Heat and Mass Transfer

Introduction

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Course: Heat and Mass Transfer

Prerequisites

Thermodynamics, Fluid Mechanics

References

- Incropera FP and Dewitt DP, *Fundamentals of Heat and Mass Transfer*, Fifth edition, John Wiley and Sons, 2010.
- Cengel YA, *Heat and Mass Transfer - A Practical Approach*, Third edition, McGraw-Hill, 2010.
- Holman JP, *Heat Transfer*, McGraw-Hill, 1997.

Class Timings: ME305

Tue:	9 AM to 10 AM,	Room-107
Wed, Thu, Fri:	11 AM to 12 AM,	Room-107

Weblinks

www.iitp.ac.in/∼sudheer/teaching.html

Course Content: Heat and Mass Transfer

Introduction:

- What, How, and Where?
- Thermodynamics and Heat transfer
- Application
- Physical mechanism of heat transfer

Conduction:

- Introduction
- 1D, steady-state
- 2D, steady-state
- Transient

Convection:

- Introduction
- External and internal flows
- Free convection
- Boiling and condensation
- Heat exchangers

Radiation:

- Introduction
- View factors

Mass Transfer:

- Introduction
- Mass diffusion equation
- Transient diffusion

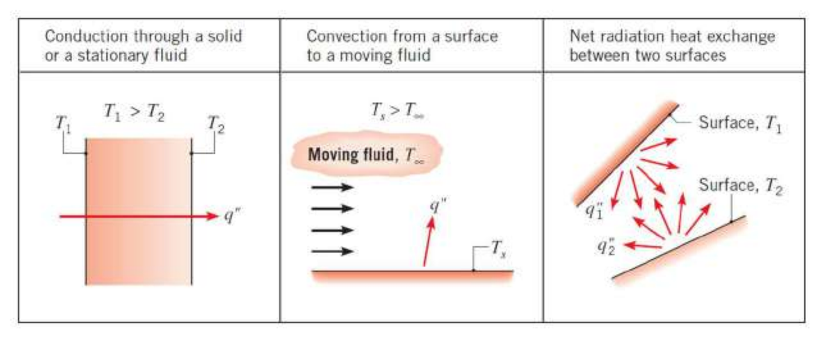
Heat Transfer - What?

The science that deals with the determination of the rates of energy transfer due to temperature difference.

Driving force

- Temperature difference
  - as the voltage difference in electric current
  - as the pressure difference in fluid flow
- Rate depends on magnitude of dT

Heat Transfer - How?



Heat Transfer - Where?

Heat Transfer - Where else?

Thermodynamics and Heat Transfer

Thermodynamics

Deals with the amount of energy (heat or work) during a process
Only considers the end states in equilibrium
Why?

Heat Transfer

Deals with the rate of energy transfer
Transient and non-equilibrium
How long?

Thermodynamics and Heat Transfer

Laws of Thermodynamics

Zeroth law - Temperature
First law Energy conserved
Second law Entropy
Third law $S \to$ constant as $T \to 0$

Laws of Heat Transfer

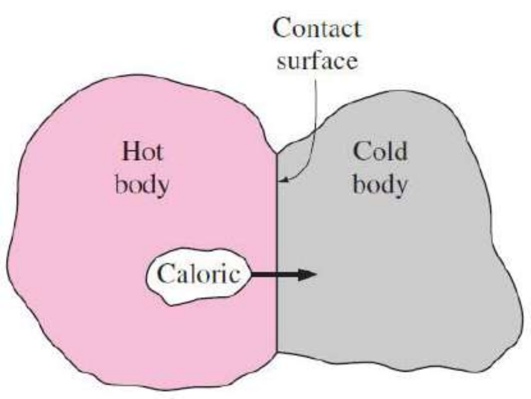
Fouriers law - Conduction
Newtons law of cooling - Convection
Stephan-Boltzmann law - Radiation

Heat Transfer - History

Caloric theory (18th Century)

Heat is a fluid like substance, '*caloric*' poured from one body into another.

Caloric: Massless, colorless, odorless, tasteless



Kinetic theory (19^{th} Century)

Molecules - tiny balls - are in motion possessing kinetic energy

Heat: The energy associated with the random motion of atoms and molecules

Heat, Rate, Flux

Heat

The amount of heat transferred during a process, Q

Heat transfer rate

The amount of heat transferred per unit time, $\dot{Q}$ or simply q

$$Q = \int_0^{\Delta t} q dt$$

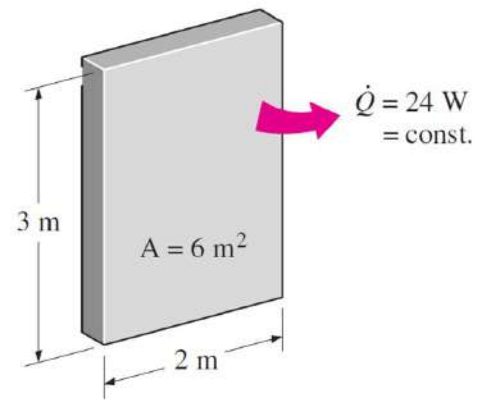
$$Q = q\Delta t, \text{ if } q \text{ is constant}$$

Heat flux

The rate of heat transfer per unit area normal to the direction of heat transfer:

$$q'' = \frac{q}{A}$$

Heat, Rate, Flux

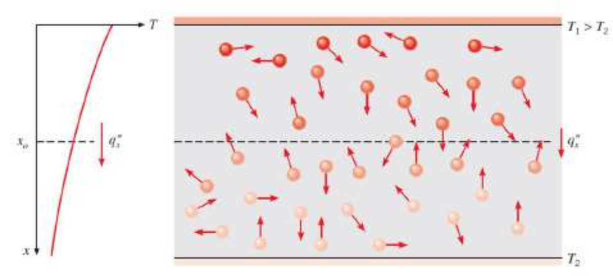


$$q'' = \frac{24 \text{ W}}{6 \text{ m}^2} = 4 \text{ W/m}^2$$

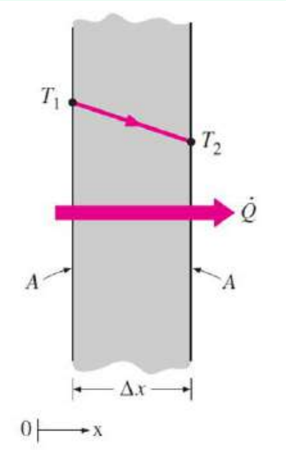
Viewed as

The transfer of energy from the more energetic to the less energetic particles of a substance due to interactions between the particles.

Net transfer by random molecules motion - *diffusion of energy*



Conduction: Fourier's Law of Heat Conduction



$$q_{cond} = -kA\frac{T_1 - T_2}{\Delta x} = -kA\frac{dT}{dx}$$

Problem: Conduction

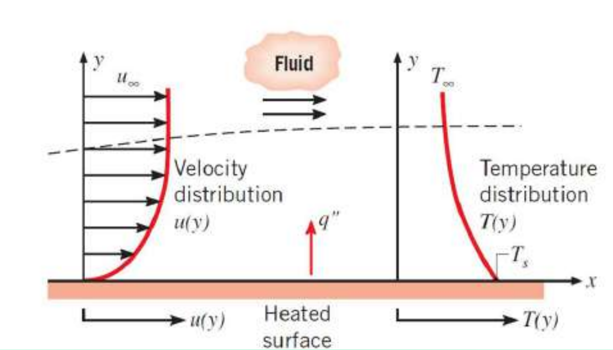
The wall of an industrial furnace is constructed from 0.15 m thick fireclay brick having a thermal conductivity of 1.7 W/m K. Measurements made during steady-state operation reveal temperatures of 1400 and 1150 K at the inner and outer surfaces, respectively. What is the rate of heat loss through a wall that is $0.5 \times 1.2\ \mathrm{m}^2$ on a side?

Ans: 1.7 kW

Comprised of two mechanisms

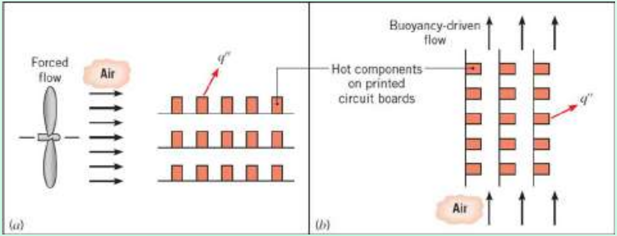
Energy transfer due to random molecular motion - *diffusion*

Energy transfer by the bulk motion of the fluid - *advection*

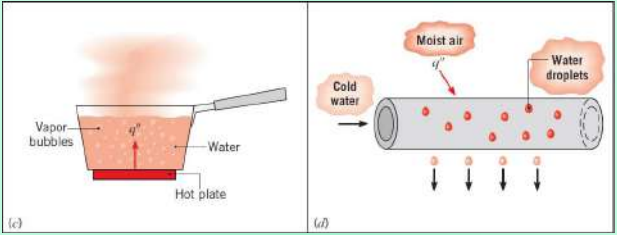


Boundary layer development in convection heat transfer

Forced and Free/Natural Convection



Boiling and Condensation



Convection: Newton's Law of Cooling

$$q_{conv} = hA_s\left(T_s - T_\infty\right)$$

Process	h **(W/m^2 K)**
Free convection	
Gases	2-25
Liquids	50-1000
Convection with phase change	
Boiling and Condensation	2500-100,000

Thermal Radiation

Radiation

Energy emitted by matter that is at a nonzero temperature

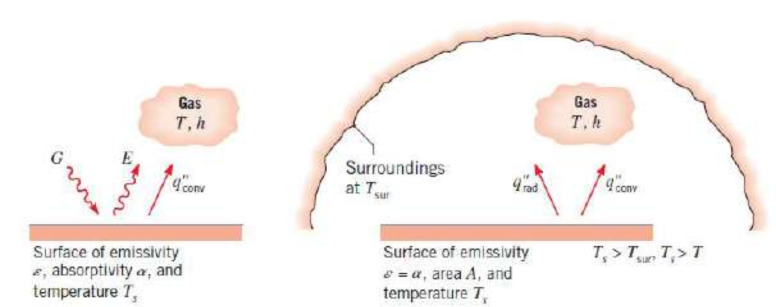
Transported by electromagnetic waves (or photons)

Medium?

Surface Emissive Power

The rate at which energy is released per unit area (W/m^2)

$$E_b = \sigma T_s^4$$



For a real surface:

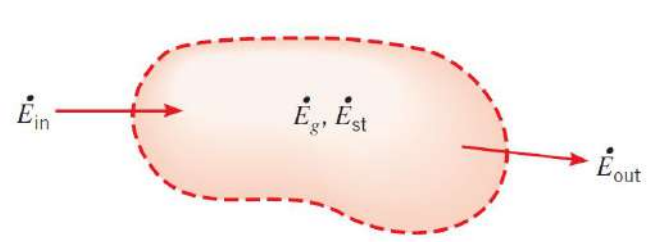
$$E = \varepsilon \sigma T_s^4$$

$$q''_{rad} = \varepsilon \sigma \left(T_s^4 - T_{sur}^4 \right)$$

First Law of Thermodynamics

$$E_{in} - E_{out} = \Delta E_{st}$$

In rate form:



$$\dot{E}_{in} - \dot{E}_{out} = \frac{dE_{st}}{dt} = \dot{E}_{st}$$

First Law of Thermodynamics

The inflow and outflow terms are *surface phenomena*.

The *energy generation term* is a volumetric phenomenon.

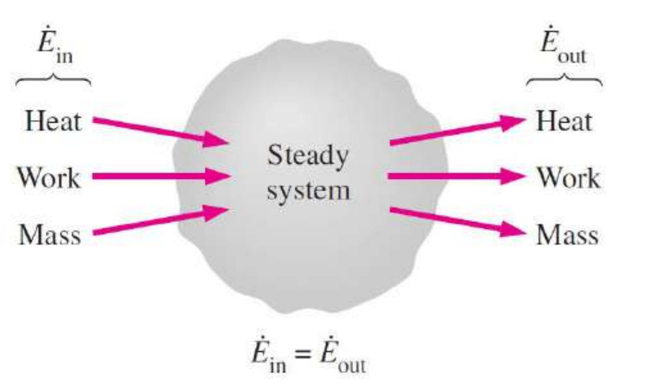
- chemical, electrical

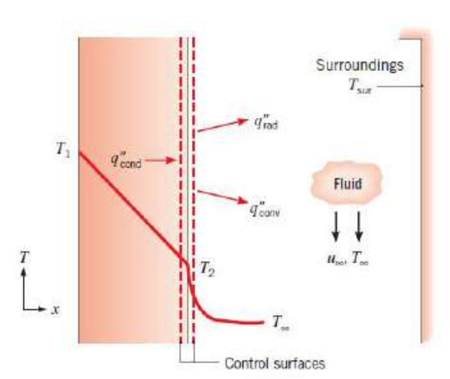
The *energy storage* is also a volumetric phenomenon.

- $\Delta U + \Delta KE + \Delta PE$
- ΔU: sensible/thermal, latent, and chemical components

First Law of Thermodynamics

Steady state with no heat generation





$$
\begin{aligned}
E_{in} - E_{out} &= 0 \\
q_{cond} - q_{conv} - q_{rad} &= 0
\end{aligned}
$$

Problem Solving: Methodology

Analysis of different problems will give a deeper appreciation for the fundamentals of the subject, and you will gain confidence in your ability to apply these fundamentals to the solution of engineering problems.

Be consistent in following these steps:

1. known
2. Find
3. Schematic
4. Assumptions
5. Properties
6. Analysis
7. Comments

Problem: Conduction

The hot combustion gases of a furnace are separated from the ambient air and its surrounding, which are at 25°C, by a brick wall 0.15 m thick. The brick has a thermal conductivity of 1.2 W/m K and a surface emissivity of 0.8. Under steady-state conditions an outer surface temperature of 100°C is measured. Free convection heat transfer to the air adjoining the surface is characterized by a convection coefficient of 20 W/m^2 K. What is the brick inner surface temperature.

Ans: 625 K

Problem: Convection

An experiment to determine the convection coefficient associated with airflow over the surface of a thick stainless steel casting involves the insertion of thermocouples into the casting at distances of 10 and 20 mm from the surface along a hypothetical line normal to the surface. The steel has a thermal conductivity of 15 W/m K. If the thermocouples measure temperatures of 50 and 40°C in the steel when the air temperature is 100°C, what is the convection coefficient?

Ans: 375 W/m^2 K

Problem: Radiation

The roof of a car in a parking lot absorbs a solar radiant flux of 800 W/m^2, and the underside is perfectly insulated. The convection coefficient between the roof and the ambient air is 12 W/m^2 K.

a) Neglecting radiation exchange with the surroundings, calculate the temperature of the roof under steady-state conditions if the ambient air temperature is 20°C.

b) For the same ambient air temperature, calculate the temperature of the roof if its surface emissivity is 0.8.

c) The convection coefficient depends on air flow conditions over the roof, increasing with increasing air speed. Compute and plot the roof temperature as a function of h for $2 \leq h \leq 200$ W/m^2 K.

Ans: 86.7°C

Heat and Mass Transfer

Heat Diffusion Equation

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Steady-state vs Transient

Fourier's law of heat conduction

$$q_{cond} = -kA\frac{dT}{dx}$$

transient

multidimensional - complex geometries

Steady-state heat transfer

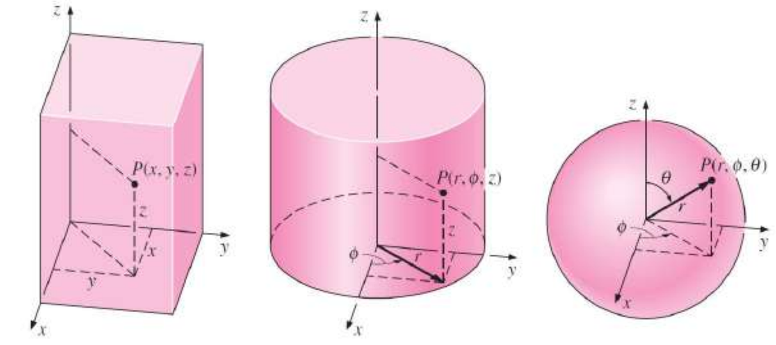
- No change with time at any point within the medium
- T and q'' remains unchanged with time
- $T = T(x, y, z)$
- Usually no but assumed

Transient heat transfer

- Time dependence
- $T = T(x, y, z, t)$
- Special case - *lumped* - T changes with time but not with location:

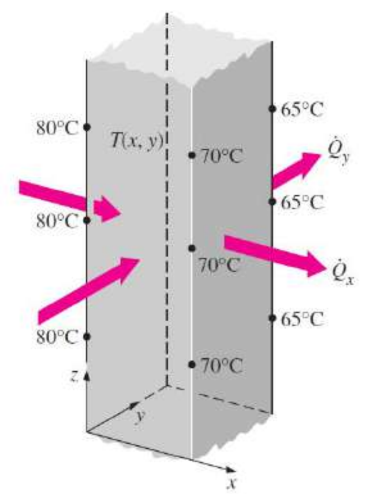
  $T = T(t)$

Coordinate System



(a) Rectangular coordinates (b) Cylindrical coordinates (c) Spherical coordinates

Multidimensional Heat Transfer



Heat Flux Direction

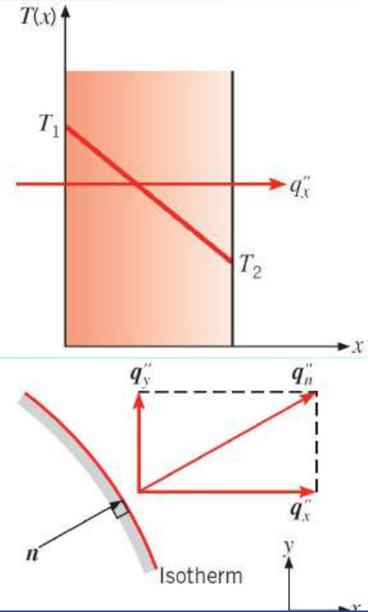
The direction of heat flow will always be normal to a surface of constant temperature, called an isothermal surface.

$$q''_x = -k\frac{\partial T}{\partial x}; q''_y = -k\frac{\partial T}{\partial y}; q''_z = -k\frac{\partial T}{\partial z}$$

$$\begin{aligned} q''_n &= q''_x\vec{i} + q''_y\vec{j} + q''_z\vec{k} \\ &= -k\left(\frac{\partial T}{\partial x}\vec{i} + \frac{\partial T}{\partial y}\vec{j} + \frac{\partial T}{\partial z}\vec{k}\right) \\ &= -k\nabla T \end{aligned}$$

where n is the normal of the isothermal surface and

$$q''_n = -k\frac{\partial T}{\partial n}$$



Thermal Conductivity

Thermal conductivity

$$k = \frac{q''}{(\partial T/\partial x)}$$

The rate of heat transfer through a unit thickness of the material per unit area per unit temperature difference.

Specific heat, C_p

Ability to store thermal energy.
At room temperature,

$$\begin{aligned} C_p &= 4.18 \text{ kJ/kg K, water} \\ &= 0.45 \text{ kJ/kg K, iron} \end{aligned}$$

Thermal conductivity, k

Material's ability to conduct heat
At room temperature,

$$\begin{aligned} k &= 0.607 \text{ W/m K, water} \\ &= 80.2 \text{ W/m K, iron} \end{aligned}$$

Thermal Conductivity

- Transport property
- Indication of the rate at which energy is transferred by the diffusion process
- Depends on the physical structure of matter, atomic and molecular, related to the state of the matter
- *Isotropic* material - k is independent of the direction of transfer, $k_x = k_y = k_z$

  Laminated composite materials and wood

  k across grain is different than that parallel to grain

Kinetic theory of gases:

$$v_{rms} = \sqrt{\frac{3RT}{M}}$$

$T \uparrow \quad k \uparrow$

$M \uparrow \quad k \downarrow$

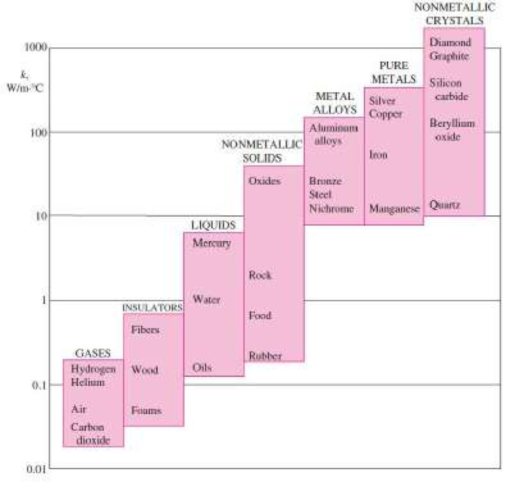
He(4), Air(29)

Liquids: Strong intermolecular forces

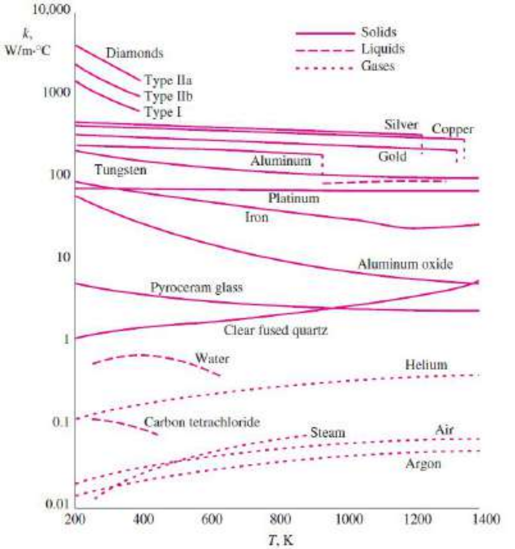
Most liquids: $T \uparrow \quad k \downarrow$

$M \uparrow \quad k \downarrow$

Except water: Not a linear trend



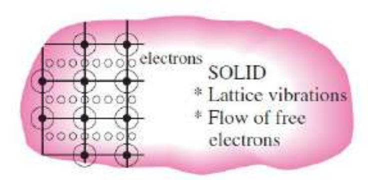
- Temp. dependency causes considerable complexity in conduction analysis
- $k_{average}$



k for Solids

$$k = k_l + k_e$$

High k for pure metals is due to k_e
k_l depends on the way the molecules are arranged



- **Diamond** - highly ordered crystalline solid
  - **Highest** known k
  - However a poor electric conductor (even semiconductors like silicon)
  - *Diamond heat sinks* - cooling electronic components

k for Alloys

Pure metals have high k

$$k_{iron} = 83 \text{ W/m K}$$
$$k_{chromium} = 95 \text{ W/m K}$$

Steel is iron + 1% chrome

$$k_{steel} = 62 \text{ W/m K}$$

Alloy of two metals k_1 and $k_2 < k_1$ and k_2

Thermal Diffusivity α

- Thermophysical properties

  k Transport property

  ρ, C_p Thermodynamic properties
- ρC_p is volumetric heat capacity (J/m^3 K)

- High α: faster propagation of heat into the medium
- Small α: heat is mostly absorbed by the material and a small amount of heat is conducted further

Heat Diffusion Equation: Application

Problem and application

- Determine *temperature distribution* in a medium resulting from conditions imposed on its boundaries
- The conduction heat flux at any point in the medium or on the surface may be computed from Fourier's law
- This information could be used to determine thermal stresses, expansions, deflections
- Temperature distribution may also be used to optimize the thickness of an insulating material or to determine the compatibility of special coatings or adhesives used with the material

Control Volume

Assumptions

Homogeneous medium | No bulk motion (advection)

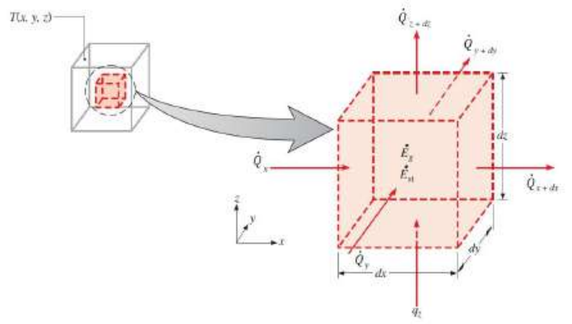
Schematic

Consider an infinitesimally small (differential) CV, $dx.dy.dz$

$$q_{x+dx} = q_x + \frac{\partial q_x}{\partial x} dx$$

$$q_{y+dy} = q_y + \frac{\partial q_y}{\partial y} dy$$

$$q_{z+dz} = q_z + \frac{\partial q_z}{\partial z} dz$$



Volumetric Properties

Generation

$$\dot{E}_g = \dot{e}_g dxdydz$$

$\dot{e}_g$ is in W/m^3

Storage

$$\dot{E}_{st} = \underbrace{\rho C_p \frac{\partial T}{\partial t}}_{\downarrow} dxdydz$$

Rate of change of the sensible/thermal energy of the medium/volume

Heat Diffusion Equation

Governing Equation

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = \dot{E}_{st}$$

$$q_x + q_y + q_z - q_{x+dx} - q_{y+dy} - d_{z+dz} + \dot{e}_g dxdydz = \rho C_p \frac{\partial T}{\partial t} dxdydz$$

$$-\frac{\partial q_x}{\partial x}dx - \frac{\partial q_y}{\partial y}dy - \frac{\partial q_z}{\partial z}dz + \dot{e}_g dxdydz = \rho C_p \frac{\partial T}{\partial t} dxdydz$$

However,

$$q_x = -kdydz\frac{\partial T}{\partial x}; \quad q_y = -kdxdz\frac{\partial T}{\partial y}; \quad q_z = -kdxdy\frac{\partial T}{\partial z}$$

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{e}_g = \rho C_p \frac{\partial T}{\partial t}$$

Special Cases

Fourier-Biot equation - Isotropic

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial x^2} + \frac{\dot{e}_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Diffusion equation - Transient, no heat generation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Poisson equation - Steady-state

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial x^2} + \frac{\dot{e}_g}{k} = 0$$

Laplace equation - Steady-state, no heat generation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial x^2} = 0$$

Coordinate Systems

Cartesian coordinates $T(x, y, z)$

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{e}_g = \rho C_p \frac{\partial T}{\partial t}$$

Cylindrical coordinates $T(r, \phi, z)$

$$\frac{1}{r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{e}_g = \rho C_p \frac{\partial T}{\partial t}$$

Spherical coordinates $T(r, \phi, \theta)$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)$$
$$+\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right) + \dot{e}_g = \rho C_p \frac{\partial T}{\partial t}$$

Boundary and Initial Conditions

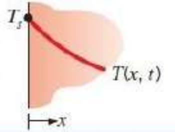
- Necessary to solve the appropriate form of the heat equation
  - Depends on the physical conditions at boundaries
  - On time
- Boundary conditions can be simply expressed in mathematical form
  - Second order in space, two boundary conditions for each coordinate needed to describe the system
  - First order in time, only one condition, *initial condition* must be specified

Boundary Conditions at $x = 0$

1. Constant surface temperature

$$T(0, t) = T_s$$

Dirichlet Condition
***BC* of first kind**

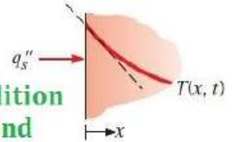


2. Constant surface heat flux

(a) Finite heat flux

$$-k\frac{\partial T}{\partial x}\bigg|_{x=0} = q_s''$$

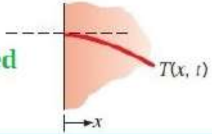
Neumann Condition
***BC* of second kind**



(b) Adiabatic or insulated surface

$$\frac{\partial T}{\partial x}\bigg|_{x=0} = 0$$

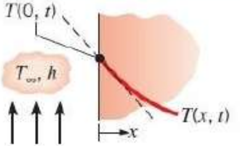
Perfectly insulated
Adiabatic



3. Convection surface condition

$$-k\frac{\partial T}{\partial x}\bigg|_{x=0} = h[T_\infty - T(0, t)]$$

BC of third kind



Problem: Diffusion Equation

A long copper bar of rectangular cross section, whose width w is much greater than its thickness L, is maintained in contact with a heat sink at its lower surface, and the temperature throughout the bar is approximately equal to that of the sink, T_o. Suddenly, an electric current is passed through the bar and an airstream of temperature T_∞ is passed over the top surface, while the bottom surface continues to be maintained at T_o. Obtain the differential equation and the boundary and initial conditions that could be solved to determine the temperature as a function of position and time in the bar.

