Quantum Walk Algorithms for 2D Spatial Search

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Abstract: Quantum algorithms is a field that is largely unexplored and has much scope for development. In this field, the optimal search algorithm is one under much pursuit. The basic quantum walk-based algorithms for finding a desired item out of N items in a 2D lattice of size $\sqrt{N}X\sqrt{N}$ take $O(\sqrt{NlogN})$ steps to solve the problem. But the algorithm proposed by Avatar Tulsi does the above in $O(\sqrt{NlogN})$ using an ancilla qubit. This project will explore the above algorithm along with the ones it is based on.

INTRODUCTION:

Consider a two dimensional lattice of size $\sqrt{N}X\sqrt{N}$ with sites labeled by x and y coordinates as $|x, y\rangle$. Classically, an algorithm to find a marked item would take $O(N)$ time steps. However, since the emergence of quantum computing, many algorithms have been developed that are more efficient in terms of computational complexity. While the hunt for the best quantum walk-based algorithm for two dimensional spatial search is still on, in this paper, we will be discussing the progress in the area in chronological fashion, as well as looking at attempts at implementation of some of these algorithms.

In the quantum scenario, the problem can be looked at as follows: There are N items on a two-dimensional lattice of size $\sqrt{N}X\sqrt{N}$, with sites denoted similarly. In this case, the coordinates correspond to basis elements of an N dimensional Hilbert Space. Let f(x,y) be a binary function, with f(x,y)=1 if the item at $|x, y\rangle$ is marked, and 0 otherwise. The problem is to find the distinct marked item $|m\rangle = |x, y\rangle$ with the least number of time steps possible, given the constraints that one time step we can either examine the current site (perform an oracle query) or move to a neighbouring lattice site.

The Grover's search algorithm provides a quadratic speedup in the 1 dimensional case, with a complexity of $O(\sqrt{N})$. However, in the 2D case, as shown by Benioff, the algorithm is not faster than classical ones. Aaronson and Ambainis proposed a divide-and-conquer based algorithm on 2D connected graphs to show that search is possible in $O(\sqrt{N}log^{3/2}N)$.

Ambainis, Kempe and Rivosh (AKR) improved on Aaronson and Ambainis' algorithm using a quantum walk search algorithm that exploits judicious 'coin-flip' operations. Similarly, Childs and Goldstone proposed a continuous time algorithm analogue of AKR's discrete algorithm. Both of these achieve a complexity of $O(\sqrt{NlogN})$.

As an improvement to AKR, Dr. Avatar Tulsi has developed a quantum walk-based search algorithm that uses an ancilla qubit to control the walk. It has a complexity of $O(\sqrt{NlogN})$.

THESIS STATEMENT:

In this project, I have reviewed Grover's algorithm, AKR's discrete quantum walk algorithm and Avatar Tulsi's controlled quantum walk algorithm. I have also implemented Grover's algorithm for the 1D case.

METHODOLOGY:

Grover's Algorithm

This algorithm speeds up unstructured search quadratically using an 'amplitude amplification trick'. Essentially, it is an $O(\sqrt{N})$ algorithm to find a marked item from a list of N items. A brief explanation of the algorithm in 1D case is as follows:

1. Uniform superposition of all items is carried out by $|s\rangle = \frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}|x\rangle$

This is implemented in a circuit using Hadamard gates.

2. Oracle reflection U_f is performed on state $|s\rangle$. The oracle is a diagonal matrix with all diagonal elements equal to 1 other than the one corresponding to the marked item. Geometrically, this step leads to sign flip of the marked item, hence lowering the amplitude of the average.

3. The operation $U_s = 2\,|s\rangle\,\langle s| - 1$ is performed. This completes amplitude amplification as only the marked item is raised to a higher amplitude.

After roughly $\sqrt{N}$ repetitions, the marked item's location can be determined with high accuracy.

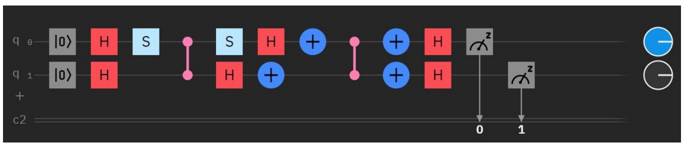

FIG. 1: Circuit for detecting marked item $|w\rangle = |01\rangle$

The Grover's algorithm has a computational complexity of $O(N)$ in the 2D case as the step involving reflection about the superposition takes $O(\sqrt{N})$ time steps as each time step allows movement only to neigbouring sites so $\sqrt{N}$ steps are required.

AKR's Discrete Quantum Walk Algorithm

This algorithm is based on quantum walks on an undirected graph. Unlike classical random walks, quantum ones require an additional 'coin' register to store the direction in which the walk is moving.

Discrete Quantum Walk: It is an alternation of coin flip and moving given by $U = S.C$ where
S: $|i\rangle \otimes|\text{x}\rangle \rightarrow|\pi(i)\rangle \otimes|\tilde{x}\rangle$
This basically corresponds to permutation of the d basis states of the coin space H_d. The coin flip C_0 is given by $C_0 = 2\,|s\rangle\,\langle s| - I_d$ where $|s\rangle$ is the superimposed state.

