

# Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE



<http://algs4.cs.princeton.edu>

## GEOMETRIC APPLICATIONS OF BSTs

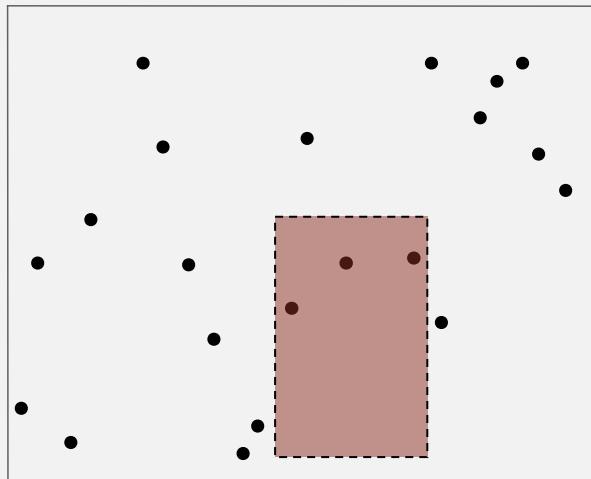
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- ▶ *1d range search*
- ▶ *line segment intersection*
- ▶ *kd trees*
- ▶ *interval search trees*
- ▶ *rectangle intersection*

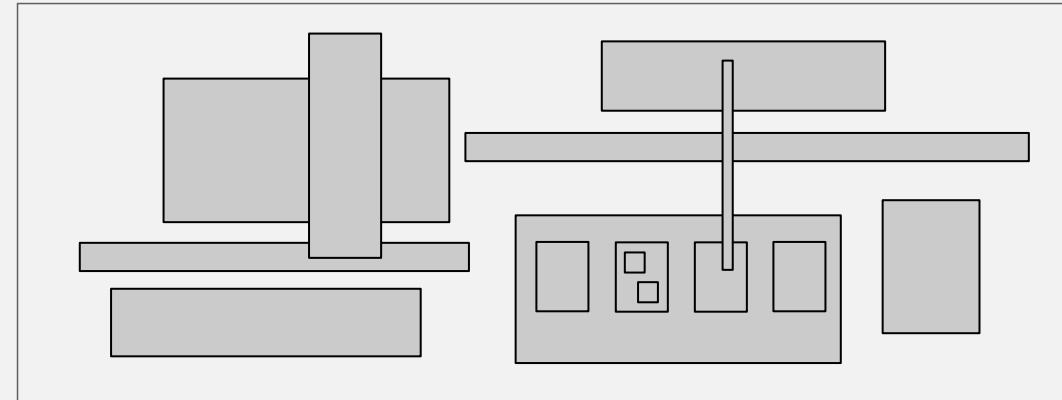
# Overview

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This lecture. Intersections among **geometric objects**.



2d orthogonal range search



orthogonal rectangle intersection

**Applications.** CAD, games, movies, virtual reality, databases, GIS, ....

**Efficient solutions.** **Binary search trees** (and extensions).

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# 1d range search

Extension of ordered symbol table.

- Insert key-value pair.
- Search for key  $k$ .
- Delete key  $k$ .
- Range search: find all keys between  $k_1$  and  $k_2$ .
- Range count: number of keys between  $k_1$  and  $k_2$ .

Application. Database queries.

Geometric interpretation.

- Keys are point on a line.
- Find/count points in a given 1d interval.

• • • • [ • • • ] • • • •

insert B	B
insert D	B D
insert A	A B D
insert I	A B D I
insert H	A B D H I
insert F	A B D F H I
insert P	A B D F H I P
count G to K	2
search G to K	H I

# 1d range search: elementary implementations

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Unordered list. Fast insert, slow range search.

Ordered array. Slow insert, binary search for  $k_1$  and  $k_2$  to do range search.

order of growth of running time for 1d range search

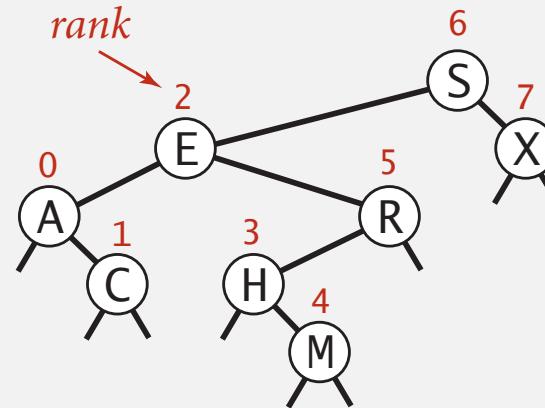
data structure	insert	range count	range search
unordered list	1	N	N
ordered array	N	$\log N$	$R + \log N$
goal	$\log N$	$\log N$	$R + \log N$

$N$  = number of keys

$R$  = number of keys that match

## 1d range count: BST implementation

1d range count. How many keys between  $l_o$  and  $h_i$  ?



```
public int size(Key lo, Key hi)
{
    if (contains(hi)) return rank(hi) - rank(lo) + 1;
    else                  return rank(hi) - rank(lo);
}
```

number of keys < h<sub>i</sub>

Proposition. Running time proportional to  $\log N$ .

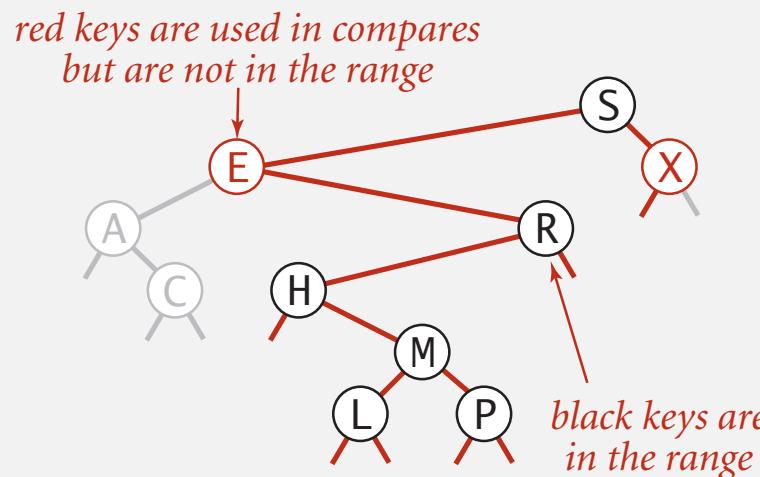
Pf. Nodes examined = search path to  $l_o$  + search path to  $h_i$ .

## 1d range search: BST implementation

1d range search. Find all keys between  $lo$  and  $hi$ .

- Recursively find all keys in left subtree (if any could fall in range).
- Check key in current node.
- Recursively find all keys in right subtree (if any could fall in range).

searching in the range  $[F .. T]$



Proposition. Running time proportional to  $R + \log N$ .

Pf. Nodes examined = search path to  $lo$  + search path to  $hi$  + matches.

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## GEOMETRIC APPLICATIONS OF BSTs

---

- ▶ *1d range search*
- ▶ *line segment intersection*
- ▶ *kd trees*
- ▶ *interval search trees*
- ▶ *rectangle intersection*

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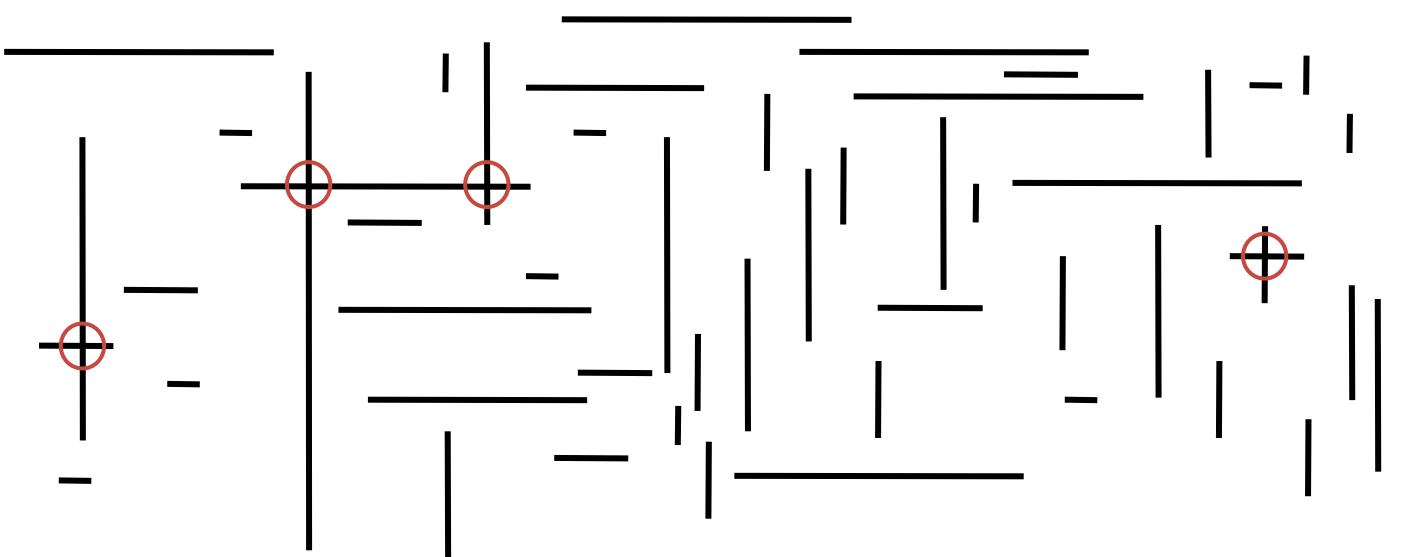
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# Orthogonal line segment intersection

Given  $N$  horizontal and vertical line segments, find all intersections.



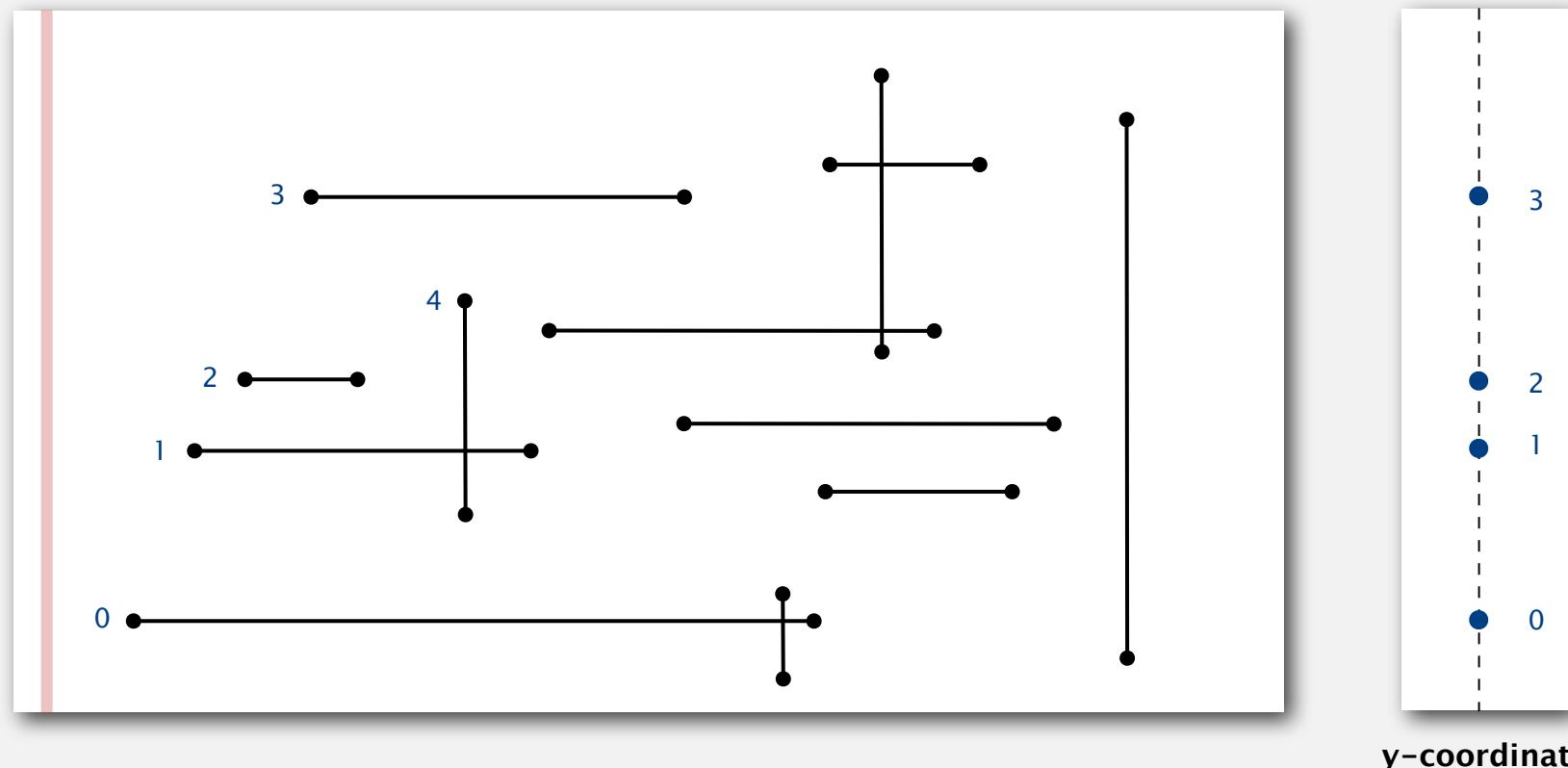
Quadratic algorithm. Check all pairs of line segments for intersection.

