Kartik Bhatia 19103294 B9

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Assignment - 1

A1. $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x^2 & 0 \le x < \pi \end{cases}$

$0 \le x < \pi \quad f(x) = x^2$ = even function

$$a_0 = \frac{1}{\pi} \int_0^{\pi} x^2 \, dx = \frac{x^3}{3} \Big]_0^{\pi} \cdot \frac{1}{\pi}$$

$$= \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx \, dx =$$

$$= \frac{x^2 \sin nx}{n} - \int_0^{\pi} 2x \frac{\sin nx}{n} dx$$

$$= \frac{x^2 \sin nx}{n} - \left[\frac{-2x \cos nx}{n} - \int \frac{2 \cos nx}{n} \right]$$

$$= \frac{x^2 \sin nx}{n} + \frac{2x \cos nx}{n} + \frac{2}{n^2} \sin nx \Big]_0^{\pi}$$

$$= \frac{4}{n^2} (-1)^n$$

$$\therefore f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos nx$$

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$$f(x) = \frac{\pi^2}{3} + 4\left[\frac{-\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \dots\right]$$

$f(x) = x^2$

Put $x = \pi$

$$\pi^2 = \frac{\pi^2}{3} + 4\left[\frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right]$$

$$\frac{\pi^2}{6} = 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right] \quad -(1)$$

Put $x = 0$.

$$-\frac{\pi^2}{3} = 4\left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \dots\right]$$

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} + \dots \quad -(2)$$

(1) + (2)

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

A2. a) $u_{xx} - x u_{xy} + 3u_x + 5 = 0$

$A = 1 \quad B = -1 \quad C = 0$

$B^2 - 4AC = 1$

$B^2 - 4AC > 0$

Hence PDE is hyperbolic

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b) $xu_{xx} - 2u_{xy} + xu_{yy} + u_x = 0$

$A = x$

$B = -2$

$C = x$

$B^2 - 4AC = 4 - 4x^2 = 4(1-x^2)$

$x^2 \in [0,\infty)$

$-x^2 \in (-\infty,0]$

$1-x^2 \in (-\infty,1]$

for $x = 1$ it is parabolic

for other values it is elliptical

c) $u_{yy} + 2u_x + u_y + 3 = 0$

$C = 1$

$A = 0$

$B = 0$

$B^2 - 4AC = 0$

Hence PDE is Parabolic as $B^2 - 4AC = 0$

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A3. One dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad -(1)$$

In steady state

$$\frac{\partial u}{\partial t} = 0$$

From equation (1)

$$c^2 \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0$$

$$\frac{\partial u}{\partial x} = A \Rightarrow u(x) = Ax + B \quad -(2)$$

$\therefore$ Boundary Conditions $u(0,t) = 0$
$u(l,t) = 100$

$u(0) = B \quad \therefore B = 0$

$u(l) = Al + B \Rightarrow 100 = Al \Rightarrow A = \frac{100}{l}$

$$u(x,t) = \frac{100x}{l} = f(x)$$

After steady state rod has put on $25^0 c$ at end A and $75^0 c$ at end B

Boundary Conditions are $u(0,t) = 25$
$u(l,t) = 75$

Initial Condition $u(x,0) = \frac{100x}{l} = f(x)$

Since Boundary Conditions are non-homogeneous

$\therefore$ Solution of heat equation is :-

$$u(x,t) = u_{ss}(x) + u_{tr}(x,t) \quad -(3)$$

$u_{ss}(0) = 25$

$u_{ss}(l) = 75$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{has Solution as}$$

$$u_{ss}(x) = Ax + B$$

$$u_{ss}(0) = B \Rightarrow B = 25$$

$$u_{ss}(l) = Al + B$$

$$75 = Al + 25 \quad \Rightarrow \quad A = \frac{50}{l}$$

$$u_{ss} = \frac{50x}{l} + 25 \quad -(4)$$

Now from equation -3)

$$u(x,t) = u_{ss}(x) + u_{tr}(x,t)$$

$$u(0,t) = u_{ss}(0) + u_{tr}(0,t)$$

$$25 = 25 + u_{tr}(0,t)$$

$$\boxed{u_{tr}(0,t) = 0}$$

$$u(l,t) = u_{ss}(l) + u_{tr}(l,t)$$

$$75 = 75 + u_{tr}(l,t)$$

$$u_{tr}(l,x) = 0$$

For u_{tr} Boundary conditions are homogeneous

$u_{tr}(0,t) = 0 = u_{tr}(l,t)$ $\rightarrow$ BC for u_{tr}

Now finding initial conditions:

