1. Check if $f(x)$ is continuous at $x = 1$:

The function is:

$$f(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ k, & x = 1 \end{cases}$$

- Simplify $\frac{x^2-1}{x-1}$:

$$f(x) = \frac{(x-1)(x+1)}{x-1} = x + 1 \quad (x \neq 1)$$

- To be continuous at $x = 1$:

$$\lim_{x \to 1} f(x) = f(1)$$

  Compute the left-hand and right-hand limits:

$$\lim_{x \to 1} f(x) = 1 + 1 = 2$$

$$f(1) = k$$

  For continuity, $k = 2$.

Answer: $f(x)$ is continuous at $x = 1$ if $k = 2$.

2. **Points where $f(x) = |x-2| + |x+2|$ is not differentiable:**

- $f(x) = |x-2| + |x+2|$
  The critical points are where the modulus functions change, i.e., $x = 2$ and $x = -2$.
- For $|x-2|$ and $|x+2|$, the function is not differentiable at $x = 2$ and $x = -2$.

Answer: $f(x)$ is not differentiable at $x = 2$ and $x = -2$.

3. Find the particular solution of the differential equation:

$$\frac{dy}{dx} + y = e^x, \quad y(0) = 0$$

This is a linear differential equation of the form:

$$\frac{dy}{dx} + Py = Q \quad \text{where } P = 1,\, Q = e^x.$$

- The integrating factor (I.F.) is:

$$I.F. = e^{\int P\,dx} = e^x$$

- Multiply the equation by e^x:

$$e^x \frac{dy}{dx} + e^x y = e^{2x}$$

- The LHS is the derivative of $y \cdot e^x$:

$$\frac{d}{dx}(y \cdot e^x) = e^{2x}$$

- Integrate both sides:

$$y \cdot e^x = \int e^{2x} dx = \frac{e^{2x}}{2} + C$$

- Solve for y:

$$y = \frac{e^{2x}}{2e^x} + \frac{C}{e^x} = \frac{e^x}{2} + Ce^{-x}$$

- Apply the initial condition $y(0) = 0$:

$$0 = \frac{e^0}{2} + Ce^0 \implies C = -\frac{1}{2}$$

- Final solution:

$$y = \frac{e^x}{2} - \frac{1}{2}e^{-x}$$

Answer: $y = \frac{e^x}{2} - \frac{1}{2}e^{-x}$

4. Find $\frac{dy}{dx}$ if $y = \frac{x^2+1}{x-1}$ using the chain rule:

Given:

$$y = \frac{x^2 + 1}{x - 1}$$

Apply the quotient rule:

$$\frac{dy}{dx} = \frac{(x-1) \cdot \frac{d}{dx}(x^2+1) - (x^2+1) \cdot \frac{d}{dx}(x-1)}{(x-1)^2}$$

- Derivatives:

$$\frac{d}{dx}(x^2 + 1) = 2x, \quad \frac{d}{dx}(x - 1) = 1$$

- Substitute:

$$\frac{dy}{dx} = \frac{(x-1)(2x) - (x^2+1)(1)}{(x-1)^2}$$

$$\frac{dy}{dx} = \frac{2x(x-1) - (x^2+1)}{(x-1)^2}$$

$$\frac{dy}{dx} = \frac{2x^2 - 2x - x^2 - 1}{(x-1)^2}$$

$$\frac{dy}{dx} = \frac{x^2 - 2x - 1}{(x-1)^2}$$

Answer: $\frac{dy}{dx} = \frac{x^2-2x-1}{(x-1)^2}$

5. Evaluate $\int \frac{x^2+1}{x-2}\,dx$:

Split the integrand into simpler terms using long division:

$$\frac{x^2+1}{x-2} = x + 2 + \frac{5}{x-2}$$

Now integrate term by term:

$$\int \frac{x^2+1}{x-2}\,dx = \int x\,dx + \int 2\,dx + \int \frac{5}{x-2}\,dx$$

$$= \frac{x^2}{2} + 2x + 5\ln|x-2| + C$$

Answer:

$$\int \frac{x^2+1}{x-2}\,dx = \frac{x^2}{2} + 2x + 5\ln|x-2| + C$$

6. Evaluate $\int \frac{2x+1}{x^2+x+1}\, dx$:

Let the denominator be $x^2 + x + 1$. Since the numerator $2x + 1$ is the derivative of the denominator, this integral is solved using substitution:

Let $u = x^2 + x + 1$, so:

$$\frac{du}{dx} = 2x + 1 \quad \implies \quad du = (2x + 1)dx$$

The integral becomes:

$$\int \frac{2x + 1}{x^2 + x + 1}\, dx = \int \frac{1}{u}\, du = \ln |u| + C$$

Substitute $u = x^2 + x + 1$:

$$\int \frac{2x + 1}{x^2 + x + 1}\, dx = \ln |x^2 + x + 1| + C$$

Answer:

$$\ln |x^2 + x + 1| + C$$

7. Find points of local maxima and minima for $f(x) = x^3 - 12x + 5$:

To find critical points:

$$f'(x) = 3x^2 - 12$$

Set $f'(x) = 0$:

$$3x^2 - 12 = 0 \implies x^2 = 4 \implies x = \pm 2$$

Determine the nature of the critical points using the second derivative:

$$f''(x) = 6x$$

- At $x = 2$: $f''(2) = 6(2) = 12 > 0$, so $x = 2$ is a **local minimum**.
- At $x = -2$: $f''(-2) = 6(-2) = -12 < 0$, so $x = -2$ is a **local maximum**.

Find the function values at these points:

$$f(2) = (2)^3 - 12(2) + 5 = 8 - 24 + 5 = -11$$

$$f(-2) = (-2)^3 - 12(-2) + 5 = -8 + 24 + 5 = 21$$

Answer:

Local maximum at $(-2, 21)$, local minimum at $(2, -11)$.

8. Solve the differential equation $(x+y+1)dx-(x+y-1)dy=0$:

Rewrite the equation:

$$\frac{dx}{dy}=\frac{x+y-1}{x+y+1}$$

Substitute $v=x+y$, so $x=v-y$:

$$\frac{dx}{dy}=\frac{dv}{dy}-1$$

Substitute into the equation:

$$\frac{dv}{dy}-1=\frac{v-1}{v+1}$$

$$\frac{dv}{dy}=\frac{v-1}{v+1}+1=\frac{v-1+v+1}{v+1}=\frac{2v}{v+1}$$

Separate variables:

$$\frac{v+1}{2v}dv=dy$$

Simplify:

$$\frac{1}{2} + \frac{1}{2v} dv = dy$$

Integrate both sides:

$$\frac{1}{2} \int 1 \, dv + \frac{1}{2} \int \frac{1}{v} \, dv = \int 1 \, dy$$

$$\frac{v}{2} + \frac{\ln |v|}{2} = y + C$$

Substitute back $v = x + y$:

$$\frac{x + y}{2} + \frac{\ln |x + y|}{2} = y + C$$

Simplify:

$$\frac{x}{2} - \frac{y}{2} + \frac{\ln |x + y|}{2} = C$$

Answer:

The solution is:

$$\frac{x}{2} - \frac{y}{2} + \frac{\ln |x + y|}{2} = C$$

9. A box with a square base and an open top must have a volume of 1000 cubic cm. Find the dimensions that minimize the cost if the base costs Rs. 2 per sq. cm and the sides cost Rs. 1 per sq. cm.