Nondegeneracy assumption. All  $x$ - and  $y$ -coordinates are distinct.

# Orthogonal line segment intersection: sweep-line algorithm

Sweep vertical line from left to right.

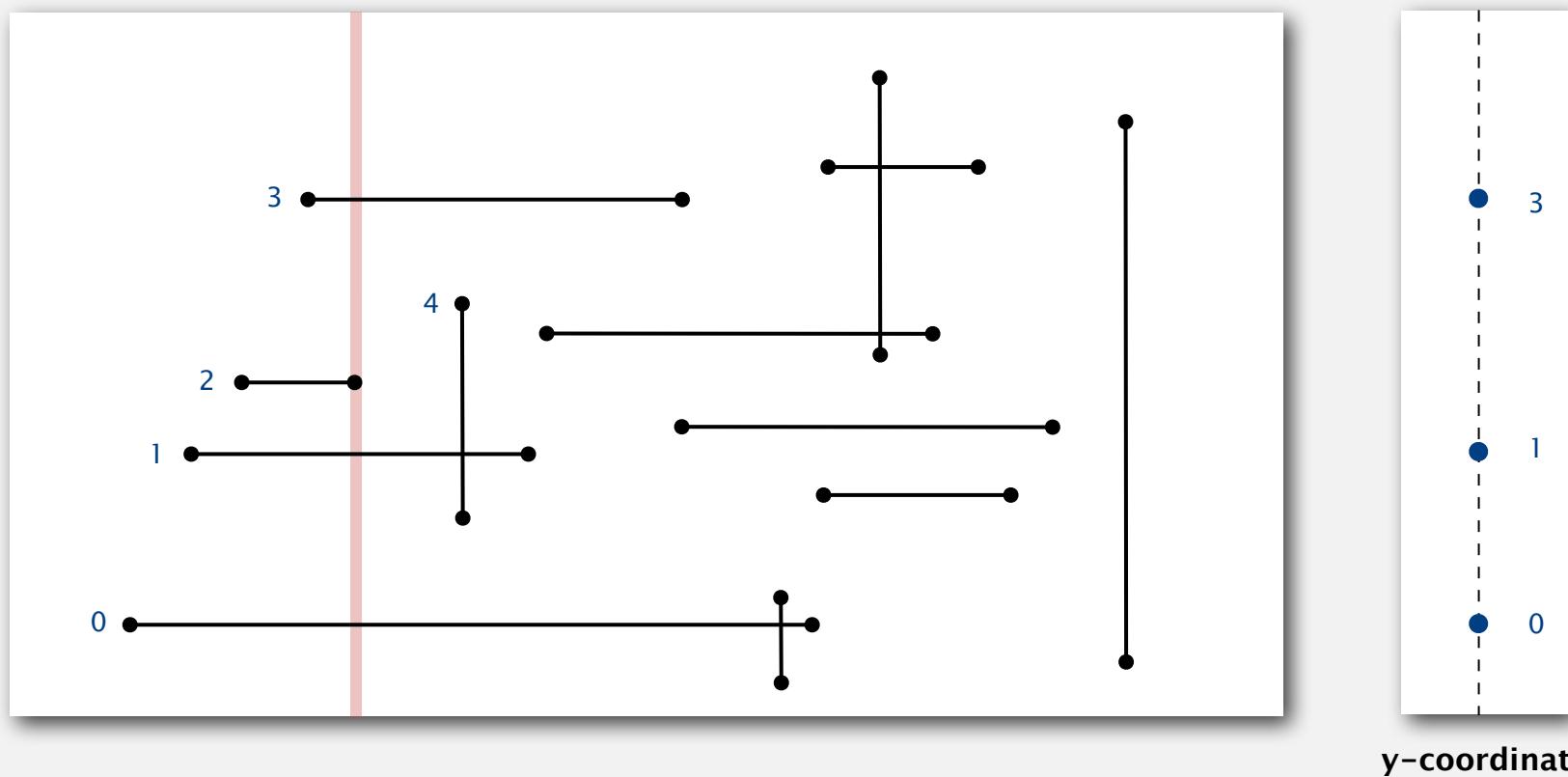
- $x$ -coordinates define events.
- $h$ -segment (left endpoint): insert  $y$ -coordinate into BST.



# Orthogonal line segment intersection: sweep-line algorithm

Sweep vertical line from left to right.

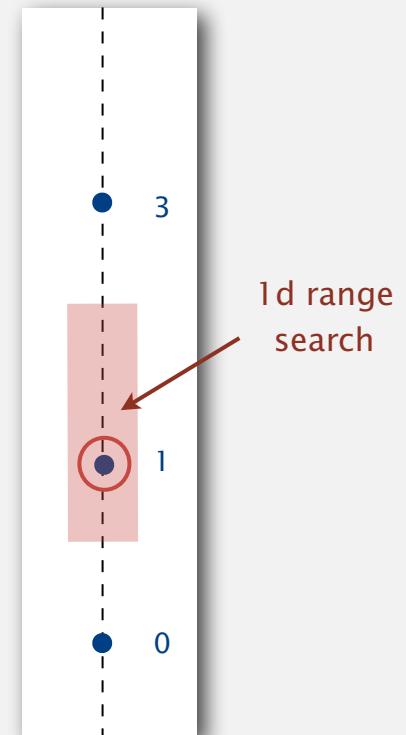
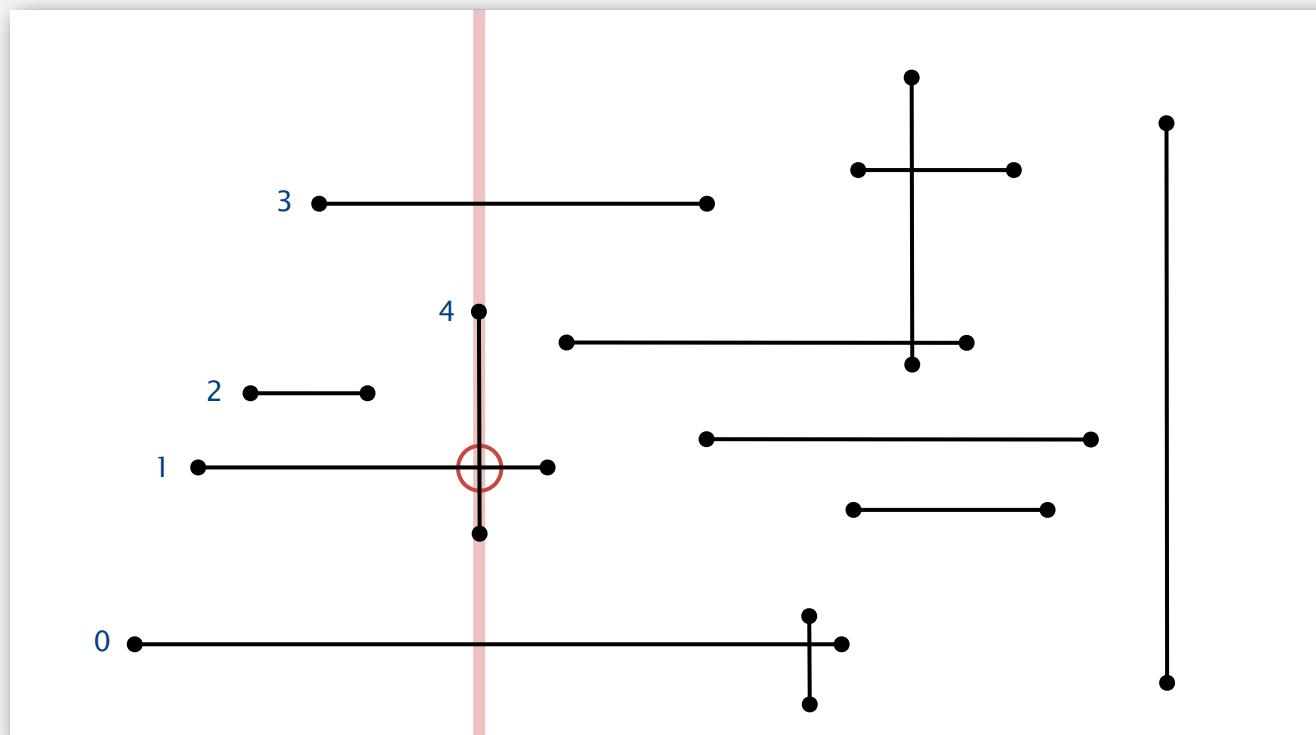
- $x$ -coordinates define events.
- $h$ -segment (left endpoint): insert  $y$ -coordinate into BST.
- $h$ -segment (right endpoint): remove  $y$ -coordinate from BST.



# Orthogonal line segment intersection: sweep-line algorithm

Sweep vertical line from left to right.

- $x$ -coordinates define events.
- $h$ -segment (left endpoint): insert  $y$ -coordinate into BST.
- $h$ -segment (right endpoint): remove  $y$ -coordinate from BST.
- $v$ -segment: range search for interval of  $y$ -endpoints.



$y$ -coordinates

## Orthogonal line segment intersection: sweep-line analysis

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**Proposition.** The sweep-line algorithm takes time proportional to  $N \log N + R$  to find all  $R$  intersections among  $N$  orthogonal line segments.

Pf.

- Put  $x$ -coordinates on a PQ (or sort).  $\leftarrow N \log N$
- Insert  $y$ -coordinates into BST.  $\leftarrow N \log N$
- Delete  $y$ -coordinates from BST.  $\leftarrow N \log N$
- Range searches in BST.  $\leftarrow N \log N + R$

**Bottom line.** Sweep line reduces 2d orthogonal line segment intersection search to 1d range search.

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## 2-d orthogonal range search

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Extension of ordered symbol-table to 2d keys.

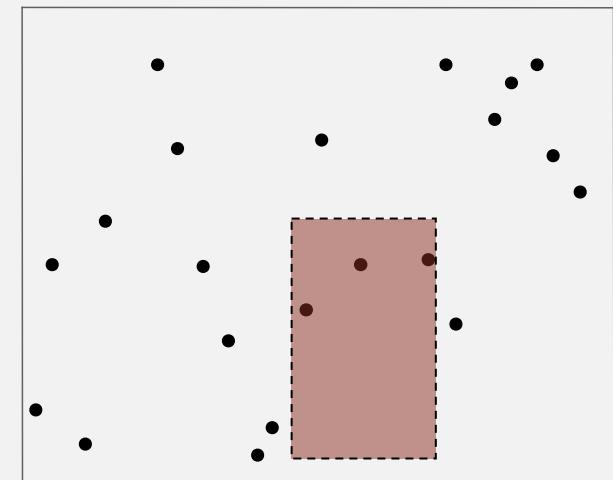
- Insert a 2d key.
- Delete a 2d key.
- Search for a 2d key.
- Range search: find all keys that lie in a 2d range.
- Range count: number of keys that lie in a 2d range.

Applications. Networking, circuit design, databases, ...

Geometric interpretation.

- Keys are point in the plane.
- Find/count points in a given *h-v* rectangle

↑  
rectangle is axis-aligned

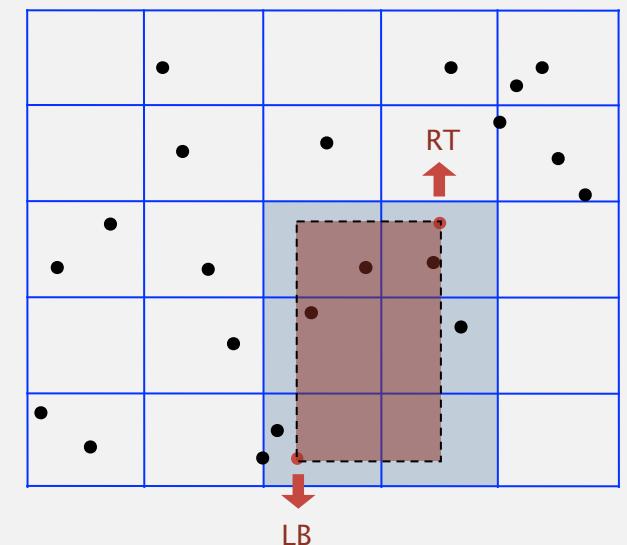


## 2d orthogonal range search: grid implementation

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### Grid implementation.

- Divide space into  $M$ -by- $M$  grid of squares.
- Create list of points contained in each square.
- Use 2d array to directly index relevant square.
- Insert: add  $(x, y)$  to list for corresponding square.
- Range search: examine only squares that intersect 2d range query.



## 2d orthogonal range search: grid implementation analysis

Space-time tradeoff.

- Space:  $M^2 + N$ .
- Time:  $1 + N/M^2$  per square examined, on average.