From equation $u(x,0) = u_{st}(x) + u_{tr}(x,0)$

$$\frac{100x}{l} = \frac{50x}{l} + 25 + u_{tr}(x,0)$$

$$\boxed{\frac{50x}{l} - 25 = u_{tr}(x,0)}$$ initial condition for u_{tr}

Hence we have to find the solution of following heat equation.

$$\frac{\partial u_{tr}}{\partial t} = c^2 \frac{\partial^2 u_{tr}}{\partial x^2} \quad -(5)$$

with BC : $u_{tr}(0,t) = 0 = u_{tr}(l,t)$

initial condition : $u_{tr}(x,0) = \left(\frac{50x}{l} - 25\right) = f(x)$

Solution of given heat equation (5) is

$$u_{tr}(x,t) = \sum_{n=1}^{\infty} d_n \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n\pi c}{l}\right)^2 t}$$

where $d_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right) dx$

$$d_n = \frac{2}{l}\int_0^l \left(\frac{50x}{l} - 25\right) \sin\left(\frac{n\pi x}{l}\right) dx$$

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$$= \frac{2}{l}\left[\left(\frac{50x}{l} - 25\right)\left(\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right) - \int \frac{50}{l}\left(\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right) dx\right]_0^l$$

$$= \frac{2}{l}\left[\left(-\frac{50x}{l} + 25\right)\left\{\frac{l}{n\pi}\cos\left(\frac{n\pi x}{l}\right)\right\} + \frac{50}{n\pi}\sin\left(\frac{\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\left\{(25 - 50)\left(\frac{l}{n\pi}\cos n\pi\right) + \frac{50}{n\pi}\cdot\frac{l}{n\pi}\sin n\pi\right\}\right.$$

$$\left. - \left\{25 \times \frac{1}{\frac{n\pi}{l}}\cos 0 + \frac{50}{n\pi}\cdot\frac{l}{n\pi}\sin 0\right\}\right]$$

$$= \frac{2}{l}\left[25 \cdot \frac{l}{n\pi}(-1)^{n+1} + 0 - \frac{25l}{n\pi}\right]$$

$$= \frac{2}{l} \times \frac{25l}{n\pi}\left[(-1)^{n+1} - 1\right]$$

$$d_n = \frac{50}{n\pi}\left[(-1)^{n+1} - 1\right] = \begin{cases} 0 & n \text{ is odd} \\ \frac{-100}{n\pi} & n \text{ is even} \end{cases}$$

$$u_{tr}(x,t) = \sum_{n=2,4,\ldots}^{\infty} \frac{-100}{n\pi}\sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n\pi c}{l}\right)^2 t}$$

$$= \sum_{m=1}^{\infty} \frac{-100}{2m\pi}\sin\left(\frac{2m\pi x}{l}\right) e^{-\left(\frac{2m\pi c}{l}\right)^2 t}$$

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$$= \sum_{n=1}^{\infty} \left(\frac{-50}{n\pi}\right) \sin\left(\frac{2n\pi x}{l}\right) e^{\frac{-4n^2\pi^2c^2t}{l^2}}$$

Hence Complete Solution is

$$u(x,t) = u_{ss}(x) + u_{tr}(x,t)$$

$$u(x,t) = \frac{50x}{l} + 25 + \sum_{n=1}^{\infty} \left(\frac{-50}{n\pi}\right) . \sin\left(\frac{2n\pi x}{l}\right) e^{\frac{-4n^2\pi^2c^2t}{l^2}}$$

Q4. $$u(5,80) = \frac{50\times5}{20} + 25 + \sum_{n=1}^{\infty} \left(\frac{-50}{n\pi}\right) \sin\left(\frac{2\times n\pi\times5}{20}\right) \times e^{\frac{-4\pi^2n^2\times(0.2089)\times80}{20}}$$

$$= \frac{75}{2} + \left(\frac{-50}{\pi}\right) \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}\right) e^{-n^2\pi^2\times3.3424}$$

$$= \frac{75}{2} - \frac{50}{\pi}\left[e^{-8.320} + \frac{1}{2}\times0 + \frac{1}{3}(-1)e^{-9\times8\times3239} \cdots\cdots \right]$$

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