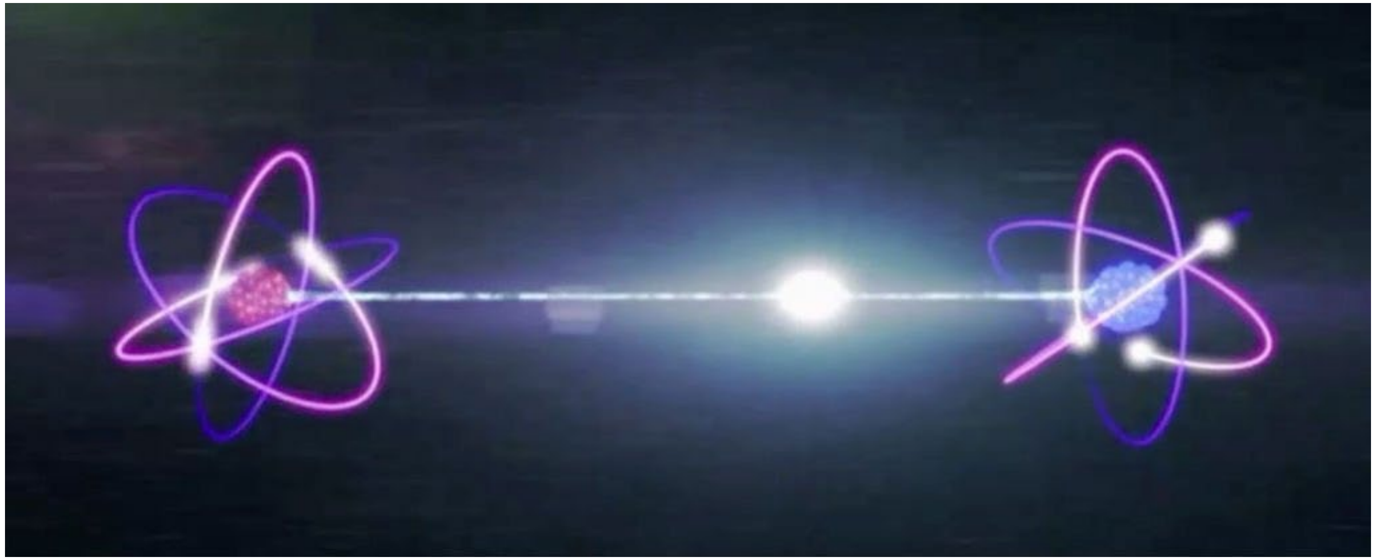
13. Entanglement



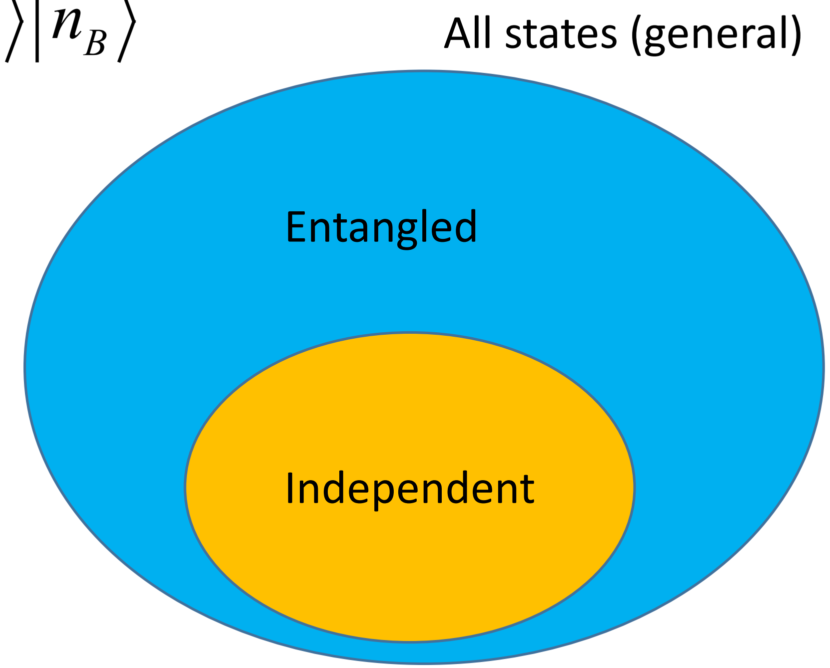
Entangled states

The most general quantum state for two composite systems is

$$\left|\Psi_{general}\right\rangle = \sum_{n_A=0}^{D-1}\sum_{n_B=0}^{D-1} a_{n_A n_B}\left|n_A\right\rangle \otimes \left|n_B\right\rangle$$

Meanwhile, for independent states, we have

$$\left|\Psi_{independent}\right\rangle = \left(\sum_{n_A=0}^{D-1} a_{n_A}\left|n_A\right\rangle\right)\left(\sum_{n_B=0}^{D-1} b_{n_B}\left|n_B\right\rangle\right) = \sum_{n_A=0}^{D-1}\sum_{n_B=0}^{D-1} a_{n_A} b_{n_B}\left|n_A\right\rangle\left|n_B\right\rangle$$



The independent states is a more restricted form of wavefunction.

The states that cannot be written in the product form $a_{n_A} b_{n_B}$ have some correlations

“Entangled” states

Example: entangled qubits

Most general two qubit state

$$\left|\Psi_{general}\right\rangle = a_{00}\left|0\right\rangle\left|0\right\rangle + a_{01}\left|0\right\rangle\left|1\right\rangle + a_{10}\left|1\right\rangle\left|0\right\rangle + a_{11}\left|1\right\rangle\left|1\right\rangle$$

Independent two qubit state

$$\left|\Psi_{independent}\right\rangle = \left(a_0\left|0\right\rangle + a_1\left|1\right\rangle\right)\otimes\left(b_0\left|0\right\rangle + b_1\left|1\right\rangle\right)$$

$$= a_0 b_0\left|0\right\rangle\left|0\right\rangle + a_0 b_1\left|0\right\rangle\left|1\right\rangle + a_1 b_0\left|1\right\rangle\left|0\right\rangle + a_1 b_1\left|1\right\rangle\left|1\right\rangle$$

There are some states that can be written in the first form but not the second.