Solution: Diffusion Equation

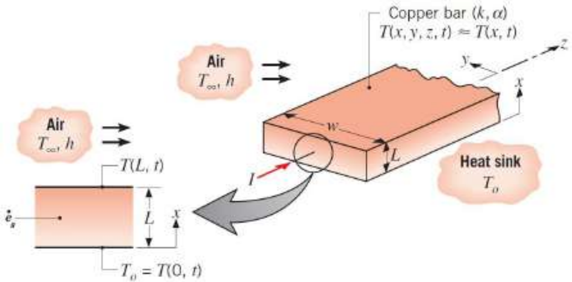
Known

- Copper bar initial temperature, T_o
- Suddenly heated up by electric current, $\dot{e}_g$
- Airstream, h, T_∞

Find

Differential equation;
Boundary conditions;
Initial condition

Schematic



Solution: Diffusion Equation

Assumptions

- $w \ll L$ - side effects are neglected. Thus heat transfer is primarily one dimensional (x)
- Uniform volumetric heat generation, $\dot{e}_g$
- Constant properties

Analysis

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{e}_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Boundary conditions

$$T(0,t) = T_o$$

$$-k\frac{\partial T}{\partial x}|_{x=L} = h\left[T(L,t) - T_\infty\right]$$

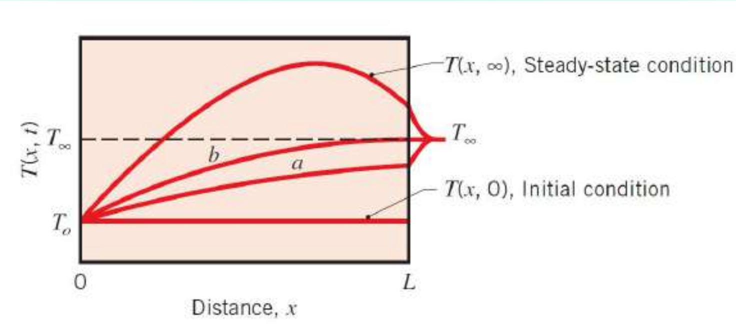
Initial condition

$$T(x,0) = T_o$$

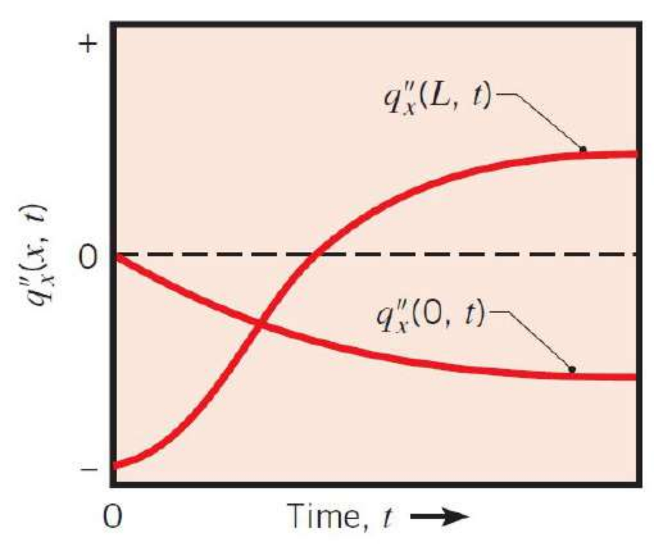
Solution: Diffusion Equation

Comments

1. If $T_o, T_\infty, \dot{e}_g$, and h are known, then the equations can be solved to obtain the $T(x,t)$
2. Top surface, $T(L,t)$ will change with time. This is unknown and may be obtained after finding $T(x,t)$



Solution: Diffusion Equation



Heat and Mass Transfer

One-Dimensional, Steady-State Conduction

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- Temp. gradients exist along only a single coordinate direction
- Heat transfer occurs exclusively in that direction
- Temp. at each point is independent of time

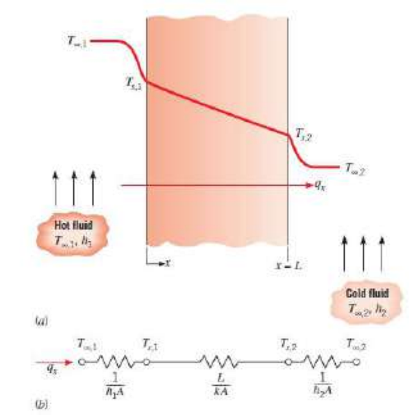
We will see:

- Temp. distribution & heat transfer rate in common (planar, cylindrical and spherical) geometries
- Thermal resistance
  - Thermal circuits to model heat flow
  - Electrical circuits to current flow

Cartesian Coordinates: $T(x)$

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0$$

For 1-D, steady-state conduction in a plane wall with no heat generation, heat flux is a constant, independent of x.



Plane Wall

If k is constant then, $T(x) = C_1 x + C_2$

$$T(0) = T_{s,1} \quad \text{and} \quad T(L) = T_{s,2}$$

$$T(x) = (T_{s,2} - T_{s,1}) \frac{x}{L} + T_{s,1}$$

$$q_x = -kA \frac{dT}{dx} = \frac{kA}{L} (T_{s,2} - T_{s,1})$$

$$q''_x = \frac{k}{L} (T_{s,2} - T_{s,1})$$

Thermal Resistance

Ratio of driving potential to the corresponding transfer rate

$$R_{t,cond} = \frac{(T_{s,1} - T_{s,2})}{q_x} = \frac{L}{kA}$$

$$R_e = \frac{E_{s,1} - E_{s,2}}{I}$$

$$R_{t,conv} = \frac{(T_s - T_\infty)}{q} = \frac{1}{hA}$$

Under steady state conditions:

$$\left|\begin{array}{l}\text{Convection rate}\\ \text{into the wall}\end{array}\right| = \left|\begin{array}{l}\text{Conduction rate}\\ \text{through the wall}\end{array}\right| = \left|\begin{array}{l}\text{Convection rate}\\ \text{from the wall}\end{array}\right|$$

$$q_x = \frac{T_{\infty,1} - T_{s,1}}{1/h_1 A} = \frac{T_{s,1} - T_{s,2}}{L/kA} = \frac{T_{s,2} - T_{\infty,2}}{1/h_2 A}$$

$$q_x = \frac{T_{\infty,1} - T_{\infty,2}}{R_{tot}} \qquad R_{tot} = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A}$$

Thermal Resistance: Radiation

The **thermal resistance for radiation** - radiation exchange between the surface and its surroundings:

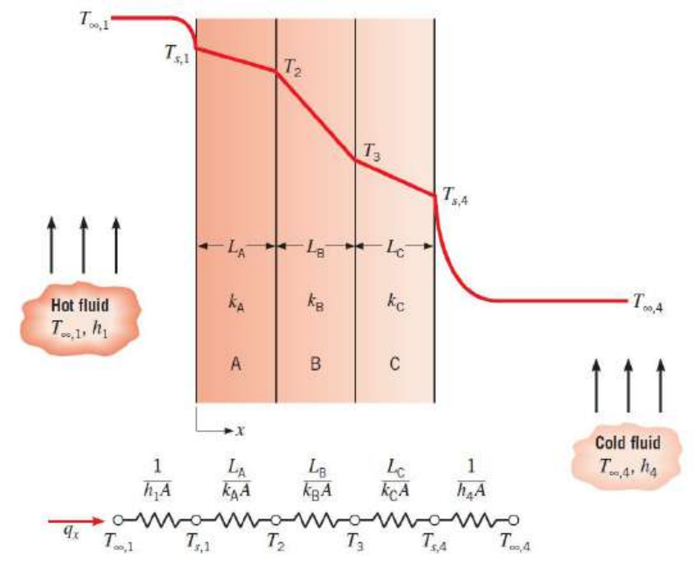
$$R_{t,rad} = \frac{T_s - T_{sur}}{q_{rad}} = \frac{1}{h_r A}$$

$$q_{rad} = h_r A \left(T_s - T_{sur}\right)$$

The radiation heat transfer coefficient, h_r:

$$h_r = \varepsilon\sigma \left(T_s + T_{sur}\right)\left(T_s^2 + T_{sur}^2\right)$$

The Composite Wall



The Composite Wall

$$q_x = \frac{T_{\infty,1} - T_{\infty,4}}{\sum R_t}$$

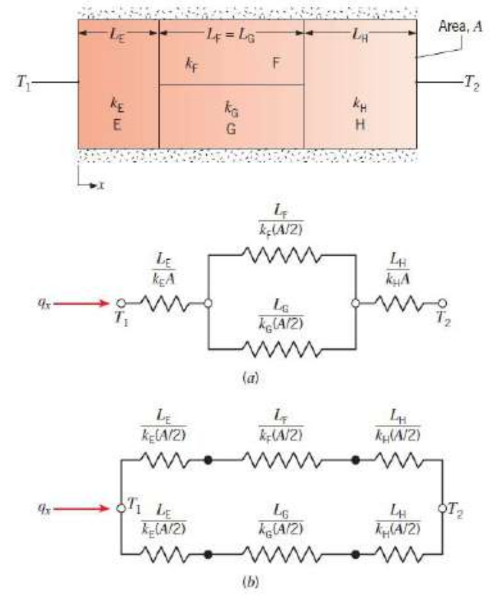
$$R_{tot} = \frac{1}{h_1 A} + \frac{L_A}{k_A A} + \frac{L_B}{k_B A} + \frac{L_C}{k_C A} + \frac{1}{h_4 A}$$

If U is the **overall heat transfer coefficient**

$$q_x = UA\Delta T$$

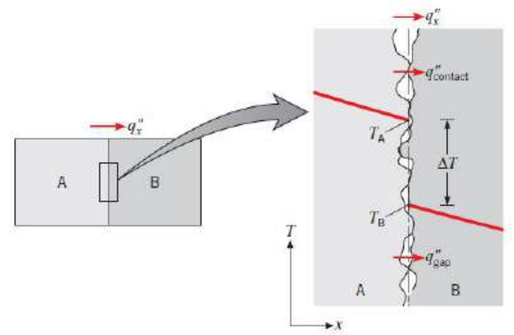
$$U = \frac{1}{R_{tot} A}$$

Series-Parallel Composite Wall



Contact Resistance

$$R''_{t,c} = \frac{T_A - T_B}{q''_x}$$



Thermal Resistance, $R''_{t,c} \times 10^4$ ($m^2 \cdot K/W$)				
(a) Vacuum Interface			**(b) Interfacial Fluid**	
Contact pressure	100 kN/m^2	10,000 kN/m^2	Air	2.75
Stainless steel	6–25	0.7–4.0	Helium	1.05
Copper	1–10	0.1–0.5	Hydrogen	0.720
Magnesium	1.5–3.5	0.2–0.4	Silicone oil	0.525
Aluminum	1.5–5.0	0.2–0.4	Glycerine	0.265

Thermal Resistance of Solid/Solid Interfaces

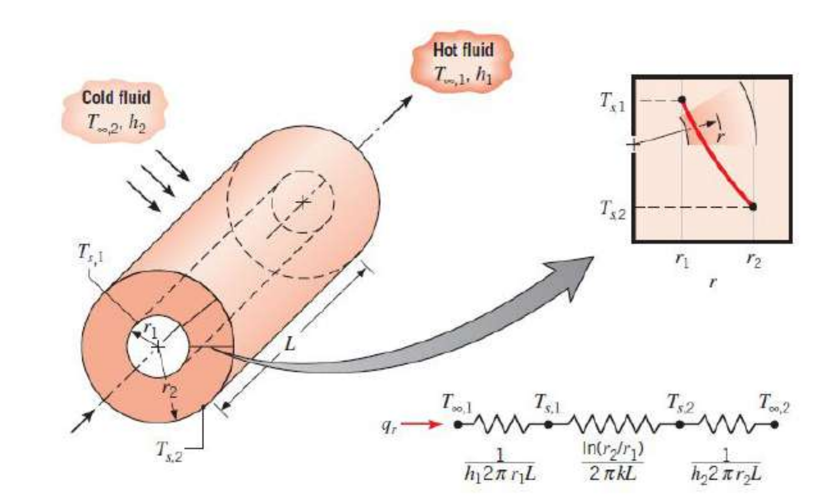
Interface	$R''_{t,c} \times 10^4$ $(m^2 \cdot K/W)$
Silicon chip/lapped aluminum in air (27–500 kN/m^2)	0.3–0.6
Aluminum/aluminum with indium foil filler (~100 kN/m^2)	~0.07
Stainless/stainless with indium foil filler (~3500 kN/m^2)	~0.04
Aluminum/aluminum with metallic (Pb) coating	0.01–0.1
Aluminum/aluminum with Dow Corning 340 grease (~100 kN/m^2)	~0.07
Stainless/stainless with Dow Corning 340 grease (~3500 kN/m^2)	~0.04
Silicon chip/aluminum with 0.02-mm epoxy	0.2–0.9
Brass/brass with 15-μm tin solder	0.025–0.14

Porous Medium

For a porous medium, an effective conductivity is considered.
Assuming, there is no fluid bulk motion and if $T_1 > T_2$

$$q_x = \frac{k_{eff} A}{L}(T_1 - T_2)$$

The Cylinder



The Cylinder

The governing equation for 1D, steady state conduction in cylindrical coordinates:

$$\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right) = 0$$

The heat flux by Fourier's law of conduction,

$$q_r = -kA\frac{dT}{dr} = -k(2\pi rL)\frac{dT}{dr}$$

- Here, $A = 2\pi rL$ is the area normal to the direction of heat transfer.
- The quantity $\frac{d}{dr}\left(kr\frac{dT}{dr}\right)$ is independent of r
- The conduction heat transfer rate q_r (not the heat flux, q_r'') is a constant in the radial direction

The Cylinder

Temperature distribution and heat transfer rate

$$T(r) = \frac{T_{s,1} - T_{s,2}}{\ln(r_1/r_2)} \ln\left(\frac{r}{r_2}\right) + T_{s,2}$$

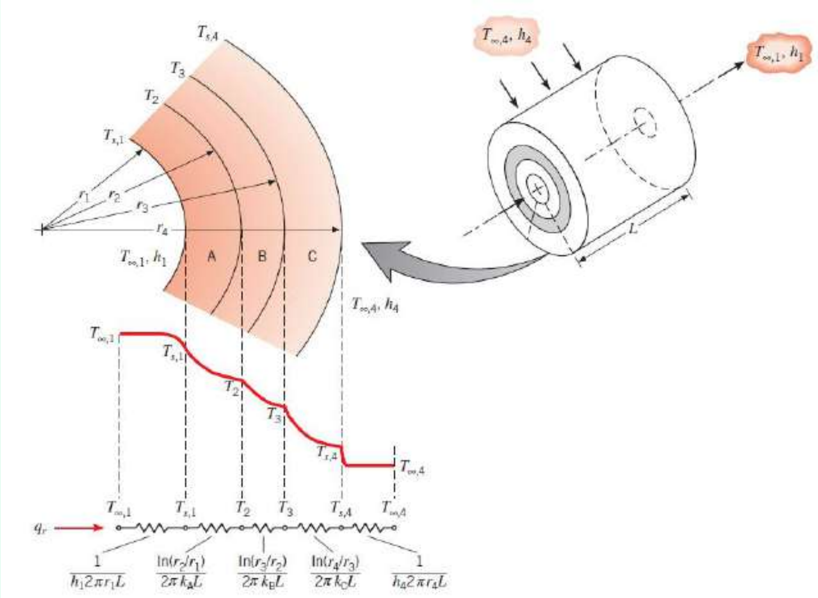
Note that the temperature distribution associated with radial conduction through a cylindrical wall is logarithmic, not linear, as it is for the plane wall.

$$q_r = \frac{2\pi Lk\left(T_{s,1} - T_{s,2}\right)}{\ln(r_2/r_1)}$$

Note that q_r is independent of r.

$$R_{t,cond} = \frac{\ln(r_2/r_1)}{2\pi Lk}$$

The Cylinder



The Cylinder

$$q_r = \frac{(T_{\infty,1} - T_{\infty,2})}{\frac{1}{2\pi r_1 L h_1} + \frac{\ln(r_2/r_1)}{2\pi k_A L} + \frac{\ln(r_3/r_2)}{2\pi k_B L} + \frac{\ln(r_4/r_3)}{2\pi k_C L} + \frac{1}{2\pi r_4 L h_4}}$$

$$q_r = \frac{(T_{\infty,1} - T_{\infty,2})}{R_{tot}} = UA\,(T_{\infty,1} - T_{\infty,2})$$

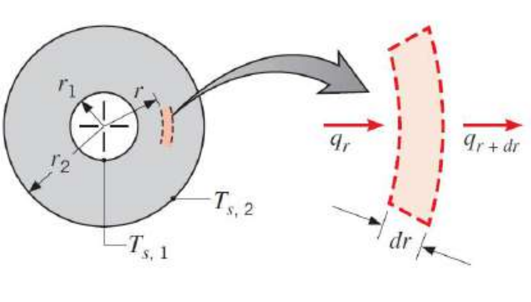
If U is defined in terms of the inside area, $A_1 = 2\pi r_1 L$, then:

$$U = \frac{1}{\frac{1}{h_1} + \frac{r_1}{k_A}\ln(\frac{r_2}{r_1}) + \frac{r_1}{k_B}\ln(\frac{r_3}{r_2}) + \frac{r_1}{k_C}\ln(\frac{r_4}{r_3}) + \frac{r_1}{r_4}\frac{1}{h_4}}$$

- UA is constant, while U is not
- In radial systems, q is constant, while q'' is not

The Sphere

Consider a hollow sphere, whose inner and outer surfaces are exposed to fluids at different temperatures.



$$\frac{1}{r^2}\frac{d}{dr}\left(kr^2\frac{dT}{dr}\right) = 0$$

$$q_r = -k(4\pi r^2)\frac{dT}{dr}$$

The Sphere

Temperature distribution and heat transfer rate

$$T(r) = T_{s,1} + \frac{T_{s,1} - T_{s,2}}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)} \left[\frac{1}{r_1} - \frac{1}{r}\right]$$

$$q_r = \frac{4\pi k \left(T_{s,1} - T_{s,2}\right)}{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}$$

$$R_{t,cond} = \frac{1}{4\pi k}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Problem: Sphere

A spherical thin walled metallic container is used to store liquid nitrogen at 80 K. The container has a diameter of 0.5 m and is covered with an evacuated, reflective insulation composed of silica powder. The insulation is 25 mm thick, and its outer surface is exposed to ambient air at 310 K. The convection coefficient is known to be 20 W/m^2 K. The latent heat of vaporization and the density of the liquid nitrogen are 2×10^5 J/kg and 804 kg/m^3, respectively. Thermal conductivity of evacuated silica powder (300 K) is 0.0017 W/m K.

1. What is the rate of heat transfer to the liquid nitrogen?
2. What is the rate of liquid boil-off?

Solution: Sphere

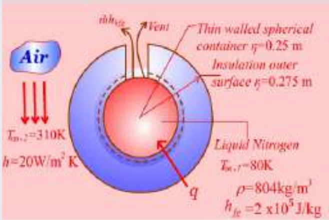
known

Liquid nitrogen is stored in a spherical container that is insulated and exposed to ambient air.

Find

- The rate of heat transfer to the nitrogen.
- The mass rate of nitrogen boil-off.

Schematic

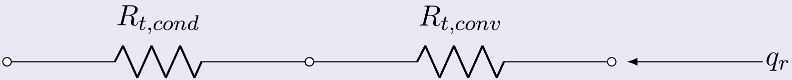


Solution: Sphere

Assumptions

1. Steady state conditions
2. One-dimensional transfer in the radial direction
3. Negligible resistance to heat transfer through the container wall and from the container to the nitrogen
4. Constant properties
5. Negligible radiation exchange between outer surface of insulation and surroundings

Analysis



Solution: Analysis

$$R_{t,cond} = \frac{1}{4\pi k}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$R_{t,conv} = \frac{1}{h\left(4\pi r_2^2\right)} q_r = \frac{(T_{\infty,2} - T_{\infty,1})}{R_{t,cond} + R_{t,conv}} = 12.88W$$

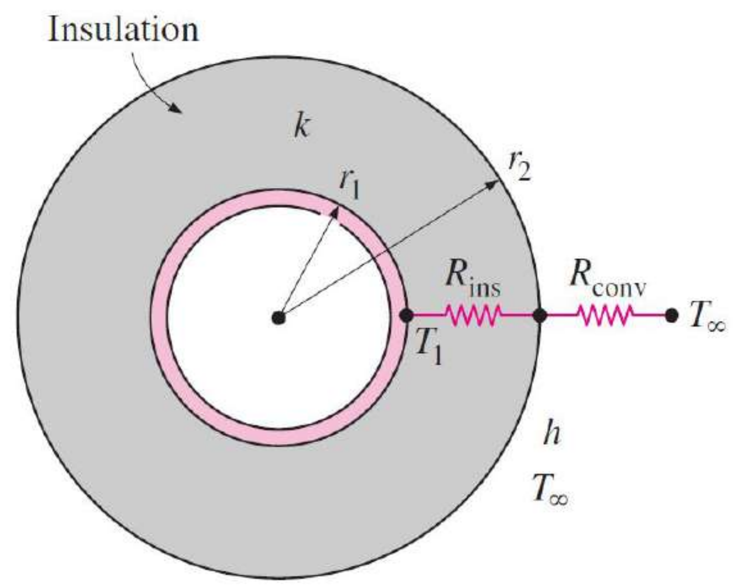
Energy balance for a control surface about the nitrogen:

$$\begin{aligned}
\dot{E}_{in} - \dot{E}_{out} &= 0 \\
q - \dot{m}h_{fg} &= 0 \\
\implies \dot{m} &= 6.44 \times 10^{-5} \text{ kg/W} \\
&= 5.56 \text{ kg/day} \\
&= \frac{\dot{m}}{\rho} = 0.00692 \text{ m}^3\text{/day} \\
&\boxed{= 6.92 \text{ liters/day}}
\end{aligned}$$

The Critical Radius of Insulation

- We know that by adding more insulation to a wall always decreases heat transfer.
- This is expected, since the heat transfer area A is constant, and adding insulation will always increase the thermal resistance of the wall without affecting the convection resistance.
- However, adding insulation to a cylindrical piece or a spherical shell, is a different matter.
- The additional insulation increases the conduction resistance of the insulation layer but it also decreases the convection resistance of the surface because of the increase in the outer surface area for convection.
- Therefore, the heat transfer from a pipe may increase or decrease, depending on which effect dominates.

The Critical Radius of Insulation



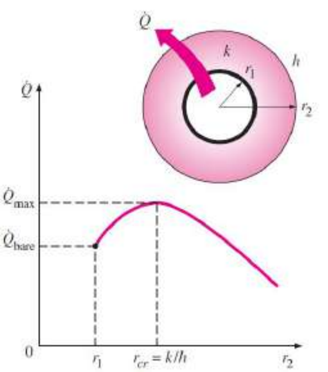
The Critical Radius of Insulation

The rate of heat transfer from the insulated pipe to the surrounding air can be expressed as:

$$q_r = \frac{(T_1 - T_\infty)}{\frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi L k} + \frac{1}{h(2\pi r_2 L)}}$$

The value of r_2 at which heat transfer rate reaches max. is determined from the requirement that $\frac{dq_r}{dr}$ (zero slope):

$$r_{cr,cylinder} = \frac{k}{h}$$

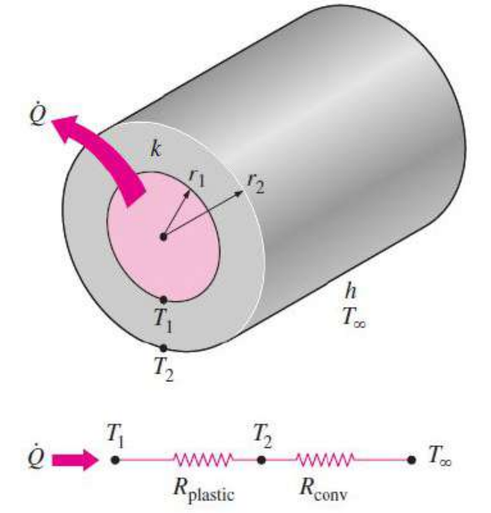


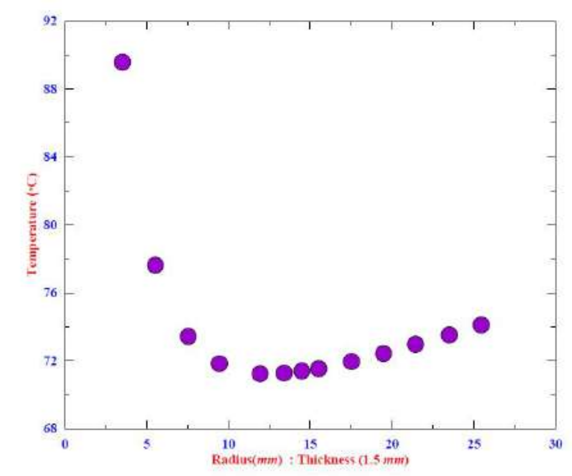
Problem: Critical Radius

A 3 mm diameter and 6 m long electric wire is tightly wrapped with a 2 mm thick plastic cover whose thermal conductivity is $k = 0.15$ W/m K. Electrical measurements indicate that a current of 10 A passes through the wire and there is a voltage drop of 8 V along the wire. If the insulated wire is exposed to a medium at 27°C with a heat transfer coefficient of $h = 12$ W/m^2 K, determine the temperature at the interface of the wire and the plastic cover in steady operation. Also determine whether doubling the thickness of the plastic cover will increase or decrease this interface temperature. Ans: 89.5°C, 77.5°C

Hint: $q_r = VI$

Problem: Critical Radius




92
88
84
80
76
72
68
Temperature (°C)
0
5
10
15
20
25
30
Radius(mm) : Thickness (1.5 mm)

	Plane Wall	Cylindrical Wall[a]	Spherical Wall[a]
Heat equation	$\frac{d^2T}{dx^2} = 0$	$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$	$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0$
Temperature distribution	$T_{s,1} - \Delta T\frac{x}{L}$	$T_{s,2} + \Delta T\frac{\ln(r/r_2)}{\ln(r_1/r_2)}$	$T_{s,1} - \Delta T\left[\frac{1-(r_1/r)}{1-(r_1/r_2)}\right]$
Heat flux (q'')	$k\frac{\Delta T}{L}$	$\frac{k\,\Delta T}{r\ln(r_2/r_1)}$	$\frac{k\,\Delta T}{r^2[(1/r_1)-(1/r_2)]}$
Heat rate (q)	$kA\frac{\Delta T}{L}$	$\frac{2\pi Lk\,\Delta T}{\ln(r_2/r_1)}$	$\frac{4\pi k\,\Delta T}{(1/r_1)-(1/r_2)}$
Thermal resistance ($R_{t,\text{cond}}$)	$\frac{L}{kA}$	$\frac{\ln(r_2/r_1)}{2\pi Lk}$	$\frac{(1/r_1)-(1/r_2)}{4\pi k}$

[a]The critical radius of insulation is $r_{cr} = k/h$ for the cylinder and $r_{cr} = 2k/h$ for the sphere.

Conduction with Thermal Energy Generation

A very common thermal energy generation process involves the conversion from electrical to thermal energy in a current carrying medium (resistance heating). The rate at which energy is generated by passing a current through a medium of electrical resistance R_e is:

$$\dot{E}_g = I^2 R_e$$

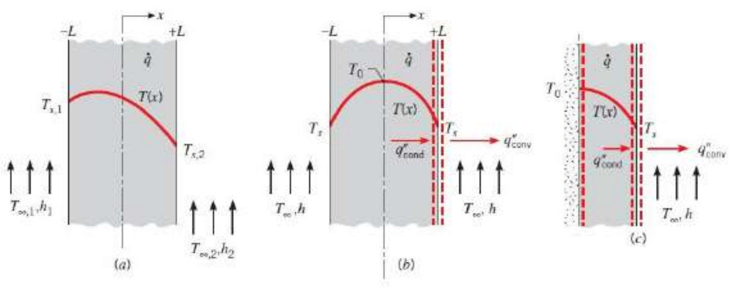
If this power generation occurs uniformly throughout the medium of volume V, the volumetric generation rate (W/m^3) is:

$$\dot{q} = \frac{\dot{E}_g}{V} = \frac{I^2 R_e}{V}$$

Conduction with $\dot{E}_g$ in a Plane Wall

Consider a plane wall, in which there is uniform energy generation per unit volume ($\dot{q}$ is constant) and the surfaces are maintained at $T_{s,1}$ and $T_{s,2}$. The appropriate form of the heat equation:

$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0 \qquad \left| \begin{aligned} T(-L) &= T_{s,1} \\ T(L) &= T_{s,2} \end{aligned} \right.$$



Conduction with $\dot{E}_g$ in a Plane Wall

$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + \frac{T_{s,2} - T_{s,1}}{2}\frac{x}{L} + \frac{T_{s,1} + T_{s,2}}{2}$$

The heat flux at any point in the wall may be determined by Fourier's Law. The heat flux is not independent of x.