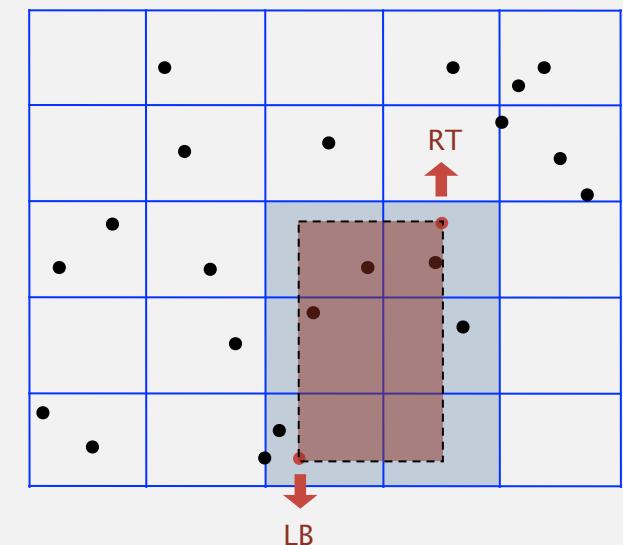
Choose grid square size to tune performance.

- Too small: wastes space.
- Too large: too many points per square.
- Rule of thumb:  $\sqrt{N}$ -by- $\sqrt{N}$  grid.

Running time. [if points are evenly distributed]

- Initialize data structure:  $N$ .
- Insert point: 1.
- Range search: 1 per point in range.

choose  $M \sim \sqrt{N}$



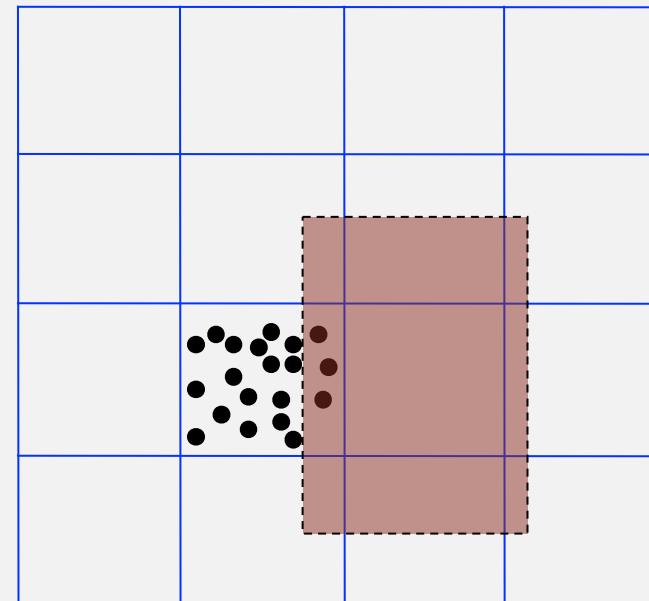
# Clustering

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Grid implementation. Fast, simple solution for evenly-distributed points.

**Problem.** Clustering a well-known phenomenon in geometric data.

- Lists are too long, even though average length is short.
- Need data structure that adapts gracefully to data.



# Clustering

---

Grid implementation. Fast, simple solution for evenly-distributed points.

Problem. Clustering a well-known phenomenon in geometric data.

Ex. USA map data.



13,000 points, 1000 grid squares



half the squares are empty

half the points are  
in 10% of the squares

# Space-partitioning trees

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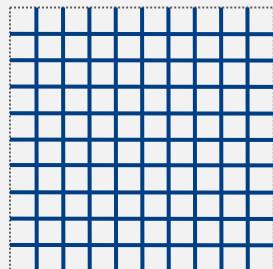
Use a **tree** to represent a recursive subdivision of 2d space.

**Grid.** Divide space uniformly into squares.

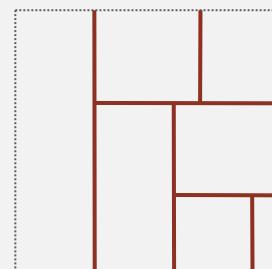
**2d tree.** Recursively divide space into two halfplanes.

**Quadtree.** Recursively divide space into four quadrants.

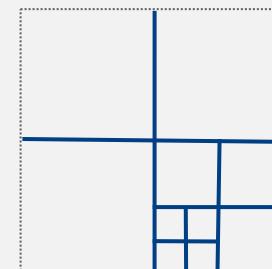
**BSP tree.** Recursively divide space into two regions.



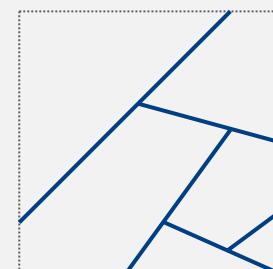
Grid



2d tree



Quadtree

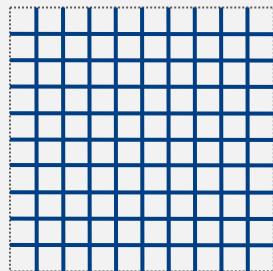


BSP tree

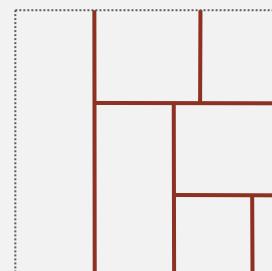
# Space-partitioning trees: applications

## Applications.

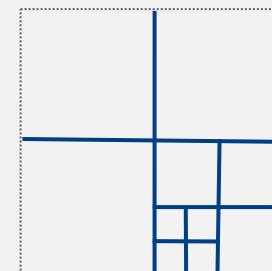
- Ray tracing.
- **2d range search.**
- Flight simulators.
- N-body simulation.
- Collision detection.
- Astronomical databases.
- **Nearest neighbor search.**
- Adaptive mesh generation.
- Accelerate rendering in Doom.
- Hidden surface removal and shadow casting.



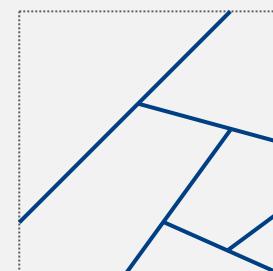
Grid



2d tree



Quadtree

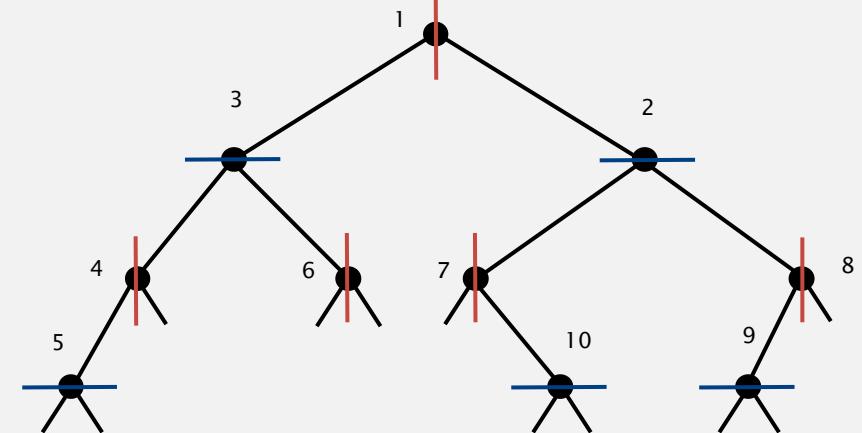
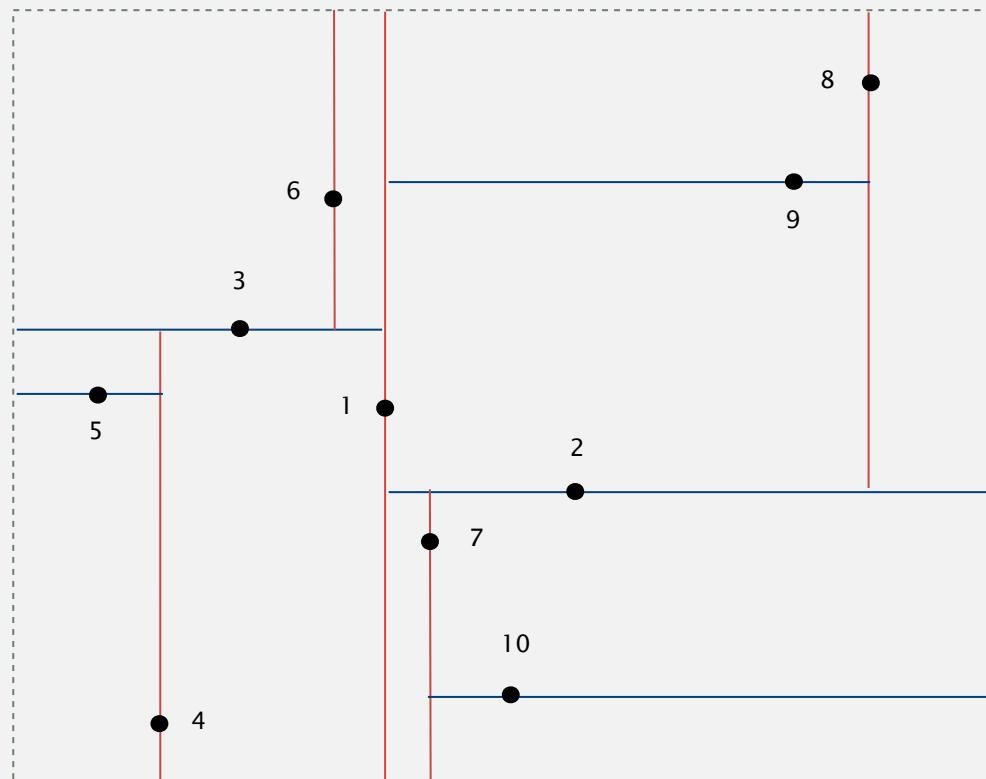


BSP tree

## 2d tree construction

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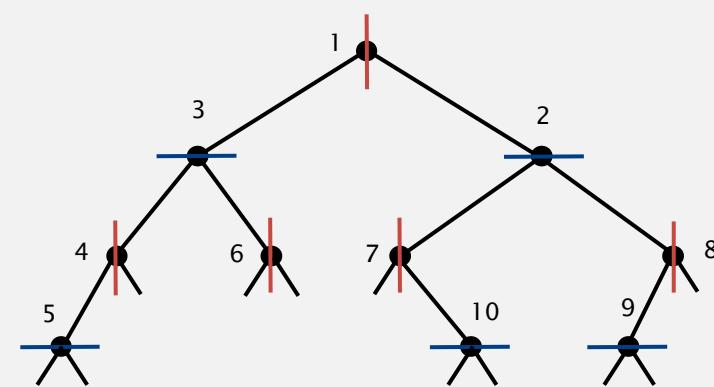
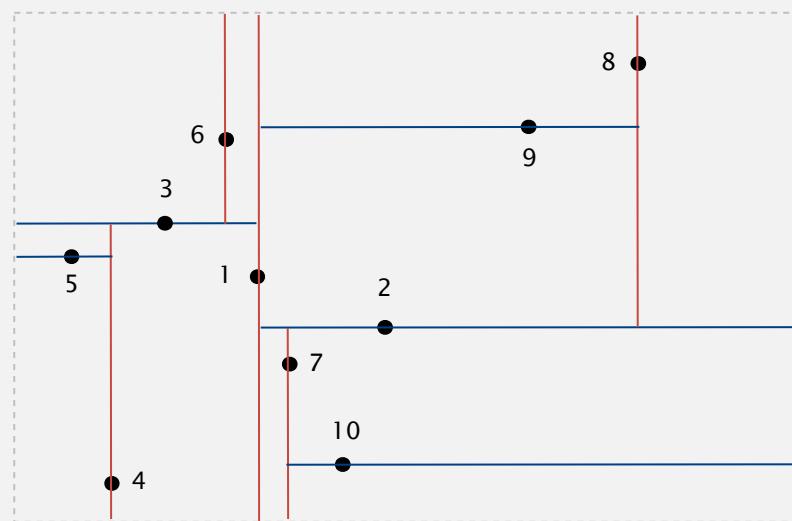
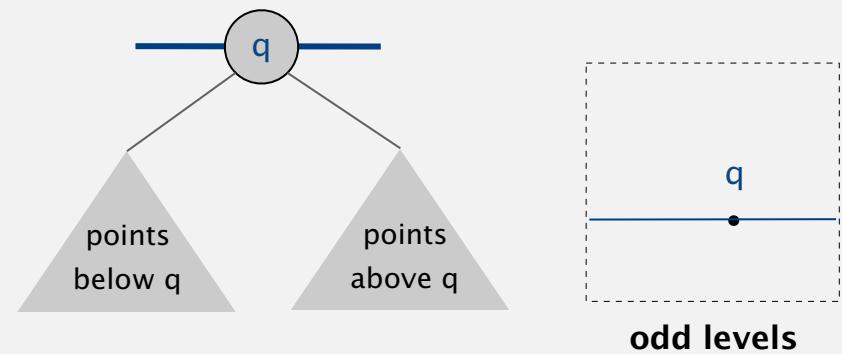
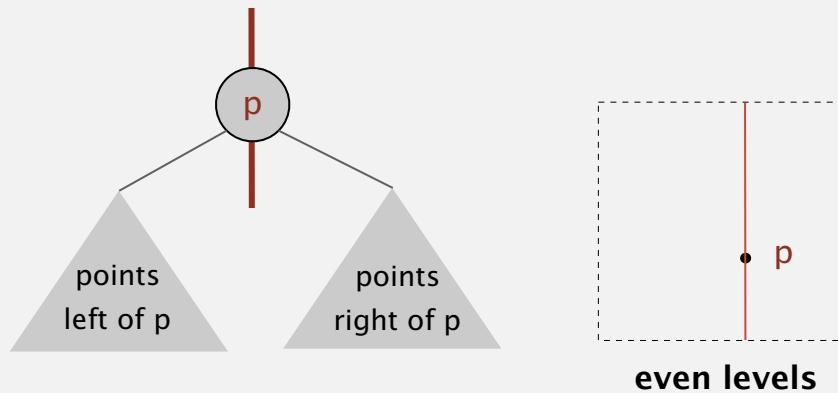
Recursively partition plane into two halfplanes.