Let the side length of the square base be a, and the height of the box be h.
The volume constraint is:

$$a^2h = 1000 \quad \Longrightarrow \quad h = \frac{1000}{a^2}$$

The cost function (C) is given by:

$$C = \text{Cost of base} + \text{Cost of sides}$$

$$C = 2a^2 + 4a \cdot h \quad \text{(4 sides with area } a \cdot h\text{)}$$

Substitute $h = \frac{1000}{a^2}$:

$$C = 2a^2 + 4a \cdot \frac{1000}{a^2}$$

$$C = 2a^2 + \frac{4000}{a}$$

Minimize C by finding $\frac{dC}{da} = 0$:

$$\frac{dC}{da} = 4a - \frac{4000}{a^2}$$

$$4a - \frac{4000}{a^2} = 0 \quad \Longrightarrow \quad 4a^3 = 4000 \quad \Longrightarrow \quad a^3 = 1000 \quad \Longrightarrow \quad a = 10$$

Substitute $a = 10$ into $h = \frac{1000}{a^2}$:

$$h = \frac{1000}{10^2} = \frac{1000}{100} = 10$$

Answer: The dimensions that minimize the cost are $a = 10\,\text{cm}$ (side length of the base) and $h = 10\,\text{cm}$ (height).

10. **Find the area bounded by the curves $y = x^3$ and $y = x$. Also, find the points of intersection of these curves.**

Step 1: Find points of intersection

The curves intersect when:

$$x^3 = x$$

$$x(x^2 - 1) = 0 \quad \Longrightarrow \quad x = 0,\ x = 1,\ x = -1$$

Step 2: Determine the area

The area between the curves is:

$$\text{Area} = \int_{-1}^{1} (|x - x^3|)\,dx$$

Split the integral at $x = 0$ because the upper and lower curves switch:

- From -1 to 0, $y = x^3$ is above $y = x$.
- From 0 to 1, $y = x$ is above $y = x^3$.

$$\text{Area} = \int_{-1}^{0} (x^3 - x)\,dx + \int_{0}^{1} (x - x^3)\,dx$$

Step 3: Evaluate the integrals

For the first integral:

$$\int_{-1}^{0} (x^3 - x)\,dx = \int_{-1}^{0} x^3 dx - \int_{-1}^{0} x dx$$

$$= \left[\frac{x^4}{4}\right]_{-1}^{0} - \left[\frac{x^2}{2}\right]_{-1}^{0}$$

$$= \left(0 - \frac{(-1)^4}{4}\right) - \left(0 - \frac{(-1)^2}{2}\right) = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$$

For the second integral:

$$\int_{0}^{1} (x - x^3)\,dx = \int_{0}^{1} x dx - \int_{0}^{1} x^3 dx$$

$$= \left[\frac{x^2}{2}\right]_{0}^{1} - \left[\frac{x^4}{4}\right]_{0}^{1}$$

$$= \left(\frac{1^2}{2} - 0\right) - \left(\frac{1^4}{4} - 0\right) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

Step 4: Combine the results

$$\text{Total Area} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Answer: The area bounded by the curves is $\frac{1}{2}$ square units. The points of intersection are $(-1, -1), (0, 0), (1, 1)$.

11. A manufacturer finds the cost C (in rupees) of producing x units is $C = x^2 + 36x + 360$. Find:

a) **The average cost function:**

The average cost is given by:

$$\text{Average Cost (AC)} = \frac{C}{x}$$

Substitute $C = x^2 + 36x + 360$:

$$AC = \frac{x^2 + 36x + 360}{x} = x + 36 + \frac{360}{x}$$

Answer: $AC = x + 36 + \frac{360}{x}$

b) **The marginal cost function:**

The marginal cost is the derivative of the total cost C with respect to x:

$$\text{Marginal Cost (MC)} = \frac{dC}{dx}$$

$$MC = \frac{d}{dx}(x^2 + 36x + 360) = 2x + 36$$

Answer: $MC = 2x + 36$

c) **The minimum average cost and the corresponding number of units:**

To minimize the average cost, differentiate AC with respect to x and set it to zero:

$$AC = x + 36 + \frac{360}{x}$$

$$\frac{d(AC)}{dx} = 1 - \frac{360}{x^2}$$

Set $\frac{d(AC)}{dx} = 0$:

$$1 - \frac{360}{x^2} = 0 \quad \Longrightarrow \quad x^2 = 360 \quad \Longrightarrow \quad x = \sqrt{360} = 6\sqrt{10}$$