Special case: $T_{s,1} = T_{s,2} = T_s$

The temperature distribution is then symmetrical about the central plane:

$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + T_s$$

The maximum temperature exists at the central plane:

$$T(0) = T_0 = \frac{\dot{q}L^2}{2k} + T_s$$

Conduction with $\dot{E}_g$ in a Plane Wall

$$\frac{T(x) - T_0}{T_s - T_0} = \left(\frac{x}{L}\right)^2$$

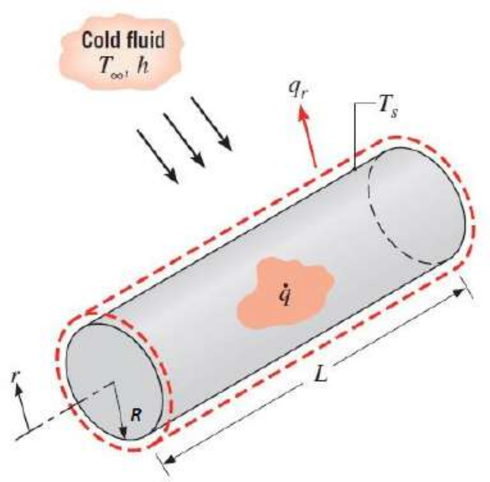
- It is important to note that the temperature gradient $\frac{dT}{dx}\,|_{x=0} = 0$ at the plane of symmetry.
- No heat transfer across this plane - adiabatic surface.
- The above equation can also be applied to plane walls that are perfectly insulated on one side ($x = 0$) and a fixed T_s on the other side ($x = L$).

However, in most of the cases, T_s is an unknown. It is computed from the energy balance at the surface to the adjoining fluid:

$$-k\,\frac{dT}{dx}\bigg|_{x=L} = h(T_s - T_\infty)$$

$$\implies T_s = T_\infty + \frac{\dot{q}L}{h}$$

Consider a long, solid cylinder (may be a current carrying wire).
For steady state conditions: $\dot{E}_g = q_{conv}$.



Conduction with $\dot{E}_g$ in Radial Systems

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{\dot{q}}{k} = 0$$

Boundary conditions:

$$\left.\frac{dT}{dr}\right|_{r=0} = 0, \quad T(R) = T_s \quad \text{and} \quad T(r=0) = T_0$$

$$\implies T(r) = T_s + \frac{\dot{q}R^2}{4k}\left(1 - \frac{r^2}{R^2}\right)$$

$$\text{or} \qquad \frac{T(r) - T_s}{T_0 - T_s} = 1 - \left(\frac{r}{R}\right)^2$$

T_s can be obtained from the energy balance at the surface:

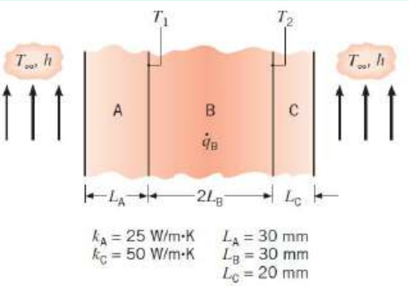
$$\dot{q}\ \pi R^2 L = h\ 2\pi RL\ (T_s - T_\infty)$$

$$T_s = T_\infty + \frac{\dot{q}R}{2h}$$

Problem

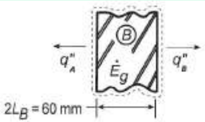
Consider one-dimensional conduction in a plane composite wall. The outer surfaces are exposed to a fluid at 25°C and a convection heat transfer coefficient of 1000 W/m^2 K. The middle wall B experiences uniform heat generation $\dot{q}_B$, while there is no generation in walls A and C. The temperatures at the interfaces are $T_1 = 261$°C and $T_2 = 211$°C. Assuming negligible contact resistance at the interfaces, determine $\dot{q}_B$ and k_B.

Ans: 4×10^6 W/m^3, 15.3 W/m K.



From an energy balance on wall B,

$$\dot{E}_in - \dot{E}_out + \dot{E}_g = \dot{E}_{st}$$



$$-q_A - q_C + \dot{q_B}V = 0$$

$$\implies -q''_A - q''_C + \dot{q_B}\ 2L_B = 0$$

$$\dot{q}_B = \frac{q_A + q_C}{2L_B}$$

Heat flow across ambient and wall A:

$$q''_A = \frac{T_1 - T_\infty}{\left(\frac{1}{h} + \frac{L_A}{k_A}\right)} = \frac{261 - 25}{\frac{1}{1000} + \frac{30\times10^{-3}}{25}} = 107272.7 \text{ W/m}^2$$

Solution

Heat flow across ambient and wall C:

$$q''_C = \frac{T_1 - T_\infty}{\left(\frac{1}{h} + \frac{L_C}{k_C}\right)} = \frac{211 - 25}{\frac{1}{1000} + \frac{20\times10^{-3}}{50}} = 132857.1 \text{ W/m}^2$$

$$\boxed{\dot{q}_B = \frac{q_A + q_C}{2L_B} = \frac{107272.7 + 132857.1}{60 \times 10^{-3}} = 4 \times 10^6 \text{ W/m}^3}$$

Solution

$$T(x) = \frac{\dot{q}_B L_B^2}{2k_B}\left(1 - \frac{x^2}{L_B^2}\right) + \frac{T_2 - T_1}{2}\frac{x}{L_B} + \frac{T_1 + T_2}{2}$$

$$q''_B(x) = -k_B\left[\frac{\dot{q}_B}{2k_B}(-2x) + \frac{T_2 - T_1}{2L_B}\right]$$

$$q''_B\Big|_{x=-L_B} = -\dot{q}_B L_B - \frac{T_2 - T_1}{2L_B}k_B$$

$$k_B = \frac{-q''_A + \dot{q}_B L_B}{(T_1 - T_2)/2L_B} = 15.35 \text{ W/m K}$$

Problem

A plane wall is a composite of two materials, A and B. The wall of material A has uniform heat generation $\dot{q} = 1.5 \times 10^6$ W/m^3, $k_A = 75$ W/m K, and thickness $L_A = 50$ mm. The wall material B has no generation with $k_B = 150$ W/m K, and thickness $L_B = 20$ mm. The inner surface of material A is well insulated, while the outer surface of material B is cooled by a water stream with $T_\infty = 30°$C and $= 1000$ W/m^2 K.

- Sketch the temperature distribution that exists in the composite under steady state conditions.
- Determine the temperature T_0 of the insulated surface and the temperature T_2 of the cooled surface.

Heat and Mass Transfer

1D Heat Transfer from Extended Surfaces - Fins

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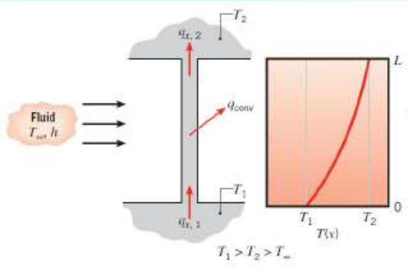
Heat Transfer from Extended Surfaces

Extended surface: solid that experiences energy transfer by conduction within its boundaries, as well as energy transfer by convection and/or radiation between its boundaries and the surround-

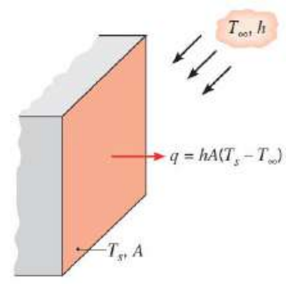


ings. A strut is used to provide mechanical support to two walls at different T. A temperature gradient in the x-direction sustains heat transfer by conduction internally, at the same time there is energy transfer by convection from the surface.

The most frequent application is one in which an extended surface is used specifically to enhance the heat transfer rate between a solid and an adjoining fluid - called as **fin**

Consider a plane wall:

$$q_{conv} = hA(T_s - T_\infty)$$

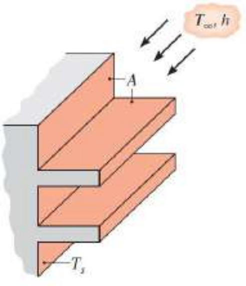


Heat Transfer from Extended Surfaces

For fixed T_s, 2 ways to enhance the rate of heat transfer:

- Increase the fluid velocity: cost of blower or pump power
- T_∞ could be reduced: impractical

Limitations: Many situations would be encountered in which increasing h to the max. possible value is either insufficient to obtain the desired heat transfer rate or the associated costs are prohibitively high.



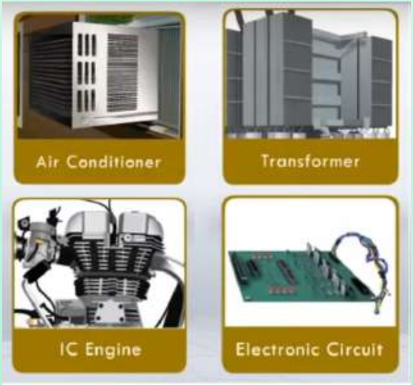
How about increasing surface area for convection?
By providing fins that extend from the wall into the surrounding fluid.

Fin Material

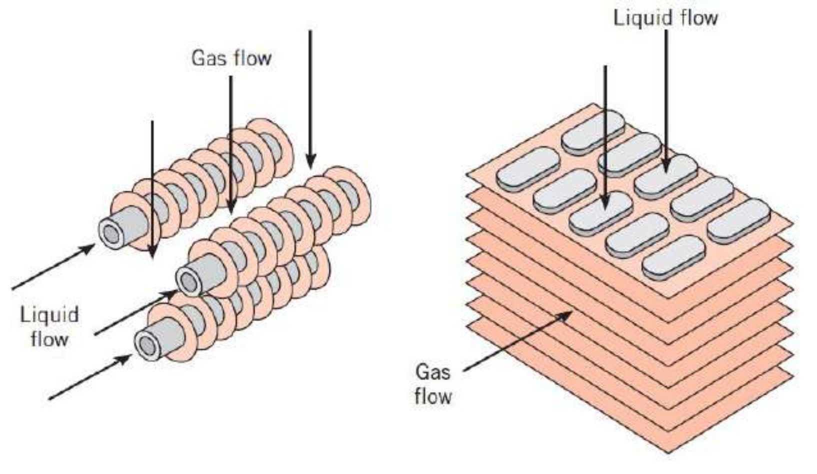
- k of the fin material has a strong effect on the temperature distribution along the fin and therefore influences the degree to which the heat transfer rate is enhanced.
- Ideally, the fin material should have a large k to minimize temperature variations from its base to its tip.
- In the limit of infinite thermal conductivity, the entire fin would be at the temperature of the base surface, thereby providing the maximum possible heat transfer enhancement.

Application of Fins

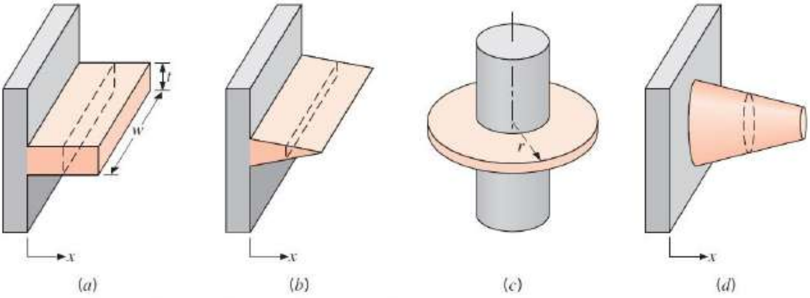
- The arrangement for cooling engine heads on motorcycles and lawn-mowers
- For cooling electric power transformers
- The tubes with attached fins used to promote heat exchange between air and the working fluid of an air conditioner



Typical Finned Tube Heat Exchangers



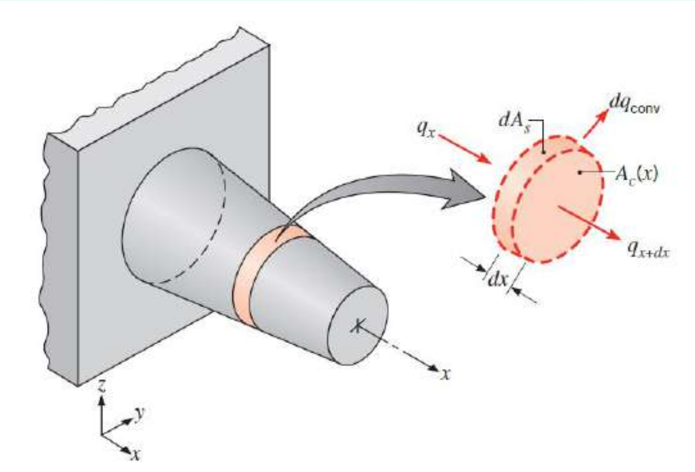
Fin Configurations



(a) Straight fin of uniform cross section. (b) Straight fin of nonuniform cross section. (c) Annular fin. (d) Pin fin.

For an extended surface, the direction of heat transfer from the boundaries is perpendicular to the principal direction of heat transfer in the solid.

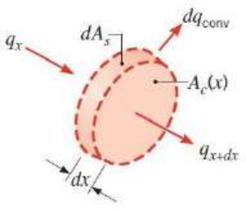
To determine the heat transfer rate associated with a fin, we must first obtain the temperature distribution along the fin.



Assumptions

- 1-D heat transfer (longitudinal x direction). In practice the fine is thin and the temperature changes in the longitudinal direction are much larger than those in the transverse direction.
- Steady state
- k is constant
- No heat generation
- Negligible radiation from the surface
- The rate at which the energy is convected to the fluid from any point on the fin surface must be balanced by the rate at which the energy reaches that point due to conduction in the transverse (y, z) direction.
- h is uniform over the surface

Derivation for fin



$$q_x = q_{x+dx} + dq_{conv}$$

However,

$$q_x = -kA_c\frac{dT}{dx}$$

A_c may vary with x.

$$q_{x+dx} = q_x + \frac{dq_x}{dx}dx$$

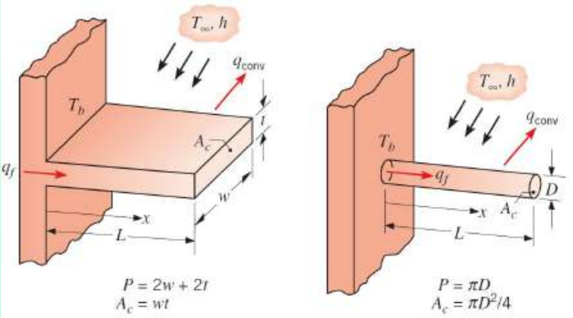
$$q_{conv} = hdA_s(T - T_\infty)$$

dA_s is the surface area of dx

$$\implies k\frac{d}{dx}\left(A_c\frac{dT}{dx}\right)dx - hdA_s(T - T_\infty)$$

$$\boxed{\frac{d^2T}{dx^2} + \left(\frac{1}{A_c}\frac{dA_c}{dx}\right)\frac{dT}{dx} - \left(\frac{1}{A_c}\frac{h}{k}\frac{dA_s}{dx}\right)(T - T_\infty)}$$

Fins of Uniform Cross-Sectional Area



- $T(0) = T_b$
- A_c is constant, $dA_c/dx = 0$
- $A_s = Px$ where x is measured from base, P is fin perimeter
- $dA_s/dx = P$

$$\frac{d^2T}{dx^2} - \frac{hP}{kA_c}(T - T_\infty) = 0$$

Fins of Uniform Cross-Sectional Area

Excess temperature, θ

$$\theta(x) = T(x) - T_\infty$$

$$d\theta/dx = dT/dx$$

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0$$

where $m^2 = \frac{hP}{kA_c}$

The above equation is a linear, homogeneous, second-order differential equation with constant coefficients. The general solution is of the form:

$$\boxed{\theta(x) = C_1 e^{mx} + C_2 e^{-mx}}$$

It is necessary to specify appropriate BCs for C_1 and C_2.

Fins of Uniform Cross-Sectional Area

One such condition may be specified in terms of the temperature at the base of the fin $(x = 0)$:

$$\theta(0) = T_b - T_\infty = \theta_b$$

The second condition, specified at the fin tip $(x = L)$, may correspond to any one of the four different physical conditions:

A. h at the fin tip
B. Adiabatic condition at the fin tip
C. Prescribed temperature maintained at the fin tip
D. Infinite fin (very long fin)

Fins of Uniform Cross-Sectional Area

A. Infinite fin (very long fin): As $L \to \infty, \theta_L \to 0$

B. Adiabatic condition at the fin tip

$$\left.\frac{d\theta}{dx}\right|_{x=L} = 0$$

C. h at the fin tip

$$hA_c[T(L) - T_\infty] = -kAc\left.\frac{dT}{dx}\right|_{x=L} \implies h\theta(L) = -k\left.\frac{d\theta}{dx}\right|_{x=L}$$

D. Prescribed temperature maintained at the fin tip: $\theta(L) = \theta_L$

Case A: Infinite fin (very long fin)

As $L \to \infty, \theta_L \to 0$ and $e^{-mL} \to 0$

$$\theta|_{x=0} = \theta_b; \qquad \theta|_{x=L} = 0$$

$$\theta_b = C_1 e^{mx} + C_2 e^{-mx}; \qquad C_1 e^{mL} + C_2 e^{-mL} = 0$$

Equation for infinite fin

$$\frac{\theta}{\theta_b} = e^{-mx}$$

$$q_f = -kA_c \left.\frac{d\theta}{dx}\right|_{x=0} = \sqrt{hPkA_c}\,\theta_b$$

Case B: Adiabatic Condition at the Fin Tip

$$\theta\big|_{x=0} = \theta_b; \qquad \left.\frac{d\theta}{dx}\right|_{x=L} = 0$$

$$\theta_b = C_1 e^{mx} + C_2 e^{-mx}; \qquad C_1 e^{mx} - C_2 e^{-mx} = 0$$

$$\frac{\theta}{\theta_b} = \frac{e^{mx}}{1 + e^{2mL}} + \frac{e^{-mx}}{1 + e^{-2mL}}$$

Equation for adiabatic condition

$$\frac{\theta}{\theta_b} = \frac{\cosh[m(L - x)]}{\cosh mL}$$

Note: $\cosh A = \dfrac{e^{A} + e^{-A}}{2}$

Case B: Adiabatic Condition at the Fin Tip

$$
\begin{aligned}
q_f = -kA_c \left.\frac{dT}{dx}\right|_{x=0} &= -kA_c \left.\frac{d\theta}{dx}\right|_{x=0} \\
&= -kA_c \quad m\theta_b \left[\frac{1}{1+e^{2mL}} - \frac{1}{1+e^{-2mL}}\right] \\
&= \sqrt{hPkA_c}\theta_b \left[\frac{e^{mL}-e^{-mL}}{e^{mL}+e^{-mL}}\right]
\end{aligned}
$$

Rate of heat transfer: Adiabatic Condition

$$q_f = \sqrt{hPkA_c}\theta_b \tanh mL$$

Case C: h from the Fin Tip

A practical way is to account for the heat loss from the fin tip is to replace the *fin length* L in the relation for the adiabatic tip case by a corrected length.

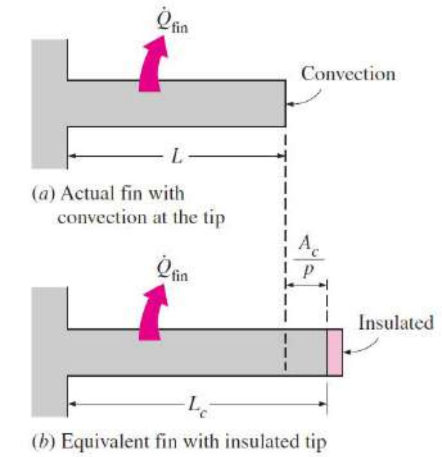
$$L_c = L + \frac{A_c}{P}$$

$$\boxed{\frac{\theta}{\theta_b} = \frac{\cosh[m(L_c - x)]}{\cosh mL_c}}$$

$$\boxed{q_f = \sqrt{hPkA_c}\theta_b \tanh mL_c}$$

$$L_{c,rectanular\ fin} = L + \frac{t}{2}$$

$$L_{c,cylindrical\ fin} = L + \frac{D}{4}$$



Temperature distribution & heat loss for fins of uniform cross-section

Tip Cond.	at $x = L$	$\frac{\theta}{\theta_b}$	q_f
Infinite fin	$\theta(L) = 0$	e^{-mx}	M
Adiabatic	$\frac{d\theta}{dx}\Big\vert_{x=L} = 0$	$\frac{\cosh[m(L-x)]}{\cosh mL}$	$M \tanh mL$
Convection	$h\theta_L = -k \, \frac{d\theta}{dx}\Big\vert_{x=L_c}$	$\frac{\cosh[m(L_c-x)]}{\cosh mL_c}$	$M \tanh mL$

$$m = \sqrt{\frac{hP}{kA_c}}; \qquad M = \sqrt{hPkA_c}\,\theta_b; \qquad L_c = L + \frac{A_c}{P}$$

Problem

A very long rod 5 mm in diameter has one end maintained at 100°C. The surface of the rod is exposed to ambient air at 25°C with a convection heat transfer coefficient of 100 W/m^2 K.

- Determine the temperature distributions along rods constructed from pure copper, 2024 aluminium alloy and type AISI 316 stainless steel. What are the corresponding heat losses from the rods? Ans: 8.3 W, 5.6 W and 1.6 W
- Estimate how long the rods must be for the assumption of infinite length to yield an accurate estimate of the heat loss.

At $T = (T_b + T_\infty)/2 = 62.5$°C $= 335\ K$:

$$k_{copper} = 398\ W/m\ K$$
$$k_{aluminium} = 180\ W/m\ K$$
$$k_{stainless\ steel} = 14\ W/m\ K$$

Hint: For an infinitely long fin: $\theta/\theta_b = e^{-mx}$

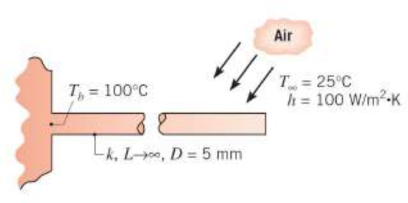
Solution

Find

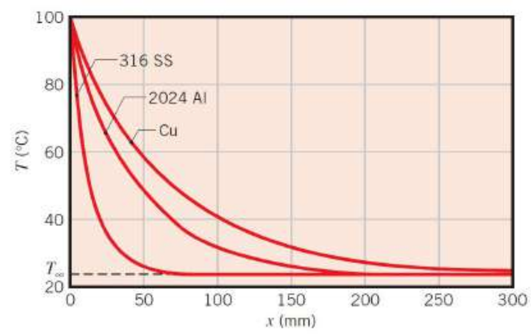
- $T(x)$ and heat loss when rod is Cu, Al, SS.
- How long rods must be to assume infinite length.

Assumptions

- Steady state conditions, 1-D along the rod
- Constant properties and uniform h
- Negligible radiation exchange with surroundings



$$T = T_\infty + (T_b - T_\infty)e^{-mx}$$



There is little additional heat transfer associated with lengths more than 50 mm (SS), 200 mm (Al), and 300 mm (Cu).

$$q_f = \sqrt{hPkA_c}\theta_b$$

Since there is no heat loss from the tip of an infinitely long rod, an estimate of the validity of the approximation may be made by comparing q_f for infinitely long fin and adiabatic fin tip.

$$\sqrt{hPkA_c}\theta_b = \sqrt{hPkA_c}\theta_b \tanh mL$$

$$\tanh 4 = 0.999 \qquad \text{and} \qquad \tanh 2.5 = 0.987$$

$$\implies \qquad mL \geq 2.5$$

$$L \geq \frac{2.65}{m} = 2.5\sqrt{\frac{kA_c}{hP}}$$

$$L_{Cu} = 0.18\ m; \quad L_{Al} = 0.12\ m; \quad \text{and} \quad L_{SS} = 0.033\ m$$

Solution: Comments

Comments

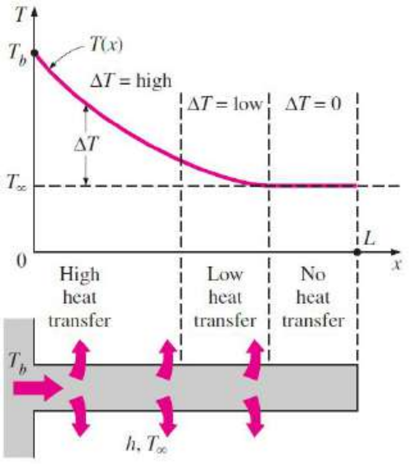
- The above results suggest that the fin heat transfer rate may accurately be predicted from the infinite fin approximation if $mL \geq 2.5$
- For more accuracy, if $mL \geq 4.6$:

  $L_\infty =$ 0.33 m (Cu), 0.23 m (Al) and 0.07 m (SS)

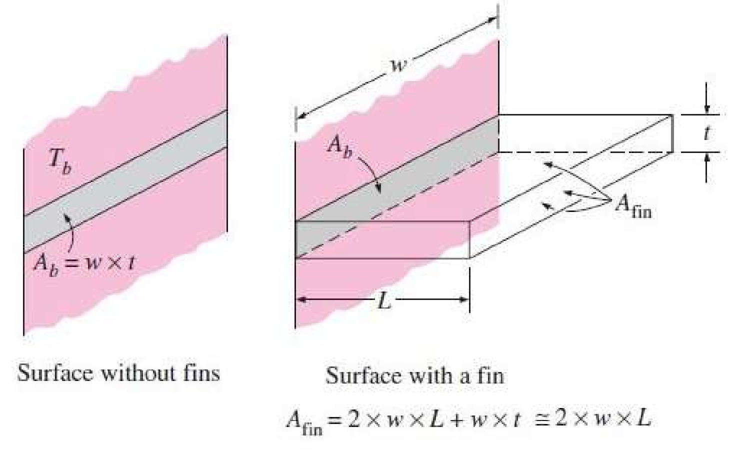
Proper Length of a Fin

$$\frac{q_{fin}}{q_{long\ fin}} = \tanh mL$$

mL	$\tanh mL$
0.1	0.100
0.2	0.197
0.5	0.462
1.0	0.762
1.5	0.905
2.0	0.964
2.5	**0.987**
3.0	0.995
4.0	0.999
5.0	**1.000**



Fin Efficiency



No fin: $q_{conv} = hA_b(T_b - T_\infty)$

Fin Efficiency

- The temperature of the fin will be T_b at the fin base and gradually decrease towards the fin tip.
- Convection from the fin surface causes the temperature at any cross-section to drop somewhat from the midsection toward the outer surfaces.
- However, the cross-sectional area of the fins is usually very small, and thus the temperature at any cross-section can be considered to be uniform.
- Also, the fin tip can be assumed for convenience and simplicity to be adiabatic by using the corrected length for the fin instead of the actual length.

In the limiting case of zero thermal resistance or infinite k, the temperature of fin will be uniform at the value of T_b. The heat transfer from the fin will be maximum in this case ($k \rightarrow \infty$):

$$q_{fin,max} = hA_{fin}(T_b - T_\infty)$$

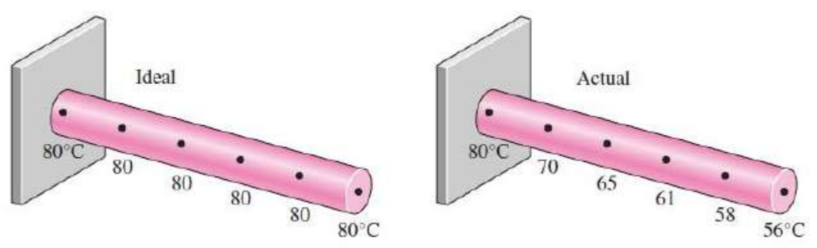
Fin Efficiency

In reality, however, the temperature of the fin will drop along the fin and thus the heat transfer from the fin will be less because of the decreasing $[T(x) - T_\infty]$ toward the fin tip.

