Solutions to Durrett's Probability: Theory and Examples (Edition 4.1)

Chapter 1

1.1.1 *Proof.* We need to show if $\mathcal{F}_i, i \in I$ are σ algebras, $\cap_{i\in I}\mathcal{F}_i$ is also a *sigma* algebra

(a) Since $\Omega \in \mathcal{F}_i$, $\forall i \in I$ from the definition of σ algebra, we have $\Omega \in \cap_{i\in I}\mathcal{F}_i$

(b) If $A \in \cap_{i\in I}\mathcal{F}_i$, then A must be in each of the $\mathcal{F}_i s$ i.e. $A \in \mathcal{F}_i$, $\forall i$. Since $\mathcal{F}_i s$ are σ algebras, we have $A^c \in \mathcal{F}_i$, $\forall i$. Therefore $A^c \in \cap_{i\in I}\mathcal{F}_i$

(c) Let $A_1, A_2, \cdots \in \cap_{i\in I}\mathcal{F}_i$. Then $A_n \in \mathcal{F}_i$, $\forall i$, $\forall n$. Since $\mathcal{F}_i s$ are *sigma* algebras, we have $\cup_n A_n \in \mathcal{F}_i$, $\forall i$. Hence we have $\cup_n A_n \in \cap_{i\in I}\mathcal{F}_i$

This proof is the same even if the intersection is over an arbitrary index set because of the definition of intersection.

To show that given Ω and a collection of subsets $\mathcal{A}$, there exists a smallest σ algebra.

Let $\mathbb{F}$ be the collection of all sigma algebras $\mathcal{F}_i$, $i \in I$ containing $\mathcal{A}$. Define $\sigma(\mathcal{A}) = \cap_{i\in I}\mathcal{F}_i$. We have already shown that this is a σ algebra. Clearly $\sigma(\mathcal{A}) \subset \mathcal{F}_i$, $\forall i$. Suppose there exists some $\mathcal{G} \subset \sigma(\mathcal{A})$ and which contains $\mathcal{A}$, then surely $\mathcal{G} \in \mathbb{F}$ and therefore $\mathcal{G} = \mathcal{F}_i$ for some i which implies $\sigma(\mathcal{A}) \subset \mathcal{G}$. Hence we have $\sigma(\mathcal{A}) = \cap_{i\in I}\mathcal{F}_i$ to be the smallest σ algebra generated by $\mathcal{A}$ ■

1.1.2 *Proof.* To show $(\Omega, \mathcal{F}, \mathcal{P})$ is a probability space, we need to show $\mathcal{F} = \{A : A \text{ is countable or } A^c \text{ is countable}\}$ is a sigma algebra and $\mathcal{P}$ is a probability measure.

To show $\mathcal{F}$ is a sigma algebra, observe that:

(a) $\Omega \in \mathcal{F}$, since $\emptyset$ is countable by definition.

(b) if $A \in \mathcal{F}$ then A must be either countable, in which case $(A^c)^c = A$ is countable and therefore A^c belongs to $\mathcal{F}$, or A^c is countable and so A^c is also in $\mathcal{F}$.

(c) if $A_n \in \mathcal{F}$, $\forall n \in \mathcal{N}$, then $A \triangleq \cup_{n=1}^{\infty} A_n$ is either countable, in which case it belongs to $\mathcal{F}$, or uncountable. If A is uncountable then there exists an $m \in \mathcal{N}$ such that A_m is uncountable and A_m^c is countable. Since, $A^c = (\cap_{n=1,n\neq m}^{\infty} A_n^c) \cap A_m^c$ and intersection of any set with a countable set is countable, A^c is countable and thus A belongs to $\mathcal{F}$.

Thus $\mathcal{F}$ is a sigma algebra.

To show $\mathcal{P}$ is a probability algebra, observe that:

(a) Since Ω is uncountable $P(\Omega) = 1$.

