12. Composite systems

$$|\Psi\rangle$$

Tensor product

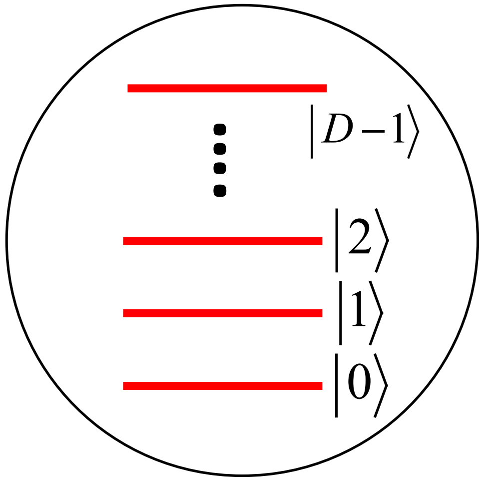
Up to now, we have talked about quantum systems, described by D levels.

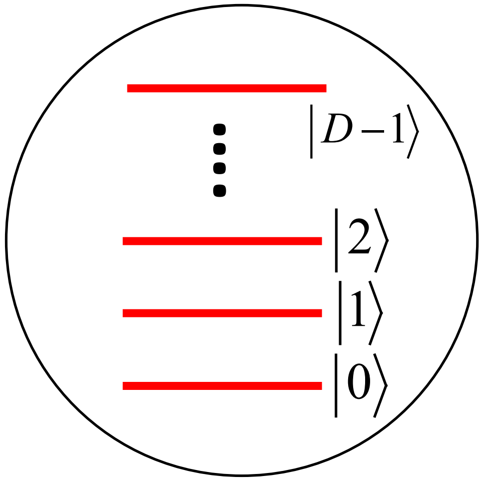
To recap: a quantum state can be described by any superposition

$$|\psi\rangle = \sum_{n=0}^{D-1} a_n |n\rangle \qquad\qquad \sum_{n=0}^{D-1} |a_n|^2 = 1$$

What if there are more than one of these systems?

Alice

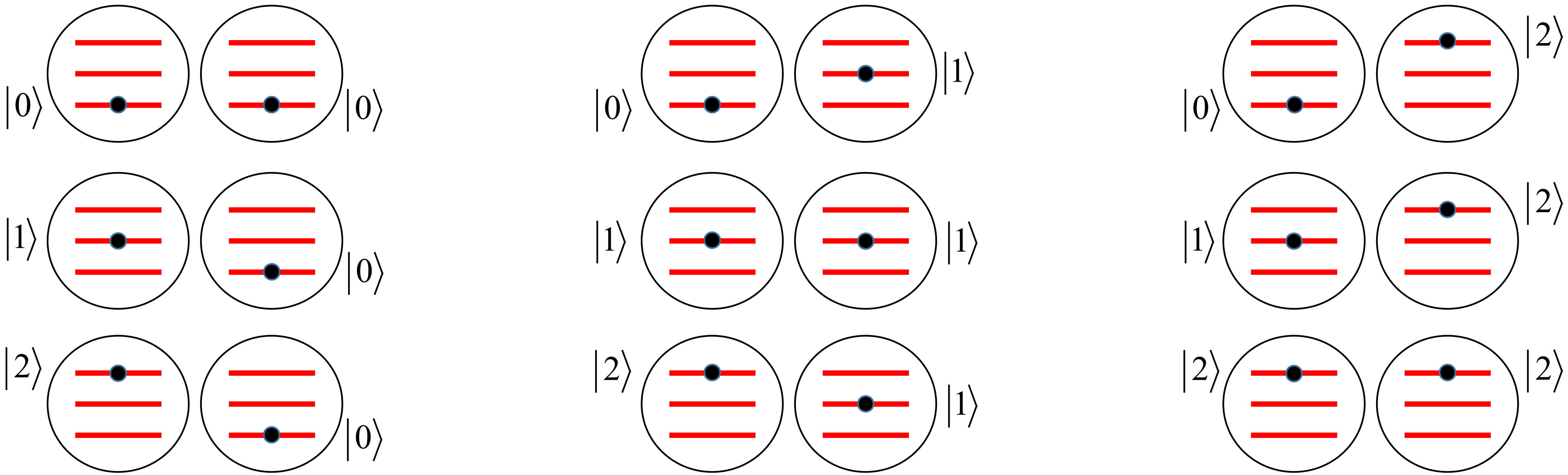




Bob

The same principle applies as before, where each distinct state can be in a superposition.

