Solutions to Durrett's Probability: Theory and Examples (Edition 4.1)

Chapter 1

1.1.2 *Proof.* To show $(\Omega, \mathcal{F}, \mathcal{P})$ is a probability space, we need to show $\mathcal{F} = \{A : A \text{ is countable or } A^c \text{ is countable}\}$ is a sigma algebra and $\mathcal{P}$ is a probability measure.

To show $\mathcal{F}$ is a sigma algebra, observe that:

(a) $\Omega \in \mathcal{F}$, since $\emptyset$ is countable by definition.

(b) if $A \in \mathcal{F}$ then A must be either countable, in which case $(A^c)^c = A$ is countable and therefore A^c belongs to $\mathcal{F}$, or A^c is countable and so A^c is also in $\mathcal{F}$.

(c) if $A_n \in \mathcal{F}, \ \forall n \in \mathcal{N}$, then $A \triangleq \cup_{n=1}^{\infty} A_n$ is either countable, in which case it belongs to $\mathcal{F}$, or uncountable. If A is uncountable then there exists an $m \in \mathcal{N}$ such that A_m is uncountable and A_m^c is countable. Since, $A^c = (\cap_{n=1, n\neq m}^{\infty} A_n^c) \cap A_m^c$ and intersection of any set with a countable set is countable, A^c is countable and thus A belongs to $\mathcal{F}$.

Thus $\mathcal{F}$ is a sigma algebra.

To show $\mathcal{P}$ is a probability algebra, observe that:

(a) Since Ω is uncountable $P(\Omega) = 1$.

(b) if $A_i \in \mathcal{F}$ are disjoint sets, then it is easy to see that either all the sets are countable or exactly one set is uncountable (because, if there are two disjoint sets A_n and A_m in $\mathcal{F}$ which are uncountable, then since $A_m \subseteq A_n^c$ and A_n^c is countable, A_m should be countable leading to contradiction). Thus, $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=0}^{\infty} P(A_i) = 0$ if all A_i's are countable and is equal to 1 if one of them is uncountable.

Thus $\mathcal{P}$ is a Probability measure. ∎