For example,

$$\left|\Psi_{\text{entangled}}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|0\right\rangle\left|0\right\rangle + \left|1\right\rangle\left|1\right\rangle\right)$$

So

$$a_0 b_1 = 0$$
$$a_1 b_0 = 0$$

Then

$$a_0 = 0 \Rightarrow a_0 b_0 = 0 \neq \frac{1}{\sqrt{2}}$$

Bell states

For qubits actually there are 4 orthogonal states that are "maximally entangled"

$$\frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle+|1\rangle|1\rangle\right),\frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle-|1\rangle|1\rangle\right),\frac{1}{\sqrt{2}}\left(|0\rangle|1\rangle+|1\rangle|0\rangle\right),\frac{1}{\sqrt{2}}\left(|0\rangle|1\rangle-|1\rangle|0\rangle\right)$$

These are called the "Bell states" after John Bell who found that quantum states had more correlations than classical states.

Quantum correlations

Now let's consider the situation that we have this entangled state and the two qubits were on the opposite sides of the Earth.

$$|\Psi_{\text{entangled}}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + |1\rangle|1\rangle\right)$$

Say now we measure the state, and we do it in two steps, where particle 1 is measured first, then particle 2.

$$\frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + |1\rangle|1\rangle\right)$$

$|0\rangle_A$

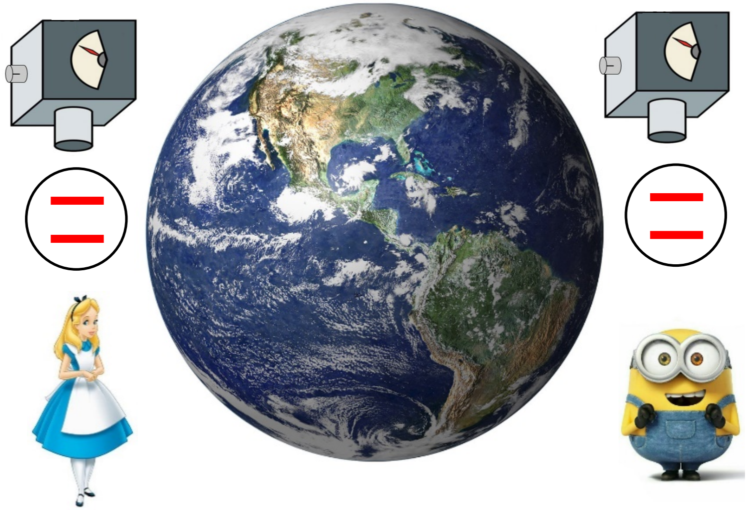
$|0\rangle_B \quad p_{00} = 1/2$

$|1\rangle_B \quad p_{01} = 0$

$|1\rangle_A$

$|0\rangle_B \quad p_{10} = 0$

$|1\rangle_B \quad p_{11} = 1/2$



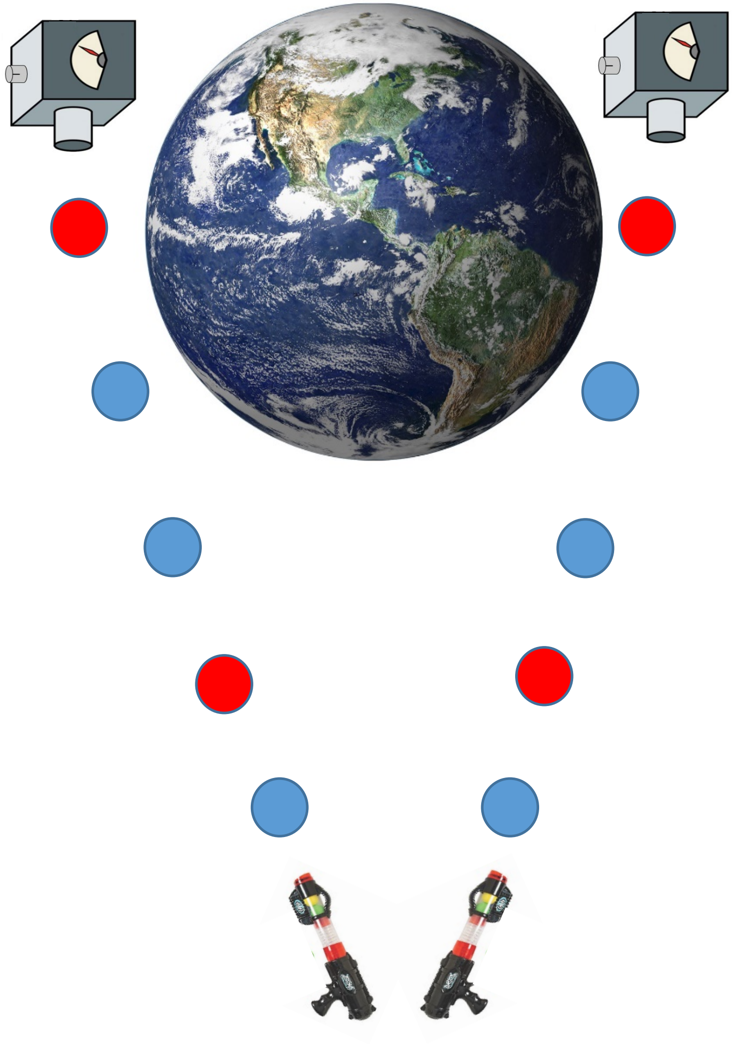
So the particles come out always in the same state even thought they are on the opposite sides of the Earth.

AMAZING!?

NO! This is not amazing.

You can do the same thing with usual probabilities.

All you need to do is to randomly prepare particles in pairs, and then send them to the opposite sides of the Earth.