For example, for two D=3 systems:



There are $D^2 = 3 \times 3 = 9$ distinct states. Each of these can be distinguished from each other (e.g. by a measurement).

Mathematically, each combination is formed by a "tensor product" of the states. $|n_A\rangle \otimes |n_B\rangle$

This is often abbreviated as $|n_A\rangle |n_B\rangle$ or $|n_A n_B\rangle$

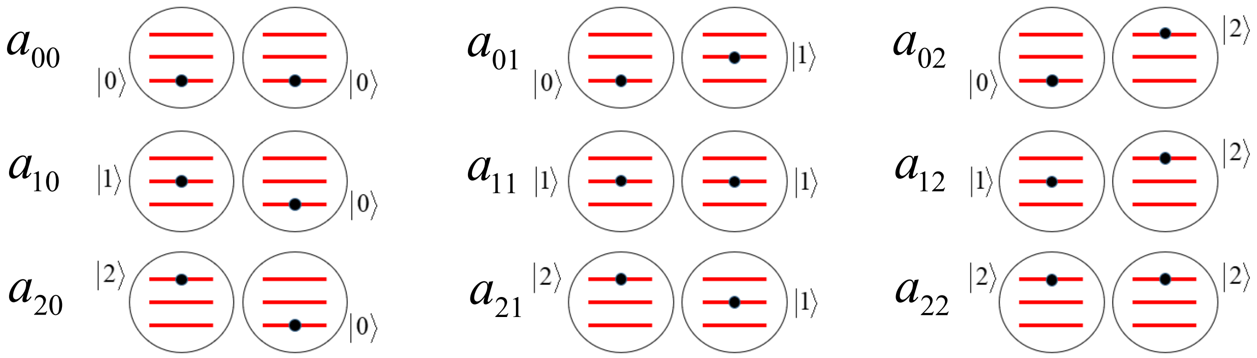
The ordering is fixed and should not be changed: Alice has first state, Bob second.

Quantum state of two composite systems

A general quantum state can then by made by a superposition of the tensor product states.

$$|\Psi\rangle = \sum_{n_A=0}^{D-1}\sum_{n_B=0}^{D-1} a_{n_A n_B} |n_A\rangle \otimes |n_B\rangle$$

The coefficients are for each combination of states:



For the state to be normalized $\langle \Psi | \Psi \rangle = 1$

$$\sum_{n_A=0}^{D-1}\sum_{n_B=0}^{D-1} |a_{n_A n_B}|^2 = 1$$

Again of this is abbreviated as

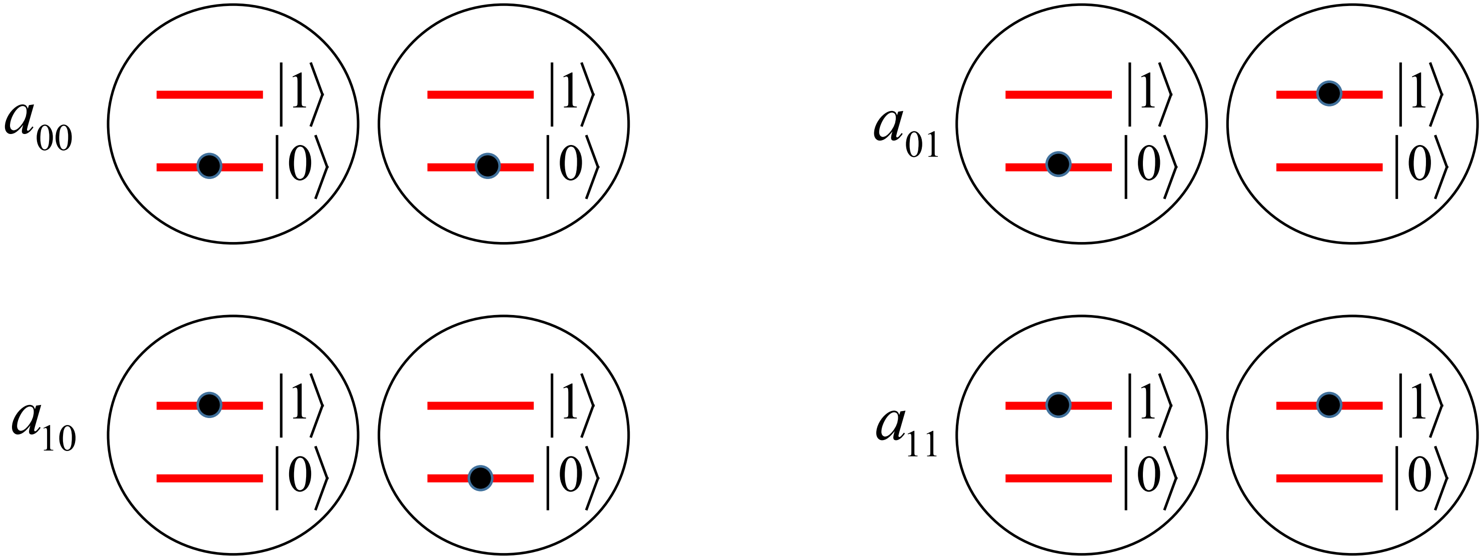
$$|\psi\rangle = \sum_{n_A}\sum_{n_B} a_{n_A n_B} |n_A\rangle |n_B\rangle$$

or

$$|\psi\rangle = \sum_{n_A}\sum_{n_B} a_{n_A n_B} |n_A n_B\rangle$$

Example: two qubits

For two qubits, there are $D^2 = 2\times 2 = 4$ states



$$|\Psi\rangle = \sum_{n_A=0}^{D-1}\sum_{n_B=0}^{D-1} a_{n_A n_B} |n_A\rangle|n_B\rangle = a_{00}|0\rangle|0\rangle + a_{01}|0\rangle|0\rangle + a_{10}|0\rangle|0\rangle + a_{11}|0\rangle|0\rangle$$

So that the state is normalized, we also need $$\sum_{n_A=0}^{D-1}\sum_{n_B=0}^{D-1} |a_{n_A n_B}|^2 = |a_{00}|^2 + |a_{01}|^2 + |a_{10}|^2 + |a_{11}|^2 = 1$$

Special case: Independent states

Say there are two qubits on opposite ends of the Earth, and they have never interacted. Can we form a composite system between them?

Yes. Which system to include in your composite system is a matter of your choice.

In this case the state would be written

$$|\Psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$$

where

$$|\psi_A\rangle = \sum_{n_A=0}^{D-1} a_{n_A} |n_A\rangle \qquad |\psi_B\rangle = \sum_{n_B=0}^{D-1} b_{n_B} |n_B\rangle$$

Then

$$|\Psi\rangle = \left(\sum_{n_A=0}^{D-1} a_{n_A} |n_A\rangle \right) \otimes \left(\sum_{n_B=0}^{D-1} b_{n_B} |n_B\rangle \right) = \sum_{n_A=0}^{D-1} \sum_{n_B=0}^{D-1} \; a_{n_A} b_{n_B} |n_A\rangle |n_B\rangle$$

$|\psi_A\rangle$ $|\psi_B\rangle$

e.g. for qubits

$$= \left(a_0 |0\rangle + a_1 |1\rangle \right) \otimes \left(b_0 |0\rangle + b_1 |1\rangle \right)$$