Perturbed quantum walk: This is the quantum walk modified for including a marked vertex v. It is given by $U' = S.C'$ where
$C' = C - (C_0 - C_1)\otimes|\text{v}\rangle\,\langle v|$

The steps in AKR's algorithm are as follows:
1. Obtain uniform superposition of states.
2. Apply the perturbed quantum walk U' T times, where $T = O(\sqrt{NlogN})$
3. Measure the position register.
4. Check if the measured vertex is marked.

The shifting operator for 2 dimensions is given below:

$$
\begin{aligned}
S_{ff}: \quad & |\rightarrow\rangle \otimes |x,y\rangle \longrightarrow |\leftarrow\rangle \otimes |x+1,y\rangle \\
& |\leftarrow\rangle \otimes |x,y\rangle \longrightarrow |\rightarrow\rangle \otimes |x-1,y\rangle \\
& |\uparrow\rangle \otimes |x,y\rangle \longrightarrow |\downarrow\rangle \otimes |x,y+1\rangle \\
& |\downarrow\rangle \otimes |x,y\rangle \longrightarrow |\uparrow\rangle \otimes |x,y-1\rangle
\end{aligned}
$$

FIG. 2: Flip-flop shift operator

Note that there is an alternate shift operator, the 'moving shift' given as

$$
\begin{aligned}
S_{m}: \quad & |\rightarrow\rangle \otimes |x,y\rangle \longrightarrow |\rightarrow\rangle \otimes |x+1,y\rangle \\
& |\leftarrow\rangle \otimes |x,y\rangle \longrightarrow |\leftarrow\rangle \otimes |x-1,y\rangle \\
& |\uparrow\rangle \otimes |x,y\rangle \longrightarrow |\uparrow\rangle \otimes |x,y+1\rangle \\
& |\downarrow\rangle \otimes |x,y\rangle \longrightarrow |\downarrow\rangle \otimes |x,y-1\rangle
\end{aligned}
$$

FIG. 3: Moving shift operator

However, this operator does not speed up search, and AKR show in their paper that it only provides a lower bound of $\Omega(N)$.

Controlled Quantum Walk Algorithm

This algorithm, proposed by Dr. Avatar Tulsi is an improvement of AKR's algorithm. It has a computational complexity of $O(\sqrt{NlogN})$ and uses an ancilla qubit to control the discrete quantum walk. The joint Hilbert space in this case is 8N dimensional and given by $H = H_b \otimes H_c \otimes H_N$.

The steps involved in the algorithm are:
1. Obtain uniform superposition of states. This takes $O(\sqrt{N})$ time steps.
2. Apply the operator U_C given by

$$U_C = (\bar{Z}_b).c_1 W.(X_\delta^\dagger)_b.c_1 \bar{R}_{uc,m}.(X_\delta)_b$$

Here, W is the walk operator defined in AKR's algorithm, $R_{uc,m}$ is the negative of the reflection about $|1\rangle|u_c\rangle|m\rangle$ state. Z and X_δ are the single qubit gates given by

$$X_\delta = \begin{pmatrix} cos\delta & sin\delta \\ -sin\delta & cos\delta \end{pmatrix}, \quad \bar{Z} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

FIG. 4: Single qubit gates Z and X_δ used in the controlled quantum walk algorithm

The choice of δ is quite important. The optimal choice is $cos\delta = \Theta(\sqrt{1/logN})$, while for $\delta = 0$, the ancilla qubit is redundant and the algorithm reduces to AKR's discrete quantum walk algorithm. The marked item can be found with high accuracy when the operator is applied iteratively $O(\sqrt{NlogN})$ times.

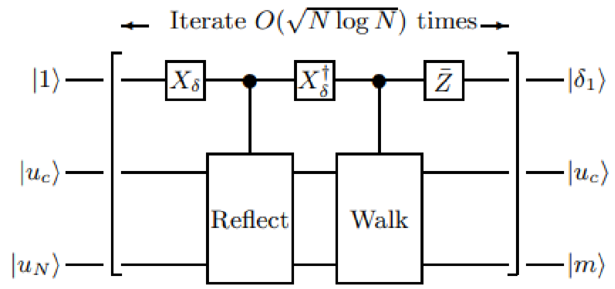

FIG. 5: Circuit for controlled quantum walk algorithm for 2D spatial search

Above is the circuit for the controlled quantum walk algorithm. The walk W and reflection $R_{uc,m}$ are performed if and only if the ancilla qubit is $|1\rangle$.

RESULTS:

Unfortunately, due to limitations in qiskit (only 5 qubits permitted), implementation of 2D search algorithms has not been possible.

DISCUSSION AND CONCLUSION:

We have seen that over the years, quantum algorithms in the area of 2 dimensional spatial search have progressed significantly in terms of computational complexity. However, despite numerous innovations, it is still an open question if the lower bound of $\Omega(N)$ can be reached.
An interesting area to explore as pointed out by Dr. Saikat Ghosh is the space allotment efficiency and resource theory involved. Apart from this, there is also much potential in the area of implementation of these algorithms and application to real life problems.
Finally, another point to note is that it is quite easy to generalize these algorithms to multiple marked items and higher dimensions. Interestingly, AKR's algorithm completes search in $c\sqrt{N}$ time steps for higher dimensions, which is the lower bound for quantum search algorithms. Dr. Tulsi's algorithm does improve on this as well, but only in the constant factor c.

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