Solutions to Durrett's Probability: Theory and Examples (Edition 4.1)

Chapter 1

1.1.1 *Proof.* We need to show if $\mathcal{F}_i, i \in I$ are σ algebras, $\cap_{i\in I}\mathcal{F}_i$ is also a *sigma* algebra

(a) Since $\Omega \in \mathcal{F}_i$, $\forall i \in I$ from the definition of σ algebra, we have $\Omega \in \cap_{i\in I}\mathcal{F}_i$

(b) If $A \in \cap_{i\in I}\mathcal{F}_i$, then A must be in each of the $\mathcal{F}_i s$ i.e. $A \in \mathcal{F}_i$, $\forall i$. Since $\mathcal{F}_i s$ are σ algebras, we have $A^c \in \mathcal{F}_i$, $\forall i$. Therefore $A^c \in \cap_{i\in I}\mathcal{F}_i$

(c) Let $A_1, A_2, \cdots \in \cap_{i\in I}\mathcal{F}_i$. Then $A_n \in \mathcal{F}_i$, $\forall i$, $\forall n$. Since $\mathcal{F}_i s$ are *sigma* algebras, we have $\cup_n A_n \in \mathcal{F}_i$, $\forall i$. Hence we have $\cup_n A_n \in \cap_{i\in I}\mathcal{F}_i$

This proof is the same even if the intersection is over an arbitrary index set because of the definition of intersection.

To show that given Ω and a collection of subsets $\mathcal{A}$, there exists a smallest σ algebra.

Let $\mathbb{F}$ be the collection of all sigma algebras $\mathcal{F}_i$, $i \in I$ containing $\mathcal{A}$. Define $\sigma(\mathcal{A}) = \cap_{i\in I}\mathcal{F}_i$. We have already shown that this is a σ algebra. Clearly $\sigma(\mathcal{A}) \subset \mathcal{F}_i$, $\forall i$. Suppose there exists some $\mathcal{G} \subset \sigma(\mathcal{A})$ and which contains $\mathcal{A}$, then surely $\mathcal{G} \in \mathbb{F}$ and therefore $\mathcal{G} = \mathcal{F}_i$ for some i which implies $\sigma(\mathcal{A}) \subset \mathcal{G}$. Hence we have $\sigma(\mathcal{A}) = \cap_{i\in I}\mathcal{F}_i$ to be the smallest σ algebra generated by $\mathcal{A}$

$\blacksquare$

1.1.2 *Proof.* To show $(\Omega, \mathcal{F}, \mathcal{P})$ is a probability space, we need to show $\mathcal{F} = \{A : A \text{ is countable or } A^c \text{ is countable}\}$ is a sigma algebra and $\mathcal{P}$ is a probability measure.

To show $\mathcal{F}$ is a sigma algebra, observe that:

(a) $\Omega \in \mathcal{F}$, since $\emptyset$ is countable by definition.

(b) if $A \in \mathcal{F}$ then A must be either countable, in which case $(A^c)^c = A$ is countable and therefore A^c belongs to $\mathcal{F}$, or A^c is countable and so A^c is also in $\mathcal{F}$.

(c) if $A_n \in \mathcal{F}$, $\forall n \in \mathcal{N}$, then $A \triangleq \cup_{n=1}^{\infty} A_n$ is either countable, in which case it belongs to $\mathcal{F}$, or uncountable. If A is uncountable then there exists an $m \in \mathcal{N}$ such that A_m is uncountable and A_m^c is countable. Since, $A^c = (\cap_{n=1,n\neq m}^{\infty} A_n^c) \cap A_m^c$ and intersection of any set with a countable set is countable, A^c is countable and thus A belongs to $\mathcal{F}$.

Thus $\mathcal{F}$ is a sigma algebra.

To show $\mathcal{P}$ is a probability algebra, observe that:

(a) Since Ω is uncountable $P(\Omega) = 1$.

(b) if $A_i \in \mathcal{F}$ are disjoint sets, then it is easy to see that either all the sets are countable or exactly one set is uncountable (because, if there are two disjoint sets A_n and A_m in $\mathcal{F}$ which are uncountable, then since $A_m \subseteq A_n^c$ and A_n^c is countable, A_m should be countable leading to contradiction). Thus, $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=0}^{\infty} P(A_i) = 0$ if all A_i's are countable and is equal to 1 if one of them is uncountable.

Thus $\mathcal{P}$ is a Probability measure. ■

1.1.3 *Proof.* We shall show $\sigma(\mathcal{S}_d) = \mathcal{R}^d$ in multiple steps.

(a) observe that,

$$(a_, b_1) \times (a_2, b_2)... \times (a_d, b_d) = \cup_{n \geq 1} (a_, b_1 - 1/n] \times (a_2, b_2 - 1/n]... \times (a_d, b_d - 1/n],$$

which implies that open rectangles $(a_, b_1) \times (a_2, b_2)... \times (a_d, b_d) \in \sigma(\mathcal{S}_d)$.

(b)

Claim 1. Every open set in $\mathbb{R}^d$ is a countable union of open rectangles

Proof. Associate with every internal point x in the open set, an open rectangle with rational end points such that the open rectangle is a proper subset of the open set. This is possible as the rational points are dense in $\mathbb{R}$. Thus the claim is true as there are only countable such open rectangles and their union is equal to the open set. ■

(c) Now, since every open set is a countable union of open rectangles, implying $\mathcal{R}^d \subseteq \sigma(\mathcal{S}_d)$.

(d) Observe that,

$$(a_, b_1] \times (a_2, b_2]... \times (a_d, b_d] = \cap_{n \geq 1} (a_, b_1 + 1/n) \times (a_2, b_2 + 1/n)... \times (a_d, b_d + 1/n).$$

Since any open rectangle is also an open set and hence in $\mathcal{R}^d$, the above observation shows that a set of the form $(a_, b_1] \times (a_2, b_2]... \times (a_d, b_d] \in \mathcal{R}^d$, which together proves $\sigma(\mathcal{S}_d) \subseteq \mathcal{R}^d$

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