Composite system measurements

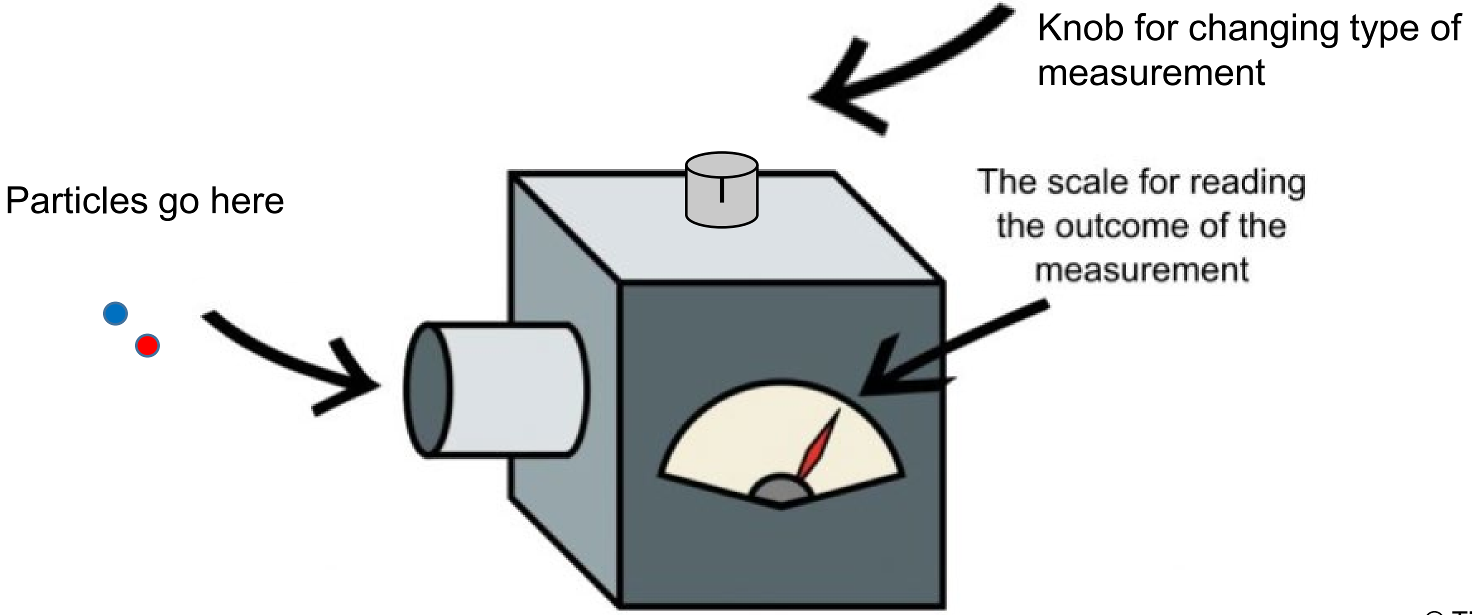
Firstly, in principle a measurements can be made in any basis on the composite system.

e.g. for 2 qubits, there are 4 states in total, so there are 4 measurement outcomes $|s0\rangle, |s1\rangle, |s2\rangle, |s3\rangle$

