

Solutions to Durrett's Probability: Theory and Examples (Edition 4.1)

Chapter 1

1.1.2 *Proof.* To show $(\Omega, \mathcal{F}, \mathcal{P})$ is a probability space, we need to show $\mathcal{F} = \{A : A \text{ is countable or } A^c \text{ is countable}\}$ is a sigma algebra and \mathcal{P} is a probability measure.

To show \mathcal{F} is a sigma algebra, observe that:

- (a) $\Omega \in \mathcal{F}$, since \emptyset is countable by definition.
- (b) if $A \in \mathcal{F}$ then A must be either countable, in which case $(A^c)^c = A$ is countable and therefore A^c belongs to \mathcal{F} , or A^c is countable and so A^c is also in \mathcal{F} .
- (c) if $A_n \in \mathcal{F}$, $\forall n \in \mathcal{N}$, then $A \triangleq \cup_{n=1}^{\infty} A_n$ is either countable, in which case it belongs to \mathcal{F} , or uncountable. If A is uncountable then there exists an $m \in \mathcal{N}$ such that A_m is uncountable and A_m^c is countable. Since, $A^c = (\cap_{n=1, n \neq m}^{\infty} A_n^c) \cap A_m^c$ and intersection of any set with a countable set is countable, A^c is countable and thus A belongs to \mathcal{F} .

Thus \mathcal{F} is a sigma algebra.

To show \mathcal{P} is a probability algebra, observe that:

- (a) Since Ω is uncountable $P(\Omega) = 1$.
- (b) if $A_i \in \mathcal{F}$ are disjoint sets, then it is easy to see that either all the sets are countable or exactly one set is uncountable (because, if there are two disjoint sets A_n and A_m in \mathcal{F} which are uncountable, then since $A_m \subseteq A_n^c$ and A_n^c is countable, A_m should be countable leading to contradiction). Thus, $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=0}^{\infty} P(A_i) = 0$ if all A_i 's are countable and is equal to 1 if one of them is uncountable.

Thus \mathcal{P} is a Probability measure. ■

1.1.3 *Proof.* We shall show $\sigma(\mathcal{S}_d) = \mathcal{R}^d$ in multiple steps.

- (a) observe that,

$$(a, b_1) \times (a_2, b_2) \dots \times (a_d, b_d) = \cup_{n \geq 1} (a, b_1 - 1/n] \times (a_2, b_2 - 1/n] \dots \times (a_d, b_d - 1/n],$$

which implies that open rectangles $(a, b_1) \times (a_2, b_2) \dots \times (a_d, b_d) \in \sigma(\mathcal{S}_d)$.

- (b)

Claim 1. Every open set in \mathbb{R}^d is a countable union of open rectangles

Proof. Associate with every internal point x in the open set, an open rectangle with rational end points such that the open rectangle is a proper subset of the open set. This is possible as the rational points are dense in \mathbb{R} . Thus the claim is true as there are only countable such open rectangles and their union is equal to the open set. ■

- (c) Now, since every open set is a countable union of open rectangles, implying $\mathcal{R}^d \subseteq \sigma(\mathcal{S}_d)$.
- (d) Observe that,

$$(a, b_1] \times (a_2, b_2] \dots \times (a_d, b_d] = \cap_{n \geq 1} (a, b_1 + 1/n) \times (a_2, b_2 + 1/n) \dots \times (a_d, b_d + 1/n).$$

Since any open rectangle is also an open set and hence in \mathcal{R}^d , the above observation shows that a set of the form $(a, b_1] \times (a_2, b_2] \dots \times (a_d, b_d] \in \mathcal{R}^d$, which together proves $\sigma(\mathcal{S}_d) \subseteq \mathcal{R}^d$

■