## 2d tree implementation

**Data structure.** BST, but alternate using  $x$ - and  $y$ -coordinates as key.

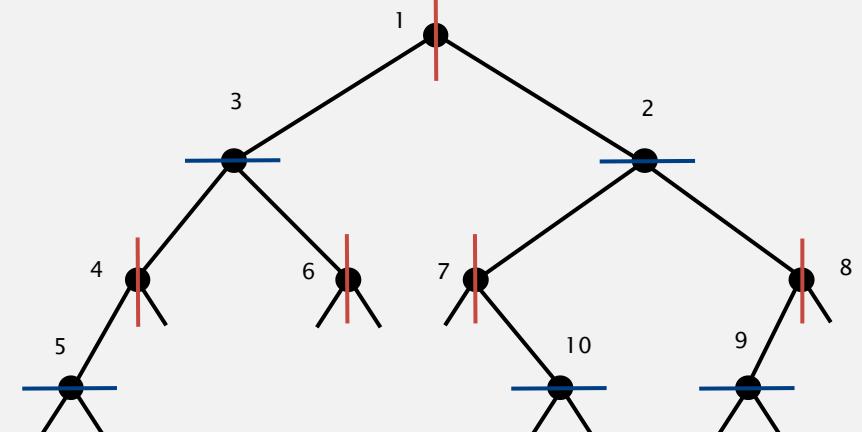
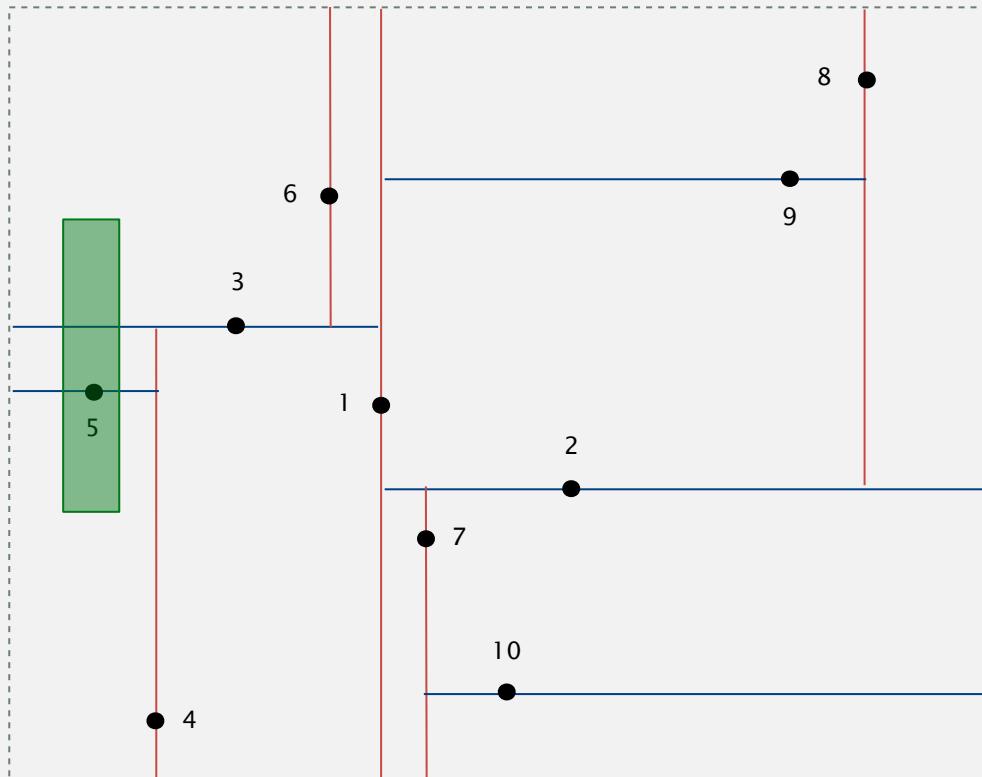
- Search gives rectangle containing point.
- Insert further subdivides the plane.



# Range search in a 2d tree demo

**Goal.** Find all points in a query axis-aligned rectangle.

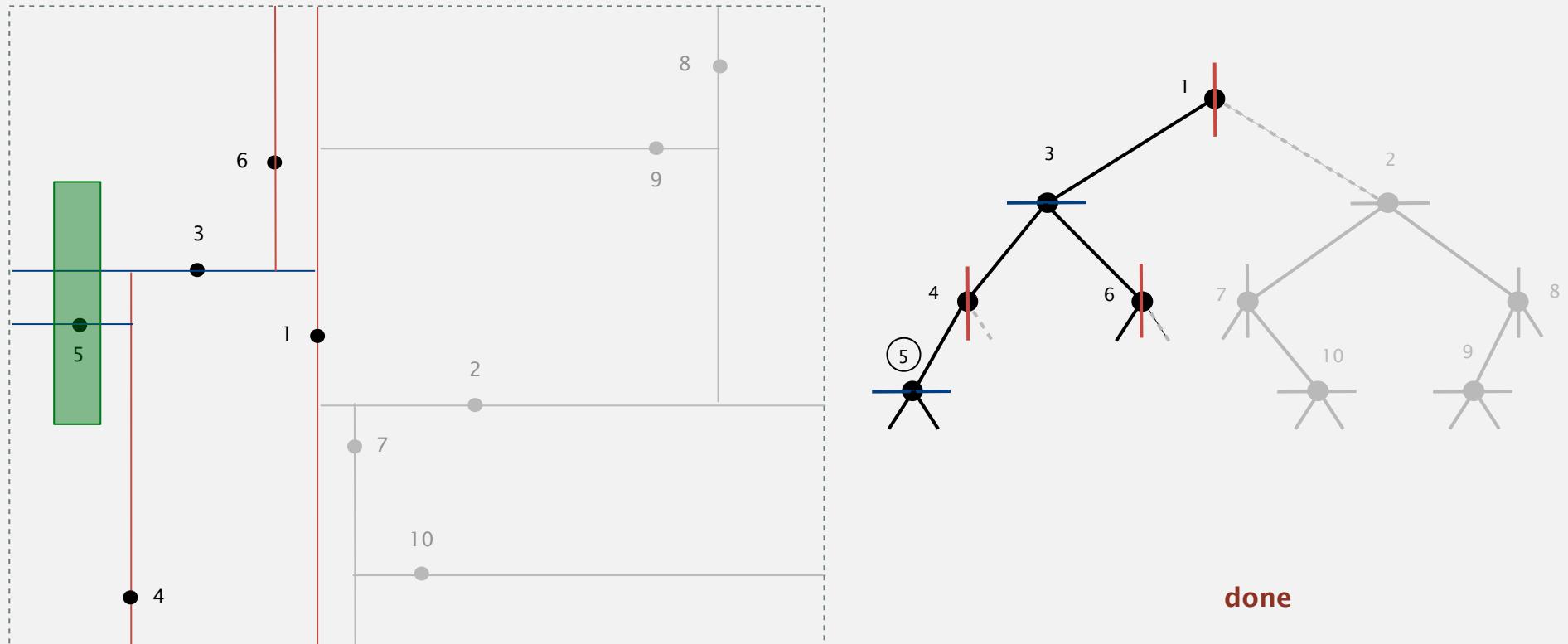
- Check if point in node lies in given rectangle.
- Recursively search left/bottom (if any could fall in rectangle).
- Recursively search right/top (if any could fall in rectangle).



# Range search in a 2d tree demo

**Goal.** Find all points in a query axis-aligned rectangle.

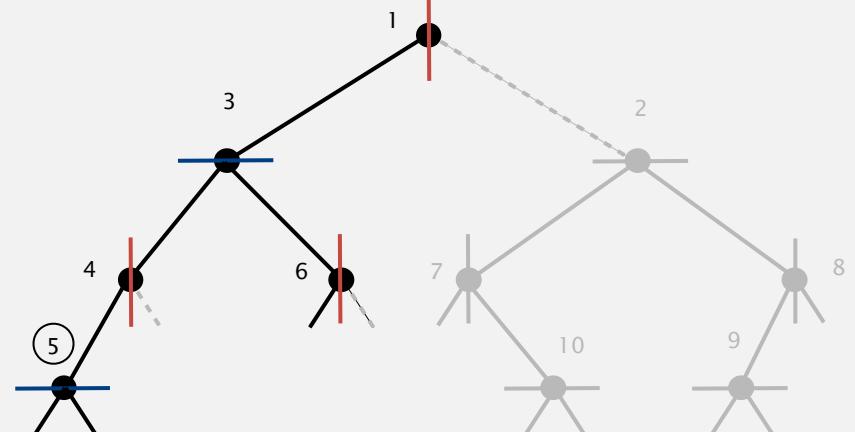
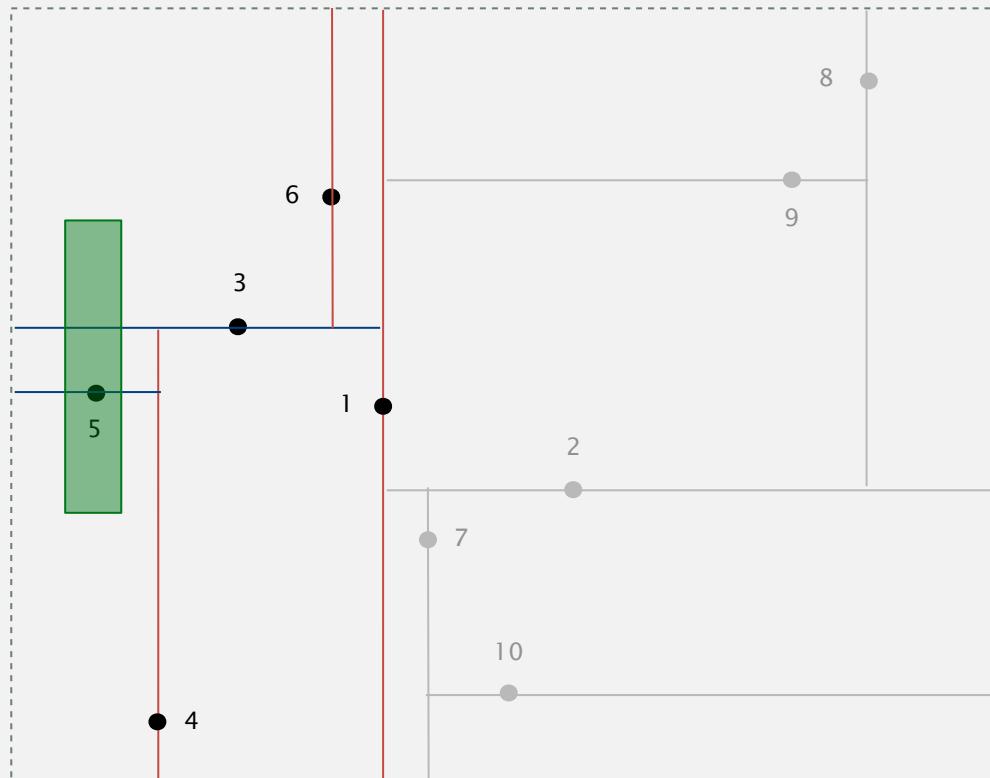
- Check if point in node lies in given rectangle.
- Recursively search left/bottom (if any could fall in rectangle).
- Recursively search right/top (if any could fall in rectangle).