(b) if $A_i \in \mathcal{F}$ are disjoint sets, then it is easy to see that either all the sets are countable or exactly one set is uncountable (because, if there are two disjoint sets A_n and A_m in $\mathcal{F}$ which are uncountable, then since $A_m \subseteq A_n^c$ and A_n^c is countable, A_m should be countable leading to contradiction). Thus, $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=0}^{\infty} P(A_i) = 0$ if all A_i's are countable and is equal to 1 if one of them is uncountable.

Thus $\mathcal{P}$ is a Probability measure. ∎

1.1.3 *Proof.* We shall show $\sigma(\mathcal{S}_d) = \mathcal{R}^d$ in multiple steps.

(a) observe that,

$$(a_,b_1) \times (a_2, b_2)... \times (a_d, b_d) = \cup_{n\geq 1}(a_,b_1 - 1/n] \times (a_2, b_2 - 1/n]... \times (a_d, b_d - 1/n],$$

which implies that open rectangles $(a_,b_1) \times (a_2, b_2)... \times (a_d, b_d) \in \sigma(\mathcal{S}_d)$.

(b)

Claim 1. Every open set in $\mathbb{R}^d$ is a countable union of open rectangles

Proof. Associate with every internal point x in the open set, an open rectangle with rational end points such that the open rectangle is a proper subset of the open set. This is possible as the rational points are dense in $\mathbb{R}$. Thus the claim is true as there are only countable such open rectangles and their union is equal to the open set. ∎

(c) Now, since every open set is a countable union of open rectangles, implying $\mathcal{R}^d \subseteq \sigma(\mathcal{S}_d)$.

(d) Observe that,

$$(a_,b_1] \times (a_2, b_2]... \times (a_d, b_d] = \cap_{n\geq 1}(a_,b_1 + 1/n) \times (a_2, b_2 + 1/n)... \times (a_d, b_d + 1/n).$$

Since any open rectangle is also an open set and hence in $\mathcal{R}^d$, the above observation shows that a set of the form $(a_,b_1] \times (a_2, b_2]... \times (a_d, b_d] \in \mathcal{R}^d$, which together proves $\sigma(\mathcal{S}_d) \subseteq \mathcal{R}^d$

∎

1.1.5 *Proof.* To show that if $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ are σ algebras, then $\bigcup_i \mathcal{F}_i$ is an algebra

(a) Since $\Omega \in \mathcal{F}_i, \forall i, \Omega \in \bigcup_i \mathcal{F}_i$

(b) Suppose $A \in \bigcup_i \mathcal{F}_i$. Then $A \in \mathcal{F}_k$ for some k. Since $\mathcal{F}_k$ is a σ algebra, $A^c \in \mathcal{F}_k$. Hence $A^c \in \bigcup_i \mathcal{F}_i$

(c) Suppose $A_1, A_2, \cdots, A_n \in \bigcup_i \mathcal{F}_i$. Then $A_i \in \mathcal{F}_{k_i}$ for some $k_i \in \mathbb{N}$. Thus $\bigcup_i A_i \in \bigcup_{k_i} \mathcal{F}_{k_i} \subset \mathcal{F}_i$

Hence $\bigcup_i \mathcal{F}_i$ is an algebra.
Counter-example for not being a σ algebra.
Consider $\Omega = \mathbb{N}$ and let $\mathcal{F}_1 = \sigma(\{1\}), \mathcal{F}_2 = \sigma(\{1\},\{2\})$ and similarly $\mathcal{F}_n = \sigma(\{1\},\{2\},\cdots\{n\})$ for any n. Clearly $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$. Let $A_i = \{2i\}$. Clearly $A_i \in \bigcup_i \mathcal{F}_i$. But $\bigcup_i A_i =\{$set of all even numbers$\} \notin \bigcup_i \mathcal{F}_i$ since none of the sets A_i contain all of the even numbers. In other words, since we know that $\mathcal{F} =\{A$ is finite or A^c is finite$\}$ is an algebra and not a sigma algebra, we should arrive at a sequence of sets which results in this as the union. Defining as before we can now see that the set of even numbers and its complement the set of odd numbers are both not finite which shows that the countable union property necessary for it to be a σ algebra does not hold. ∎