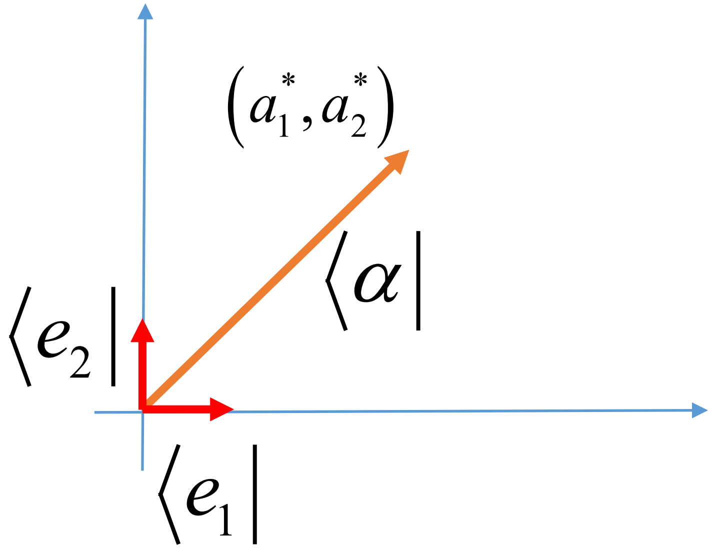
Conjugate vector

Define the conjugate vector as

$$\langle \alpha | = a_1^* \langle e_1 | + a_2^* \langle e_2 | + \ldots + a_n^* \langle e_n |$$

These can be represented by row vectors, with elements that are complex conjugates of the original vector

$$\langle \alpha | = \begin{pmatrix} a_1^* & \cdots & a_n^* \end{pmatrix}$$

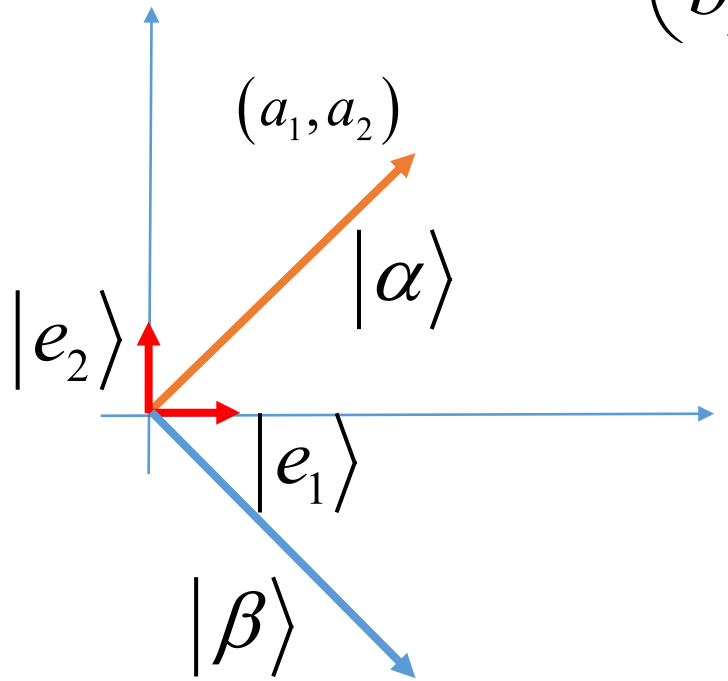


Inner products

Generalization of the dot product, which takes two vectors and gives a scalar

Component form definition

$$\langle\alpha|\beta\rangle = a_1^* b_1 + a_2^* b_2 + \cdots + a_n^* b_n = \begin{pmatrix} a_1^* & \cdots & a_n^* \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$



If the vectors are orthogonal, then the inner product is zero $\langle\alpha|\beta\rangle = 0$

Magnitude of inner product tells you how similar the vectors are. "Fidelity" $F = |\langle\alpha|\beta\rangle|^2$

Norm of a vector

Norm: $\|\alpha\| \equiv \sqrt{\langle\alpha|\alpha\rangle}$

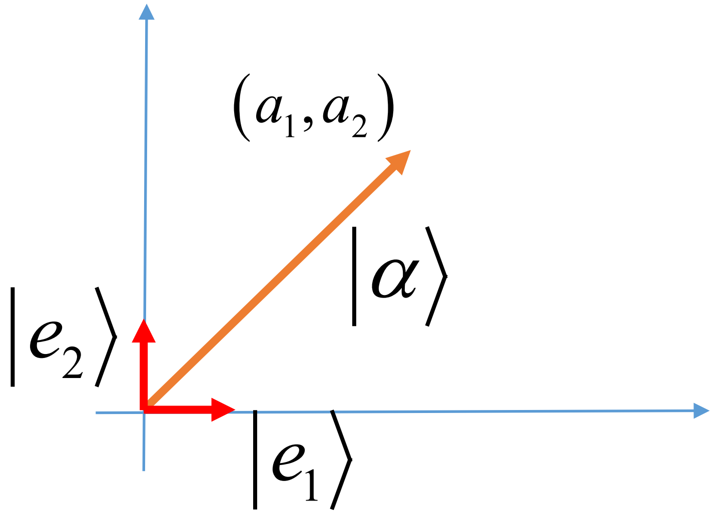
("length of a vector")

$$\langle\alpha|\alpha\rangle = |a_1|^2 + |a_2|^2 + \cdots + |a_n|^2$$

Unit vector: A vector with norm = 1 $\langle e_i | e_j \rangle = \delta_{ij}$

Components of the vector $a_i = \langle e_i|\alpha\rangle$

Normalized vector $|\alpha'\rangle = \dfrac{|\alpha\rangle}{\sqrt{\langle\alpha|\alpha\rangle}}$ $\langle\alpha'|\alpha'\rangle = 1$