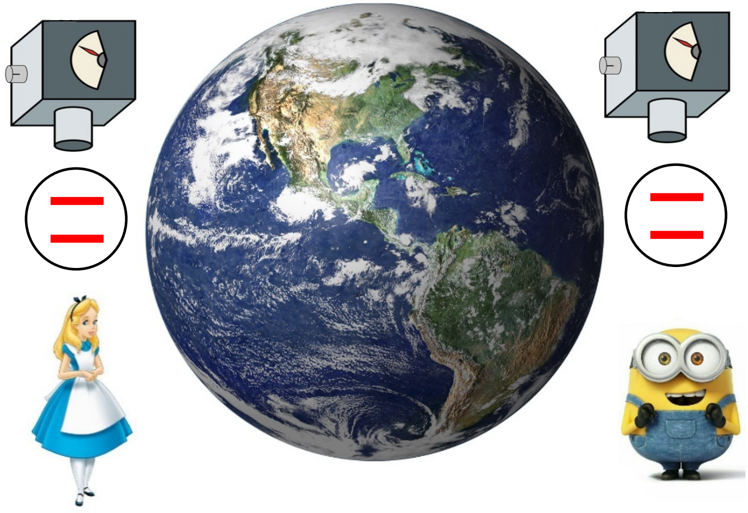
Quantum mechanics has another trick: measuring in different bases.

The key thing is to notice that the entangled state can be equally written as

$$\left|\Psi_{\text{entangled}}\right\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + |1\rangle|1\rangle\right)$$
$$= \frac{1}{2}\left[\left(|+\rangle + |-\rangle\right)\left(|+\rangle + |-\rangle\right) + \left(|+\rangle - |-\rangle\right)\left(|+\rangle - |-\rangle\right)\right]$$
$$= \frac{1}{\sqrt{2}}\left(|+\rangle|+\rangle + |-\rangle|-\rangle\right)$$

Say now we measure the state in the $|+\rangle, |-\rangle$ basis. We do it in two steps as before, where particle 1 is measured first, then particle 2.

$$\frac{1}{\sqrt{2}}\left(|+\rangle|+\rangle + |-\rangle|-\rangle\right) \rightarrow |+\rangle_A \rightarrow |+\rangle_B \quad p_{++} = 1/2$$
$$|+\rangle_A \rightarrow |-\rangle_B \quad p_{+-} = 0$$
$$\frac{1}{\sqrt{2}}\left(|+\rangle|+\rangle + |-\rangle|-\rangle\right) \rightarrow |-\rangle_A \rightarrow |+\rangle_B \quad p_{-+} = 0$$
$$|-\rangle_A \rightarrow |-\rangle_B \quad p_{--} = 1/2$$

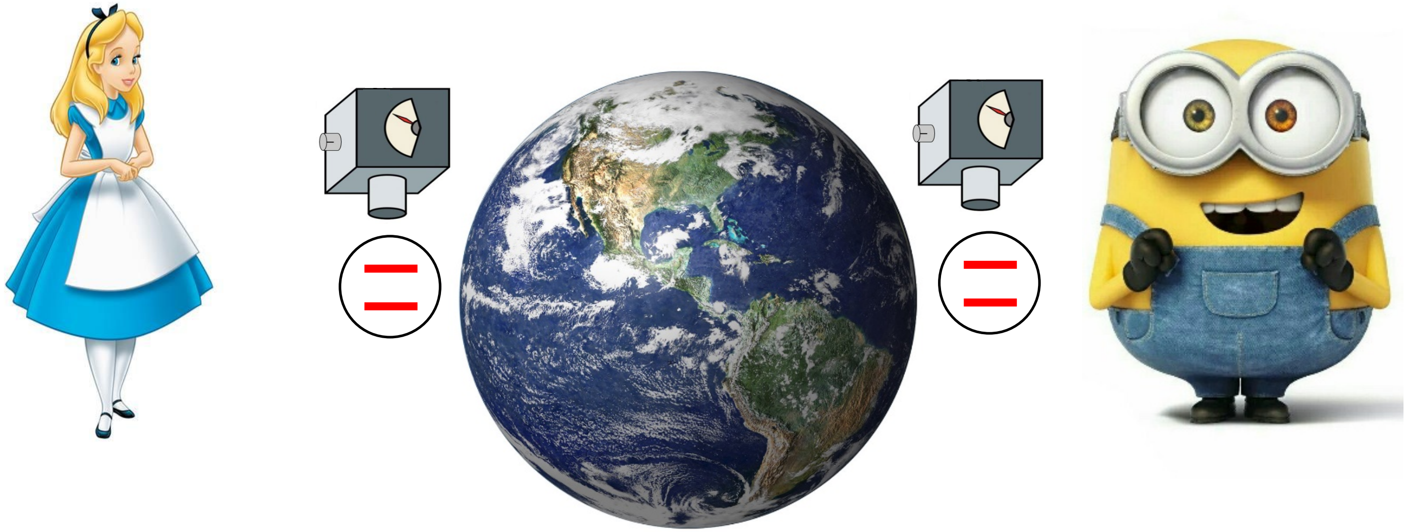


So the particles are correlated also in the $|+\rangle, |-\rangle$ basis.

AMAZING?!

This time it IS amazing.

Say Alice decides at the last minute whether she will measure in the $|0\rangle, |1\rangle$ basis or $|+\rangle, |-\rangle$ basis.



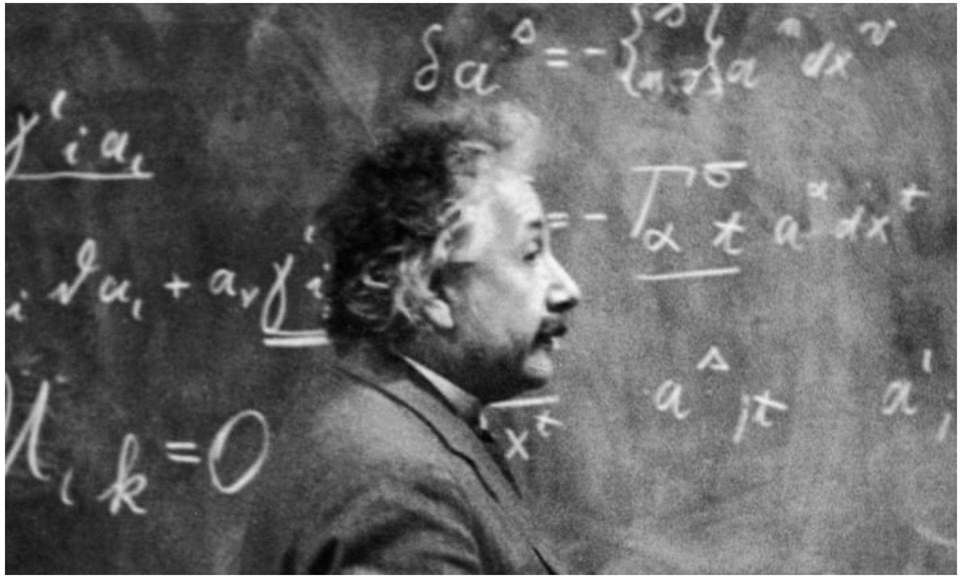
If Alice chooses $|0\rangle, |1\rangle$ basis: Bob gets $|0\rangle$ or $|1\rangle$ randomly, but perfectly correlated with Alice.