To account for the effect of this decrease in temperature on heat transfer, we define fin efficiency as:

$$\eta_{fin} = \frac{q_{fin}}{q_{fin,max}} = \frac{\text{Actual heat transfer rate from the fin}}{\begin{array}{c}\text{Ideal heat transfer rate from the fin} \\ \text{if the entire fin were at base temperature}\end{array}}$$

Fin Efficiency



Fin Efficiency: Uniform Cross-Sectional Area

Case A: Infinitely long fins

$$\eta_{\text{long fin}} = \frac{q_{fin}}{q_{fin,max}} = \frac{1}{mL}$$

$$\because A_{fin} = pL$$

Case B: Adiabatic tip

$$\eta_{\text{adiabatic}} = \frac{q_{fin}}{q_{fin,max}} = \frac{\tanh mL}{mL}$$

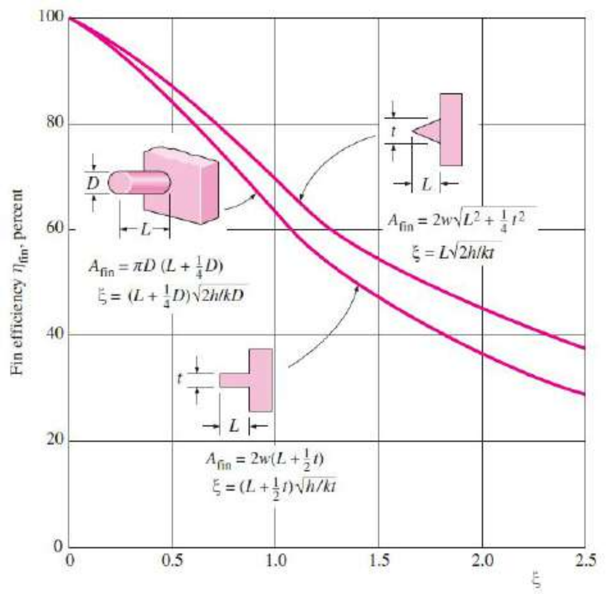
Case C: Convection at tip

$$\eta_{h \text{ at tip}} = \frac{q_{fin}}{q_{fin,max}} = \frac{\tanh mL_c}{mL_c}$$

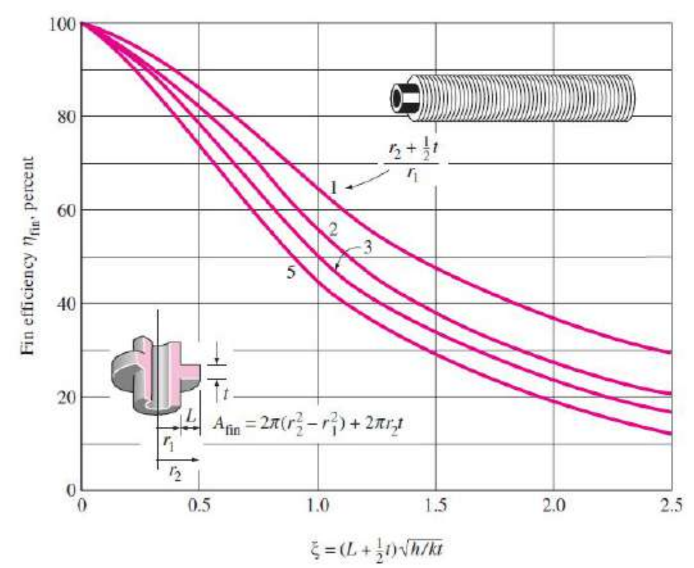
Fin Efficiency: Proper Length of a Fin

- An important consideration in the design of finned surace is the selection of the proper fin length, L.
- Normally the longer the fin, the larger the heat transfer and thus the higher the rate of heat transfer from the fin.
- But also the larger the fin, the bigger the mass, the higher the price, and the larger the fluid friction.
- Therefore, increasing the length of the fin beyond a certain value cannot be justified unless the added benefits outweigh the added cost.
- Also, η_{fin} decreases with increasing fin length because of the decrease in fin temperature with length.
- Fin lengths that cause the fin efficiency to drop below 60% usually cannot be justified economically and should be avoided.
- η of most fins used in practice is $> 90\%$.

η of Rectangular, Triangular, Parabolic Profiles



η of Annular fins of constant thickness, t

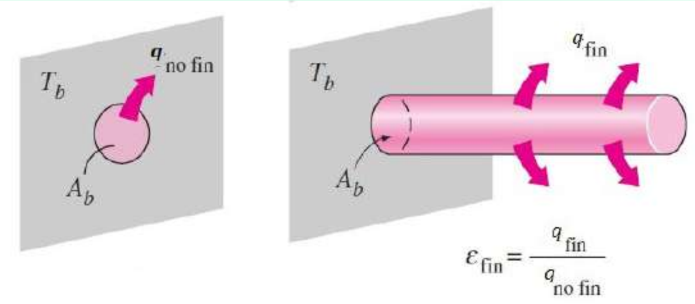


Fin Effectiveness

The performance of the fins is judged on the basis of enhancement of heat transfer relative to the no fin case.

$$\varepsilon_{fin} = \frac{q_{fin}}{q_{no\ fin}} = \frac{q_f}{hA_b(T_b - T_\infty)}$$

where A_b is the fin cross-sectional area at the base.



Fin Effectiveness: Physical Significance

- $\boldsymbol{\varepsilon_{fin} = 1}$ indicates that the addition of fins to the surface does not affect heat transfer at all. That is, heat conducted to the fin through the base area A_b is equal to the heat transferred from the same area A_b to the surrounding medium.
- $\boldsymbol{\varepsilon_{fin} < 1}$ indicates that the fin actually acts as insulation, slowing down the heat transfer from the surface. This situation can occur when fins made of low k are used.
- $\boldsymbol{\varepsilon_{fin} > 1}$ indicates that the fins are enhancing heat transfer from the surface. However, the use of fins cannot be justified unless ε_{fin} is sufficiently larger than 1 (≥ 2). Finned surfaces are designed on the basis of maximizing effectiveness of a specified cost or minimizing cost for a desired effectiveness.

Efficiency and Effectiveness

η_{fin} and ε_{fin} are related to performance of the fin, but they are different quantities.

$$\begin{aligned}
\varepsilon_{fin} &= \frac{q_{fin}}{q_{no\ fin}} \\
&= \frac{q_{fin}}{hA_b(T_b - T_\infty)} \\
&= \frac{\eta_{fin} h A_{fin}(T_b - T_\infty)}{hA_b(T_b - T_\infty)} \\
\implies & \boxed{\varepsilon_{fin} = \frac{\eta_{fin} A_{fin}}{A_b}}
\end{aligned}$$

Therefore, η_{fin} can be determined easily when ε_{fin} is known, or vice versa.

ε for a Long Uniform Cross-Section Fin

$$\varepsilon_{fin} = \frac{q_{fin}}{q_{no\ fin}} = \frac{\sqrt{hPkA_c}\theta_b}{hA_b(T_b - T_\infty)} = \sqrt{\frac{kP}{hA_c}}$$

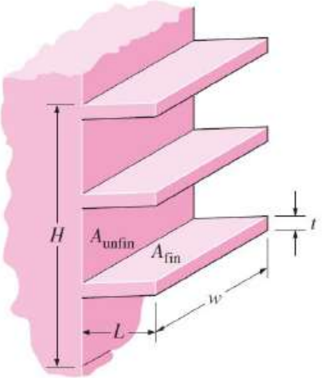
$$\because A_c = A_b \text{ and } \theta_b = T_b - T_\infty$$

- k of fin should be high. Ex: Cu, Al, Fe. Aluminium is low cost, weight, and resistant to corrosion.
- P/A_c should be high. Thin plates or slender pin fins
- h should be low. Gas instead of liquid; Natural convection instead of forced convection. Therefore, in liquid-to-gas heat exchangers (car radiators), fins are placed on the gas side.

Multiple Fins

Heat transfer rate for a surface containing n fins:

$$\begin{aligned} q_{tot,fin} &= q_{unfin} + q_{fin} \\ &= hA_{unfin}(T_b - T_\infty) \\ &\qquad + \eta_{fin}A_{fin}(T_b - T_\infty) \end{aligned}$$



$$\begin{aligned} A_{no\ fin} &= w \times H \\ A_{unfin} &= w \times H - 3 \times (t \times w) \\ A_{fin} &= 2 \times L \times w + t \times w \text{ (one fin)} \\ &\approx 2 \times L \times w \end{aligned}$$

$$\boxed{q_{tot,fin} = h(A_{unfin} + \eta_{fin}A_{fin})(T_b - T_\infty)}$$

Overall Effectiveness

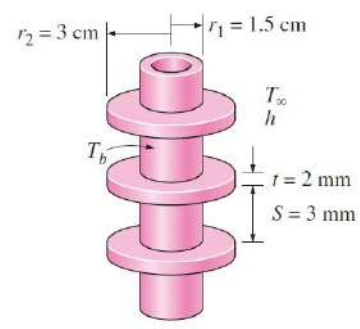
We can also define an overall effectiveness for a finned surface as the ratio of the total q from the finned surface to the q from the same surface if there were no fins:

$$\varepsilon_{fin,overall} = \frac{q_{fin}}{q_{nofin}}$$
$$= \frac{h(A_{unfin} + \eta_{fin} A_{fin})(T_b - T_\infty)}{h A_{nofin}(T_b - T_\infty)}$$

A_{nofin} is the area of the surface when there are no fins
A_{fin} is the total surface area of all the fins on the surface
A_{unfin} is the area of the unfinned portion of the surface.

$\varepsilon_{fin,overall}$ depends on number of fins per unit length as well as ε_{fin} of individual fins.
$\varepsilon_{fin,overall}$ is a better measure of the performance than ε_{fin} of individual fins.

Steam in a heating system flows through tubes: outer diameter is $D_1 = 3$ cm and whose walls are maintained at 125°C. Circular aluminium fins ($k = 180$ W/m K) of $D_2 = 6$ cm, $t = 2$ mm are attached. The space between the fins is 3 mm, and thus there are 200 fins per meter length of the tube. Surrounded air: $T_\infty = 27$°C, $h = 60$ W/m^2 K. Determine the increase in heat transfer from the tube per meter of its length as a result of adding fins.



Solution

Known

Properties of the fin, ambient conditions, heat transfer coefficient, dimensions of the fin.

Find

Increase in heat transfer from the tube per meter of its length as a result of adding fins.

Assumptions

- Steady state conditions, 1-D along the rod
- Constant properties and uniform h
- Negligible radiation exchange with surroundings

Solution: Analysis

In case of no fins (per unit length, $l = 1$ m:

$$A_{nofin} = \pi D_1 l = 0.0942 \text{ m}^2$$

$$q_{nofin} = h A_{nofin}(T_b - T_\infty) = 554 \text{ W}$$

$$r_1 = D_1/2 = 0.015 \text{ m}$$

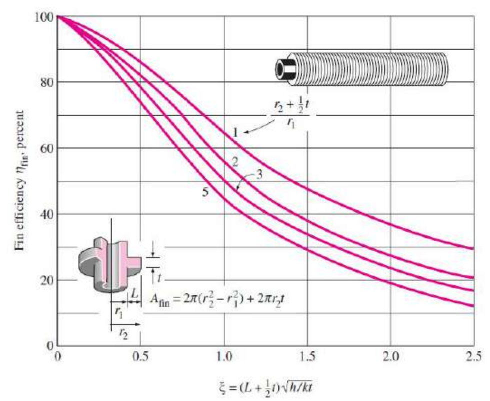
$$r_2 = D_2/2 = 0.03 \text{ m}$$

$$\frac{r_2 + \frac{t}{2}}{r_1} = \mathbf{2.07}$$

$$L = r_2 - r_1 = 0.015 \text{ m}$$

$$\xi = \left(L + \frac{t}{2}\right)\sqrt{\frac{h}{kt}} = \mathbf{0.207}$$

$$\Longrightarrow \boxed{\eta_{fin} = 0.95}$$



Solution: Analysis

q from finned portion

$$A_{fin} = 2\pi\left(r_2^2 - r_1^2\right) + 2\pi r_2 t$$
$$= 0.00462 \text{ m}^2$$
$$q_{fin} = \eta_{fin} q_{fin,max}$$
$$= \eta_{fin} h A_{fin}(T_b - T_\infty)$$
$$= 27.81 \text{ W}$$

q from unfinned portion of tube

$$A_{unfin} = 2\pi r_1 S$$
$$= 0.000283 \text{ m}^3$$
$$q_{unfin} = h A_{unfin}(T_b - T_\infty)$$
$$= 1.67 \text{ W}$$

There are 200 fins per meter length of the tube. The total heat transfer from the finned tube:

$$q_{tot,fin} = n(q_{fin} + q_{unfin}) = 5896 \text{ W}$$

$\therefore$ the increase in heat transfer from the tube per meter of its length as a result of the addition of fins is:

$$q_{increase} = q_{tot,fin} - q_{nofin} = \mathbf{5342\ W} \qquad \text{per meter tube length}$$

Solution: Comments

Effectiveness

The overall effectiveness of the finned tube is:

$$\varepsilon_{fin,overall} = \frac{q_{tot,fin}}{q_{tot,nofin}} = 10.6$$

That is, the rate of heat transfer from the steam tube increases by a factor of 10 as a result of adding fins.

Heat and Mass Transfer

Transient Heat Conduction

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Time dependent conduction - Temperature history inside a conducting body that is immersed suddenly in a bath of fluid at a different temperature.

Ex: Quenching of special alloys, heat treatment of bearings

The temperature of such a body varies with time as well as position.

$$T(x, y, z, t)$$

Transient Heat Conduction

A body is exposed to ambient

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

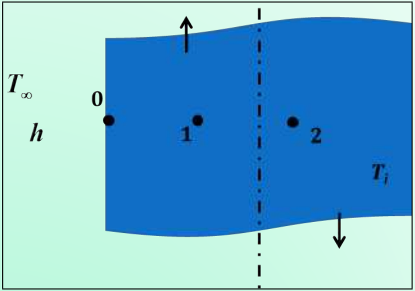
No heat generation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

$\alpha \rightarrow$ Thermal diffusivity (m^2/s)

It appears only in the transient conduction

$$T = f(x, t)$$



$$T_{t=0} = T_i$$

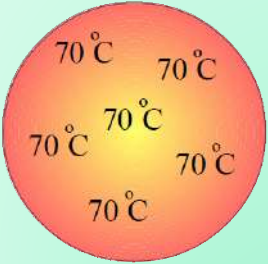
$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$$

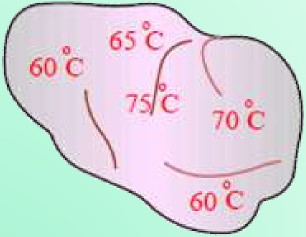
$$q_{x=\pm L} = h(T_\infty - T)$$

Lumped Capacitance Model

Lumped: Temperature is essentially uniform throughout the body.

$$T(x, y, z, t) = T(t)$$

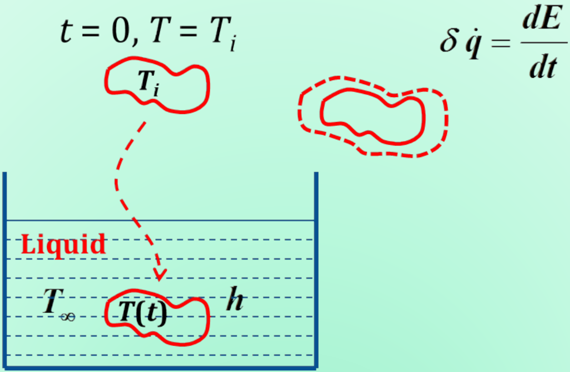




Copper ball with uniform temperature   Pot

Lumped Capacitance Model

Hot forging that is initially at uniform temperature, T_i and is quenched by immersing it in a liquid of lower temperature $T_\infty < T_i$



$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_{gen} = \dot{E}_{st}$$

Lumped Capacitance Model

$$-hA(T - T_\infty) = \rho V C_p \frac{dT}{dt}$$

$$\int\limits_{T=T_i}^{T} \frac{dT}{T - T_\infty} = -\frac{hA}{\rho V C_p} \int\limits_{t=0}^{t} dt$$

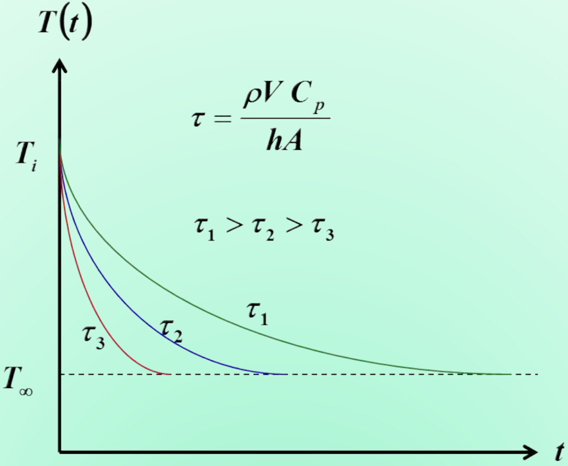
$$\boxed{\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = e^{-\frac{t}{\tau}} \qquad \tau = \frac{\rho V C_p}{hA}}$$

$$\tau = \left(\frac{1}{hA}\right)(\rho V C_p) = R_t C_t$$

R_t - Resistance to convection heat transfer

C_t - Lumped thermal capacitance of the solid

Time Constant



$$\left.\frac{\theta}{\theta_i}\right|_{t=\tau} = 0.368$$

Total Energy Transfer

The rate of convection heat transfer between the body and its environment at any time: $q = hA[T(t) - T_\infty]$
Total energy transfer occurring up to sometime, t:

$$
\begin{aligned}
Q &= \int_{t=0}^{t} q dt \\
&= \int_{t=0}^{t} hA[T(t) - T_\infty] dt \\
&= hA(T_i - T_\infty) \int_{t=0}^{t} e^{-\frac{t}{\tau}}
\end{aligned}
$$

$$
\boxed{Q = \rho V C_p (T_i - T_\infty) \left[1 - e^{-\frac{t}{\tau}}\right]}
$$

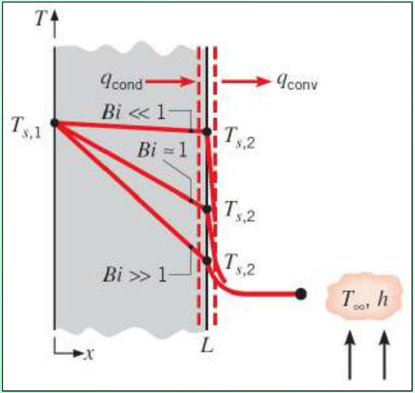
Consider a body exposed to ambient

$$q_{conv} = q_{cond}$$

$$h(T_{s,2} - T_\infty) = k\frac{T_{s,1} - T_{s,2}}{L_c}$$

$$\frac{T_{s,1} - T_{s,2}}{T_{s,2} - T_\infty} = \frac{hL_c}{k} = \text{Bi}$$

$$L_c = \frac{V}{A}$$



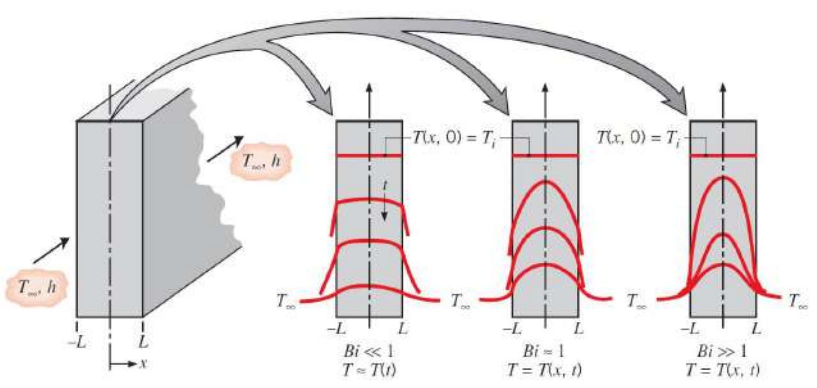
Biot Number

$$\begin{aligned}\text{Bi} &= \frac{hL_c}{k} \\ &= \frac{h\Delta T}{k\Delta T/L_c} \\ &= \frac{\text{Conv. at the surface of the body}}{\text{Conduction within the body}}\end{aligned}$$

$$\begin{aligned}\text{Bi} &= \frac{L_c/k}{1/h} \\ &= \frac{\text{Conduction resistance within the body}}{\text{Conv. resistance at the surface}}\end{aligned}$$

Generally accepted, $\boxed{\text{Bi} \leq 0.1}$ for assuming lumped.

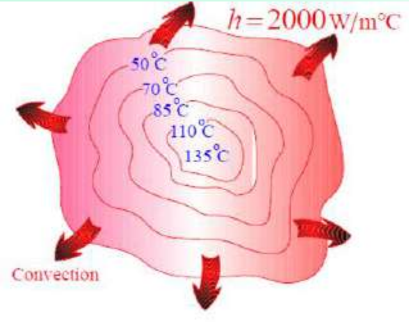
Biot Number



T_∞, h
T_∞, h
$-L$
L
x
$T(x, 0) = T_i$
$T(x, 0) = T_i$
t
T_∞
T_∞
T_∞
T_∞
$-L$
L
$Bi \ll 1$
$T \approx T(t)$
$-L$
L
$Bi \approx 1$
$T = T(x, t)$
$-L$
L
$Bi \gg 1$
$T = T(x, t)$

Small bodies with higher k and low h are most likely satisfy Bi ≤ 0.1.

When k is low and h is high, large temperature differences occur between the inner and outer regions of the body.



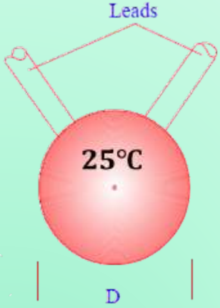
Professor Jean-Baptiste Biot

Jean-Baptiste Biot
(1774-1862)

- French physicist, astronomer, and mathematician born in Paris, France.
- Professor of mathematical physics at Collège de France .
- At the age of 29, he worked on the analysis of heat conduction even earlier than Fourier did (unsuccessful). After 7 years, Fourier read Biot's work.
- Awarded the Rumford Medal of the Royal Society in 1840 for his contribution in the field of Polarization of light.

Problem: Thermocouple Diameter

Determine the thermocouple junction diameter needed to have a time constant of one second.
Ambient: $T_\infty = 200°$C, $h = 400$ W/m^2 K
Material
properties: $k = 20$ W/m K, $C_p = 400$ J/kg K, $\rho = 8500$ kg/m^{3}Ans:



0.706 mm

Problem: Solution

Known

Thermo-physical properties of the thermocouple junction used to measure the temperature of a gas stream.
Thermal environmental conditions.

Find

Junction diameter needed for a time constant of 1 second.

Assumptions

- Temperature of the junction is uniform at any instant.
- Radiation exchange with the surroundings is negligible.
- Losses by conduction through the leads is negligible.
- Constant properties.

Problem: Analysis

$$L_c = \frac{V}{A} = \frac{\pi D^3/6}{\pi D^2} = \frac{D}{6}$$

$$\tau = \frac{\rho C_p V}{hA} = \frac{\rho C_p D}{6h}$$

$$\boxed{D = 0.706 \text{ mm}}$$

$$\text{Bi} = \frac{hL_c}{k} = 2.35 \times 10^{-3} < 0.1$$

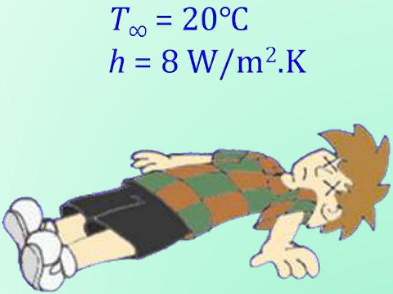
Criterion for using the lumped capacitance model is satisfied and the lumped capacitance method may be used to an excellent approximation.

Comments

Heat transfer due to radiation exchange between the junction and the surroundings and conduction through the leads would affect the time response of the junction and would, in fact, yield an equilibrium temperature that differs from T_∞.

Problem: Predict the Time of Death

A person is found dead at 5 PM in a room. The temperature of the body is measured to be 25°C when found. Estimate the time of death of that person.



Known

T of the person at 5 PM.
Thermal environmental conditions.

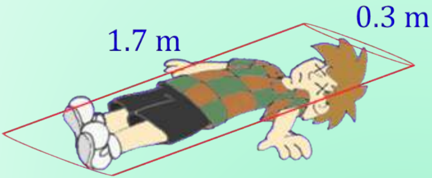
Find

The time of death of the person is to be estimated.

Problem: Solution

Assumptions

- The body can be modeled as a cylinder.
- Radiation exchange with the surroundings is negligible.
- The initial temperature of the person is 37°C.
- Assuming properties of water.



Water at (37 + 25)/2 = 31°C
$k = 0.617$ W/m.K
$Cp = 4.178$ kJ/kg.K
$\rho = 996$ kg/m^3

$$L_c = \frac{V}{A} = \frac{(\pi D^2/4)L}{\pi DL + 2(\pi D^2/4)} = 0.069 \text{ m}$$

Problem: Solution

$$\text{Bi} = \frac{hL_c}{k} = 0.9 > 0.1$$

Comment: Criterion for using the lumped capacitance model is not satisfied. However, let us get a rough estimate.

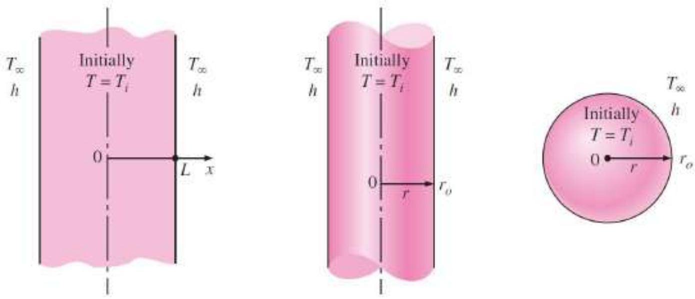
$$\tau = \frac{\rho C_p V}{hA} = \frac{\rho C_p V}{hA} = 35891 \text{ s}$$

$$\frac{T - T_\infty}{T_i - T_\infty} = \frac{25 - 20}{37 - 20} = e^{-\frac{t}{\tau}}$$

$$\boxed{t = 43923 \text{ s} = 12.2 \text{ hours}}$$

Therefore, the person would have died around 5 AM.

A large plane wall — A long cylinder — A sphere



$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$T(x,0) = T_i \quad \text{Initial}$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0 \quad \text{Symmetry}$$

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h\left[T(L,t) - T_\infty\right] \quad \text{Boundary}$$

Transient Conduction with Spatial Effects - 1D

In all these three cases posses geometric and thermal symmetry:

the plane wall is symmetric about its center plane ($x = 0$),

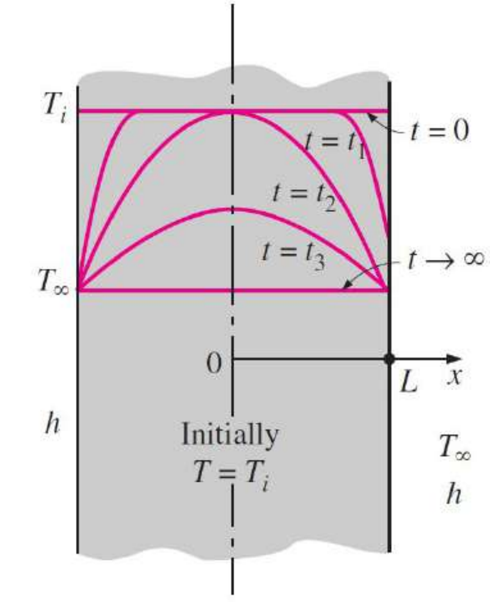
the cylinder is symmetric about its center line($r = 0$), and

the sphere is symmetric about its center point ($r = 0$)

Neglect q_{rad} or incorporate as h_r.

The solution, however, involves infinite series, which are inconvenient and time consuming to evaluate.

Transient Temperature Profiles



Reduction of Parameters

It involves the parameters, $x, L, t, k, \alpha, h, T_i,$ and T_∞ which are too many to make any graphical presentation of the results practical.

$$\text{Dimensionless temperature:} \quad \theta(x,t) = \frac{T(x,t) - T_\infty}{T_i - T_\infty}$$

$$\text{Dimensionless distance from center:} \quad X = \frac{x}{L}$$

$$\text{Dimensionless } h \text{ (Biot number):} \quad \mathrm{Bi} = \frac{hL}{k}$$

$$\text{Dimensionless time (Fourier number):} \quad \mathrm{Fo} = \frac{\alpha t}{L^2} = \tau$$

The non-dimensionalization enables us to present the temperature in terms of three parameters only: $\theta = f(X, \mathrm{Bi}, \mathrm{Fo})$.

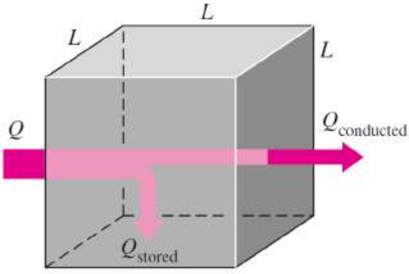
In case of lumped system analysis, $\theta = f(\mathrm{Bi}, \mathrm{Fo})$.