Question

1) Find $\langle \alpha | \beta \rangle$

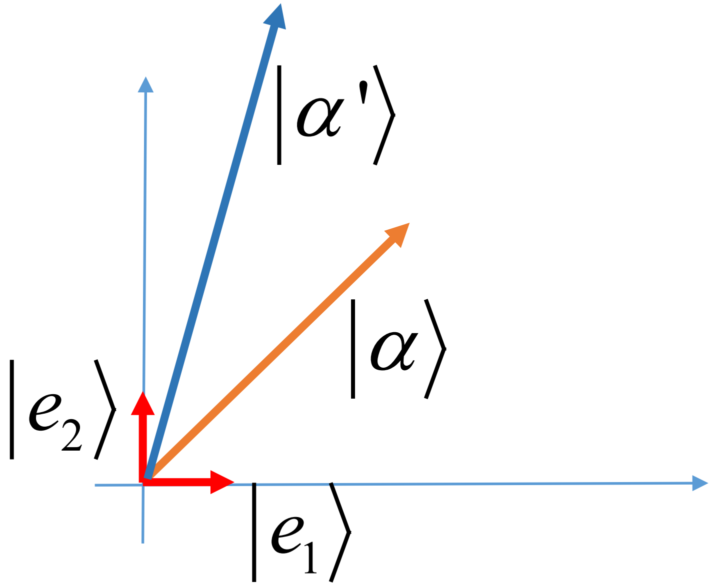
2) Normalize $|\alpha\rangle$

$$|\alpha\rangle = 2|e_1\rangle + i|e_2\rangle$$
$$|\beta\rangle = (1-i)|e_1\rangle + |e_2\rangle$$

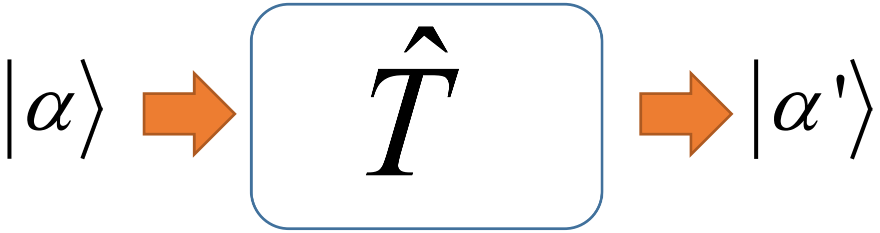
Operators

Linear transformation: Takes a vector and transforms it to another vector $|\alpha\rangle \rightarrow |\alpha'\rangle = \hat{T}|\alpha\rangle$

"Linear" means $\hat{T}(a|\alpha\rangle + b|\beta\rangle) = a(\hat{T}|\alpha\rangle) + b(\hat{T}|\beta\rangle)$



Operators are some kind of "machines" that turn one vector into another



Operators and matrices

Linear operators can always be represented by matrices $|\alpha'\rangle = \hat{T}|\alpha\rangle$

$$\mathbf{T} = \begin{pmatrix} T_{11} & T_{12} & \dots & T_{1n} \\ T_{21} & T_{22} & \dots & T_{2n} \\ \vdots & \vdots & & \vdots \\ T_{n1} & T_{n2} & \dots & T_{nn} \end{pmatrix} \qquad \mathbf{a} \equiv \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \qquad \mathbf{a}' = \mathbf{Ta} \qquad a_i' = \sum_{j=1}^{n} T_{ij} a_j .$$

The elements of the matrices are $T_{ij} = \langle e_i | \hat{T} | e_j \rangle$

The operator itself can be written $\hat{T} = \sum_{ij} T_{ij} \left| e_i \right\rangle \left\langle e_j \right|$

Outer product

We can also take a vector and conjugate vector and make an operator:

$$S = |\alpha\rangle\langle\beta| \quad = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \begin{pmatrix} b_1^* & \cdots & b_n^* \end{pmatrix} = \begin{pmatrix} a_1 b_1^* & & a_1 b_n^* \\ & & \\ a_n b_1^* & & a_n b_n^* \end{pmatrix}$$

A simple example is the outer product of two unit vectors

$$|e_1\rangle\langle e_3| = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This explains why we can write the operator T as

$$\hat{T} = \sum_{ij} T_{ij} |e_i\rangle\langle e_j|$$

$$\mathbf{T} = \begin{pmatrix} T_{11} & T_{12} & \dots & T_{1n} \\ T_{21} & T_{22} & \dots & T_{2n} \\ \vdots & \vdots & & \vdots \\ T_{n1} & T_{n2} & \dots & T_{nn} \end{pmatrix}$$

Question

1) Find $Z|\alpha\rangle$

2) Use the above result to find $\langle\beta|Z|\alpha\rangle$

$$|\alpha\rangle = \frac{|e_1\rangle + |e_2\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$|\beta\rangle = \frac{|e_1\rangle - |e_2\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$