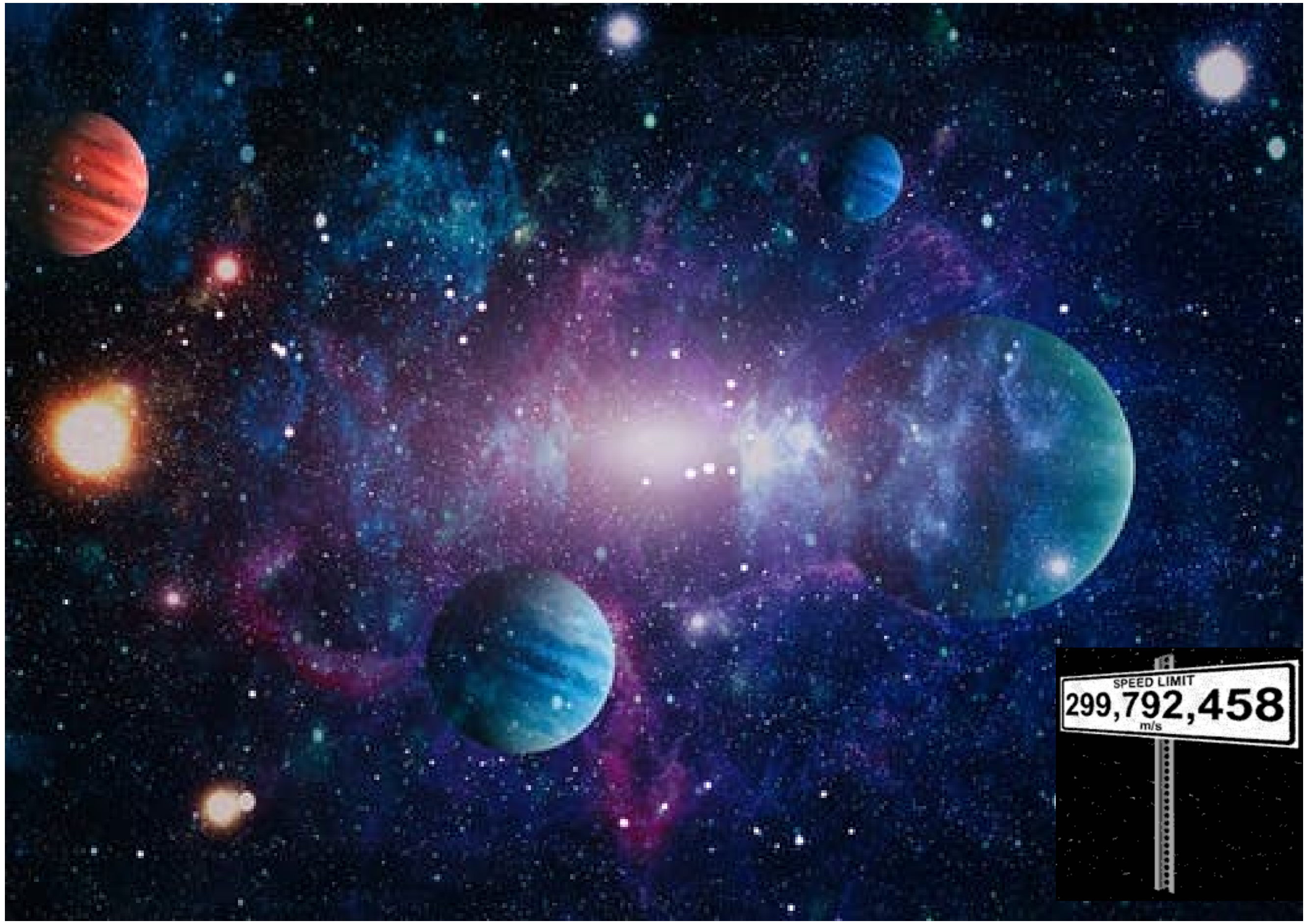
If Alice chooses $|+\rangle, |-\rangle$ basis: Bob gets $|+\rangle$ or $|-\rangle$ randomly, but perfectly correlated with Alice.

Bob gets the state INSTANTANEOSLY, faster than the speed of light.