# Range search in a 2d tree analysis

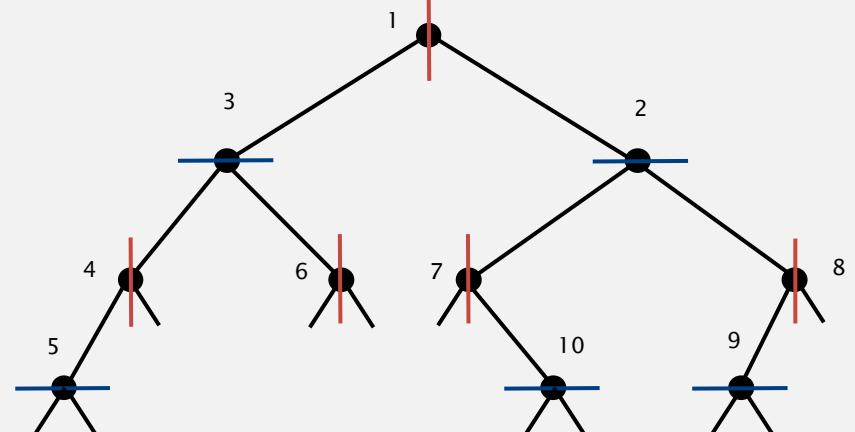
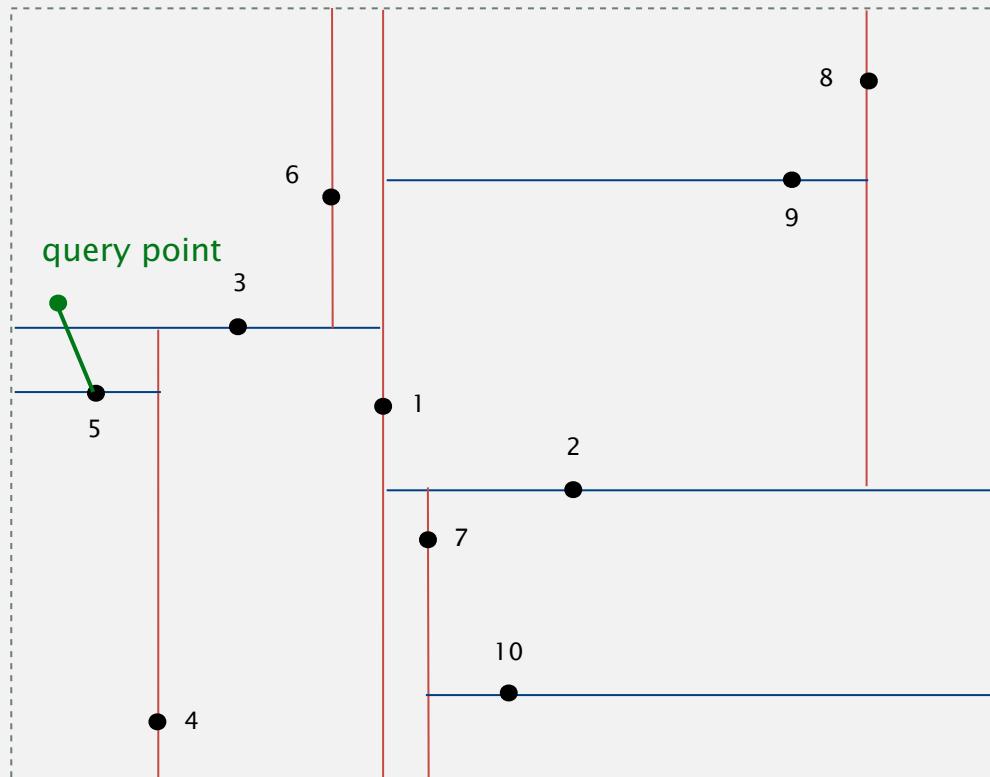
Typical case.  $R + \log N$ .

Worst case (assuming tree is balanced).  $R + \sqrt{N}$ .



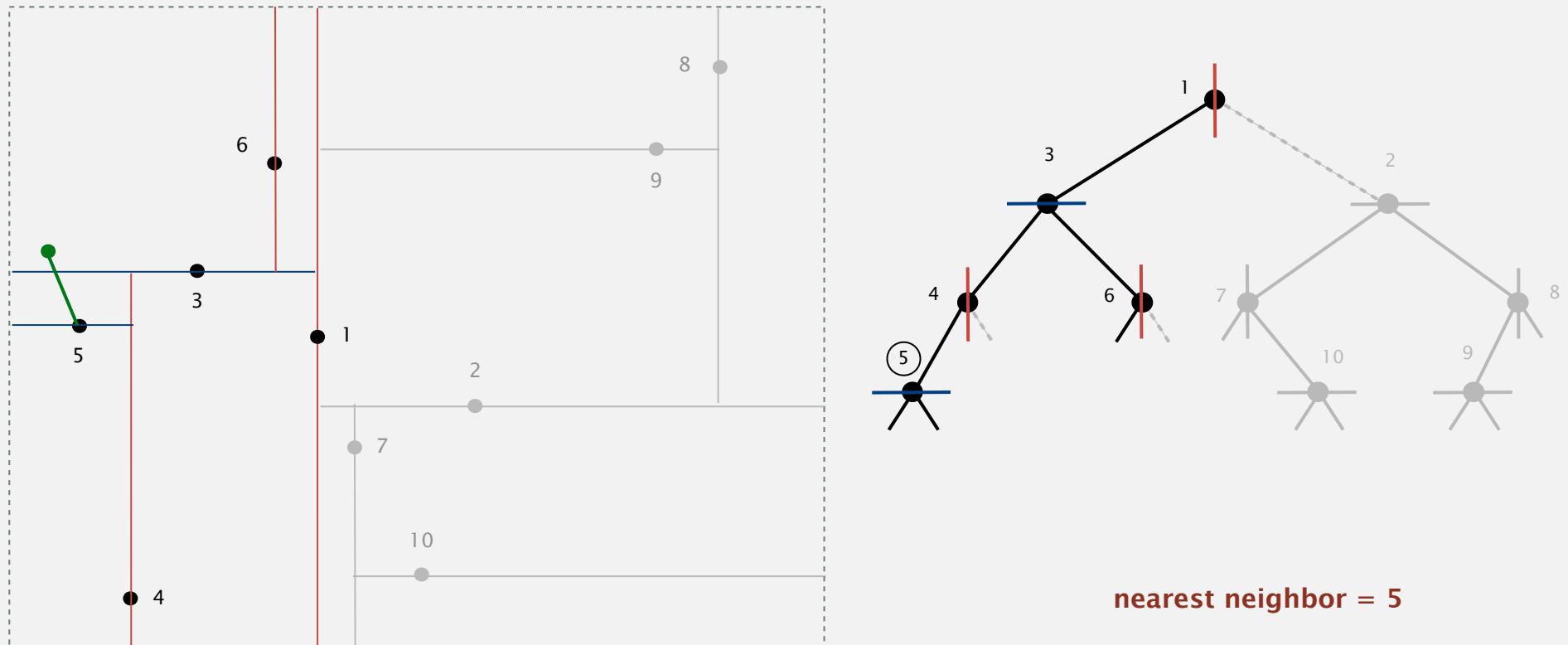
# Nearest neighbor search in a 2d tree demo

Goal. Find closest point to query point.



## Nearest neighbor search in a 2d tree demo

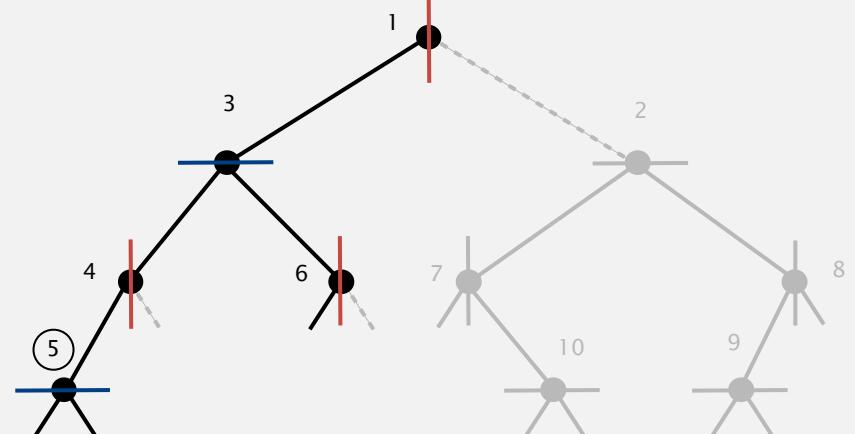
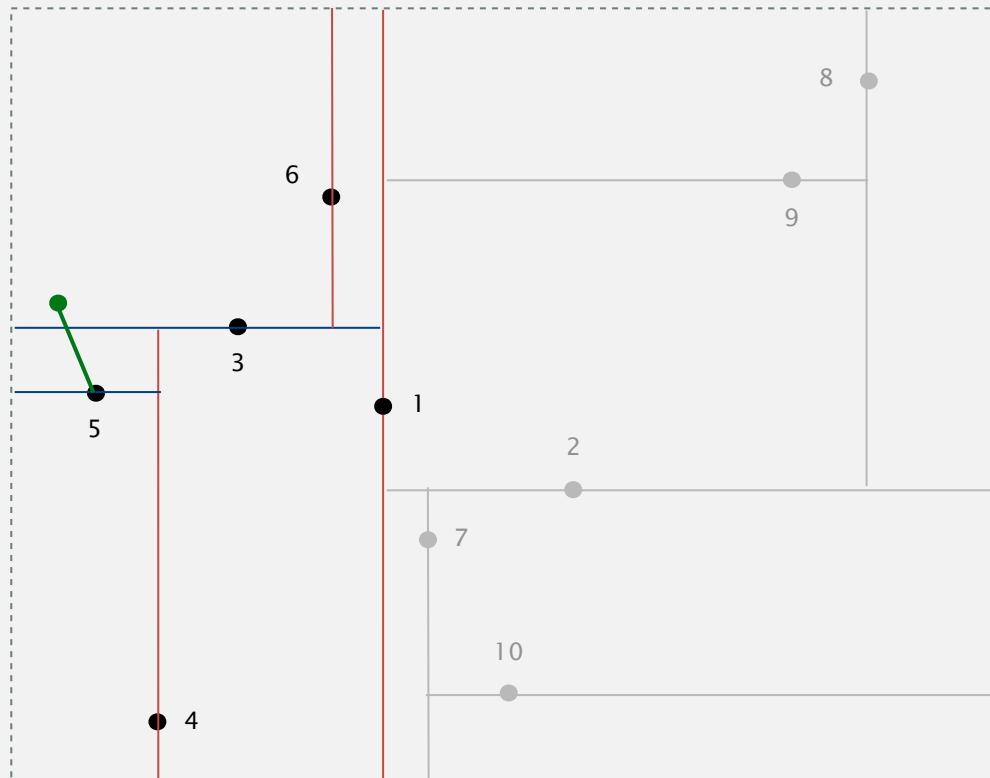
- Check distance from point in node to query point.
- Recursively search left/bottom (if it could contain a closer point).
- Recursively search right/top (if it could contain a closer point).
- Organize method so that it begins by searching for query point.



# Nearest neighbor search in a 2d tree analysis

Typical case.  $\log N$ .

Worst case (even if tree is balanced).  $N$ .



nearest neighbor = 5

## Flocking birds

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Q. What "natural algorithm" do starlings, migrating geese, starlings, cranes, bait balls of fish, and flashing fireflies use to flock?



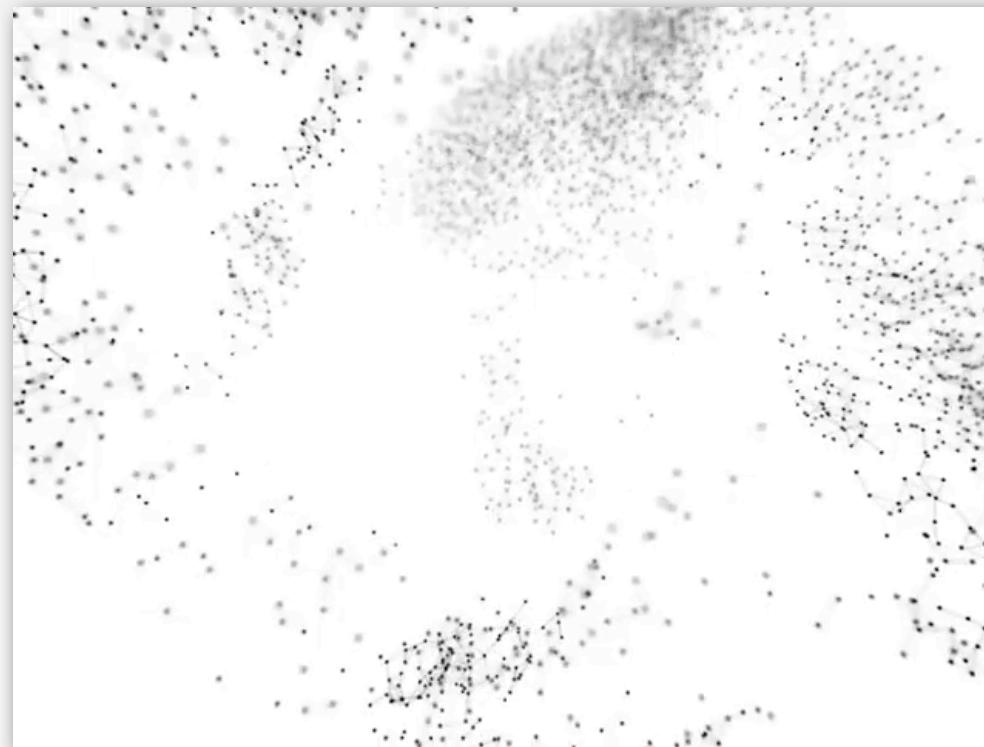
<http://www.youtube.com/watch?v=XH-groCeKbE>

## Flocking boids [Craig Reynolds, 1986]

---

Boids. Three simple rules lead to complex emergent flocking behavior:

- Collision avoidance: point away from **k nearest** boids.
- Flock centering: point towards the center of mass of **k nearest** boids.
- Velocity matching: update velocity to the average of **k nearest** boids.

