1) a)

$$LHS = (C \setminus (A \cup B)) = (C \cap (A \cup B)^c)$$
$$= (C \cap (A^c \cap B^c))$$
$$= A^c \cap B^c \cap C$$

$$RHS = ((C \setminus A) \cap (C \setminus B))$$
$$= ((C \cap A^c) \cap (C \cap B^c))$$
$$= C \cap A^c \cap C \cap B^c$$
$$= A^c \cap B^c \cap C$$
$$= LHS$$

b) let $A = \{1, 2, 3\}$,
$B = \{2, 4\}$

$A \setminus B = \{1, 3\}$
$B \setminus A = \{4\}$ $\therefore$ $A \setminus B \neq B \setminus A$

2)
a) Injective:
$$\forall x_1, x_2 \in X, (x_1 \neq x_2) \Rightarrow (f(x_1) \neq f(x_2))$$
Surjective:
$$\forall y \in Y (\exists x \in X (f(x) = y))$$
Bijective:
injective & surjective

b) $f(x) = 1 - x^4$

c)


$f(x) \leq 1$
$\therefore$ not surj
$\therefore$ not bij

$f(1) = 0$
$f(-1) = 0$ $f(1) = f(-1) = 0$, $1 \neq -1$ $\therefore$ not injective

3) $\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 5 & 4 \end{pmatrix}$

a) $\alpha = \sigma_1 ; \sigma_2$

$\alpha(1\ 2\ 3\ 4\ 5) = \sigma_2(\sigma_1(1\ 2\ 3\ 4\ 5))$
$= \sigma_2(2\ 3\ 1\ 5\ 4) = (1\ 3\ 2\ 4\ 5)$

$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 & 5 \end{pmatrix}$ //

$\beta = \sigma_2 ; \sigma_1$

$\beta(1\ 2\ 3\ 4\ 5) = \sigma_1(\sigma_2(1\ 2\ 3\ 4\ 5))$
$= \sigma_1(2\ 1\ 3\ 5\ 4) = (3\ 2\ 1\ 4\ 5)$

$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 4 & 5 \end{pmatrix}$ //

b) $sgn(\sigma_1) = (-1)^3 = -1$
$sgn(\sigma_2) = (-1)^2 = 1$
$sgn(\alpha) = (-1)^1 = -1$

$sgn(f) = (-1)^k$
s.t. k is the number of different values for which $f(x) > f(y)$ when $x < y$

$sgn(\sigma_1) \cdot sgn(\sigma_2) = -1 \times 1 = -1 = sgn(\alpha)$

$\therefore$ correct

c) $\sigma_1(1\,2\,3\,4\,5) = (2\,3\,1\,5\,4)$

$\sigma_1^2(1\,2\,3\,4\,5) = (3\,1\,2\,4\,5)$

$\sigma_1^3(1\,2\,3\,4\,5) = (1\,2\,3\,5\,4)$

$\sigma_1^4(1\,2\,3\,4\,5) = (2\,3\,1\,4\,5)$

$\sigma_1^5(1\,2\,3\,4\,5) = (3\,1\,2\,5\,4)$

$\sigma_1^6(1\,2\,3\,4\,5) = (1\,2\,3\,4\,5)$

$\sigma_2(1\,2\,3\,4\,5) = (2\,1\,3\,5\,4)$

$\sigma_2^2(1\,2\,3\,4\,5) = (1\,2\,3\,4\,5)$

order $\sigma_1 = 6$

order $\sigma_2 = 2$ //

4) $11 = 5(2) + 1$

a) $5 = 5(1) + 0$ $\therefore$ gcd $= 1$

$11 - 5(2) = 1$

$\therefore k_1 = 1, \; k_2 = -2, \; d = 1$ //

b) $\varphi(187) = \varphi(17)\varphi(11) = 16 \times 10$

$= 160$ //

c) $7^{162} \pmod{187} = 7^2 \cdot 7^{160} \pmod{187}$

$= 7^2 \cdot 7^{\varphi(187)} \pmod{187}$

$= 7^2 \cdot 1 \pmod{187}$

$= 49 \pmod{187}$

//

5)

a) $\begin{pmatrix} 5 & a \\ a & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 & a \\ a & 5 \end{pmatrix}^{-1} \begin{pmatrix} 6 \\ 6 \end{pmatrix}$$

$$= \frac{1}{a^2-25} \begin{pmatrix} 5 & -a \\ -a & 5 \end{pmatrix} \begin{pmatrix} 6 \\ 6 \end{pmatrix}$$

$$= \frac{1}{a^2-25} \begin{pmatrix} 30-6a \\ 30-6a \end{pmatrix}$$

$$= \frac{1}{(a+5)(a-5)} \begin{pmatrix} 6(5-a) \\ 6(5-a) \end{pmatrix}$$

$$= \frac{(5-a)}{(a+5)(a-5)} \begin{pmatrix} 6 \\ 6 \end{pmatrix}$$

$$= \frac{-1}{a+5} \begin{pmatrix} 6 \\ 6 \end{pmatrix}$$

$a \neq -5, 5$

b) $a = -5$ (see (a))

c) $a = 5$ (see (a))

d) $A = \begin{pmatrix} 5 & 7 \\ 7 & 5 \end{pmatrix}$ $\quad \det(A - \lambda I) = 0$

$$\begin{vmatrix} 5-\lambda & 7 \\ 7 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 49$$

$$= 25 + \lambda^2 - 10\lambda - 49$$

$$= \lambda^2 - 10\lambda - 24 = 0$$

$$(\lambda - 12)(\lambda + 2) = 0$$

$$\lambda_1 = 12, \quad \lambda_2 = -2 \quad //$$

these are the eigenvalues

Eigenvectors:

$A v = \lambda v$, v is the eigenvector

$\Rightarrow A v - \lambda v = \underline{0}$

$\Rightarrow (A - \lambda I) v = \underline{0}$ & solve for v

let $v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Method 1 (Gaussian elimination):

$$\lambda_1: \begin{pmatrix} 5-\lambda_1 & 7 \\ 7 & 5-\lambda_1 \end{pmatrix} = \begin{pmatrix} -7 & 7 \\ 7 & -7 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -7 & 7 & 0 \\ 7 & -7 & 0 \end{array}\right) \xrightarrow{R_2' = R_2 + R_1} \left(\begin{array}{cc|c} -7 & 7 & 0 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1' = \frac{1}{7}R_1} \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow -x_1 + x_2 = 0$$

$$\Rightarrow x_1 = x_2 \qquad \text{let } x_1 = 1 \qquad \therefore v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} //$$

Method 2 (Normal simultaneous eqs):

these are the eigenvectors

$$\lambda_2: \begin{pmatrix} 5-\lambda_2 & 7 \\ 7 & 5-\lambda_2 \end{pmatrix} = \begin{pmatrix} 7 & 7 \\ 7 & 7 \end{pmatrix}$$

$$\begin{pmatrix} 7 & 7 \\ 7 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \left.\begin{array}{l} 7x_1 + 7x_2 = 0 \\ 7x_1 + 7x_2 = 0 \end{array}\right\} \Rightarrow x_1 + x_2 = 0$$

$$\Rightarrow x_1 = -x_2$$

$$\text{let } x_1 = 1, \qquad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} //$$

$$v_1 \cdot v_2 = 0 \Rightarrow v_1 \perp v_2$$

$$v_1 \cdot v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1(1) + 1(-1) = 1 - 1 = 0$$

$\therefore$ v_1 & v_2 are orthogonal //