Spooky action at a distance

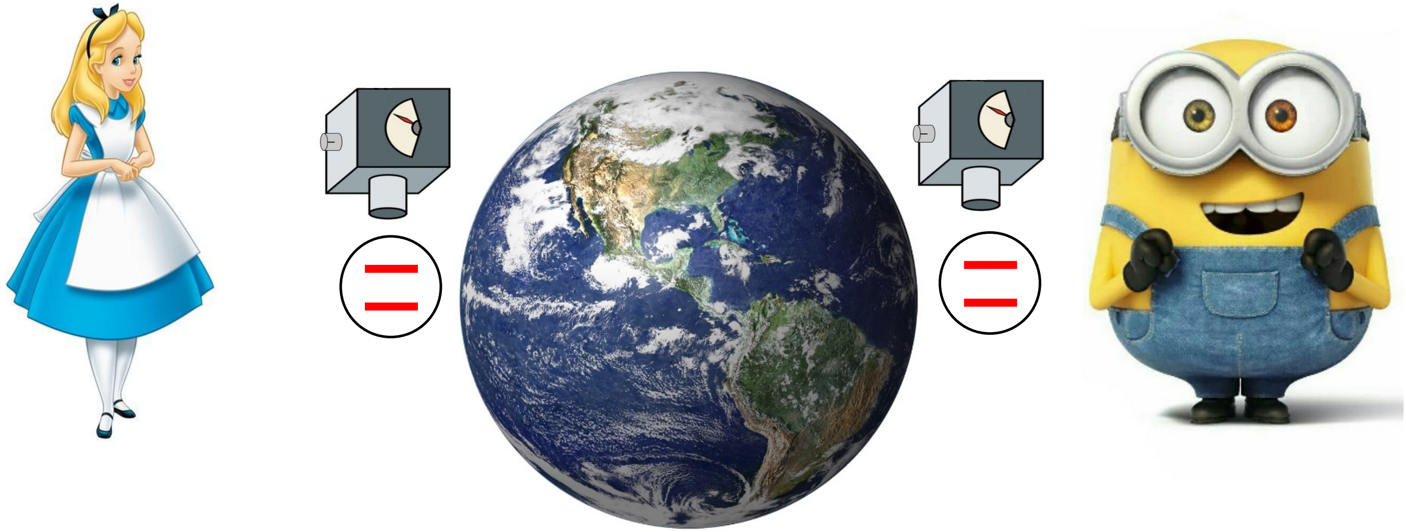
But Einstein says that nothing can travel faster than the speed of light.





How can quantum mechanics allow for the physically different states to be created at Bob’s side, affected by Alice’s choice?   Einstein: “Spooky action at a distance”

Why is it still ok?



If Alice chooses $|0\rangle, |1\rangle$ basis:

Bob gets $|0\rangle$ or $|1\rangle$ each with probability p=0.5 (correlated with Alice)

$$\langle X \rangle = \langle Y \rangle = \langle Z \rangle = 0$$

If Alice chooses $|+\rangle, |-\rangle$ basis:

Bob gets $|+\rangle$ or $|-\rangle$ each with probability p=0.5 (correlated with Alice)

$$\langle X \rangle = \langle Y \rangle = \langle Z \rangle = 0$$

On average Bob cannot actually tell which setting Alice chose. Superluminal communication is still impossible!

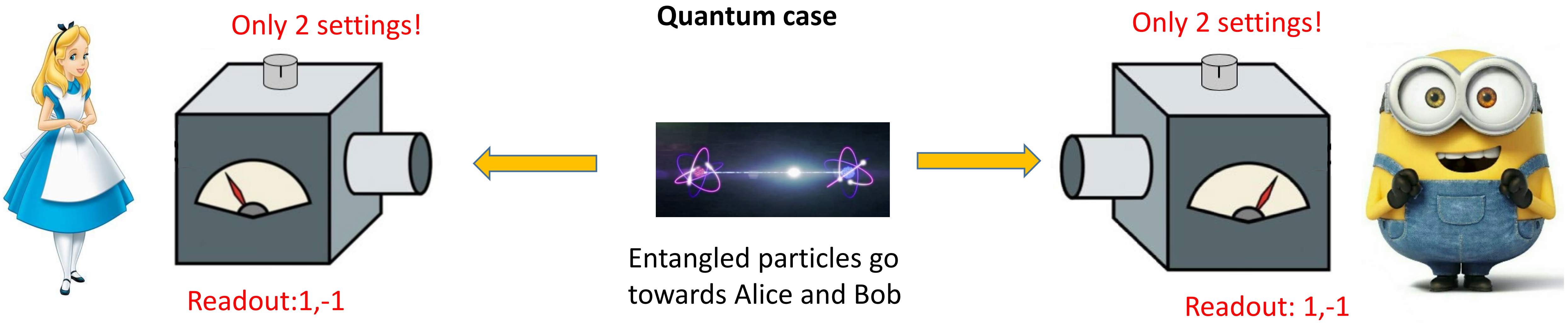
Bell’s inequality

John Bell quantified this idea that quantum states have more correlations than possible with classical physics.

This ruled out the “realist” interpretation of quantum mechanics, and has been said that it is "the most profound discovery of science.“ (Henry Stapp)

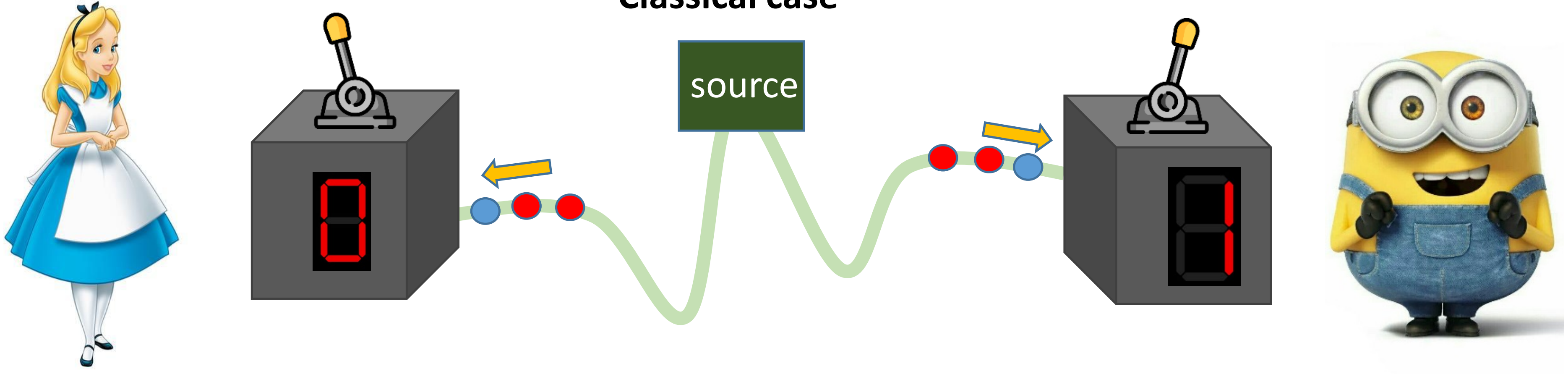
Bell then compared the following two situations.