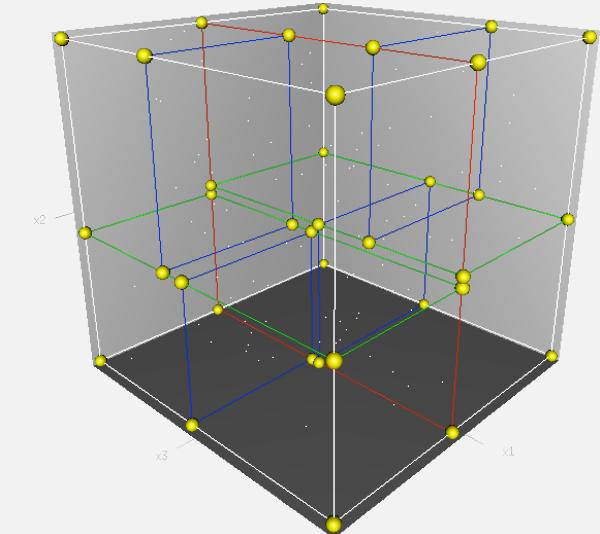
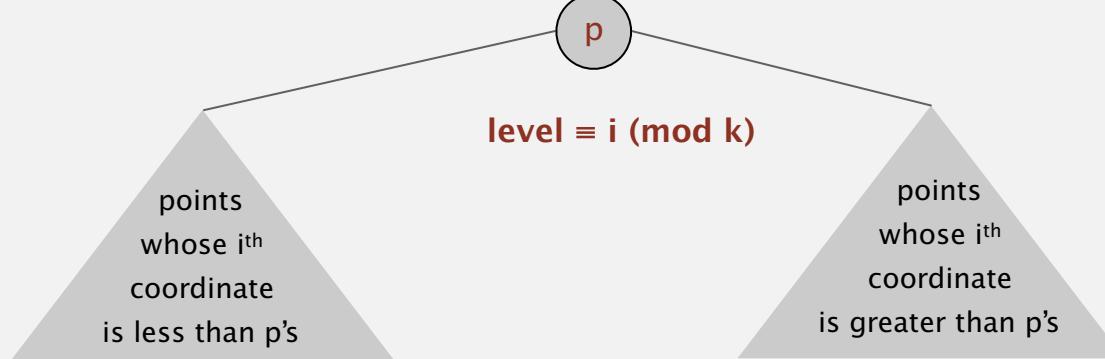


# Kd tree

---

Kd tree. Recursively partition  $k$ -dimensional space into 2 halfspaces.

Implementation. BST, but cycle through dimensions ala 2d trees.



Efficient, simple data structure for processing  $k$ -dimensional data.

- Widely used.
- Adapts well to high-dimensional and clustered data.
- Discovered by an undergrad in an algorithms class!



Jon Bentley

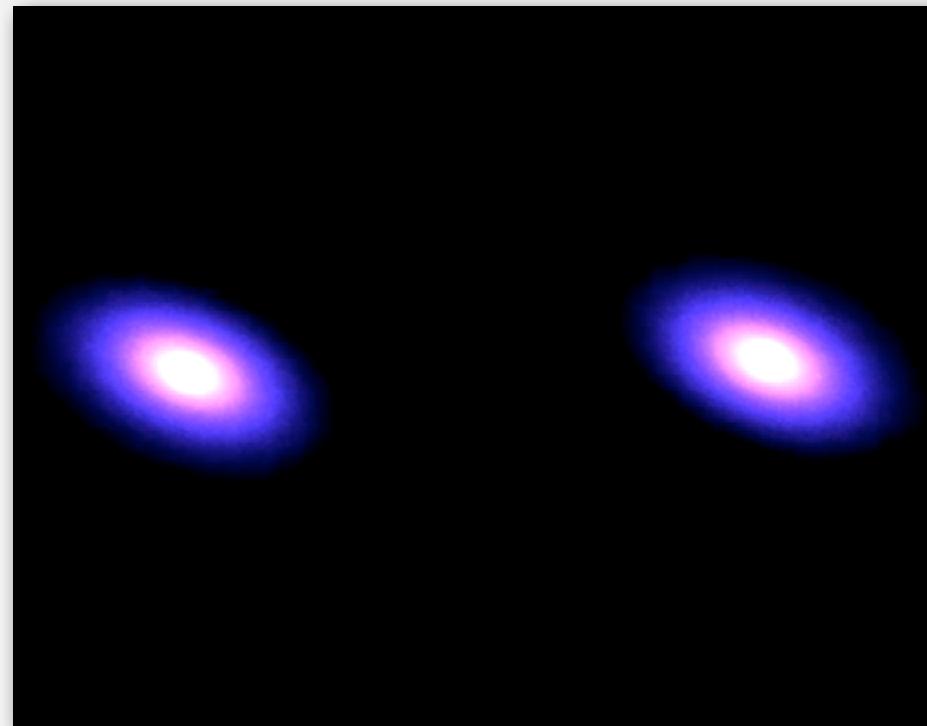
## N-body simulation

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**Goal.** Simulate the motion of  $N$  particles, mutually affected by gravity.

**Brute force.** For each pair of particles, compute force:  $F = \frac{G m_1 m_2}{r^2}$

**Running time.** Time per step is  $N^2$ .



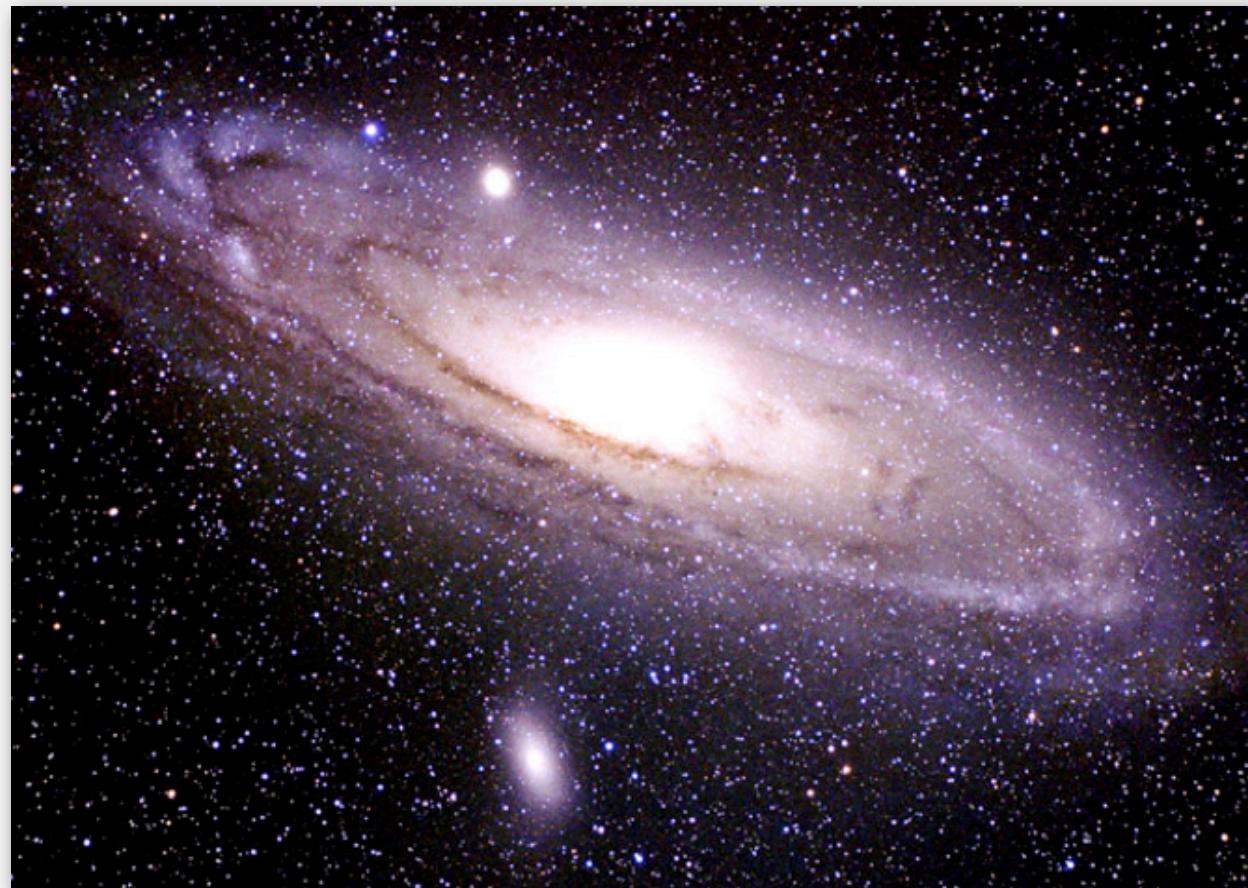
[http://www.youtube.com/watch?v=ua7YlN4eL\\_w](http://www.youtube.com/watch?v=ua7YlN4eL_w)

# Appel's algorithm for N-body simulation

---

**Key idea.** Suppose particle is far, far away from cluster of particles.

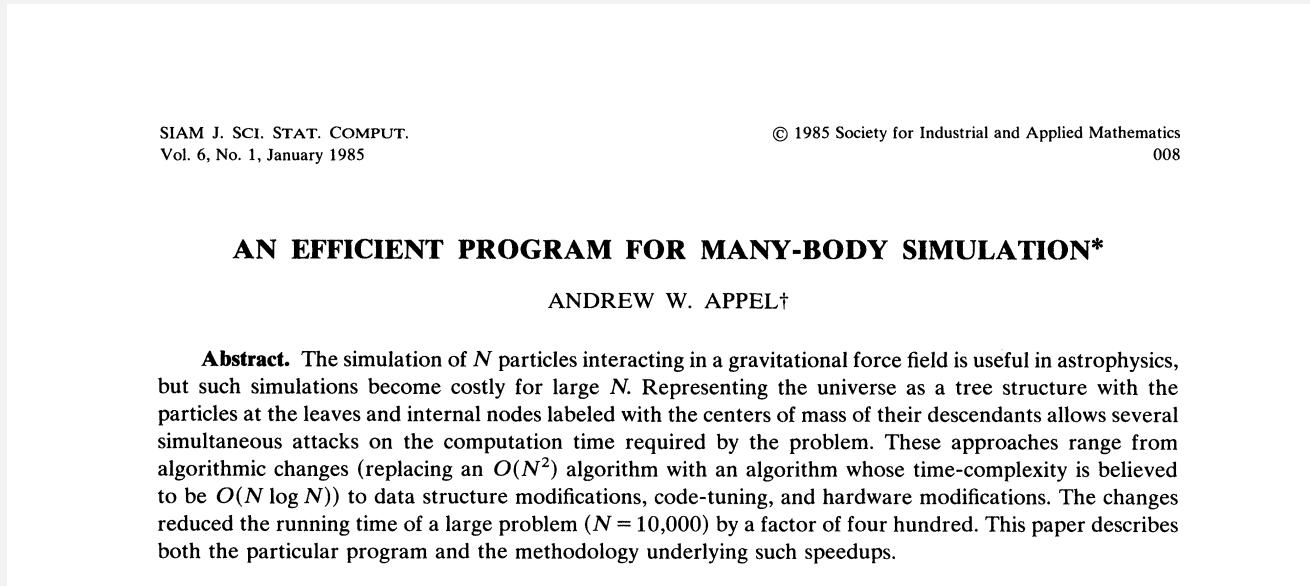
- Treat cluster of particles as a single aggregate particle.
- Compute force between particle and **center of mass** of aggregate.



# Appel's algorithm for N-body simulation

---

- Build 3d-tree with  $N$  particles as nodes.
- Store center-of-mass of subtree in each node.
- To compute total force acting on a particle, traverse tree, but stop as soon as distance from particle to subdivision is sufficiently large.



**Impact.** Running time per step is  $N \log N \Rightarrow$  enables new research.

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## 1d interval search

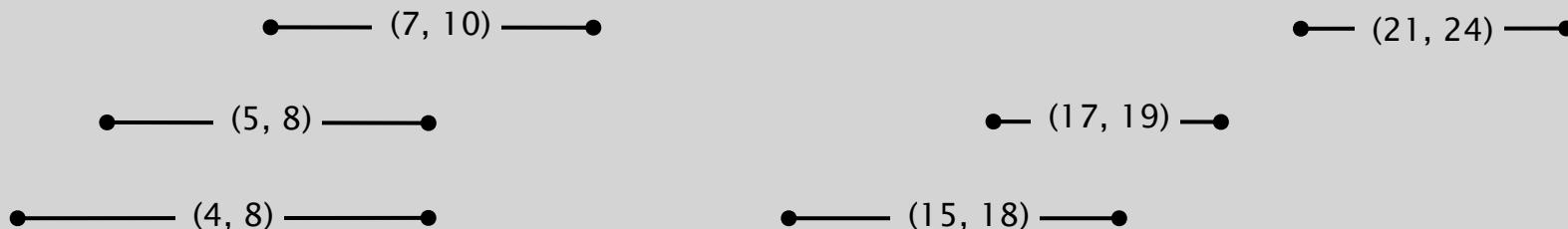
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**1d interval search.** Data structure to hold set of (overlapping) intervals.