Fourier Number: Physical Significance

$$\text{Fo} = \frac{\alpha t}{L^2} = \frac{\left(kL^2 \frac{\Delta T}{L}\right)}{\left(\rho L^3 C_p \frac{\Delta T}{t}\right)}$$

$$= \frac{\text{Rate of heat conducted across } L \text{ of a body of volume } L^3}{\text{Rate of heat stored in a body of volume } L^3}$$

$$\text{Fo} = \frac{Q_{conducted}}{Q_{stored}}$$



Exact Solution: One-Term Approximation

The exact solution involves infinite series. However, the terms in terms in the solutions converge rapidly with increasing time, and for $\boldsymbol{\tau > 0.2}$, keeping the first term and neglecting all the remaining terms in the series results in an error under 2%.

$$\text{Plane Wall:} \quad \theta(x,t)_{wall} = \frac{T(x,t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \cos(\lambda_1 x/L)$$

$$\text{Cylinder:} \quad \theta(r,t)_{cyl} = \frac{T(r,t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} J_0(\lambda_1 r/r_0)$$

$$\text{Sphere:} \quad \theta(r,t)_{sph} = \frac{T(r,t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \frac{\sin(\lambda_1 r/r_0)}{\lambda_1 r/r_0}$$

A_1 and λ_1 are functions of the Bi number only.

Note: $\cos(0) = J_0(0) = 1$ and the limit of $(\sin x)/x = 1$.

$$\theta_0 = \frac{T_0 - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau}$$

Exact Solution: One Term Approximation

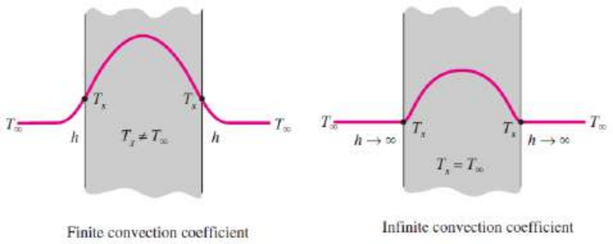
Coefficients used in the one-term approximate solution of transient one-dimensional heat conduction in plane walls, cylinders, and spheres ($Bi = hL/k$ for a plane wall of thickness $2L$, and $Bi = hr_o/k$ for a cylinder or sphere of radius r_o)

	Plane Wall		Cylinder		Sphere	
Bi	λ_1	A_1	λ_1	A_1	λ_1	A_1
0.01	0.0998	1.0017	0.1412	1.0025	0.1730	1.0030
0.02	0.1410	1.0033	0.1995	1.0050	0.2445	1.0060
0.04	0.1987	1.0066	0.2814	1.0099	0.3450	1.0120
0.06	0.2425	1.0098	0.3438	1.0148	0.4217	1.0179
0.08	0.2791	1.0130	0.3960	1.0197	0.4860	1.0239
0.1	0.3111	1.0161	0.4417	1.0246	0.5423	1.0298
0.2	0.4328	1.0311	0.6170	1.0483	0.7593	1.0592
0.3	0.5218	1.0450	0.7465	1.0712	0.9208	1.0880
0.4	0.5932	1.0580	0.8516	1.0931	1.0528	1.1164
0.5	0.6533	1.0701	0.9408	1.1143	1.1656	1.1441
0.6	0.7051	1.0814	1.0184	1.1345	1.2644	1.1713
0.7	0.7506	1.0918	1.0873	1.1539	1.3525	1.1978
0.8	0.7910	1.1016	1.1490	1.1724	1.4320	1.2236
0.9	0.8274	1.1107	1.2048	1.1902	1.5044	1.2488
1.0	0.8603	1.1191	1.2558	1.2071	1.5708	1.2732
2.0	1.0769	1.1785	1.5995	1.3384	2.0288	1.4793
3.0	1.1925	1.2102	1.7887	1.4191	2.2889	1.6227
4.0	1.2646	1.2287	1.9081	1.4698	2.4556	1.7202
5.0	1.3138	1.2403	1.9898	1.5029	2.5704	1.7870
6.0	1.3496	1.2479	2.0490	1.5253	2.6537	1.8338
7.0	1.3766	1.2532	2.0937	1.5411	2.7165	1.8673
8.0	1.3978	1.2570	2.1286	1.5526	2.7654	1.8920
9.0	1.4149	1.2598	2.1566	1.5611	2.8044	1.9106
10.0	1.4289	1.2620	2.1795	1.5677	2.8363	1.9249
20.0	1.4961	1.2699	2.2880	1.5919	2.9857	1.9781
30.0	1.5202	1.2717	2.3261	1.5973	3.0372	1.9898
40.0	1.5325	1.2723	2.3455	1.5993	3.0632	1.9942
50.0	1.5400	1.2727	2.3572	1.6002	3.0788	1.9962
100.0	1.5552	1.2731	2.3809	1.6015	3.1102	1.9990
∞	1.5708	1.2732	2.4048	1.6021	3.1416	2.0000

The zeroth- and first-order Bessel functions of the first kind

ξ	$J_0(\xi)$	$J_1(\xi)$
0.0	1.0000	0.0000
0.1	0.9975	0.0499
0.2	0.9900	0.0995
0.3	0.9776	0.1483
0.4	0.9604	0.1960
0.5	0.9385	0.2423
0.6	0.9120	0.2867
0.7	0.8812	0.3290
0.8	0.8463	0.3688
0.9	0.8075	0.4059
1.0	0.7652	0.4400
1.1	0.7196	0.4709
1.2	0.6711	0.4983
1.3	0.6201	0.5220
1.4	0.5669	0.5419
1.5	0.5118	0.5579
1.6	0.4554	0.5699
1.7	0.3980	0.5778
1.8	0.3400	0.5815
1.9	0.2818	0.5812
2.0	0.2239	0.5767
2.1	0.1666	0.5683
2.2	0.1104	0.5560
2.3	0.0555	0.5399
2.4	0.0025	0.5202
2.6	−0.0968	−0.4708
2.8	−0.1850	−0.4097
3.0	−0.2601	−0.3391
3.2	−0.3202	−0.2613

Special Case



$$\frac{1}{\text{Bi}} = \frac{k}{hL} = 0 \text{ corresponds to } h \to \infty,$$

which corresponds to specified surface temperature, T_∞, case.

Total Energy Transferred from the Wall

The maximum amount of heat that a body can gain (or lose if $T_i > T_\infty$) is the change in the energy content of the body:

$$Q_{max} = mC_p(T_\infty - T_i) = \rho V C_p(T_\infty - T_i)$$

Q_{max} represents the amount of heat transfer for $t \to \infty$. The amount of heat transfer, Q at a finite time t is obviously less than this. It can be expressed as the sum of the internal energy changes throughout the entire geometry as:

$$Q = \int_V \rho C_p[T(x,t) - T_i]dV$$

Assuming constant properties:

$$\frac{Q}{Q_{max}} = \frac{\int_V \rho C_p[T(x,t) - T_i]dV}{\rho V C_p(T_\infty - T_i)} = \frac{1}{V}\int_V (1-\theta)dV$$

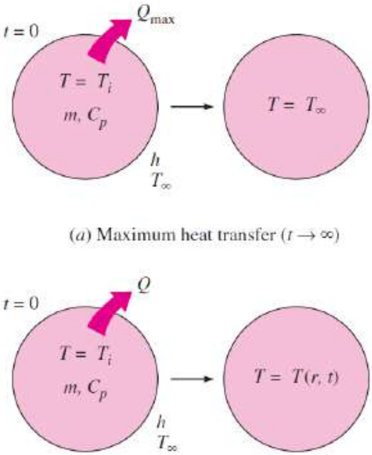
Total Energy: One Term Approximation

Plane Wall: $$\left(\frac{Q}{Q_{max}}\right)_{wall} = 1 - \theta_{0,wall}\frac{\sin\lambda_1}{\lambda_1}$$

Cylinder: $$\left(\frac{Q}{Q_{max}}\right)_{cyl} = 1 - \theta_{0,cyl}\frac{J_1(\lambda_1)}{\lambda_1}$$

Sphere: $$\left(\frac{Q}{Q_{max}}\right)_{sph} = 1 - \theta_{0,wall}\frac{\sin\lambda_1 - \lambda_1\cos\lambda_1}{\lambda_1}$$

Fraction of Total Heat Transfer: Gröber Chart



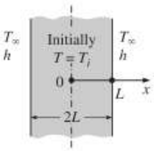
(a) Maximum heat transfer ($t \to \infty$)

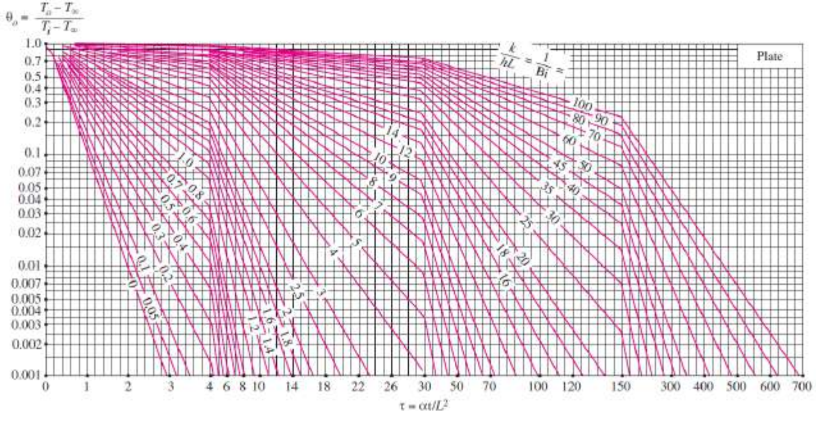
$$\left.\begin{aligned} \mathrm{Bi} &= \ldots \\ \frac{h^2 \alpha t}{k^2} &= \mathrm{Bi}^2 \tau = \ldots \end{aligned}\right\} \quad \frac{Q}{Q_{max}} = \ldots$$

(Gröber chart)

Valid for $\tau > 0.2$

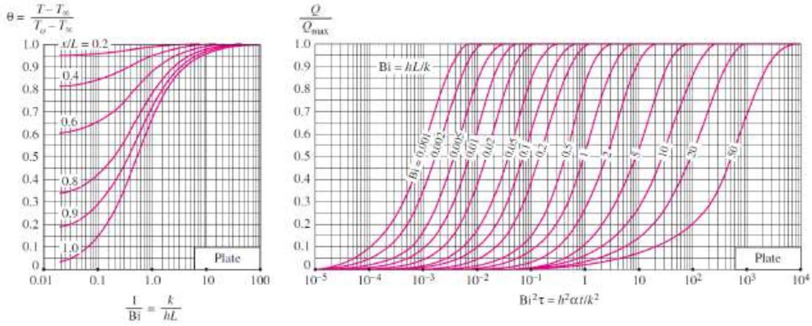
Plane wall





(a) Midplane temperature (from M. P. Heisler)

Graphical Solution: Heisler Charts



(b) Temperature distribution (from M. P. Heisler) (c) Heat transfer (from H. Gröber et al.)

Limitations

Limitations of One-term solution and Heisler/Gröber charts

- Body is initially at an uniform temperature
- T_∞ and h are constant and uniform
- No energy generation within the body

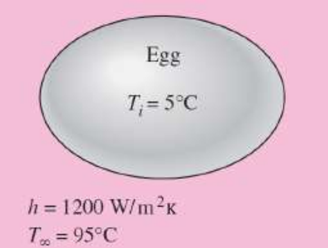
Infinitely Large or Long?

- A plate whose thickness is small relative to the other dimensions can be modeled as an infinitely large plate, except very near the outer edges.
- The edge effects on large bodies are usually negligible.
- Ex: A large plane wall such as the wall of a house.
- Similarly, a long cylinder whose diameter is small relative to its length can be analyzed as an infinitely long cylinder.

Problem

An ordinary egg can be approximated as a 5 cm diameter sphere. The egg is initially at a uniform temperature of 5°C and is dropped into boiling water at 95°C. Taking the convection heat transfer coefficient to be $h = 1200$ W/m^2 K, determine how long it will take for the center of the egg to reach 70°C.

The water content of eggs is about 74%, and thus k and α of eggs can be approximated by those of water at the average temperature $(5 + 70)/2 = 37.5$°C.



$$k = 0.627 \text{ W/m K}; \qquad \alpha = k/\rho c_p = 0.151 \times 10^{-6} \text{ m}^2/\text{s}$$

Solution

$$T_i = 5°\text{C}; \quad T_\infty = 95°\text{C} \quad T_0 = 70°\text{C} \quad h = 1200 \text{ W/m}^2 \text{ K}$$

$$k = 0.627 \text{ W/m K}; \qquad \alpha = k/\rho c_p = 0.151 \times 10^{-6} \text{ m}^2/\text{s}$$

$$\text{Bi}_{L_c} = \frac{hL_c}{k} = 16 \qquad \text{Bi}_{r_0} = \frac{hr_0}{k} = 47.8$$

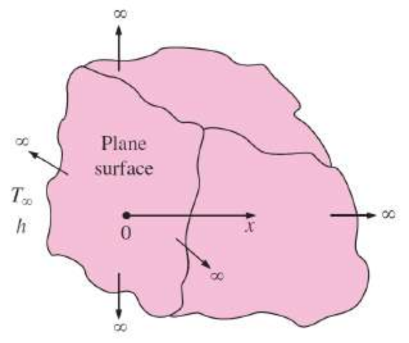
One-term approximation

$$\theta_0 = \frac{T_0 - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau}, \tau > 0.2$$

$$\tau = \frac{\alpha t}{r_0^2}$$

$$\lambda_1 = 3.0754; \qquad A_1 = 1.9958$$

$\tau = 0.209$, Ans: 14.4 min



- A single plane surface and extends to infinity in all directions.
- Ex: Earth -temperature variation near its surface
  Thick wall - temperature variation near one of its surfaces
- For short periods of time, most bodies are semi-infinite solids.
- The thickness of the body does not enter into the heat transfer analysis.

Semi-Infinite Solids

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$T(0,t) = T_s; \quad T(x \to \infty, t) = T_i$$

$$T(x,0) = T_i$$

Convert PDE into ODE by combining the two independent variables x and t into a single variable η:

$$\eta = \frac{x}{\sqrt{4\alpha t}}$$

Similarity variable, η

- $\eta = 0$ at $x = 0$
- $\eta \to \infty$ at $x \to \infty$
- $\eta \to \infty$ at $t = 0$

Semi-Infinite Solids

$$\frac{\partial T}{\partial t} = \frac{dT}{d\eta}\frac{d\eta}{dt} = -\frac{x}{2t\sqrt{4\alpha t}}\frac{dT}{d\eta}$$

$$\frac{\partial T}{\partial x} = \frac{dT}{d\eta}\frac{d\eta}{dx} = \frac{1}{\sqrt{4\alpha t}}\frac{dT}{d\eta}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{d}{d\eta}\left(\frac{\partial T}{\partial x}\right)\frac{d\eta}{dx} = \frac{1}{4\alpha t}\frac{d^2 T}{d\eta^2}$$

Convective Boundary Condition

The exact solution for a convective boundary condition:

Temperature distribution

$$1-\theta = \frac{T(x,t)-T_s}{T_\infty - T_s} = \text{efrc}\left(\frac{x}{2\sqrt{\alpha t}}\right) - \exp\left(\frac{hx}{k}+\frac{h^2\alpha t}{k^2}\right)\text{efrc}\left(\frac{x}{2\sqrt{\alpha t}}+\frac{h\sqrt{\alpha t}}{k}\right)$$

The complementary error function, erfc ξ is defined as erfc $\xi = 1 -$ erf ξ. The Gaussian error function, erf ξ, is a standard mathematical function that is tabulated.

Heat and Mass Transfer

Multidimensional Steady-State Conduction

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Two-Dimensional, Steady-State Conduction

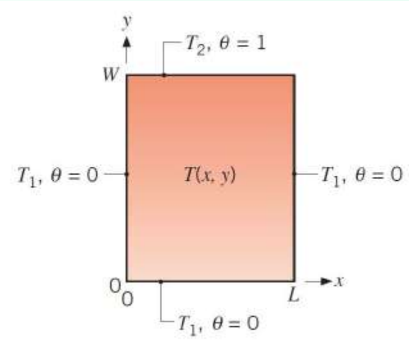
Governing equation

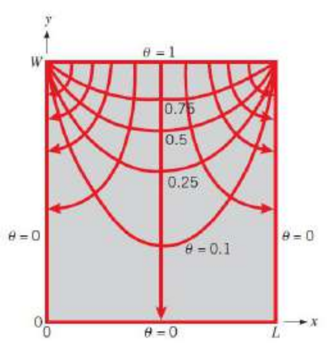
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

- Solve for $T(x, y)$
- Determine q''_x and q''_y from the rate equations

Methodologies/Approaches

- Analytical
- Graphical
- Numerical

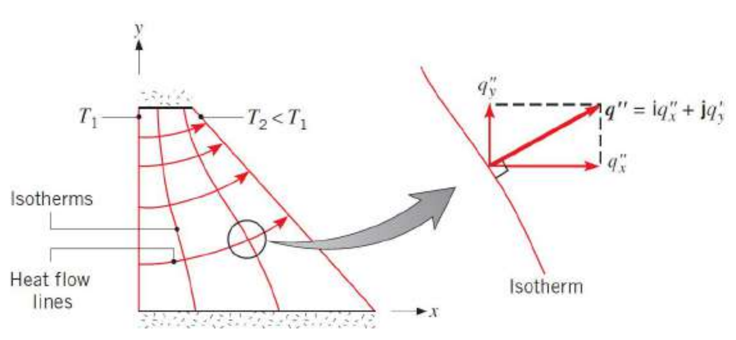




Method of separation of variables:

$$\theta(x,y) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\sin\frac{n\pi x}{L}\frac{\sinh(n\pi y/L)}{\sinh(n\pi W/l)}$$

Graphical Approach



Conduction Shape Factor

$$q = kS\Delta T_{1-2}$$

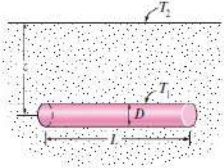
where ΔT_{1-2} is the temp. difference between boundaries.

S (m) depends on the *geometry* of the system only and $R = 1/Sk$.

Conduction Shape Factor

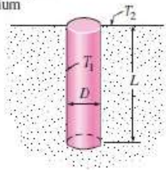
(1) Isothermal cylinder of length L buried in a semi-infinite medium ($L >> D$ and $z > 1.5D$)

$$S = \frac{2\pi L}{\ln(4z/D)}$$



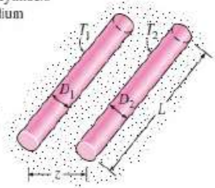
(2) Vertical isothermal cylinder of length L buried in a semi-infinite medium ($L >> D$)

$$S = \frac{2\pi L}{\ln(4L/D)}$$



(3) Two parallel isothermal cylinders placed in an infinite medium ($L >> D_1, D_2, z$)

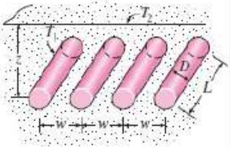
$$S = \frac{2\pi L}{\cosh^{-1}\left(\dfrac{4z^2 - D_1^2 - D_2^2}{2D_1 D_2}\right)}$$



(4) A row of equally spaced parallel isothermal cylinders buried in a semi-infinite medium ($L >> D$, z and $w > 1.5D$)

$$S = \frac{2\pi L}{\ln\left(\dfrac{2w}{\pi D}\sinh\dfrac{2\pi z}{w}\right)}$$

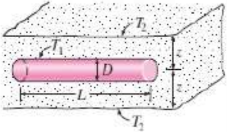
(per cylinder)



Conduction Shape Factor

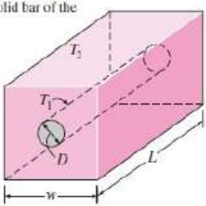
(5) Circular isothermal cylinder of length L in the midplane of an infinite wall ($z > 0.5D$)

$$S = \frac{2\pi L}{\ln(8z/\pi D)}$$



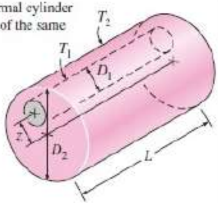
(6) Circular isothermal cylinder of length L at the center of a square solid bar of the same length

$$S = \frac{2\pi L}{\ln(1.08w/D)}$$



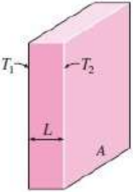
(7) Eccentric circular isothermal cylinder of length L in a cylinder of the same length ($L > D_2$)

$$S = \frac{2\pi L}{\cosh^{-1}\left(\dfrac{D_1^2 + D_2^2 - 4z^2}{2D_1D_2}\right)}$$



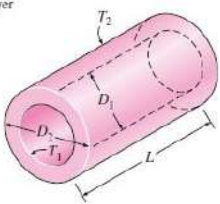
(8) Large plane wall

$$S = \frac{A}{L}$$



(9) A long cylindrical layer

$$S = \frac{2\pi L}{\ln (D_2/D_1)}$$



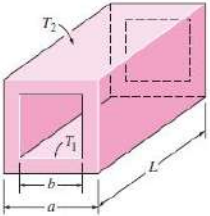
(10) A square flow passage

(*a*) For $a/b > 1.4$,

$$S = \frac{2\pi L}{0.93 \ln (0.948\, a/b)}$$

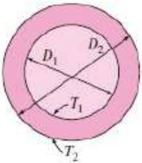
(*b*) For $a/b < 1.41$,

$$S = \frac{2\pi L}{0.785 \ln (a/b)}$$



(11) A spherical layer

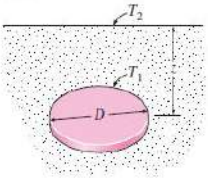
$$S = \frac{2\pi D_1 D_2}{D_2 - D_1}$$

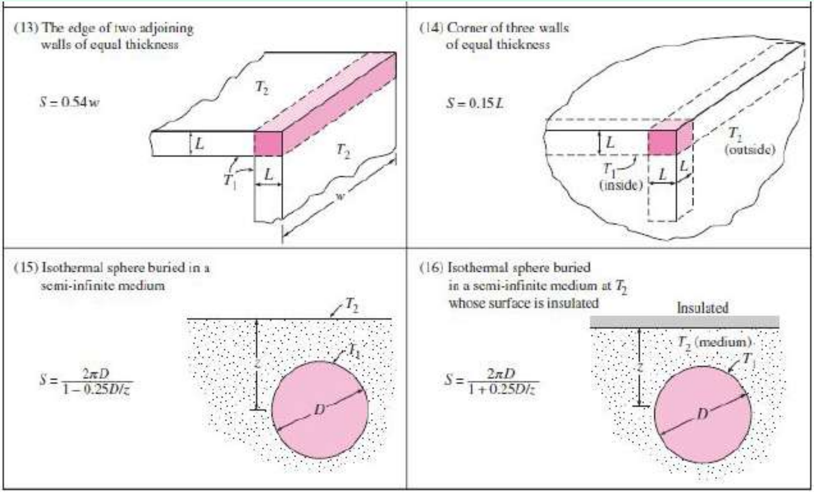


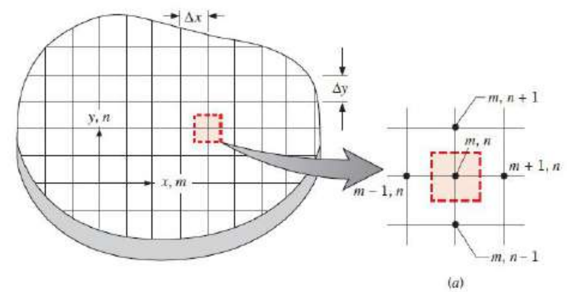
(12) Disk buried parallel to the surface in a semi-infinite medium ($z >> D$)

$S = 4D$

($S = 2D$ when $z = 0$)

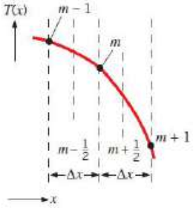



(13) The edge of two adjoining walls of equal thickness
$S = 0.54w$
T_2
L
T_1
L
T_2
w
(14) Corner of three walls of equal thickness
$S = 0.15L$
L
T_1 (inside)
L
L
T_2 (outside)
(15) Isothermal sphere buried in a semi-infinite medium
$S = \dfrac{2\pi D}{1 - 0.25D/z}$
T_2
T_1
z
D
(16) Isothermal sphere buried in a semi-infinite medium at T_2 whose surface is insulated
$S = \dfrac{2\pi D}{1 + 0.25D/z}$
Insulated
T_2 (medium)
T_1
z
D



$$\left.\frac{\partial T}{\partial x}\right|_{m-1/2,n} = \frac{T_{m,n} - T_{m-1,\,n}}{\Delta x}$$

$$\left.\frac{\partial T}{\partial x}\right|_{m+1/2,n} = \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$



(a) Nodal Network

(b) Finite-difference approximation

Heat and Mass Transfer

Numerical Methods - FDS

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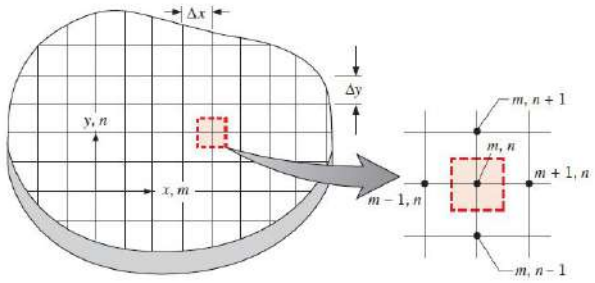
Numerical Technique

Advances in numerical computing now allow for complex heat transfer problems to be solved rapidly on computers. Some examples are:

- Finite-difference method
- Finite-element method
- Boundary-element method

In general, these techniques are routinely used to solve problems in heat transfer, fluid dynamics, stress analysis, electrostatics and magnetics, etc. Finite-difference method is ease of application.

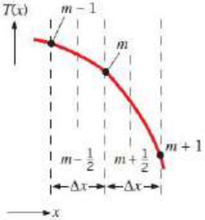
- Numerical techniques result in an approximate solution.
- Properties (e.g.,T, u) are determined at discrete points in the region of interest - referred as nodal points or nodes.



(a)

$$\left.\frac{\partial T}{\partial x}\right|_{m-1/2,n} = \frac{T_{m,n} - T_{m-1,\,n}}{\Delta x}$$

$$\left.\frac{\partial T}{\partial x}\right|_{m+1/2,n} = \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$



Finite-Difference Analysis

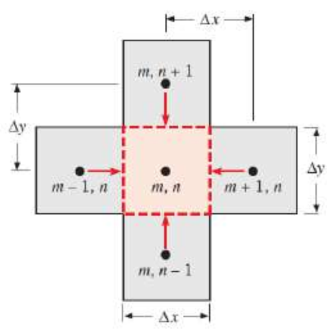
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{1}$$

$$\begin{aligned}
\left.\frac{\partial^2 T}{\partial x^2}\right|_{m,n} &\approx \frac{\left.\frac{\partial T}{\partial x}\right|_{m+1/2,n} - \left.\frac{\partial T}{\partial x}\right|_{m-1/2,n}}{\Delta x} \\
&\approx \frac{T_{m+1,n} - 2T_{m,n} + T_{m-1,n}}{(\Delta x)^2}
\end{aligned} \tag{2}$$

$$\left.\frac{\partial^2 T}{\partial y^2}\right|_{m,n} \approx \frac{T_{m+1,n} - 2T_{m,n} + T_{m-1,n}}{(\Delta y)^2} \tag{3}$$

Using a network for which $\Delta x = \Delta y$ and substituting Eqs. (2) and (3) in Eq. (1):

$$T_{m,n+1} + T_{m,n-1} + T_{m+1,n} + T_{m-1,n} - 4T_{m,n} = 0$$



Assuming that all the heat flow is into the node, $\dot{E}_{in} + \dot{E}_g = 0$:

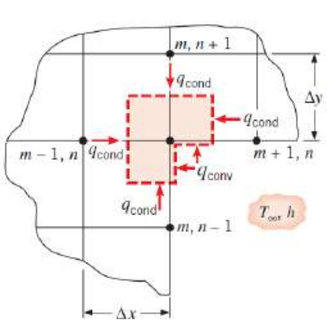
$$\sum_{i=1}^{4} q_{(i)\to(m,n)} + \dot{q}(\Delta x \cdot \Delta y \cdot 1) = 0$$

Energy Balance Method

$$q_{(m-1,n)\to(m,n)} = k(\Delta y \cdot 1)\frac{T_{m-1,n} - T_{m,n}}{\Delta x}$$
$$q_{(m+1,n)\to(m,n)} = k(\Delta y \cdot 1)\frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$
$$q_{(m,n+1)\to(m,n)} = k(\Delta x \cdot 1)\frac{T_{m,n+1} - T_{m,n}}{\Delta y}$$
$$q_{(m,n-1)\to(m,n)} = k(\Delta x \cdot 1)\frac{T_{m,n-1} - T_{m,n}}{\Delta y}$$

$$T_{m,n+1} + T_{m,n-1} + T_{m+1,n} + T_{m-1,n} + \frac{\dot{q}(\Delta x)^2}{k} - 4T_{m,n} = 0$$

Energy Balance Method



$$q_{(m-1,n)\to(m,n)} = k(\Delta y \cdot 1)\frac{T_{m-1,n} - T_{m,n}}{\Delta x}$$

$$q_{(m,n+1)\to(m,n)} = k(\Delta x \cdot 1)\frac{T_{m,n+1} - T_{m,n}}{\Delta y}$$

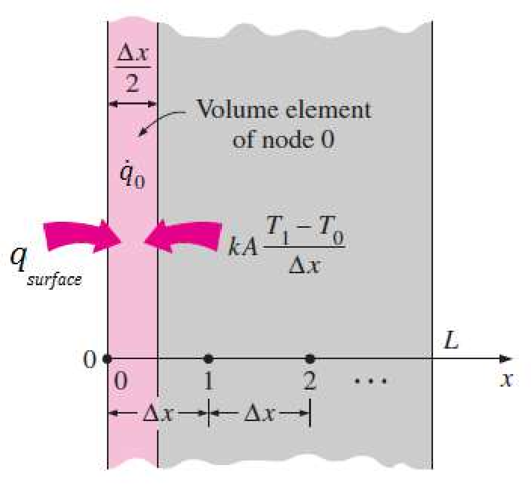
$$q_{(m+1,n)\to(m,n)} = k(\frac{\Delta y}{2} \cdot 1)\frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

$$q_{(m,n-1)\to(m,n)} = k(\frac{\Delta x}{2} \cdot 1)\frac{T_{m,n-1} - T_{m,n}}{\Delta y}$$

$$q_{(\infty)\to(m,,n)} = h\left(\frac{\Delta x}{2} \cdot 1\right)(T_\infty - T_{m,n}) + h\left(\frac{\Delta y}{2} \cdot 1\right)(T_\infty - T_{m,n})$$

$$T_{m,n+1} + T_{m,n-1} + \frac{1}{2}(T_{m+1,n} + T_{m-1,n}) + \frac{h\Delta x}{k}T_\infty - \left(3 + \frac{h\Delta x}{k}\right)T_{m,n} = 0$$

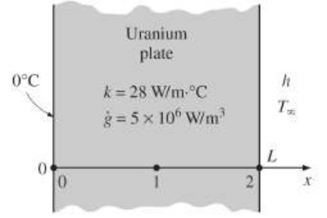
Specified Heat Flux Boundary Condition



$$q_{\text{surface}} + kA\frac{T_1 - T_0}{\delta x} + \dot{q}A\frac{\Delta x}{2} = 0$$

Problem

Consider a large uranium plate of thickness $L = 4$ cm and $k = 28$ W/m K in which heat is generated uniformly at a constant rate of $\dot{q} = 5 \times 10^6$ W/m^3. One side of the plate is maintained at $0°$C by iced water while the other side is subjected to convection to an environment at $T_\infty = 30°$C with $h = 45$ W/m^2 K. Considering a total of three equally spaced nodes in the medium, two at the boundaries and one at the middle, estimate the exposed surface temperature of the plate under steady conditions using the finite difference approach.