Alice and Bob have each a quantum measuring device, and it is a machine with only two settings and two readout values (1 and -1).



We can also consider a completely classical version, to be a completely classical black box, that just has 2 settings and 2 outcomes.

Classical case



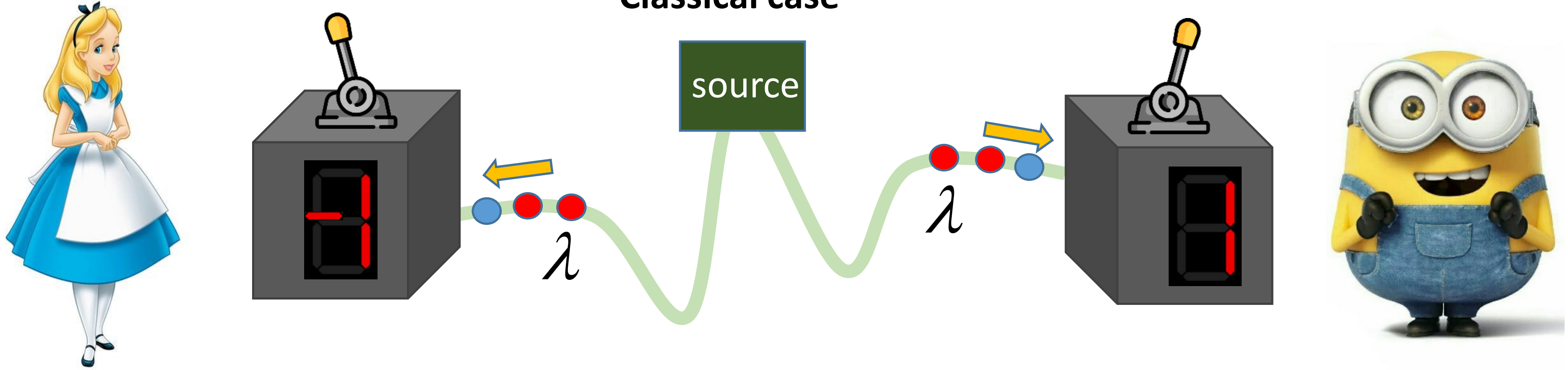
Bell then asked, what is the maximum correlation that I could possibly observe classically in this setup?

To account for the correlations, suppose there is some hidden variable λ, which is measured by both Alice and Bob's classical devices, producing some probability distribution $p(\lambda)$

e.g. the particles are red or blue in the above case. But actually λ can have an infinite number of values.

From this information, Alice and Bob has to reproduce the same statistics as the quantum measurement

Classical case



With this model, the observables for Alice and Bob are written

Setting 1:
$$\langle A \rangle = \int d\lambda\, p(\lambda) A(\lambda) \qquad \langle B \rangle = \int d\lambda\, p(\lambda) B(\lambda)$$

Setting 2:
$$\langle A' \rangle = \int d\lambda\, p(\lambda) A'(\lambda) \qquad \langle B' \rangle = \int d\lambda\, p(\lambda) B'(\lambda)$$

Here the observables only have values

$$A(\lambda), A'(\lambda), B(\lambda), B'(\lambda) = -1, 1$$

Then Bell and Clauser-Horne-Shimony-Holt (CHSH) proved that for any classical variables we must have

$$\left| \langle AB \rangle - \langle AB' \rangle + \langle A'B \rangle + \langle A'B' \rangle \right| \leq 2$$

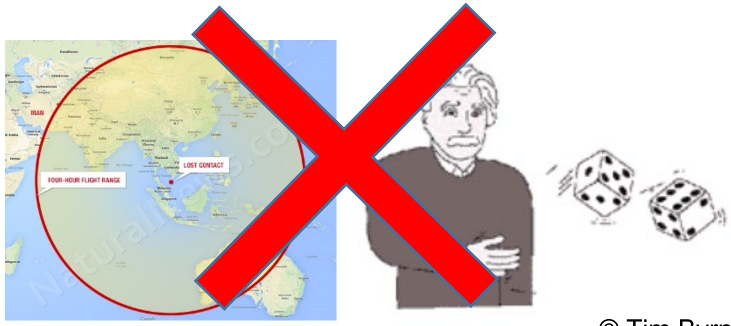
where

$$\langle AB \rangle = \int d\lambda\, p(\lambda) A(\lambda) B(\lambda)$$

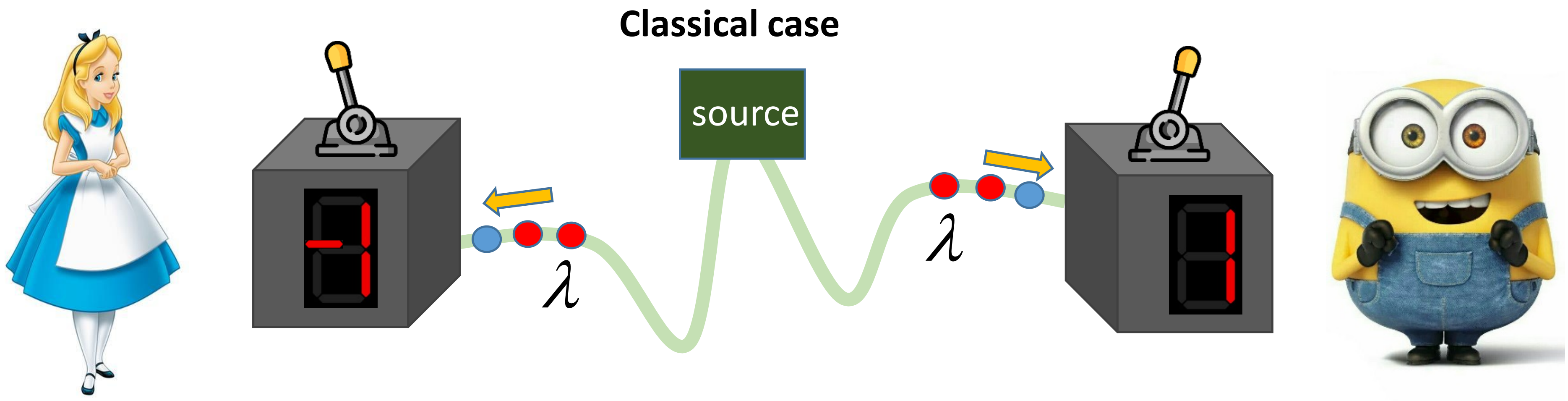
But for the quantum setup you can violate the above bound!