- Insert an interval ( $lo, hi$ ).
- Search for an interval ( $lo, hi$ ).
- Delete an interval ( $lo, hi$ ).
- **Interval intersection query:** given an interval ( $lo, hi$ ), find all intervals (or one interval) in data structure that intersects ( $lo, hi$ ).

**Q.** Which intervals intersect ( $9, 16$ )?

**A.** ( $7, 10$ ) and ( $15, 18$ ).



# 1d interval search API

---

public class IntervalST<Key extends Comparable<Key>, Value>	
IntervalST()	<i>create interval search tree</i>
void put(Key lo, Key hi, Value val)	<i>put interval-value pair into ST</i>
Value get(Key lo, Key hi)	<i>value paired with given interval</i>
void delete(Key lo, Key hi)	<i>delete the given interval</i>
Iterable<Value> intersects(Key lo, Key hi)	<i>all intervals that intersect the given interval</i>

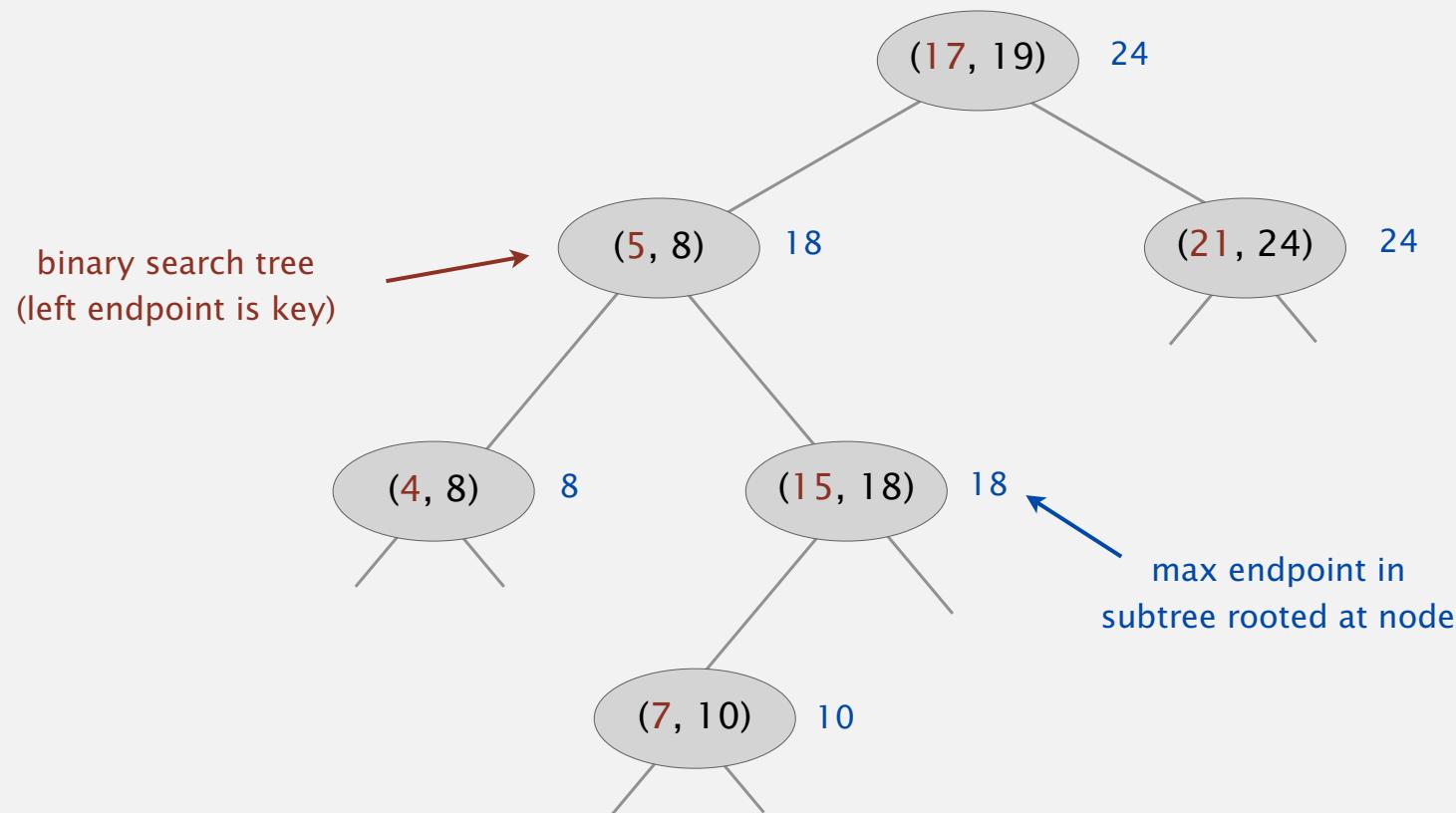
**Nondegeneracy assumption.** No two intervals have the same left endpoint.

# Interval search trees

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Create BST, where each node stores an interval ( $lo, hi$ ).

- Use left endpoint as BST **key**.
- Store **max endpoint** in subtree rooted at node.



# Interval search tree demo

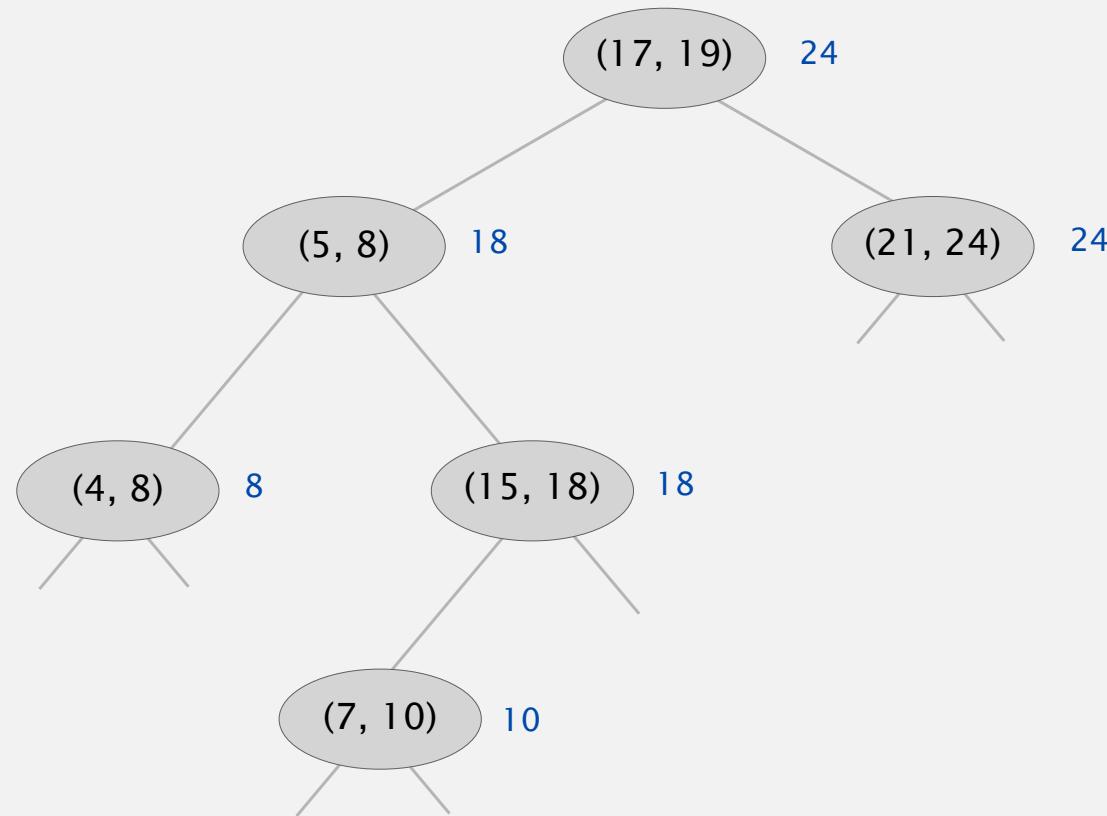
---

To insert an interval  $(lo, hi)$ :

- Insert into BST, using  $lo$  as the key.
- Update max in each node on search path.



**insert interval (16, 22)**

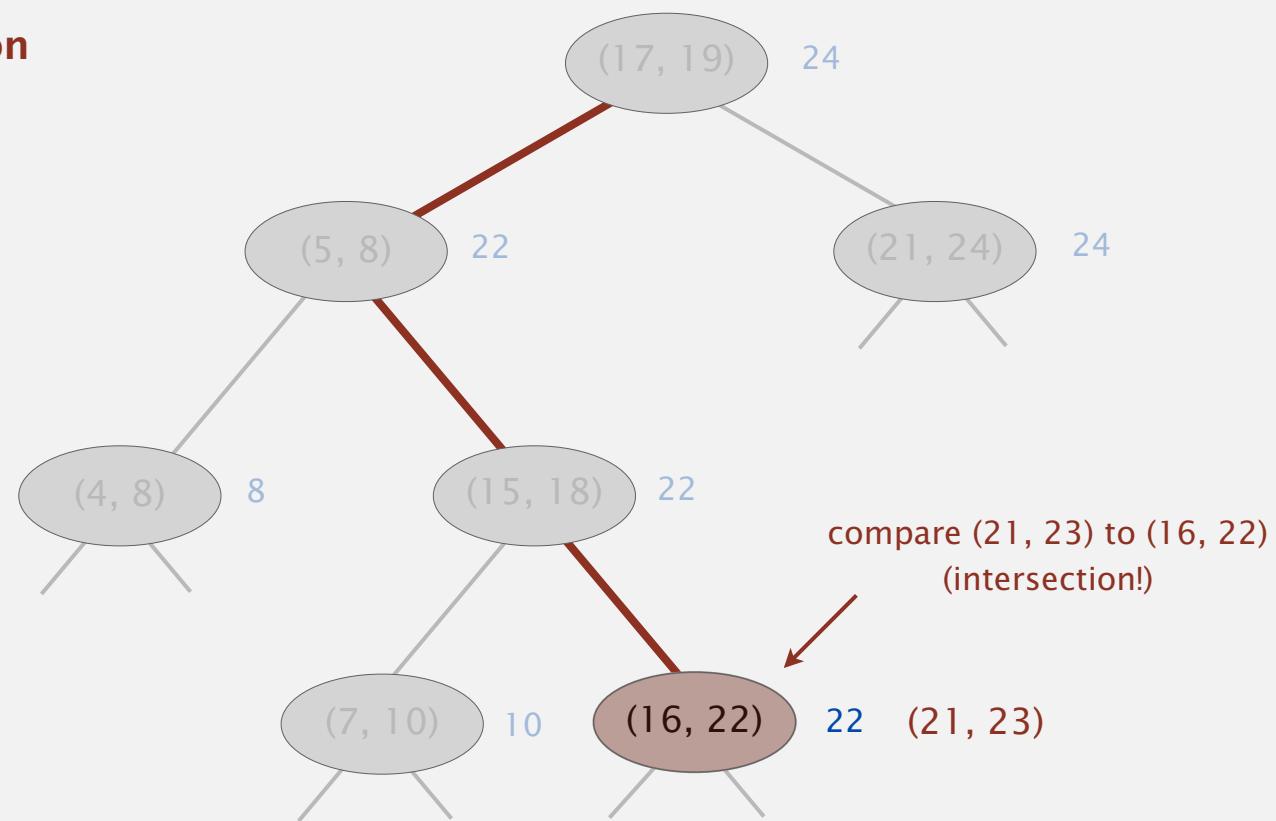


# Interval search tree demo

To search for any one interval that intersects query interval ( $lo, hi$ ):

- If interval in node intersects query interval, return it.
- Else if left subtree is null, go right.
- Else if max endpoint in left subtree is less than  $lo$ , go right.
- Else go left.

interval intersection  
search for (21, 23)



## Search for an intersecting interval implementation

---

To search for any one interval that intersects query interval ( $lo, hi$ ):

- If interval in node intersects query interval, return it.
- Else if left subtree is null, go right.
- Else if max endpoint in left subtree is less than  $lo$ , go right.
- Else go left.

```
Node x = root;
while (x != null)
{
    if      (x.interval.intersects(lo, hi)) return x.interval;
    else if (x.left == null)                  x = x.right;
    else if (x.left.max < lo)                x = x.right;
    else                                      x = x.left;
}
return null;
```

## Search for an intersecting interval analysis

To search for any one interval that intersects query interval ( $lo, hi$ ):

- If interval in node intersects query interval, return it.
- Else if left subtree is null, go right.
- Else if max endpoint in left subtree is less than  $lo$ , go right.
- Else go left.

Case 1. If search goes **right**, then no intersection in left.

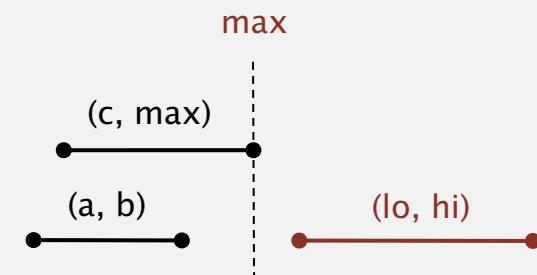
Pf. Suppose search goes right and left subtree is non empty.