$$|\Psi\rangle = \sum_{n=0}^{3} a_n |sn\rangle$$

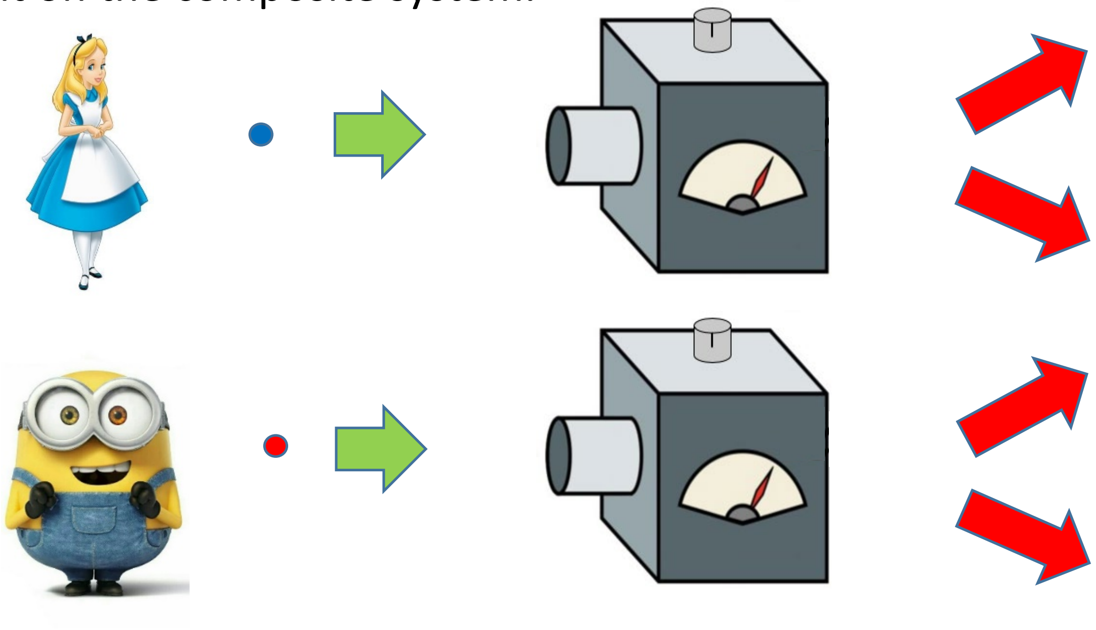
The probabilities are as usual

$$p_n = |a_n|^2$$



Local composite system measurements

But what is more common is that each particle is measured separately, and this constitutes the measurement on the composite system.



$|s0\rangle_A$ Outcome 0 for Alice's particle

$|s1\rangle_A$ Outcome 1 for Alice's particle

$|s0\rangle_B$ Outcome 0 for Bob's particle

$|s1\rangle_B$ Outcome 0 for Bob's particle

In this case the basis that is measured is

$$|s0\rangle|s0\rangle, |s0\rangle|s1\rangle, |s1\rangle|s0\rangle, |s1\rangle|s1\rangle$$

$$|\Psi\rangle = a_{00}|s0\rangle|s0\rangle + a_{01}|s0\rangle|s1\rangle + a_{10}|s1\rangle|s0\rangle + a_{11}|s1\rangle|s1\rangle$$

$$p_{00} = |a_{00}|^2 \quad p_{01} = |a_{01}|^2 \quad p_{10} = |a_{10}|^2 \quad p_{11} = |a_{11}|^2$$

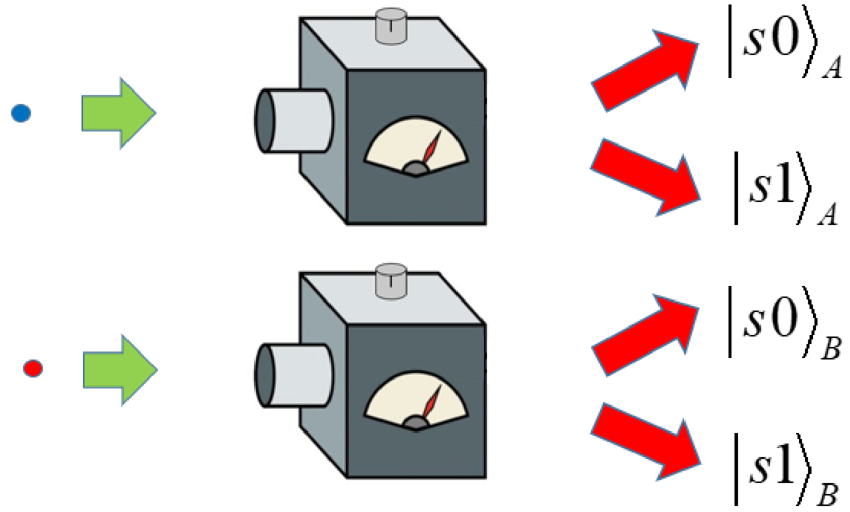
Measurement operators for composite systems

We can define the measurement operators for Alice and Bob separately as M_n^A, M_m^B

Then the measurement on the composite system is $M_n^A \otimes M_m^B$

For the qubit example before, the four measurement projectors are

$$M_n^A \otimes M_m^B = \begin{cases} |s0,s0\rangle\langle s0,s0| \\ |s0,s1\rangle\langle s0,s1| \\ |s1,s0\rangle\langle s1,s0| \\ |s1,s1\rangle\langle s1,s1| \end{cases}$$