Quantum particles can have more correlations than classical ones!

This ruled out one of the interpretations of quantum mechanics, that it is a representation of the lack of knowledge of the system.



Bell's inequality: classical case



Let's first verify that the CHSH inequality works. Suppose we have the red/blue particles and we have the following rules

Hidden variable	probability	A	A'	B	B'
red	1/2	1	-1	1	1
blue	1/2	-1	1	-1	-1

Hidden variable	probability	A	A’	B	B’
red	1/2	1	1	1	-1
blue	1/2	-1	-1	-1	1

Quiz: Now evaluate each of the terms in the CHSH inequality

$$\left|\langle AB \rangle - \langle AB' \rangle + \langle A'B \rangle + \langle A'B' \rangle\right| = 2$$

Hidden variable	probability	A	A'	B	B'
red	1/2	1	1	1	-1
blue	1/2	-1	-1	-1	1

Quiz: Now evaluate each of the terms in the CHSH inequality

$$\langle AB \rangle = 1 \qquad \langle AB' \rangle = -1 \qquad \langle A'B \rangle = 1 \qquad \langle A'B' \rangle = -1$$

(correlated) (anticorrelated) (correlated) (anticorrelated)

$$\left| \langle AB \rangle - \langle AB' \rangle + \langle A'B \rangle + \langle A'B' \rangle \right| = 2$$

Satisfies CHSH inequality

Bell's inequality: quantum case

Now let's evaluate the same thing with the entangled state $\frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle+|1\rangle|1\rangle\right)$

Choose the observables as

$$A = Z$$ (i.e. perform $|0\rangle, |1\rangle$ basis measurement)

$$A' = X$$ (i.e. perform $|+\rangle, |-\rangle$ basis measurement)

$$B = -\frac{Z+X}{\sqrt{2}}$$

$$B' = \frac{Z-X}{\sqrt{2}}$$

What kind of a measurement is $B=-\frac{Z+X}{\sqrt{2}}, B'=\frac{Z-X}{\sqrt{2}}$?

e.g. $$B=-\frac{Z+X}{\sqrt{2}}=-\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}=|5\pi/4\rangle\langle 5\pi/4|-|\pi/4\rangle\langle \pi/4|$$

$$B'=\frac{Z-X}{\sqrt{2}}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ -1 & -1\end{pmatrix}=|7\pi/4\rangle\langle 7\pi/4|-|3\pi/4\rangle\langle 3\pi/4|$$

$$|\theta\rangle=\cos\frac{\theta}{2}|0\rangle+\sin\frac{\theta}{2}|1\rangle$$

So this is again just another basis measurement, with outcomes -1,1 again.

Quiz: Now evaluate the four correlations in the CHSH inequality

$$\left|\langle AB\rangle-\langle AB'\rangle+\langle A'B\rangle+\langle A'B'\rangle\right|=?$$

What kind of a measurement is $B = -\frac{Z+X}{\sqrt{2}}, B' = \frac{Z-X}{\sqrt{2}}$?

e.g. $$B = -\frac{Z+X}{\sqrt{2}} = -\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = |5\pi/4\rangle\langle 5\pi/4| - |\pi/4\rangle\langle \pi/4|$$

$$B' = \frac{Z-X}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} = |7\pi/4\rangle\langle 7\pi/4| - |3\pi/4\rangle\langle 3\pi/4|$$

$$|\theta\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle$$

So this is again just another basis measurement, with outcomes -1,1 again.

Quiz: Now evaluate the four correlations in the CHSH inequality

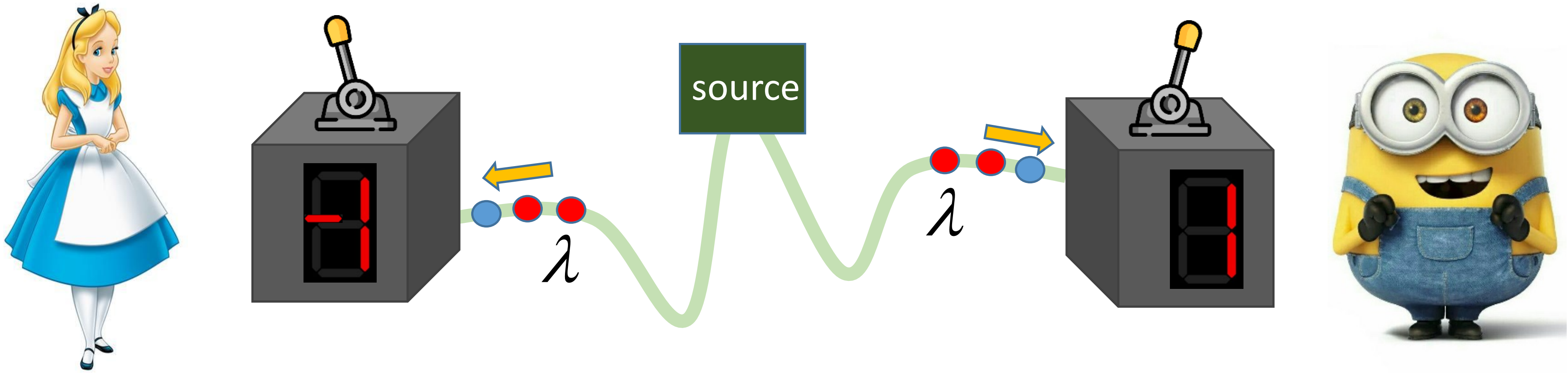
$$\langle AB \rangle = -\frac{1}{\sqrt{2}} \quad \langle AB' \rangle = \frac{1}{\sqrt{2}} \quad \langle A'B \rangle = -\frac{1}{\sqrt{2}} \quad \langle A'B' \rangle = -\frac{1}{\sqrt{2}}$$

$$\left| \langle AB \rangle - \langle AB' \rangle + \langle A'B \rangle + \langle A'B' \rangle \right| = 2\sqrt{2}$$

Violates CHSH inequality!