- Max endpoint  $max$  in left subtree is less than  $lo$ .
- For any interval  $(a, b)$  in left subtree of  $x$ ,

we have  $b \leq max < lo$ .

definition of max

reason for going right



left subtree of x

right subtree of x

## Search for an intersecting interval analysis

To search for any one interval that intersects query interval  $(lo, hi)$ :

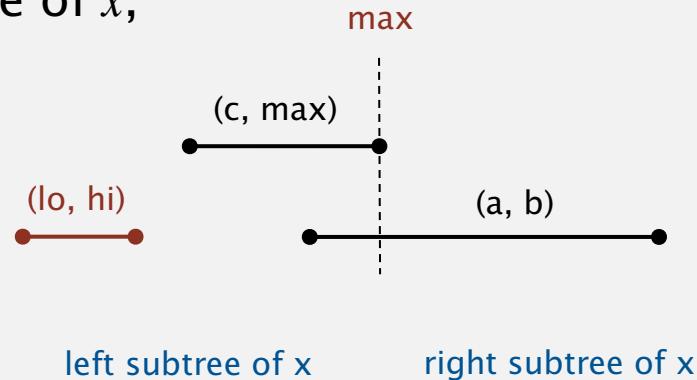
- If interval in node intersects query interval, return it.
- Else if left subtree is null, go right.
- Else if max endpoint in left subtree is less than  $lo$ , go right.
- Else go left.

Case 2. If search goes **left**, then there is either an intersection in left subtree or no intersections in either.

Pf. Suppose no intersection in left.

- Since went left, we have  $lo \leq max$ .
- Then for any interval  $(a, b)$  in right subtree of  $x$ ,  
 $hi < c \leq a \Rightarrow$  no intersection in right.

no intersections in left subtree      intervals sorted by left endpoint



# Interval search tree: analysis

Implementation. Use a red-black BST to guarantee performance.

easy to maintain auxiliary information  
using  $\log N$  extra work per op

operation	brute	interval search tree	best in theory
insert interval	1	$\log N$	$\log N$
find interval	$N$	$\log N$	$\log N$
delete interval	$N$	$\log N$	$\log N$
find any one interval that intersects $(lo, hi)$	$N$	$\log N$	$\log N$
find all intervals that intersects $(lo, hi)$	$N$	$R \log N$	$R + \log N$

order of growth of running time for  $N$  intervals

# Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

<http://algs4.cs.princeton.edu>

## GEOMETRIC APPLICATIONS OF BSTs

---

- ▶ *1d range search*
- ▶ *line segment intersection*
- ▶ *kd trees*
- ▶ *interval search trees*
- ▶ *rectangle intersection*

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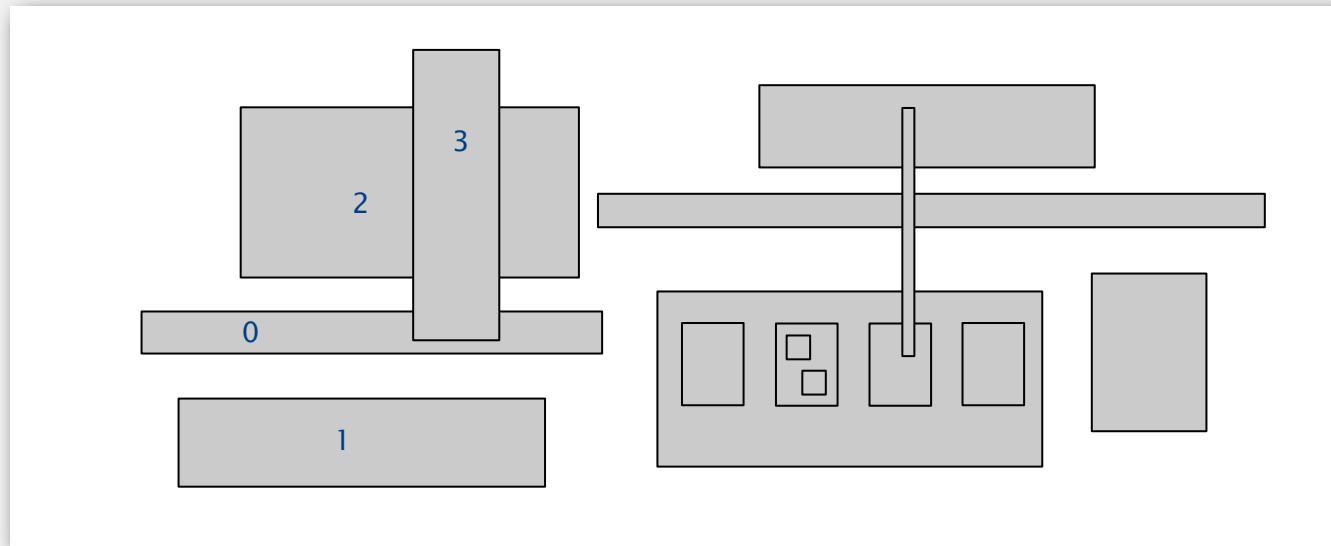
- ▶ *1d range search*
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# Orthogonal rectangle intersection

---

**Goal.** Find all intersections among a set of  $N$  orthogonal rectangles.

**Quadratic algorithm.** Check all pairs of rectangles for intersection.



**Non-degeneracy assumption.** All  $x$ - and  $y$ -coordinates are distinct.

# Microprocessors and geometry

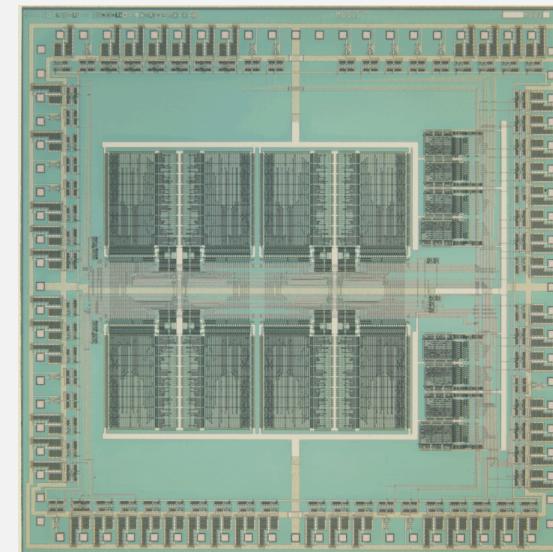
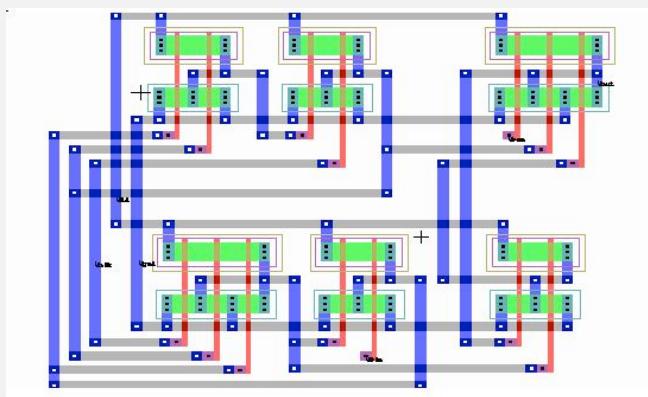
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Early 1970s. microprocessor design became a **geometric** problem.

- Very Large Scale Integration (VLSI).
- Computer-Aided Design (CAD).

Design-rule checking.

- Certain wires cannot intersect.
- Certain spacing needed between different types of wires.
- Debugging = orthogonal rectangle intersection search.



# Algorithms and Moore's law

"Moore's law." Processing power doubles every 18 months.

- $197x$ : check  $N$  rectangles.
- $197(x+1.5)$ : check  $2N$  rectangles on a  $2x$ -faster computer.



Gordon Moore

Bootstrapping. We get to use the faster computer for bigger circuits.

But bootstrapping is not enough if using a quadratic algorithm:

- $197x$ : takes  $M$  days.
- $197(x+1.5)$ : takes  $(4M)/2 = 2M$  days. (!)

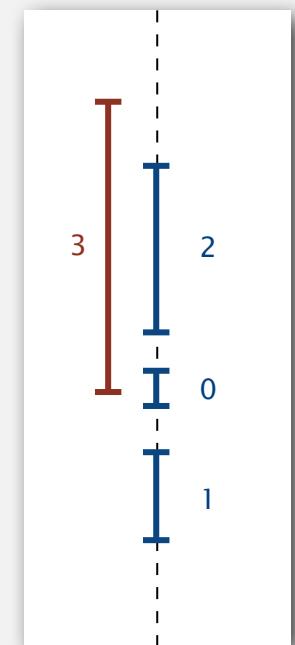
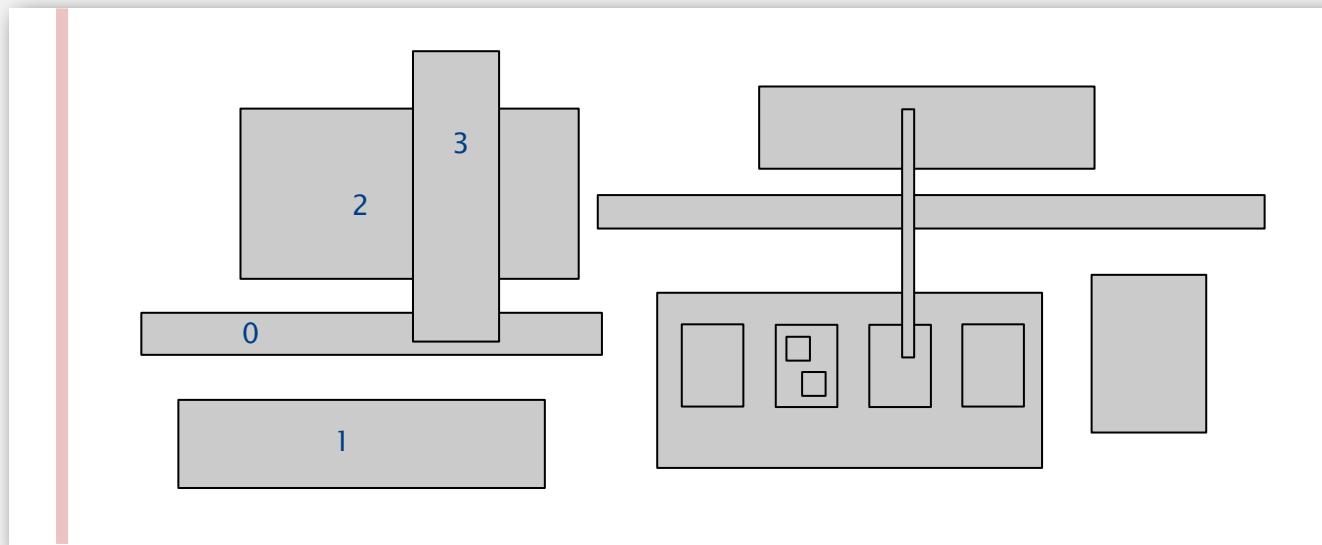


Bottom line. Linearithmic algorithm is **necessary** to sustain Moore's Law.

# Orthogonal rectangle intersection: sweep-line algorithm

Sweep vertical line from left to right.

- $x$ -coordinates of left and right endpoints define events.
- Maintain set of rectangles that intersect the sweep line in an interval search tree (using  $y$ -intervals of rectangle).
- Left endpoint: interval search for  $y$ -interval of rectangle; insert  $y$ -interval.
- Right endpoint: remove  $y$ -interval.



y-coordinates

## Orthogonal rectangle intersection: sweep-line analysis

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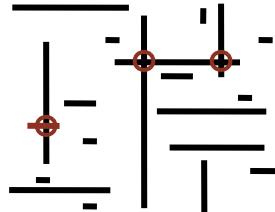
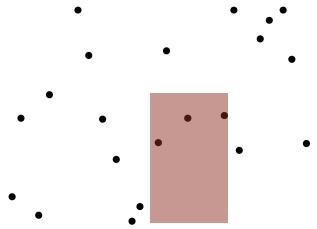
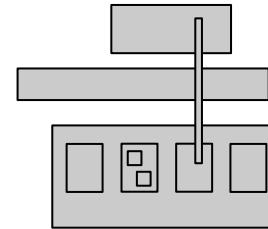
**Proposition.** Sweep line algorithm takes time proportional to  $N \log N + R \log N$  to find  $R$  intersections among a set of  $N$  rectangles.

Pf.

- Put  $x$ -coordinates on a PQ (or sort).  $\leftarrow N \log N$
- Insert  $y$ -intervals into ST.  $\leftarrow N \log N$
- Delete  $y$ -intervals from ST.  $\leftarrow N \log N$
- Interval searches for  $y$ -intervals.  $\leftarrow N \log N + R \log N$

**Bottom line.** Sweep line reduces 2d orthogonal rectangle intersection search to 1d interval search.

# Geometric applications of BSTs

problem	example	solution
1d range search	..... ..... .... ..... .....	BST
2d orthogonal line segment intersection		sweep line reduces to 1d range search
kd range search		kd tree
1d interval search		interval search tree
2d orthogonal rectangle intersection		sweep line reduces to 1d interval search

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