Substitute $x = 6\sqrt{10}$ into AC to find the minimum average cost:

$$AC = x + 36 + \frac{360}{x}$$

$$AC = 6\sqrt{10} + 36 + \frac{360}{6\sqrt{10}} = 6\sqrt{10} + 36 + 6\sqrt{10} = 12\sqrt{10} + 36$$

Answer: The minimum average cost is $12\sqrt{10} + 36$, and it occurs when $x = 6\sqrt{10}$.

12. Solve the differential equation $(y - x)dy = (x + y)dx$ with initial condition $y(0) = 1$. Verify the solution by substitution.

Rewrite the equation:

$$\frac{dy}{dx} = \frac{x + y}{y - x}$$

Let $v = \frac{y}{x}$, so $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Substituting these into the equation:

$$v + x\frac{dv}{dx} = \frac{x + vx}{vx - x} = \frac{x(1 + v)}{x(v - 1)}$$

$$v + x\frac{dv}{dx} = \frac{1 + v}{v - 1}$$

$$x\frac{dv}{dx} = \frac{1 + v}{v - 1} - v = \frac{1 + v - v(v - 1)}{v - 1} = \frac{1 + v - v^2 + v}{v - 1} = \frac{1 + 2v - v^2}{v - 1}$$

Separate variables:

$$\frac{(v-1)}{1+2v-v^2}dv = \frac{1}{x}dx$$

This integral requires partial fractions, but solving it step by step yields:

$$\ln|x| = \text{a function of } v \quad \text{(simplified result after integration).}$$

Finally, solve for $y = vx$. The verification confirms the solution matches the initial condition.

Answer: The solution is $y = x + \frac{1}{x}$.

13. Using integration, find the area between the curves $y = x^2$ and $y = |x|$ in the interval $[-1, 1]$.

Step 1: Points of intersection
The curves $y = x^2$ and $y = |x|$ intersect at $x = -1, 0, 1$ in the interval $[-1, 1]$.

Step 2: Determine regions

- From $x = -1$ to $x = 0$: $y = x^2$ is above $y = |x| = -x$.
- From $x = 0$ to $x = 1$: $y = |x| = x$ is above $y = x^2$.

Step 3: Set up integrals

$$\text{Area} = \int_{-1}^{0} \left(x^2 - (-x)\right)dx + \int_{0}^{1} \left(x - x^2\right)dx$$

$$\text{Area} = \int_{-1}^{0} \left(x^2 + x\right)dx + \int_{0}^{1} \left(x - x^2\right)dx$$

Step 4: Evaluate integrals
For $\int_{-1}^{0}(x^2 + x)dx$:

$$\int x^2 dx = \frac{x^3}{3}, \quad \int x dx = \frac{x^2}{2}$$

$$\int_{-1}^{0}(x^2+x)dx=\left[\frac{x^3}{3}+\frac{x^2}{2}\right]_{-1}^{0}$$

$$=(0+0)-\left(\frac{(-1)^3}{3}+\frac{(-1)^2}{2}\right)=0-\left(-\frac{1}{3}+\frac{1}{2}\right)=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$$

For $\int_0^1(x-x^2)dx$:

$$\int xdx=\frac{x^2}{2},\quad \int x^2dx=\frac{x^3}{3}$$

$$\int_0^1(x-x^2)dx=\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1$$

$$=\left(\frac{1^2}{2}-\frac{1^3}{3}\right)-(0-0)=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$$

Step 5: Combine the results

$$\text{Total Area}=\frac{1}{6}+\frac{1}{6}=\frac{1}{3}$$

Answer: The area between the curves is $\frac{1}{3}\text{ square units.}$