Local realism

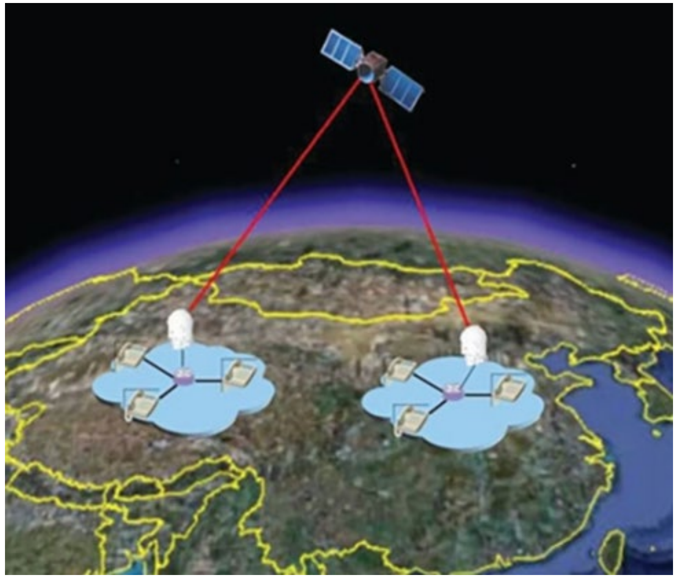
When deriving the CHSH inequality, one of the important assumptions is that only information from the local detector can be used

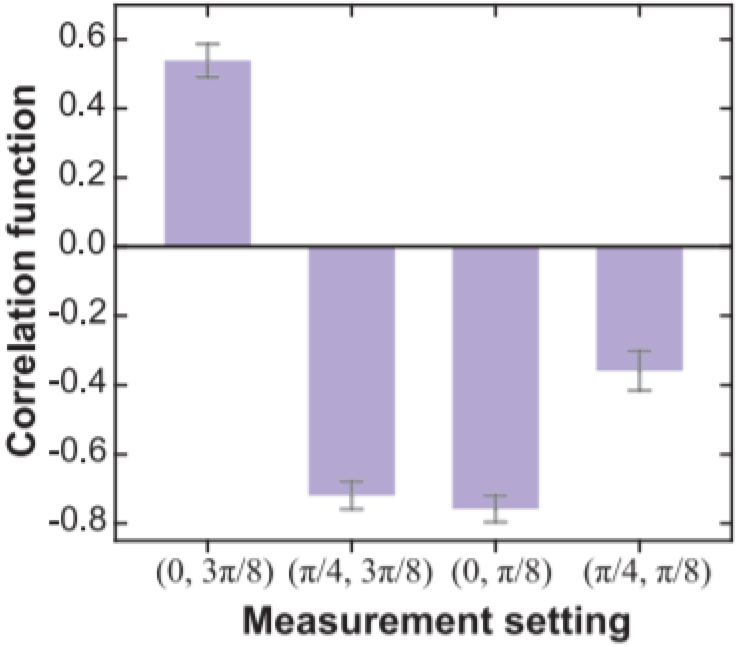


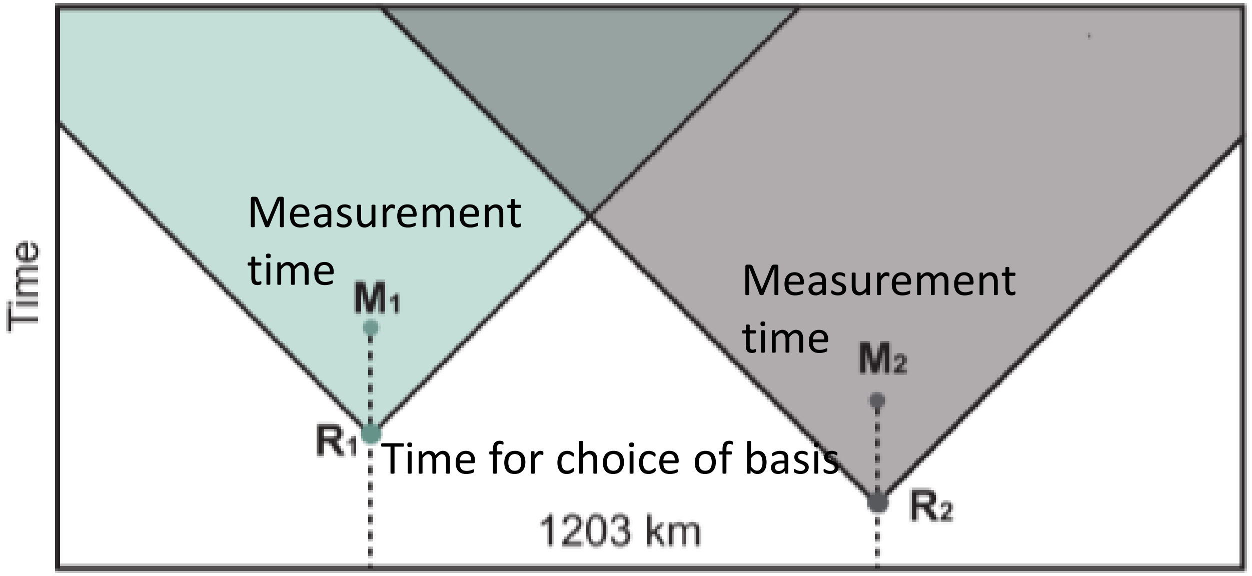
This means that Alice and Bob's choice of detector choices and the measurement outcomes should not be communicated between the two.

$$\langle AB \rangle = \int d\lambda \, p(\lambda) A(\lambda) B(\lambda) \quad \left(\neq \int d\lambda \, p(\lambda) f_{AB}(\lambda) \right)$$

In order to make sure that there is no possibility of communication between Alice and Bob, modern Bell test experiments are done far away such that the information could not travel between them, even at the speed of light.







Yin, Pan et al., Science 356, 1140–1144 (2017)

Quantum mechanics is non-local!