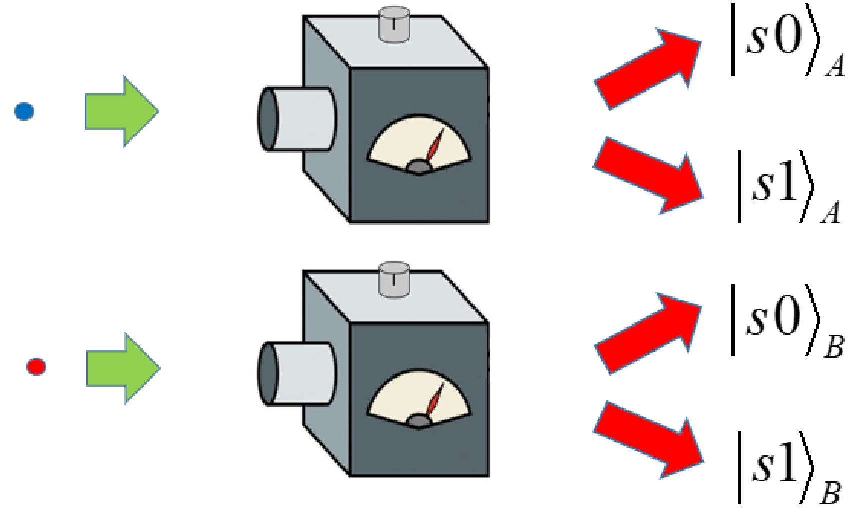


Measurement of an independent state

Now we can see why for independent states including another state is a matter of choice.

Suppose we have an independent state

$$|\Psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle = \left(a_0|0\rangle + a_1|1\rangle\right) \otimes \left(b_0|0\rangle + b_1|1\rangle\right)$$



Then the probability of getting outcome 0 for the particle 1 is

$$p_0 = p_{00} + p_{01}$$

$$|\Psi\rangle = a_0 b_0 |0\rangle|0\rangle + a_0 b_1 |0\rangle|1\rangle + a_1 b_0 |1\rangle|0\rangle + a_1 b_1 |1\rangle|1\rangle$$

$$p_{00} = |a_0 b_0|^2 \qquad p_{01} = |a_0 b_1|^2 \qquad p_0 = |a_0|^2\left(|b_0|^2 + |b_1|^2\right) = |a_0|^2$$

$\Rightarrow$ Same result as if $|\psi_B\rangle$ was not there

Measurement operators on independent states

Another way to see that Alice's measurements on an independent state, Bob's state doesn't matter

$$M_n \otimes I$$ (and $I \otimes M_n$ for Bob)

Then a measurement on the first system gives the new state

$$M_n \otimes I \left| \Psi \right\rangle = \left(M_n \left| \psi_A \right\rangle \right) \otimes \left(I \ \left| \psi_B \right\rangle \right) = \left(M_n \left| \psi_A \right\rangle \right) \otimes \left| \psi_B \right\rangle$$

The probability is

$$p_n = \left(\left\langle \Psi \right| M_n^\dagger \otimes I^\dagger \right) \left(M_n \otimes I \left| \Psi \right\rangle \right) = \left\langle \psi_A \right| M_n^\dagger M_n \left| \psi_A \right\rangle \left\langle \psi_B | \psi_B \right\rangle = \left\langle \psi_A \right| M_n^\dagger M_n \left| \psi_A \right\rangle$$

The new state is

$$\left| \Psi' \right\rangle = \frac{M_n \otimes I \left| \Psi \right\rangle}{\sqrt{p_n}} = \left(\frac{M_n \left| \psi_A \right\rangle}{\sqrt{p_n}} \right) \otimes \left| \psi_B \right\rangle$$

The second state just "goes along for the ride" and nothing happens to it.