Solution

For the interior node (1), the energy balance would result in:

$$\frac{T_0 - 2T_1 + T_2}{(\Delta x)^2} + \frac{\dot{q}}{k} = 0$$

$$\boxed{T_0 - 2T_1 + T_2 = -\frac{\dot{q}(\Delta x)^2}{k}}$$

$$2T_1 - T_2 = 71.43 \qquad (1)$$

Let us write the governing equation for the corner Node (2):

$$h(T_\infty - T_2) + kA\frac{T_1 - T_2}{\Delta x} + \dot{q}A\frac{\Delta x}{2} = 0$$

$$\boxed{T_1 - (1 + \frac{h\Delta x}{k})T_2 = -\frac{h\Delta x}{k}T_\infty - \frac{\dot{q}(\Delta x)^2}{2k}}$$

$$T_1 - 1.032T_2 = -36.68 \qquad (2)$$

Solution: Generalizing

$$T_0 - 2T_1 + T_2 = -\frac{\dot{q}(\Delta x)^2}{k}$$

$$\boxed{T_{m-1} - 2T_m + T_{m+1} = -\frac{\dot{q}(\Delta x)^2}{k}}$$

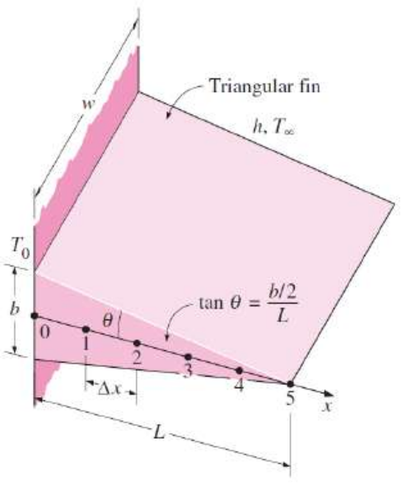
$$T_1 - (1 + \frac{h\Delta x}{k})T_2 = -\frac{h\Delta x}{k}T_\infty - \frac{\dot{q}(\Delta x)^2}{2k}$$

$$\boxed{T_{M-1} - (1 + \frac{h\Delta x}{k})T_M = -\frac{h\Delta x}{k}T_\infty - \frac{\dot{q}(\Delta x)^2}{2k}}$$

Problem

Consider an aluminum alloy fin ($k = 180$ W/m K) of triangular cross section with length $L = 5$ cm, base thickness $b = 1$ cm, and very large width w in the direction normal to the plane of paper. The base of the fin is maintained at a temperature of $T_0 = 200°$C. The fin is losing heat to the surrounding medium at $T_\infty = 25°$C with a heat transfer coefficient of $h = 15$ W/m^2 K. Using the finite difference method with six equally spaced nodes along the fin in the x-direction, determine (a) the temperatures at the nodes, (b) the rate of heat transfer from the fin for $w = 1$ m, and (c) the fin efficiency.

Problem

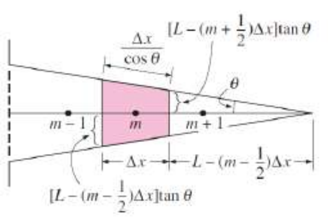


$$T_0 = 200°\text{C} \qquad k = 180 \text{ W/m K}$$

$$b = 1 \text{ cm} \qquad w = 1 \text{ cm}$$

$$T_\infty = 25°\text{C} \qquad h = 15 \text{ W/m}^2 \text{ K}$$

Solution



$$q_{\text{left}} + q_{\text{right}} + q_{\text{conv}} = 0$$

$$kA_{\text{left}}\frac{T_{m-1}-T_m}{\Delta x} + kA_{\text{right}}\frac{T_{m+1}-T_m}{\Delta x} + hA_{\text{conv}}(T_\infty - T_m) = 0$$

$$\begin{aligned} A_{\text{left}} &= 2w[L-(m-1/2)\Delta x]\tan\theta \\ A_{\text{right}} &= 2w[L-(m+1/2)\Delta x]\tan\theta \\ A_{\text{conv}} &= 2w(\Delta x/\cos\theta) \\ \tan\theta &= \frac{b/2}{L} = 0.1 \implies \theta = 5.71^\circ \end{aligned}$$

Solution

$$(5.5 - m)T_{m-1} - (10.01 - 2m)T_m + (4.5 - m)T_{m+1} = -0.209$$

Four equations for $m = 1 \to 4$. Boundary condition at node (5):

$$kA_{\mathsf{left}}\frac{T_4 - T_5}{\Delta x} + hA_{\mathsf{conv}}(T_\infty - T_5)$$

$$A_{\mathsf{left}} = 2w(\Delta x/2)\tan\theta \qquad A_{\mathsf{conv}} = 2w\frac{\Delta x/2}{\cos\theta}$$

Total 5 equations with 5 unknowns.

Solution

$$
\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} -8.01 & 3.5 & 0 & 0 & 0 \\ 3.5 & -6.01 & 2.5 & 0 & 0 \\ 0 & 2.5 & -4.01 & 1.5 & 0 \\ 0 & 0 & 1.5 & -2.01 & 0.5 \\ 0 & 0 & 0 & 1 & -1.01 \end{bmatrix}^{-1} \times \begin{bmatrix} -900.21 \\ -0.21 \\ -0.21 \\ -2.1 \\ -0.21 \end{bmatrix}
$$

$$
= \begin{bmatrix} 198.6 \\ 197.1 \\ 195.7 \\ 194.3 \\ 192.9 \end{bmatrix} {}^\circ\mathrm{C}
$$

Direct/Iterative Methods

Direct Methods

Solve in a systematic manner following a series of well-defined steps.

Iterative Methods

Start with an initial guess for the solution, and iterate until solution converges.

Gauss-Seidel/Iterative Method

Application of the Gauss-Seidel iterative method to the finite difference equations in the previous triangular fin example.

Finite difference equations in explicit form

$T_1 = 0.4371 T_2 + 112.4137$

$T_2 = 0.5826 T_1 + 0.4161 T_3 + 0.0348$

$T_3 = 0.6238 T_2 + 0.3743 T_4 + 0.0521$

$T_4 = 0.7470 T_3 + 0.2490 T_5 + 0.1041$

$T_5 = 0.9921 T_4 + 0.2073$

Iteration	Nodal Temperature, °C				
	T_1	T_2	T_3	T_4	T_5
Initial Guess	195.0	195.0	195.0	195.0	195.0
1	197.6	196.3	195.5	194.7	193.4
2	198.2	196.9	195.8	194.5	193.2
3	198.5	197.2	195.9	194.5	193.2
4	198.6	197.3	195.9	194.5	193.2
5	198.7	197.3	195.9	194.5	193.2
6	198.7	197.3	195.9	194.5	193.2
7	198.7	197.3	195.9	194.5	193.2

Transient: The Explicit Method

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$$

The problem must be discretized in time.

$$t = p\Delta t$$

$$\left.\frac{\partial T}{\partial t}\right|_{m,n} \approx \frac{T_{m,n}^{p+1} - T_{m,n}^{p}}{\Delta t}$$

In explicit method of solution, the temperatures are evaluated at the previous (p) time - forward-difference approximation to the time derivative.

$$\frac{1}{\alpha}\frac{T_{m,n}^{p+1} - T_{m,n}^{p}}{\Delta t} = \frac{T_{m+1,n}^{p} + T_{m-1,n}^{p} - 2T_{m,n}^{p}}{(\Delta x)^2} + \frac{T_{m,n+1}^{p} + T_{m,n-1}^{p} - 2T_{m,n}^{p}}{(\Delta y)^2}$$

Transient: The Explicit Method

Solving for the nodal temperature at the new $(p+1)$ time and assuming that $\Delta x = \Delta y$, it follows that:

$$T_{m,n}^{p+1} = \mathsf{Fo}(T_{m+1,n}^{p} + T_{m-1,n}^{p} + T_{m,n+1}^{p} + T_{m,n-1}^{p}) + (1 - 4\mathsf{Fo})T_{m,n}^{p}$$

$$\text{where } \mathsf{Fo} = \frac{\alpha \Delta t}{(\Delta x)^2}$$

If the system is 1D in x, the explicit form of the finite-difference equation for an interior node m reduces to:

$$T_{m}^{p+1} = \mathsf{Fo}(T_{m+1}^{p} + T_{m-1}^{p}) + (1 - 2\mathsf{Fo})T_{m}^{p}$$

Stability Criterion

- Explicit method is not unconditionally stable.
- The solution may be characterized by numerically induced oscillations, which are physically impossible
- Oscillations may become *unstable*, causing the solution to diverge from the actual steady-state conditions.

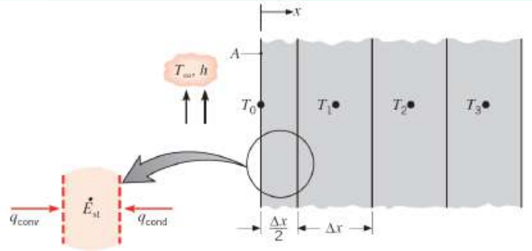
Stability Criterion

The criterion is determined by requiring that the coefficient associated with the node of interest at the previous time is greater than or equal to zero.

$$T_m^{p+1} = \mathsf{Fo}(T_{m+1}^p + T_{m-1}^p) + (1 - 2\mathsf{Fo})T_m^p$$

$$\text{1D :}(1 - 2\mathsf{Fo}) \geq 0 \implies \mathsf{Fo} \leq \frac{1}{2}$$
$$\text{2D :}(1 - 4\mathsf{Fo}) \geq 0 \implies \mathsf{Fo} \leq \frac{1}{4}$$

Energy Balance at the Boundary



$$\dot{E}_{in} + \dot{E}_g = \dot{E}_{st}$$

$$hA(T_\infty - T_0^p) + \frac{kA}{\Delta x}(T_1^p - T_0^p) = \rho A \frac{\Delta x}{2} C_p \frac{T_0^{p+1} - T_0^p}{\Delta t}$$

$$T_0^{p+1} = 2\ \mathsf{Fo}\ (T_1^p + \mathsf{Bi}\ T_\infty) + (1 - 2\ \mathsf{Fo} - 2\ \mathsf{Bi}\ \mathsf{Fo})T_0^p$$

$$\mathsf{Bi} = \frac{h\Delta x}{k}$$

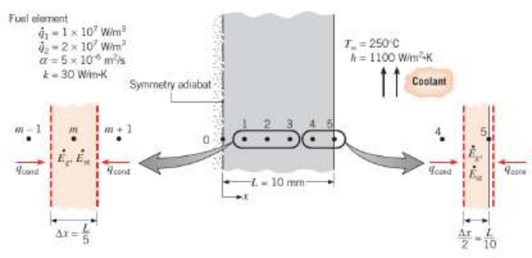
From the stability Criteria:

$$1 - 2\ \mathsf{Fo} - 2\ \mathsf{Bi}\ \mathsf{Fo} \geq 0$$

$$\boxed{\mathsf{Fo}(1 + \mathsf{Bi}) \leq \frac{1}{2}}$$

A fuel element of a nuclear reactor is in the shape of a plane wall of thickness $2L = 20$ mm and is convectively cooled at both surfaces, with $h = 1100$ W/m^2 K and $T_\infty = 250°$C. At normal operating power, heat is generated uniformly within the element at a volumetric rate of $\dot{q}_1 = 10^7$ W/m^3. A departure from the steady-state conditions associated with normal operation will occur if there is a change in the generation rate. Consider a sudden change to $\dot{q}_2 = 2 \times 10^7$ W/m^3, and determine the fuel element temperature distribution after 1.5 s.

The fuel element thermal properties are $k = 30$ W/m K and $\alpha = 5 \times 10^{-6}$ m^2/s.



Solution

For $m = 0 \to 4$:

$$kA\frac{T_{m-1}^p - T_m^p}{\Delta x} + kA\frac{T_{m+1}^p - T_m^p}{\Delta x} + \dot{q}A\Delta x = \rho A\Delta x C_p\frac{T_m^{p+1} - T_m^p}{\Delta t}$$

$$\boxed{T_m^{p+1} = \text{Fo}\left[T_{m-1}^p + T_{m+1}^p + \frac{\dot{q}(\Delta x)^2}{k}\right] + (1-2\text{Fo})T_m^p}_{m=0\to 4}$$

For $m = 5$:

$$kA\frac{T_4^p - T_5^p}{\Delta x} + hA(T_\infty - T_5^p) + \dot{q}A\frac{\Delta x}{2} = \rho A\frac{\Delta x}{2}C_p\frac{T_5^{p+1} - T_5^p}{\Delta t}$$

$$\boxed{T_5^{p+1} = 2\text{Fo}\left[T_4^p + \text{Bi}\ T_\infty + \frac{\dot{q}(\Delta x)^2}{2k}\right] + (1-2\text{Fo}-2\text{Bi Fo})T_5^p}_{m=5}$$

$$\text{Fo}(1+\text{Bi}) \leq \frac{1}{2}$$

$$\text{Bi} = 0.0733 \implies \text{Fo} \leq 0.466 \implies \boxed{\Delta t \leq 0.373 \text{ s}}$$

Solution

Assuming $\Delta t = 0.3$ s, Fo $= 0.375$,

$$T_0^{p+1} = 0.375(2T_1^p + 2.67) + 0.250T_0^p$$
$$T_1^{p+1} = 0.375(T_0^p + T_2^p + 2.67) + 0.250T_0^p$$
$$T_2^{p+1} = 0.375(T_1^p + T_3^p + 2.67) + 0.250T_0^p$$
$$T_3^{p+1} = 0.375(T_2^p + T_4^p + 2.67) + 0.250T_0^p$$
$$T_4^{p+1} = 0.375(T_3^p + T_5^p + 2.67) + 0.250T_0^p$$
$$T_5^{p+1} = 0.750(T_4^p + 19.67) + 0.195T_0^p$$

For 1D, steady state, $\dot{q}$, symmetrical about the plane:

$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + T_s \qquad T_5^{p=0} = 250 + \frac{10^7 \times 0.01}{1100} = 340.9°\text{C}$$

$$T_s = T_\infty + \frac{\dot{q}L}{h} \qquad T_m^{p=0} = 16.67(1 - 10000x^2) + 340.9$$

Solution

p	t(s)	T_0	T_1	T_2	T_3	T_4	T_5
0	0	357.58	356.91	354.91	351.58	346.91	340.91
1	0.3	358.08	357.41	355.41	352.08	347.41	341.41
2	0.6	358.58	357.91	355.91	352.58	347.91	341.88
3	0.9	359.08	358.41	356.41	353.08	348.41	342.35
4	1.2	359.58	358.91	356.91	353.58	348.89	342.82
5	1.5	360.08	359.41	357.41	354.07	349.37	343.27
∞	∞	465.15	463.82	459.82	453.15	443.82	431.82

Transient: The Implicit Method

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$$

$$\left.\frac{\partial T}{\partial t}\right|_{m,n} \approx \frac{T_{m,n}^{p+1} - T_{m,n}^{p}}{\Delta t}$$

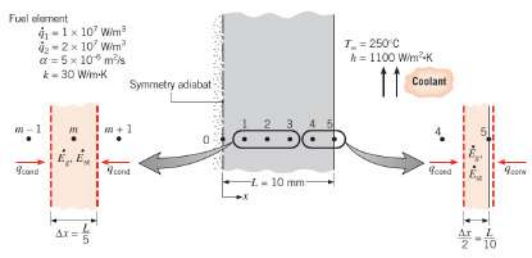
In implicit method of solution, the temperatures are evaluated at the new $(p+1)$ time - backward-difference approximation to the time derivative.

$$\frac{1}{\alpha}\frac{T_{m,n}^{p+1} - T_{m,n}^{p}}{\Delta t} = \frac{T_{m+1,n}^{p+1} + T_{m-1,n}^{p+1} - 2T_{m,n}^{p+1}}{(\Delta x)^2} + \frac{T_{m,n+1}^{p+1} + T_{m,n-1}^{p+1} - 2T_{m,n}^{p+1}}{(\Delta y)^2}$$

Problem

A fuel element of a nuclear reactor is in the shape of a plane wall of thickness $2L = 20$ mm and is convectively cooled at both surfaces, with $h = 1100$ W/m^2 K and $T_\infty = 250°$C. At normal operating power, heat is generated uniformly within the element at a volumetric rate of $\dot{q}_1 = 10^7$ W/m^3. A departure from the steady-state conditions associated with normal operation will occur if there is a change in the generation rate. Consider a sudden change to $\dot{q}_2 = 2 \times 10^7$ W/m^3, and determine the fuel element temperature distribution after 1.5 s.

The fuel element thermal properties are $k = 30$ W/m K and $\alpha = 5 \times 10^{-6}$ m^2/s.



Solution

For $m = 0 \to 4$:

$$kA\frac{T_{m-1}^{p+1} - T_m^{p+1}}{\Delta x} + kA\frac{T_{m+1}^{p+1} - T_m^{p+1}}{\Delta x} + \dot{q}A\Delta x = \rho A\Delta x C_p \frac{T_m^{p+1} - T_m^p}{\Delta t}$$

$$\boxed{\mathsf{Fo}\ T_m^{p+1} - (1 + 2\mathsf{Fo})T_m^{p+1} + \mathsf{Fo}T_{m+1}^{p+1} = -T_m^p - \mathsf{Fo}\frac{\dot{q}(\Delta x)^2}{k}}_{m=0\to 4}$$

For $m = 5$:

$$kA\frac{T_4^{p+1} - T_5^{p+1}}{\Delta x} + hA(T_\infty - T_5^{p+1}) + \dot{q}A\frac{\Delta x}{2} = \rho A\frac{\Delta x}{2}C_p\frac{T_5^{p+1} - T_5^p}{\Delta t}$$

$$\boxed{2\mathsf{Fo}T_4^{p+1} - (1 + 2\mathsf{Fo} + 2\mathsf{Bi}\ \mathsf{Fo})T_5^{p+1} = -T_5^p - Fo\frac{\dot{q}(\Delta x)^2}{k}}_{m=5}$$

Heat and Mass Transfer

Introduction to Convection

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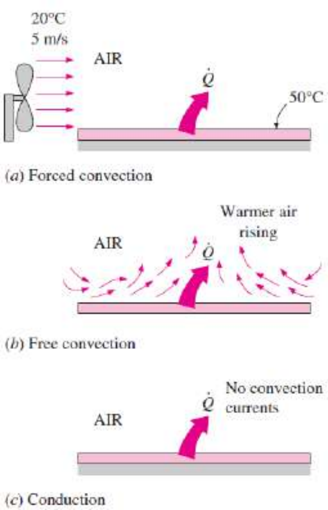
Convective Heat Transfer



Convective heat transfer involves

- fluid motion
- heat conduction

The fluid motion enhances the heat transfer, since it brings hotter and cooler chunks of fluid into contact, initiating higher rates of conduction at a greater number of sites in fluid. Therefore, the rate of heat transfer through a fluid is much higher by convection than it is by conduction.

Higher the fluid velocity, the higher the rate of heat transfer.

Convective Heat Transfer

Convection heat transfer strongly depends on

- fluid properties: μ, k, ρ, C_p
- fluid velocity: V
- geometry and the roughness of the solid surface
- type of fluid flow (laminar or turbulent)

Newton's law of cooling

$$q_{conv} = hA_s\left(T_s - T_\infty\right)$$

T_∞ is the temp. of the fluid sufficiently far from the surface

Total Heat Transfer Rate

Local heat flux

$$q''_{conv} = h_l \left(T_s - T_\infty\right)$$

h_l is the local convection coefficient

Flow conditions vary on the surface: q'', h vary along the surface.

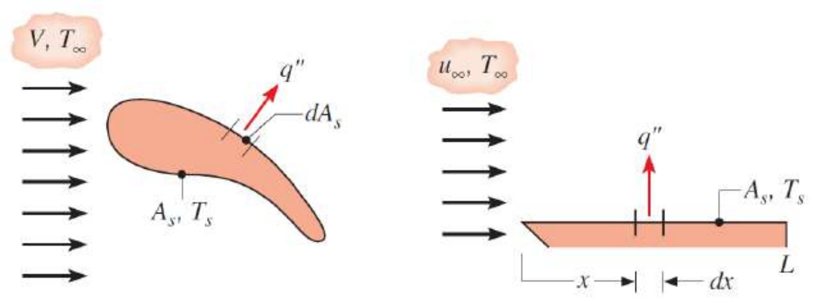
The total heat transfer rate q:

$$\begin{aligned} q_{conv} &= \int\limits_{A_s} q'' dA_s \\ &= \left(T_s - T_\infty\right) \int\limits_{A_s} h dA_s \end{aligned}$$

Total Heat Transfer Rate

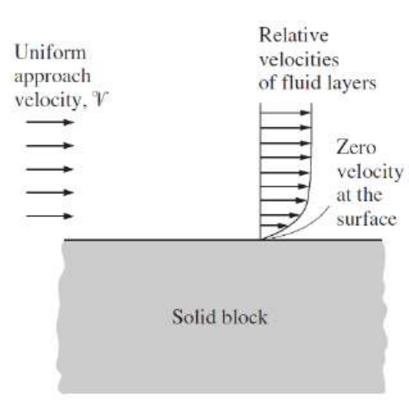
Defining an *average convection coefficient* $\bar{h}$ for the entire surface,

$$q_{conv} = \bar{h} A_s \left(T_s - T_\infty \right)$$
$$\bar{h} = \frac{1}{A_s} \int_{A_s} h dA_s$$



No-Temperature-Jump

A fluid flowing over a stationary surface - no-slip condition



A fluid and a solid surface will have the same T at the point of contact, known as no-temperature-jump condition.

No-slip, No-Temperature-Jump

With no-slip and the no-temperature-jump conditions: the heat transfer from the solid surface to the fluid layer adjacent to the surface is by pure conduction.

$$q''_{conv} = q''_{cond} = -k_{fluid} \left.\frac{\partial T}{\partial y}\right|_{y=0}$$

T represents the temperature distribution in the fluid $(\partial T/\partial y)_{y=0}$ *i.e.*, the temp. gradient at the surface.

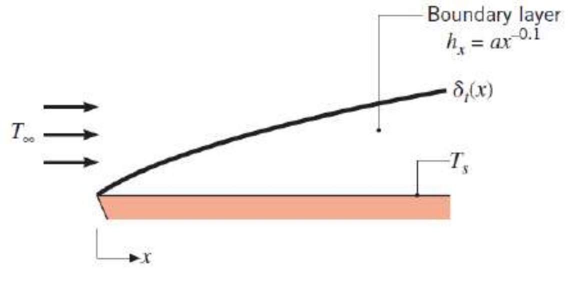
$$q''_{conv} = h(T_s - T_\infty)$$

$$h = \frac{-k_{fluid}\left(\frac{\partial T}{\partial y}\right)_{y=0}}{T_s - T_\infty}$$

Problem

Experimental results for the local heat transfer coefficient hx for flow over a flat plate with an extremely rough surface were found to fit the relation $h_x(x) = x^{-0.1}$ where x (m) is the distance from the leading edge of the plate.

- Develop an expression for the ration of the average heat transfer coefficient $\bar{h}_x$ for a path of length x to the local heat transfer coefficient h_x at x.
- Plot the variation of h_x and $\bar{h}_x$ as a function of x.



Solution

The average value of h over the region from 0 to x is:

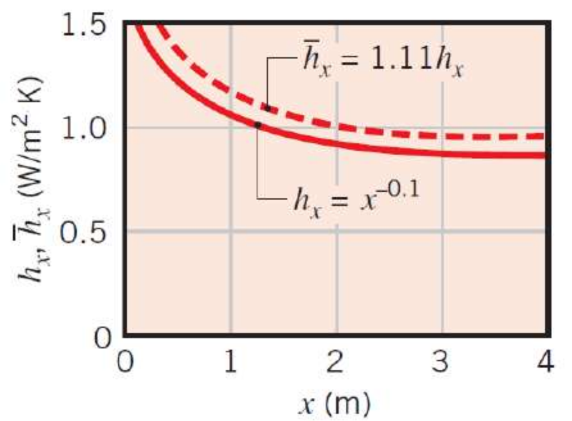
$$\bar{h}_x = = \frac{1}{x}\int_0^x h_x(x)dx$$

$$= \frac{1}{x}\int_0^x x^{-0.1}dx$$

$$= \frac{1}{x}\frac{x^{0.9}}{0.9} = 1.11x^{-0.1}$$

$$\boxed{\bar{h}_x = 1.11h_x}$$

Solution

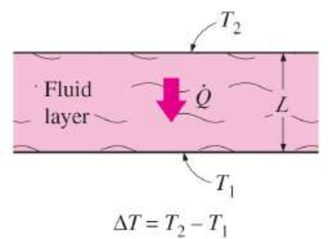


Comments

Boundary layer development causes both h_l and $\bar{h}$ to decrease with increasing distance from the leading edge. The average coefficient up to x must therefore exceed the local value at x.

Nusselt Number

$$\boxed{\text{Nu} = \frac{hL_c}{k_{fluid}}}$$



Heat transfer through the fluid layer will be by convection when the fluid involves some motion and by conduction when the fluid layer is motionless.

$$q_{conv} = h\Delta T \qquad q_{cond} = k\frac{\Delta T}{L}$$

$$\boxed{\frac{q_{conv}}{q_{cond}} = \frac{h\Delta T}{k\Delta T/L} = \frac{hL}{k} = \text{Nu}}$$

Nusselt Number

$$\boxed{\mathrm{Nu} = \frac{q_{conv}}{q_{cond}}}$$

Nusselt number: enhancement of heat transfer through a fluid layer as a result of convection relative to conduction across the same fluid layer.

$\mathrm{Nu} >> 1$ for a fluid layer - the more effective the convection

$\mathrm{Nu} = 1$ for a fluid layer - heat transfer across the layer is by pure conduction

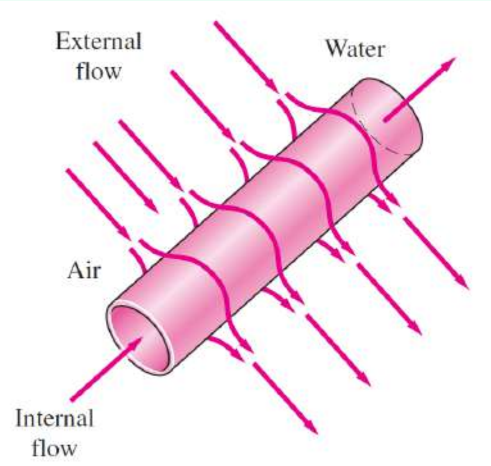
$\mathrm{Nu} < 1$???