Observables for composite systems

Observables for composite systems are in general defined according to the composite system basis

General state for composite system

$$|\psi\rangle = \sum_{n_A} \sum_{n_B} a_{n_A n_B} |n_A n_B\rangle$$

General observable on composite system

$$R = \sum_{m_A} \sum_{m_B} r_{m_A m_B} |m_A m_B\rangle \langle m_A m_B|$$

Expectation value

$$\langle R \rangle = \langle \psi | R | \psi \rangle = \sum_{m_A} \sum_{m_B} \left| b_{m_A m_B} \right|^2 r_{m_A m_B}$$

$$b_{m_A m_B} = \sum_{n_A, n_B} a_{n_A n_B} \langle m_A m_B | n_A n_B \rangle$$

Independent observables

In the same way that we can have independent states, we can have independent observables.

Let independent observables for Alice and Bob be

$$S_A = \sum_{m_A} s_{m_A} |m_A\rangle\langle m_A| \qquad\qquad S_B = \sum_{m_B} \tilde{s}_{m_B} |m_B\rangle\langle m_B|$$

Then we can make a joint observable between Alice and Bob be

$$S = S_A \otimes S_B = \sum_{m_A}\sum_{m_B} s_{m_A}\tilde{s}_{m_b} |m_A m_B\rangle\langle m_A m_B|$$

A specific case of

$$R = \sum_{m_A}\sum_{m_B} r_{m_A m_B} |m_A m_B\rangle\langle m_A m_B|$$

Qubit example 1

Evaluate $\langle Z_1 Z_2 \rangle$ for the state $|\Psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$

$$\langle Z_1 Z_2 \rangle = \langle \Psi | Z_1 Z_2 | \Psi \rangle$$

$$Z_1 Z_2 |\Psi\rangle = a_{00}|00\rangle - a_{01}|01\rangle - a_{10}|10\rangle + a_{11}|11\rangle$$

$$\langle \Psi | Z_1 Z_2 | \Psi \rangle = \left(a^*{}_{00}|00\rangle + a^*{}_{01}|01\rangle + a^*{}_{10}|10\rangle + a^*{}_{11}|11\rangle \right)\left(a_{00}|00\rangle - a_{01}|01\rangle - a_{10}|10\rangle + a_{11}|11\rangle \right)$$

$$= |a_{00}|^2 - |a_{01}|^2 - |a_{10}|^2 + |a_{11}|^2$$

Qubit example 2

Evaluate $\left\langle X_1 X_2 \right\rangle$ for the state $\left|\Psi\right\rangle = a_{00}\left|00\right\rangle + a_{01}\left|01\right\rangle + a_{10}\left|10\right\rangle + a_{11}\left|11\right\rangle$

$$\left\langle X_1 X_2 \right\rangle = \left\langle \Psi \right| X_1 X_2 \left| \Psi \right\rangle$$

$$X_1 X_2 \left|\Psi\right\rangle = a_{00}\left|11\right\rangle + a_{01}\left|10\right\rangle + a_{10} 01 + a_{11}\left|00\right\rangle$$

$$\left\langle \Psi \right| X_1 X_2 \left| \Psi \right\rangle = \left(a^*{}_{00}\left|00\right\rangle + a^*{}_{01}\left|01\right\rangle + a^*{}_{10}\left|10\right\rangle + a^*{}_{11}\left|11\right\rangle \right)\left(a_{00}\left|11\right\rangle + a_{01}\left|10\right\rangle + a_{10}\left|01\right\rangle + a_{11}\left|00\right\rangle \right)$$

$$= a_{00} a^*{}_{11} + a^*{}_{00} a_{11} + a^*{}_{01} a_{10} + a^*{}_{10} a_{01}$$

Question

Evaluate $\langle X_1 Z_2 \rangle$ For $|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|10\rangle + |00\rangle\right)$

More than two systems

What about if there are three or more systems?

The same pattern generally follows so the general state can be written

$$|\Psi\rangle = \sum_{n_A=0}^{D-1} \sum_{n_B=0}^{D-1} \sum_{n_C=0}^{D-1} a_{n_A n_B n_C} |n_A\rangle \otimes |n_B\rangle \otimes |n_C\rangle$$

For M systems, the total dimension of the composite system is D^M

An independent system would be written as

$$|\Psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \otimes |\psi_C\rangle$$

Anything that cannot be written like this would have some entanglement.