Prof. Wilhem Nußelt

- German engineer, born in Germany (1882)
- Doctoral thesis - Conductivity of Insulating Materials
- Prof. - Heat and Momentum Transfer in Tubes
- 1915 - pioneering work in basic laws of transfer
- Dimensionless groups - similarity theory of heat transfer
- Film condensation of steam on vertical surfaces
- Combustion of pulverized coal
- Analogy of heat transfer and mass transfer in evaporation

Worked till 70 years. Lived for 75 years and died in München on September 1, 1957.

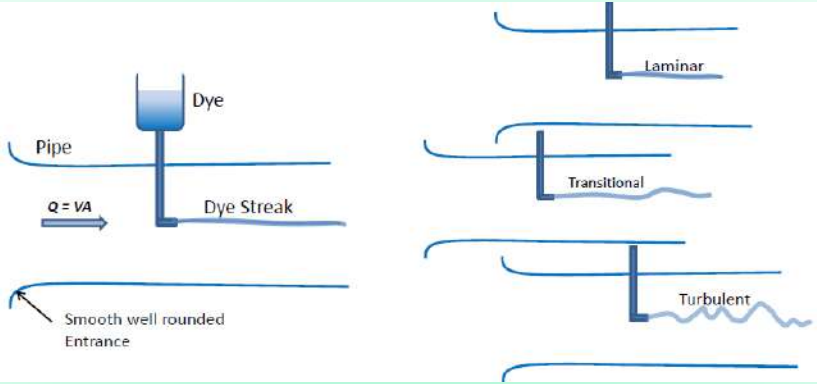
External and Internal Flows



External - flow of an unbounded fluid over a surface

Internal - flow is completely bounded by solid surfaces

Laminar and Turbulent Flows



Laminar - smooth and orderly: flow of high-viscosity fluids such as oils at low velocities

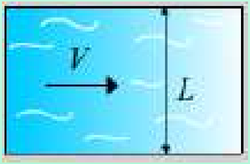
Internal - chaotic and highly disordered fluid motion: flow of low-viscosity fluids such as air at high velocities

The flow regime greatly influences the heat transfer rates and the required power for pumping.

Reynolds Number

Osborne Reynolds in 1880's, discovered that the flow regime depends mainly on the ratio of the inertia forces to viscous forces in the fluid.

Re can be viewed as the ratio of the inertia forces to the viscous forces acting on a fluid volume element.



$$\text{Re} = \frac{\text{Inertia forces}}{\text{Viscous}} = \frac{V L_c}{\nu} = \frac{\rho V L_c}{\mu}$$

The Effects of Turbulence

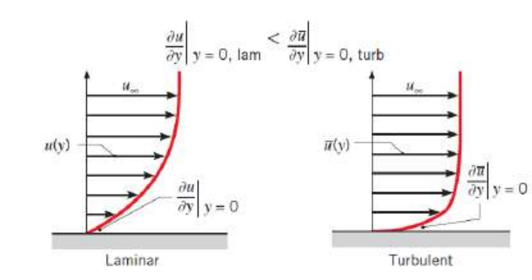
Taylor and Von Karman (1937)

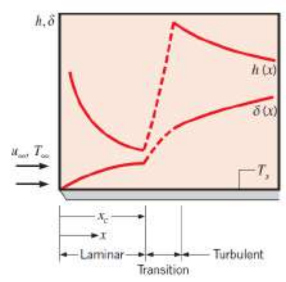
Turbulence is an irregular motion which in general makes its appearance in fluids, gaseous or liquids, when they flow past solid surfaces or even when neighboring streams of same fluid past over one another.

Turbulent fluid motion is an irregular condition of flow in which various quantities show a random variation with time and space coordinates, so that statistically distinct average values can be discerned.

The Effects of Turbulence

Because of the motion of eddies, the transport of momentum, energy, and species is enhanced.

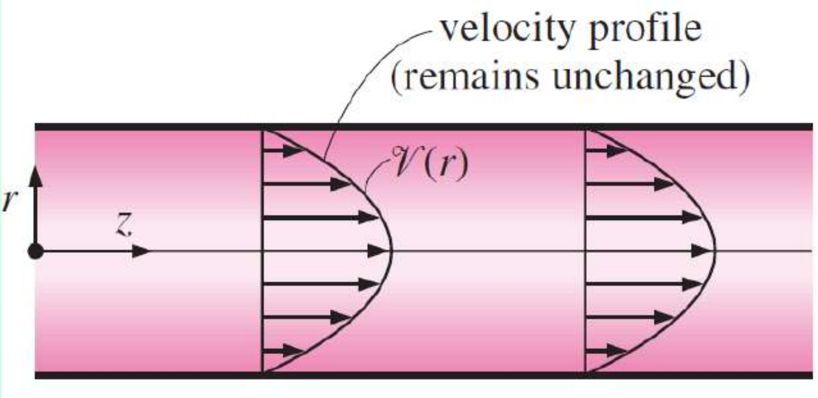




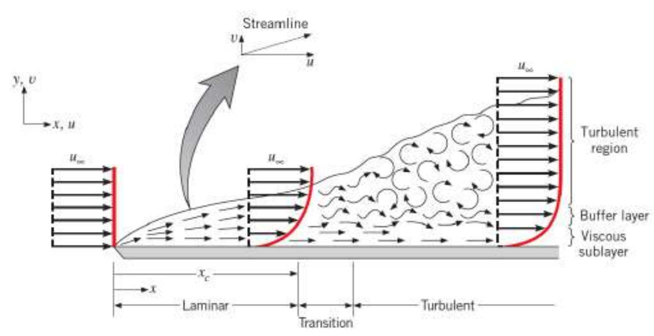
The velocity gradient at the surface, and therefore the surface shear stress, is much larger for δ_{turb} than for δ_{lam}. Similarly for temp. & conc. gradients.

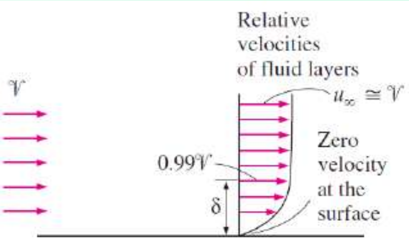
Turbulence is desirable. However, the increase in wall shear stress will have the adverse effect of increasing pump or fan power.

1-, 2-, 3- Dimensional Flows



1-D flow in a circular pipe





$$V_{x=\delta} = 0.99U_\infty$$

Wall Shear Stress

Friction force per unit area is called sheat stress

Surface shear stress

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0}$$

The determination of τ_w is not practical as it requires a knowledge of the flow velocity profile. A more practical approach in external flow is to relate τ_w to the upstream velocity U_∞ as:

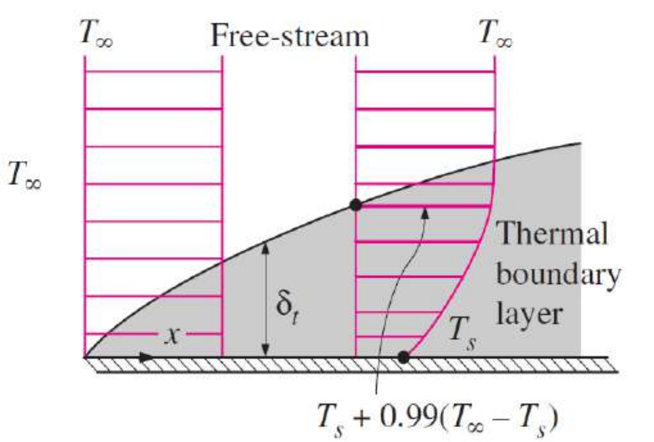
Skin friction coefficient

$$\tau_w = C_f \frac{\rho U_\infty^2}{2}$$

Friction force over the entire surface

$$F_f = C_f A_s \frac{\rho U_\infty^2}{2}$$

Thermal Boundary Layer



δ_t at any location along the surface at which
$(T - T_s) = 0.99(T_\infty - T_s)$

Prandtl Number

- Shape of the temp. profile in the thermal boundary layer dictates the convection heat transfer between a solid surface and the fluid flowing over it.
- In flow over a heated (or cooled) surface, both velocity and thermal boundary layers will develop simultaneously.
- Noting that the fluid velocity will have a strong influence on the temp. profile, the development of the velocity boundary layer relative to the thermal boundary layer will have a strong effect on the convection heat transfer.

$$\mathrm{Pr} = \frac{\text{Molecular diffusivity of momentum}}{\text{Molecular diffusivity of heat}} = \frac{\nu}{\alpha} = \frac{\mu C_p}{k}$$

Prandtl Number

Typical ranges of Pr for common fluids

Fluid	**Pr**
Liquid metals	0.004-0.030
Gases	0.7-1.0
Water	1.7-13.7
Light organic fluids	5-50
Oils	50-100,000
Glycerin	2000-100,000

Prandtl Number

$$\frac{\delta}{\delta_t} \approx \mathrm{Pr}^n$$

n is positive exponent

- $\mathrm{Pr} \cong 1$ for gases $\implies$ both momentum and heat dissipate through the fluid at about the same rate.
- Heat diffuses very quickly in liquid metals ($\mathrm{Pr} < 1$).
- Heat diffuses very slowly in oils ($\mathrm{Pr} > 1$) relative to momentum.
- Therefore, thermal boundary layer is much thicker for liquid metals and much thinner for oils relative to the velocity boundary layer.

$$\begin{aligned} \delta &= \delta_t \text{ for Pr } = 1 \\ \delta &> \delta_t \text{ for Pr } > 1 \\ \delta &< \delta_t \text{ for Pr } < 1 \end{aligned}$$

$$\mathrm{Pr} = \frac{\nu}{\alpha}$$

Prof. Ludwig Prandtl

- German Physicist, born in Bavaria (1875 - 1953)
- Father of aerodynamics
- Prof. of Applied Mechanics at Göttingen for 49 years (until his death)
- His work in fluid dynamics is still used today in many areas of aerodynamics and chemical engineering.

His discovery in 1904 of the Boundary Layer which adjoins the surface of a body moving in a fluid led to an understanding of skin friction drag and of the way in which streamlining reduces the drag of airplane wings and other moving bodies.

Heat and Mass Transfer

Convection Equations

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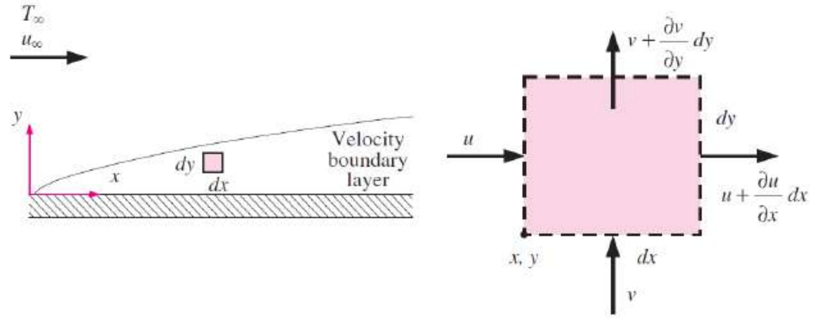
Assumptions

Assuming the flow/fluid to be:

- 2D, Steady
- Newtonian
- constant properties (ρ, μ, k, etc.)



Continuity Equation

Rate of mass flow into CV = Rate of mass flow out of CV

rate of fluid entering CV_{left}:

$$\rho u(dy \cdot 1)$$

rate of fluid leaving CV_{right}:

$$\rho(u + \frac{\partial u}{\partial x}dx)(dy \cdot 1)$$

$$\rho u(dy \cdot 1) + \rho v(dx \cdot 1) = \rho(u + \frac{\partial u}{\partial x}dx)(dy \cdot 1) + \rho(v + \frac{\partial v}{\partial y}dy)(dx \cdot 1)$$

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0}$$

Momentum Equation

Expressing Newton's second law of motion for the control volume:

(Mass) (Acceleration in x) = (Net body and surface forces in x)

$$\delta m \cdot a_x = F_{\text{surface},x} + F_{\text{body},x} \tag{3}$$

$$\delta m = \rho(dx \cdot dy \cdot 1)$$

$$\begin{aligned} a_x = \frac{du}{dt} &= \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \\ &= u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \end{aligned} \tag{4}$$

Steady state doesn't mean that acceleration is zero.
Ex: Garden hose nozzle

Momentum Equation

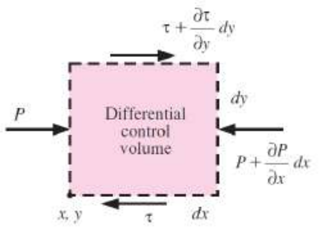
Body forces: gravity, electric and magnetic forces - $\propto$ volume.
Surface forces: pressure forces due to hydrostatic pressure and shear stresses due to viscous effects - $\propto$ surface area.

Viscous forces has two components:

1. normal to the surface- normal stress
   related to velocity gradients $\partial u/\partial x$ and $\partial v/\partial y$
2. along the wall surface - shear stress
   related to $\partial u/\partial y$

For simplicity, the normal stresses are neglected.

Momentum Equation



$$F_{\text{surface},x} = \left(\frac{\partial \tau}{\partial y} dy\right)(dx \cdot 1) - \left(\frac{\partial P}{\partial x} dx\right)(dy \cdot 1)$$

$$= \left(\frac{\partial \tau}{\partial y} - \frac{\partial P}{\partial x}\right)(dx \cdot dy \cdot 1)$$

$$\tau = \mu \left(\frac{\partial u}{\partial y}\right)$$

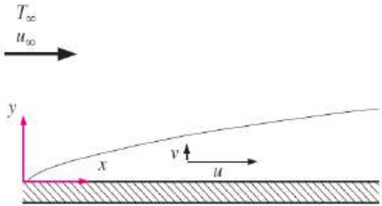
$$= \left(\mu \frac{\partial^2 u}{\partial y^2} - \frac{\partial P}{\partial x}\right)(dx \cdot dy \cdot 1) \quad (5)$$

Momentum Equation

Combining Eqs. (3), (4) and (5):

$$\boxed{\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\mu\frac{\partial^2 u}{\partial y^2}-\frac{\partial P}{\partial x}}$$

x-momentum equation



1) Velocity components:

$v \ll u$

2) Velocity grandients:

$\frac{\partial v}{\partial x} \approx 0, \frac{\partial v}{\partial y} \approx 0$

$\frac{\partial u}{\partial x} \ll \frac{\partial u}{\partial y}$

3) Temperature gradients:

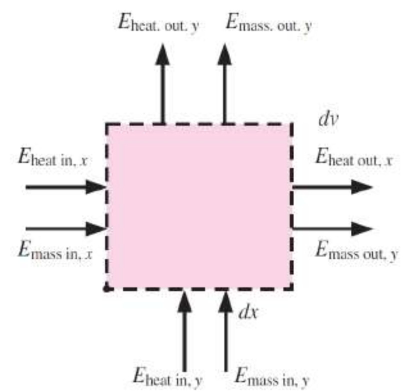
$\frac{\partial T}{\partial x} \ll \frac{\partial T}{\partial y}$

$$\boxed{\frac{\partial P}{\partial y} = 0}$$

y-momentum equation

$$P = P(x) \implies \frac{\partial P}{\partial x} = \frac{dP}{dx}$$

Energy Equation: Balance



$$(\dot{E}_{in} - \dot{E}_{out})_{\text{by mass}} + (\dot{E}_{in} - \dot{E}_{out})_{\text{by heat}} + (\dot{E}_{in} - \dot{E}_{out})_{\text{by work}} = 0 \quad (6)$$

Flowing fluid stream: is associated with enthalpy (internal energy and flow energy), potential energy (PE) and kinetic energy (KE)

The total energy of a flowing stream is:

$$\dot{m}(h + pe + ke) = \dot{m}\left(h + \cancelto{0}{gz} + \cancelto{0}{\frac{u^2 + v^2}{2}}\right) = \dot{m}C_pT$$

KE: [$\mathrm{m}^2/\mathrm{s}^2$] ≈ J/kg. h is in kJ/kg. So, KE is expressed in kJ/kg by dividing it by 1000. KE term at low velocities is negligible. By similar argument, **PE** term is negligible.

$$(\dot{E}_{in} - \dot{E}_{out})_{\text{by mass},x} = (\dot{m}C_pT)_x - \left[(\dot{m}C_pT)_x + \frac{\partial(\dot{m}C_pT)_x}{\partial x}dx\right]$$
$$= -\frac{\partial[\rho u(dy \cdot 1)C_pT]}{\partial x}dx$$

$$\boxed{(\dot{E}_{in} - \dot{E}_{out})_{\text{by mass}} = -\rho C_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right)dxdy} \qquad (7)$$

Energy Balance: by Heat

$$\begin{aligned}
(\dot{E}_{in} - \dot{E}_{out})_{\text{by heat},x} &= q_x - \left(q_x + \frac{\partial q_x}{\partial x} dx\right) \\
&= -\frac{\partial}{\partial x}\left(-k(dy \cdot 1)\frac{\partial T}{\partial x}\right) dx \\
&= k \frac{\partial^2 T}{\partial x^2} dx dy
\end{aligned}$$

$$\boxed{(\dot{E}_{in} - \dot{E}_{out})_{\text{by heat}} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) dx dy} \tag{8}$$

Energy Balance: by Work

- Work done by body forces: considered only if significant gravitational, electric, or magnetic effects exist.
- Work done by surface faces: consists of forces due to fluid pressure and viscous shear stresses.
  - Work done by pressure (the flow work) is accounted in enthalpy
  - Shear stresses that result from viscous effects are usually small (low velocities)

Energy Equation

Combining Eqs. (6), (7), and (8):

$$\boxed{\underbrace{\rho C_p \left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} \right)}_{\text{convection}} = \underbrace{k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)}_{\text{conduction}}}$$

Net energy convected by the fluid out of CV

=

Net energy transfered into CV by conduction

Energy Equation with Viscous Shear Stresses

When viscous shear stresses are not neglected, then:

$$\underbrace{\rho C_p \left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} \right)}_{\text{convection}} = \underbrace{k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)}_{\text{conduction}} + \underbrace{\mu\Phi}_{\text{viscous dissipation}}$$

where the viscous dissipation term is given as:

$$\mu\Phi = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + 2\mu \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right]$$

This accounts for the rate at which mechanical work is irreversibly converted to thermal energy due to viscous effects in the fluid.

Energy Equation with no Convection

$$\boxed{\underbrace{u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}}_{\text{advection}} = \underbrace{\alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)}_{\text{diffusion}}}$$

When the fluid is stationary, $u = v = 0$:

$$k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = 0$$

Differential Convection Equations

Steady incompressible, laminar flow of a fluid with constant properties:

Continuity:
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Momentum:
$$\rho \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right) = \mu \frac{\partial^2 u}{\partial y^2} - \frac{dP}{dx}$$

Energy:
$$\rho C_p \left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} \right) = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

with the boundary conditions

At $x = 0:$ $\quad u(0, y) = u_\infty, \quad T(0, y) = T_\infty$

At $y = 0:$ $\quad u(x, 0) = 0, \quad v(x, 0) = 0, T(x, 0) = T_s$

At $y \to \infty:$ $\quad u(x, \infty) = u_\infty, \quad T(x, \infty) = T_\infty$

Principle of Similarity

$$x^* = \frac{x}{L}, y^* = \frac{y}{L}, u^* = \frac{u}{U_\infty}, v^* = \frac{v}{U_\infty}, P^* = \frac{P}{\rho U_\infty^2}, T^* = \frac{T - T_s}{T_\infty - T_s}$$

Continuity: $$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

Momentum: $$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \frac{1}{\mathrm{Re}_L} \frac{\partial^2 u^*}{\partial y^{*2}} - \frac{dP^*}{dx^*}$$

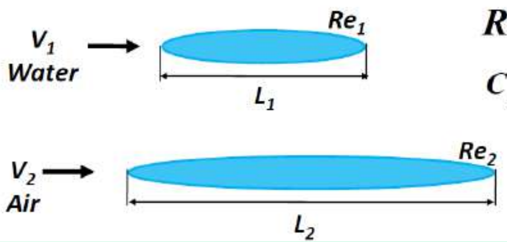
Energy: $$u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{1}{\mathrm{Re}_L \mathrm{Pr}} \frac{\partial^2 T^*}{\partial y^{*2}}$$

with the boundary conditions

At $x^* = 0$: $u^*(0, y^*) = 1,$ $T^*(0, y^*) = 1$

At $y^* = 0$: $u^*(x^*, 0) = 0,$ $v^*(x^*, 0) = 0, T(x^*, 0) = 0$

At $y^* \to \infty$: $u^*(x^*, \infty) = 1,$ $T^*(x^*, \infty) = 1$



$$Re_1 = Re_2 \quad Pr_1 = Pr_2$$

$$C_{f1} = C_{f2} \quad Nu_1 = Nu_2$$

Parameters before nondimensionalizing: $L, V, T_\infty, v, \alpha$

Parameters after nondimensionalizing: Re, PrThe number of parameters is reduced greatly by nondimensionalizing the convection equations.

Functional Forms

For a given geometry, the solution for u^* can be expressed as:

$$u^* = f_1\left(x^*, y^*, \mathrm{Re}_L\right)$$

The shear stress at the surface:

$$\begin{aligned}
\tau_w &= \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} \\
&= \frac{\mu U_\infty}{L} \left.\frac{\partial u^*}{\partial y^*}\right|_{y^*=0} \\
&= \frac{\mu U_\infty}{L} f_2\left(x^*, \mathrm{Re}_L\right)
\end{aligned}$$

Functional Forms

Now the local friction factor:

$$\begin{aligned} C_{f,x} &= \frac{\tau_w}{\rho U_\infty^2/2} \\ &= \frac{\mu U_\infty/L}{\rho U_\infty^2/2} f_2\left(x^*, \mathsf{Re}_L\right) \\ &= \frac{2}{\mathsf{Re}_L} f_2\left(x^*, \mathsf{Re}_L\right) \\ &\boxed{C_{f,x} = f_3\left(x^*, \mathsf{Re}_L\right)} \end{aligned}$$

Friction coefficient for a given geometry can be expressed in terms of the Re and the the dimensionless space variable x^* alone, instead of being expressed in terms of x, L, V, ρ and μ.

This is a very significant finding, and shows the value of nondimensionalized equations.

Functional Forms

Similary,

$$T^* = g_1(x^*, y^*, \mathrm{Re}_L, \mathrm{Pr})$$

The local convective heat coefficient becomes:

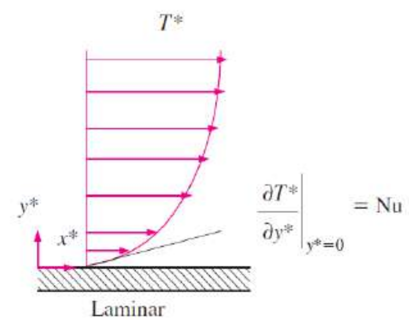
$$h_x = -\frac{k}{T_s - T_\infty} \left.\frac{\partial T}{\partial y}\right|_{y=0} = -\frac{k(T_\infty - T_s)}{L(T_s - T_\infty)} \left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0} = \frac{k}{L} \left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0}$$

Nusselt number relation gives:

$$\mathrm{Nu}_x = \frac{h_x L}{k} = \left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0} = g_2(x^*, \mathrm{Re}_L, \mathrm{Pr})$$

The Nu is equivalent to the dimensionless temp. gradient at the surface, and thus it is properly referred to as the dimensionless heat transfer coefficient.

Nusselt Number



The Nu is equivalent to the dimensionless temp. gradient at the surface, and thus it is properly referred to as the dimensionless heat transfer coefficient.

Local Nusselt number: $\mathrm{Nu}_x = f(x^*, \mathrm{Re}_L, \mathrm{Pr})$

Average Nusselt number: $\mathrm{Nu} = f(\mathrm{Re}_L, \mathrm{Pr})$

A common form: $\boxed{\mathrm{Nu} \;= C\mathrm{Re}_L^m \mathrm{Pr}^n}$

Momentum and Heat Transfer - Analogy

When $\mathrm{Pr} = 1$, and $\frac{dP^*}{dx^*} = 0$ ($\Longrightarrow$ $u = u_\infty$ in the free stream, as in flow over a flat plate), then

Momentum: $$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \frac{1}{\mathrm{Re}_L} \frac{\partial^2 u^*}{\partial y^{*2}}$$

Energy: $$u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{1}{\mathrm{Re}_L} \frac{\partial^2 T^*}{\partial y^{*2}}$$

Profile: $$u^* = T^*$$

Gradients: $$\left.\frac{\partial u^*}{\partial y^*}\right|_{y^*=0} = \left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0}$$

Analogy: $$\boxed{C_{f,x} \frac{\mathrm{Re}_L}{2} = \mathrm{Nu}_x}$$

Reynolds analogy

h for fluid with $\mathrm{Pr} \approx 1$ from C_f which is easier to measure.

Stanton Number

$$\mathsf{St} = \frac{h}{\rho C_p V} = \frac{\mathsf{Nu}}{\mathsf{Re}_L \mathsf{Pr}}$$

$$\boxed{\frac{C_{f,x}}{2} = \mathsf{St}_x} \qquad (\mathsf{Pr} = 1)$$

Stanton number is also a dimensionless heat transfer coefficient. It measures the ratio of heat transferred into a fluid to the thermal capacity of fluid.

Reynolds analogy is limited to Pr $= 1$, and $\frac{dP^*}{dx^*} = 0$.

An analogy that is applicable over a wide range of Pr by adding a Prandtl number correction.

Modified Reynolds or Chilton-Colburn Analogy

Laminar flow over a flat plate

$$C_{f,x} = 0.664\mathsf{Re}_x^{-1/2}$$

$$\mathsf{Nu} = 0.332\mathsf{Pr}^{1/3}\mathsf{Re}_x^{1/2}$$

$$C_{f,x}\frac{\mathsf{Re}_L}{2} = \mathsf{Nu}_x\mathsf{Pr}^{-1/3}$$

$$\implies \boxed{\frac{C_{f,x}}{2} = \mathsf{StPr}^{2/3} \equiv j_H} \qquad 0.6 < \mathsf{Pr} < 60$$

Here j_H is called the Colburn j-factor.

Experiments show that it may be applied for turbulent flow over a surface, even in the presence of pressure gradients ($\frac{dP^*}{dx^*} \neq 0$).

However, the analogy is not applicable for laminar flow unless $\frac{dP^*}{dx^*} = 0$. Therefore, it does not apply to laminar flow in a pipe.

Group	Definition	Interpretation
Biot number (Bi)	$\frac{hL}{k_s}$	Ratio of the internal thermal resistance of a solid to the boundary layer thermal resistance
Bond number (Bo)	$\frac{g(\rho_l - \rho_v)L^2}{\sigma}$	Ratio of gravitational and surface tension forces
Coefficient of friction (C_f)	$\frac{\tau_s}{\rho V^2/2}$	Dimensionless surface shear stress
Eckert number (Ec)	$\frac{V^2}{c_p(T_s - T_\infty)}$	Kinetic energy of the flow relative to the boundary layer enthalpy difference
Fourier number (Fo)	$\frac{\alpha t}{L^2}$	Ratio of the heat conduction rate to the rate of thermal energy storage in a solid. Dimensionless time
Friction factor (f)	$\frac{\Delta p}{(L/D)(\rho u_m^2/2)}$	Dimensionless pressure drop for internal flow
Grashof number (Gr_L)	$\frac{g\beta(T_s - T_\infty)L^3}{\nu^2}$	Measure of the ratio of buoyancy forces to viscous forces
Colburn j factor (j_H)	$St\, Pr^{2/3}$	Dimensionless heat transfer coefficient
Jakob number (Ja)	$\frac{c_p(T_s - T_{sat})}{h_{fg}}$	Ratio of sensible to latent energy absorbed during liquid–vapor phase change
Mach number (Ma)	$\frac{V}{a}$	Ratio of velocity to speed of sound
Nusselt number (Nu_L)	$\frac{hL}{k_f}$	Ratio of convection to pure conduction heat transfer
Peclet number (Pe_L)	$\frac{VL}{\alpha} = Re_L\, Pr$	Ratio of advection to conduction heat transfer rates
Prandtl number (Pr)	$\frac{c_p\mu}{k} = \frac{\nu}{\alpha}$	Ratio of the momentum and thermal diffusivities
Reynolds number (Re_L)	$\frac{VL}{\nu}$	Ratio of the inertia and viscous forces
Stanton number	$\frac{h}{\rho V c_p} = \frac{Nu_L}{Re_L\, Pr}$	Modified Nusselt number

Heat and Mass Transfer

External Flow

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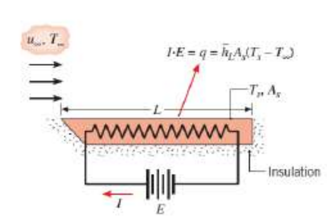
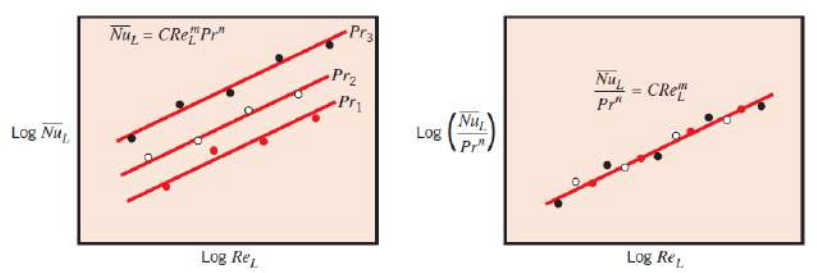
Local Nusselt number: $\text{Nu}_x = f(x^*, \text{Re}_L, \text{Pr})$

Average Nusselt number: $\text{Nu}_L = f(\text{Re}_L, \text{Pr})$

A common form: $\boxed{\text{Nu}_L = C\text{Re}_L^m \text{Pr}^n}$

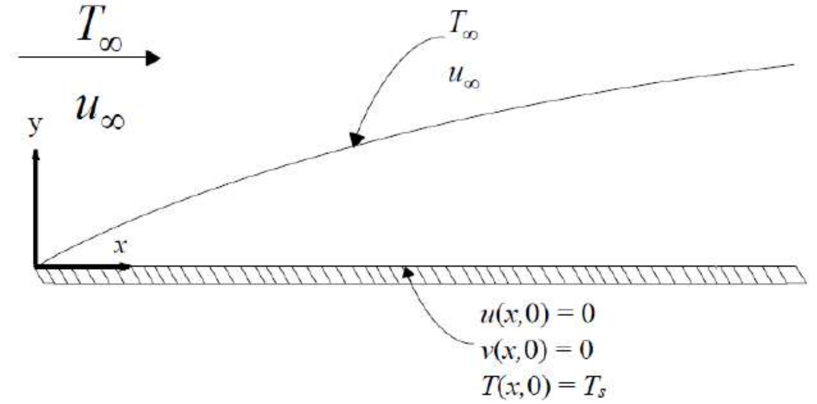
The Empirical Method





$$T_f \equiv \frac{T_s + T_\infty}{2}$$

Flat Plate in Parallel Flow



Assumptions

Steady, incompressible, laminar flow with constant fluid properties and negligible viscous dissipation.

Flat Plate in Parallel Flow

Continuity: $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Momentum: $$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

Energy: $$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$$

First solved in 1908 by German engineer H. Blasius, a student of L. Pradtl. The profile u/u_∞ remains unchanged with y/δ. A stream function $\psi(x, y)$ is defined as,

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}$$

This takes care of continuity equation.

Flat Plate in Parallel Flow

A dimensionless independent similarity variable and a dependent variable such that $u/u_\infty = f'(\eta)$,

$$\eta = y\sqrt{\frac{u_\infty}{\nu x}} \quad \text{and} \quad f(\eta) = \frac{\psi}{u_\infty\sqrt{\nu x/u_\infty}}$$

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y} = u_\infty \frac{df}{d\eta} = u_\infty f'$$

$$v = -\frac{\partial \psi}{\partial x} = \frac{1}{2}\sqrt{\frac{\nu u_\infty}{x}}\left(\eta\frac{df}{d\eta} - f\right)$$

$$(\because -2x\tfrac{\partial f}{\partial x} = \eta\tfrac{\partial f}{\partial \eta})$$

$$\boxed{2f''' + ff'' = 0}$$

The problem reduced to one of solving a nonlinear third-order ordinary differential equation.

Flat Plate in Parallel Flow

$$\boxed{2f''' + ff'' = 0}$$

A third-order nonlinear differential equation with boundary conditions:

$$u(x,0) = v(x,0) = 0 \text{ and } u(x,\infty) = u_\infty$$

$$\left.\frac{df}{\eta}\right|_{\eta=0} = f(0) = 0 \text{ and } \left.\frac{df}{\eta}\right|_{\eta\to\infty} = 1$$

The problem was first solved by Blasius using a power series expansion approach, and this original solution is known as the Blasius solution.

Flat Plate in Parallel Flow

Similarity function f and its derivatives for laminar boundary layer along a flat plate.

η	f	$\frac{df}{d\eta} = \frac{u}{u_\infty}$	$\frac{d^2 f}{d\eta^2}$
0	0	0	0.332
0.5	0.042	0.166	0.331
1.0	0.166	0.330	0.323
1.5	0.370	0.487	0.303
2.0	0.650	0.630	0.267
2.5	0.996	0.751	0.217
3.0	1.397	0.846	0.161
3.5	1.838	0.913	0.108
4.0	2.306	0.956	0.064
4.5	2.790	0.980	0.034
5.0	3.283	0.992	0.016
5.5	3.781	0.997	0.007
6.0	4.280	0.999	0.002
∞	∞	1	0

$$f' = u/u_\infty = 0.99, \text{ for } \eta = 5.0$$

$$y_{\eta=5.0} = \delta = \frac{5.0}{\sqrt{u_\infty/\nu x}} = \frac{5x}{\sqrt{\mathrm{Re}_x}}$$

As $\delta \uparrow$ with $x, \nu \uparrow$ but $\delta \downarrow$ with $u_\infty \uparrow$

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \mu u_\infty \sqrt{\frac{u_\infty}{\nu x}} \left.\frac{d^2 f}{d\eta^2}\right|_{\eta=0}$$

$$\implies \tau_w = 0.332 u_\infty \sqrt{u_\infty/\nu x}$$

$$\boxed{C_{f,x} = \frac{\tau_w}{\rho u_\infty^2/2} = 0.664 \mathrm{Re}_x^{-1/2}}$$

Unlike δ, τ_w and $C_{f,x}$ decrease along the plate as $x^{-1/2}$.

The energy equation:

$$\theta(\eta) = \frac{T(x,y) - T_s}{T_\infty - T_s}$$

$$\boxed{2\frac{d^2\theta}{d\eta^2} + \mathsf{Pr} f \frac{d\theta}{d\eta} = 0}$$

Boundary conditions:

$$\theta(0) = 0, \theta(\infty) = 1$$

For Pr $= 1$: δ and δ_t coincide. u/u_∞ and θ are identical for steady, incompressible, laminar flow of a fluid with constant properties over an isothermal flat plate.

Flat Plate in Parallel Flow

$$\boxed{2\frac{d^2\theta}{d\eta^2} + \mathsf{Pr} f \frac{d\theta}{d\eta} = 0}$$

For Pr > 0.6,

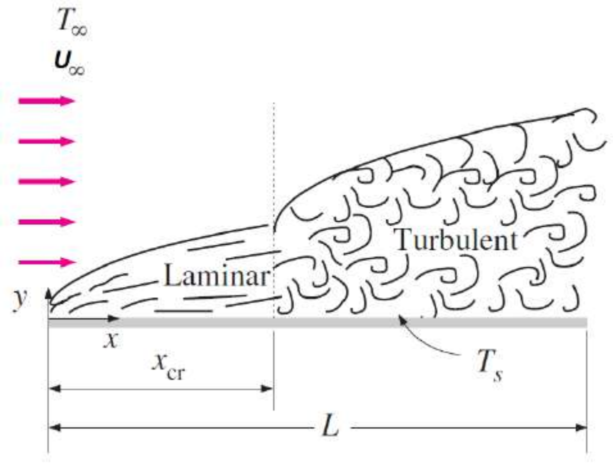
$$\left.\frac{d\theta}{d\eta}\right|_{\eta=0} = 0.332\ \mathsf{Pr}^{1/3}$$

$$\left.\frac{dT}{dy}\right|_{y=0} = 0.332\ \mathsf{Pr}^{1/3}(T_\infty - T_s)\sqrt{\frac{u_\infty}{\nu x}}$$

$$h_x = 0.332\ \mathsf{Pr}^{1/3} k\sqrt{\frac{u_\infty}{\nu x}} \qquad \boxed{\mathsf{Nu}_x = 0.332\ \mathsf{Re}_x^{1/2}\mathsf{Pr}^{1/3}}$$

$$\delta_t = \frac{\delta}{\mathsf{Pr}^{1/3}} = \frac{5x}{\mathsf{Pr}^{1/3}\sqrt{\mathsf{Re}_x}} \qquad (T_f = (T_s + T_\infty)/2)$$

Turbulent Flow



$$\text{Re}_x = \frac{\rho u_\infty x}{\mu} \qquad \text{Re}_{cr} = 5 \times 10^5$$

Laminar: $\delta_{v,x} = \dfrac{5x}{\text{Re}_x^{1/2}}$ and $C_{f,x} = \dfrac{0.664}{\text{Re}_x^{1/2}}$, $\text{Re}_x < 5 \times 10^5$

Turbulent: $\delta_{v,x} = \dfrac{0.382x}{\text{Re}_x^{1/5}}$ and $C_{f,x} = \dfrac{0.0592}{\text{Re}_x^{1/5}}$, $5 \times 10^5 \leq \text{Re}_x \leq 10^7$

Average skin friction coefficient

Laminar : $C_f = \dfrac{1}{L}\displaystyle\int_0^L C_{f,x}dx = \dfrac{1.328}{\text{Re}_L^{1/2}}$, $\text{Re}_x < 5 \times 10^5$

Turbulent : $C_f = \dfrac{0.074}{\text{Re}_L^{1/5}}$, $5 \times 10^5 \leq \text{Re}_x \leq 10^7$

Turbulent Flow

$$C_f = \frac{1}{L}\left(\int_0^{x_{cr}} C_{f,lam}dx + \int_{x_{cr}}^{L} C_{f,turb}dx\right)$$

$$\boxed{C_f = \frac{0.074}{\mathrm{Re}_L^{1/5}} - \frac{1742}{\mathrm{Re}_L}}$$

$$(5\times 10^5 \le \ \mathrm{Re}_x \le 10^7)$$

Rough surface, turbulent: $C_f = \left(1.89 - 1.62\log\frac{\varepsilon}{L}\right)^{-2.5}$

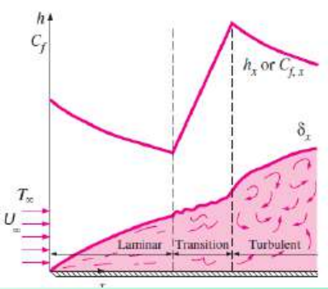
$$(\mathrm{Re} > 10^6, \varepsilon/L > 10^{-4})$$

Turbulent Flow

Laminar: $\quad \text{Nu}_x = \dfrac{h_x x}{k} = 0.332\text{Re}_x^{0.5}\ \text{Pr}^{1/3} \qquad (\text{Pr} > 0.6)$

Turbulent: $\quad \text{Nu}_x = \dfrac{h_x x}{k} = 0.0296\text{Re}_x^{0.8}\ \text{Pr}^{1/3} \qquad (0.6 \leq \text{Pr} \leq 60)$

$$(5 \times 10^5 \leq \ \text{Re}_x \leq 10^7)$$



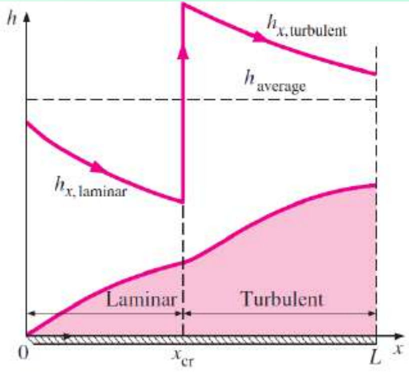
Average values:

$$\text{Nu}_{lam} = \frac{hL}{k} = 0.664\text{Re}_x^{0.5}\ \text{Pr}^{1/3}$$

$$\text{Nu}_{turb} = \frac{hL}{k} = 0.037\text{Re}_x^{0.8}\ \text{Pr}^{1/3}$$

Turbulent Flow

$$h = \frac{1}{L}\left(\int_{0}^{x_{cr}} h_{x,lam}dx + \int_{x_{cr}}^{L} h_{x,turb}dx\right)$$



$$\text{Nu} = \left(0.037\text{Re}_L^{0.8} - 871\right)\ \text{Pr}^{1/3}$$
$$(0.6 \leq \text{Pr} \leq 60)$$
$$(5 \times 10^5 \leq \ \text{Re}_L \leq 10^7)$$

Other Configurations

Liquid metals

Such as mercury have high k, very small Pr. Thus, the δ_t develops much faster than δ.

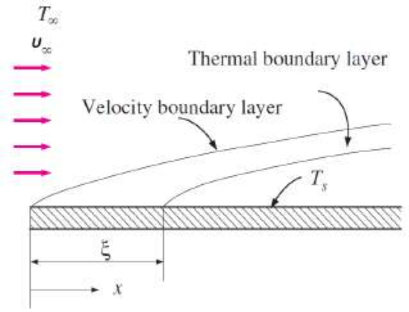
We can assume the velocity in δ_t to be constant at the free stream value and solve the energy equation.

$$\mathrm{Nu}_x = 0.565(\mathrm{Re}_x\mathrm{Pr})^{1/2} = 0.565\mathrm{Pe}_x^{1/2} \qquad (\mathrm{Pr} < 0.05, \mathrm{Pe}_x \geq 100)$$

Churchill's correlation for all Prandtl numbers

$$\mathrm{Nu}_x = \frac{0.3387\mathrm{Re}_x^{1/2}\mathrm{Pr}^{1/3}}{\left[1 + (0.0468/\mathrm{Pr})^{2/3}\right]^{1/4}} \qquad (\mathrm{Re}_x\mathrm{Pr} \geq 100)$$

Flat Plate with Unheated Starting Length



$$\text{Laminar:} \qquad \mathrm{Nu}_x = \frac{\mathrm{Nu}_x|_{\xi=0}}{\left[1-(\xi/x)^{3/4}\right]^{1/3}} = \frac{0.332\mathrm{Re}_x^{0.5}\ \mathrm{Pr}^{1/3}}{\left[1-(\xi/x)^{3/4}\right]^{1/3}}$$

$$\text{Turbulent:} \qquad \mathrm{Nu}_x = \frac{\mathrm{Nu}_x|_{\xi=0}}{\left[1-(\xi/x)^{9/10}\right]^{1/9}} = \frac{0.0296\mathrm{Re}_x^{0.8}\ \mathrm{Pr}^{1/3}}{\left[1-(\xi/x)^{9/10}\right]^{1/9}}$$

Flat Plate with Uniform Heat Flux

Laminar: $\text{Nu}_x = 0.453\text{Re}_x^{0.5}\ \text{Pr}^{1/3}$

Turbulent: $\text{Nu}_x = 0.0308\text{Re}_x^{0.8}\ \text{Pr}^{1/3}$

Net heat transfer from the surface:

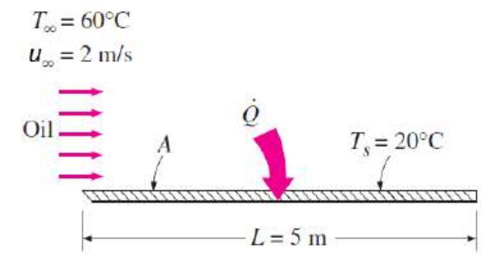
$$\dot{Q} = q_s A_s$$

$$q_s = h_x[T_s(x) - T_\infty]$$

$$\implies T_s(x) = T_\infty + \frac{q_s}{h_s}$$

Problem

Engine oil at 60°C flows over the upper surface of a 5 m long flat plate whose temperature is 20°C with a velocity of 2 m/s. Determine the total drag force and the rate of heat transfer per unit width of the entire plate.



$$T_f = 40°\text{C}$$
$$\rho = 876 \text{ kg/m}^3$$
$$\text{Pr} = 2870$$
$$k = 0.144 \text{ W/m K}$$
$$\nu = 242 \times 10^{-6} \text{ m}^2/\text{s}$$

Known: Engine oil flows over a flat plate.

Solution

Find: Total drag force, $\dot{Q}$ per unit width of plate.
Assumptions: The flow is steady, incompressible

$$\mathrm{Re}_L = \frac{u_\infty L}{\nu} = 41322.3 \qquad (< \mathrm{Re}_{cr} = 5 \times 10^5)$$

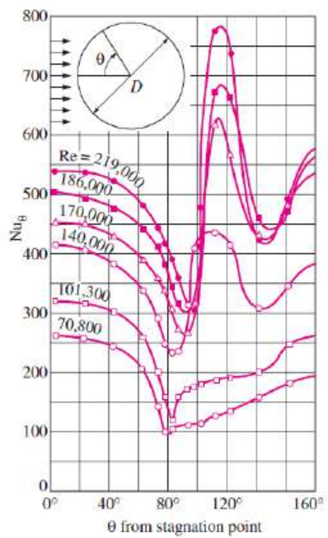
$$C_f = 1.328\mathrm{Re}_L^{-1/2} = 6.533 \times 10^{-0.3}$$

$$F_D = C_f A_s \frac{\rho u_\infty^2}{2} = 57.23 \text{ N}$$

$$\mathrm{Nu} = 0.664\mathrm{Re}_L^{1/2}\mathrm{Pr}^{1/3} = 1938.5$$

$$h = \frac{k}{L}\mathrm{Nu} = 55.98 \text{ W/m}^2 \text{ K}$$

$$\dot{Q} = hA_s(T_\infty - T_s) = 11.2 \text{ W}$$

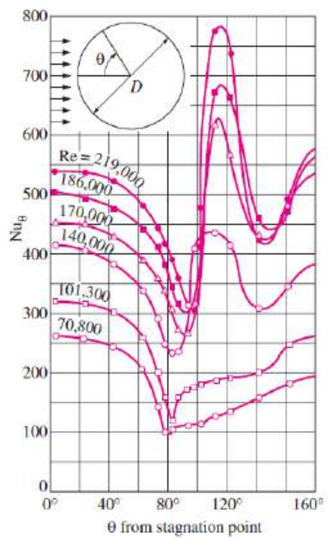


Churchill and Bernstein correlation:

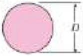
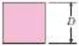
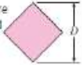
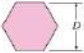
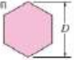
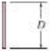
$$\text{Nu}_{cyl} = 0.3 + \frac{0.62\text{Re}^{1/2}\text{Pr}^{1/3}}{\left[1 + (0.4/\text{Pr})^{2/3}\right]^{1/4}} \times \left[1 + \left(\frac{\text{Re}}{282,000}\right)^{5/8}\right]^{4/5}$$

- Nu is relatively high at the stagnation point. Decreases with increasing θ as a result of thethickening of the laminar boundary layer.
- Minimum at 80°, which is the separation point in laminar flow.

Flow Across Cylinder

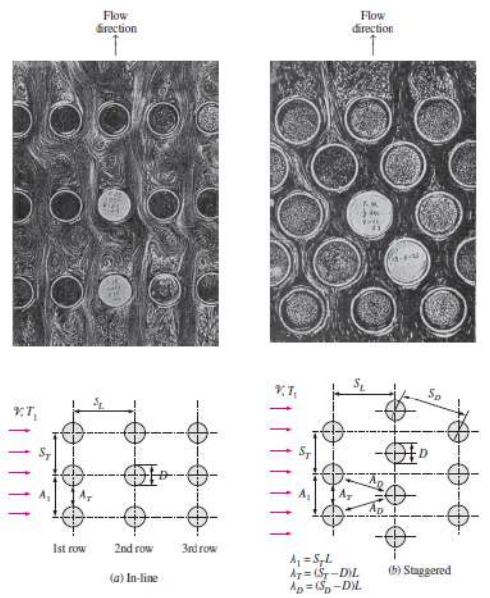


- Increases with increasing as a result of the intense mixing in the separated flow region (the wake).
- The sharp increase at about 90° is due to the transition from laminar to turbulent flow.
- The later decrease is again due to the thickening of the boundary layer.
- Nu reaches its second minimum at about 140°, which is the flow separation point in turbulent flow, and increases with as a result of the intense mixing in the turbulent wake region.

Cross-section of the cylinder	Fluid	Range of Re	Nusselt number
Circle 	Gas or liquid	0.4–4 4–40 40–4000 4000–40,000 40,000–400,000	$Nu = 0.989Re^{0.330}\,Pr^{1/3}$ $Nu = 0.911Re^{0.385}\,Pr^{1/3}$ $Nu = 0.683Re^{0.466}\,Pr^{1/3}$ $Nu = 0.193Re^{0.618}\,Pr^{1/3}$ $Nu = 0.027Re^{0.805}\,Pr^{1/3}$
Square 	Gas	5000–100,000	$Nu = 0.102Re^{0.675}\,Pr^{1/3}$
Square (tilted 45°) 	Gas	5000–100,000	$Nu = 0.246Re^{0.588}\,Pr^{1/3}$
Hexagon 	Gas	5000–100,000	$Nu = 0.153Re^{0.638}\,Pr^{1/3}$
Hexagon (tilted 45°) 	Gas	5000–19,500 19,500–100,000	$Nu = 0.160Re^{0.638}\,Pr^{1/3}$ $Nu = 0.0385Re^{0.782}\,Pr^{1/3}$
Vertical plate 	Gas	4000–15,000	$Nu = 0.228Re^{0.731}\,Pr^{1/3}$
Ellipse 	Gas	2500–15,000	$Nu = 0.248Re^{0.612}\,Pr^{1/3}$

All properties are evaluated at T_f

Flow Across Tube Banks



S_T Transverse pitch
S_L Longitudinal pitch
S_D Diagonal pitch

$$\text{Nu}_D = \frac{hD}{k} = C\ \text{Re}_D^m \text{Pr}^n (\text{Pr}/\text{Pr}_s)^{0.25}$$

Re_D is defined at u_{max} but not the u_∞.

All properties except Pr_s are evaluated at $(T_{inlet} + T_{outlet})/2$ of fluid.

Pr_s is evaluated at T_s.

Flow Across Tube Banks

Nusselt number correlations for cross flow over tube banks for $N > 16$ and $0.7 < Pr < 500$ (from Zukauskas, Ref. 15, 1987)*

Arrangement	Range of Re_D	Correlation
In-line	0–100	$Nu_D = 0.9\,Re_D^{0.4}Pr^{0.36}(Pr/Pr_s)^{0.25}$
	100–1000	$Nu_D = 0.52\,Re_D^{0.5}Pr^{0.36}(Pr/Pr_s)^{0.25}$
	$1000–2 \times 10^5$	$Nu_D = 0.27\,Re_D^{0.63}Pr^{0.36}(Pr/Pr_s)^{0.25}$
	$2 \times 10^5–2 \times 10^6$	$Nu_D = 0.033\,Re_D^{0.8}Pr^{0.4}(Pr/Pr_s)^{0.25}$
Staggered	0–500	$Nu_D = 1.04\,Re_D^{0.4}Pr^{0.36}(Pr/Pr_s)^{0.25}$
	500–1000	$Nu_D = 0.71\,Re_D^{0.5}Pr^{0.36}(Pr/Pr_s)^{0.25}$
	$1000–2 \times 10^5$	$Nu_D = 0.35(S_T/S_L)^{0.2}\,Re_D^{0.6}Pr^{0.36}(Pr/Pr_s)^{0.25}$
	$2 \times 10^5–2 \times 10^6$	$Nu_D = 0.031(S_T/S_L)^{0.2}\,Re_D^{0.8}Pr^{0.36}(Pr/Pr_s)^{0.25}$

*All properties except Pr_s are to be evaluated at the arithmetic mean of the inlet and outlet temperatures of the fluid (Pr_s is to be evaluated at T_s).

$$\mathrm{Nu}_{D,N_{L<16}} = F\,\mathrm{Nu}_D$$

Correction factor F to be used in $Nu_{D,N_L} = FNu_D$ for $N_L < 16$ and $Re_D > 1000$ (from Zukauskas, Ref 15, 1987).

N_L	1	2	3	4	5	7	10	13
In-line	0.70	0.80	0.86	0.90	0.93	0.96	0.98	0.99
Staggered	0.64	0.76	0.84	0.89	0.93	0.96	0.98	0.99

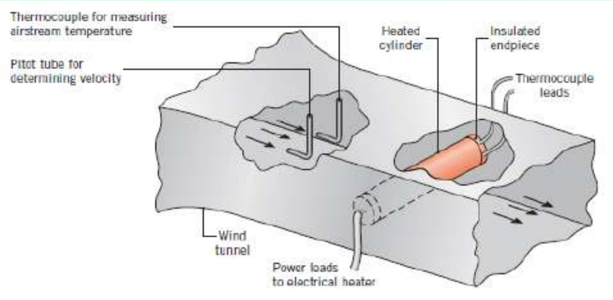
Methodology for a Convection Calculation

- Become immediately cognizant of the flow geometry.
- Specify the appropriate reference temperature and evaluate the pertinent fluid properties at that temperature.
- Calculate the Reynolds number.
- Decide whether a local or surface average coefficient is required.
- Select the appropriate correlation.

Problem: Cylinder

Experiments have been conducted on a metallic cylinder ($D = 12.7$ mm, $L = 94$ mm). The cylinder is heated internally by an electrical heater and is subjected to a cross flow of air in a low-speed wind tunnel ($V = 10$ m/s, 26.2°C). The heater power dissipation was measured to be $P = 46$ W, while $T_s = 128.4$°C. It is estimated that 15% of the power dissipation is lost through conduction and radiation.

1. Determine h from experimental observations.
2. Compare the result with appropriate correlation(s).



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$$\mathrm{Nu}_D = 0.26\ \mathrm{Re}_D^{0.6}\mathrm{Pr}^{0.37}\left(\mathrm{Pr}/\mathrm{Pr}_s\right)^{1/2} \qquad \text{(Zhukauskasa relation)}$$

Air ($T_\infty = 26.2$°C):
$\nu = 15.89 \times 10^{-6}$ m^2/s, $k = 26.3 \times 10^{-3}$ W/m K, Pr = 0.707

Air ($T_f = 77.3$°C):
$\nu = 20.92 \times 10^{-6}$ m^2/s, $k = 30 \times 10^{-3}$ W/m K, Pr = 0.700

Air ($T_s = 128.4$°C): Pr = 0.690

$$\mathrm{Nu}_D = 0.3 + \frac{0.62\mathrm{Re}_D^{1/2}\mathrm{Pr}^{1/3}}{\left[1+(0.4/\mathrm{Pr})^{2/3}\right]^{1/4}}\left[1+\left(\frac{\mathrm{Re}_D}{282,000}\right)^{5/8}\right]^{4/5}$$

(Churchill relation)

Problem: Cylinder

Experiments have been conducted on a metallic cylinder ($D = 12.7$ mm, $L = 94$ mm). The cylinder is heated internally by an electrical heater and is subjected to a cross flow of air in a low-speed wind tunnel ($V = 10$ m/s, 26.2°C). The heater power dissipation was measured to be $P = 46$ W, while $T_s = 128.4°$C. It is estimated that 15% of the power dissipation is lost through conduction and radiation.

$$\text{Nu}_D = 0.193\ \text{Re}_D^{0.618}\text{Pr}^{1/3} \qquad \text{(Hilpert correlation)}$$

Air ($T_\infty = 26.2°$C):
$\nu = 15.89 \times 10^{-6}$ m^2/s, $k = 26.3 \times 10^{-3}$ W/m K, Pr = 0.707
Air ($T_f = 77.3°$C):
$\nu = 20.92 \times 10^{-6}$ m^2/s, $k = 30 \times 10^{-3}$ W/m K, Pr = 0.700
Air ($T_s = 128.4°$C): Pr = 0.690

Problem: Sphere

The decorative plastic film on a copper sphere of 10 mm diameter is cured in an oven at 75°C. Upon removal from the oven, the sphere is subjected to an airstream at 1 atm and 23°C having a velocity of 10 m/s. Estimate how long it will take to cool the sphere to 35°C.

Copper ($T = 55$°C):

$\rho = 8933$ kg/m^3, $k = 399$ W/m K, $C_p = 387$ J/kg

Air ($T_\infty = 23$°C):

$\mu = 181.6 \times 10^{-7}$ Ns/m^2, $\nu = 15.36 \times 10^{-6}$ m^2/s, $k = 0.0258$ W/m K, Pr $= 0.709$

Air ($T_s = 55$°C): $\mu = 197.8 \times 10^{-7}$ Ns/m^2

$$\mathrm{Nu}_D = 2 + \left(0.4\ \mathrm{Re}_D^{1/2} + 0.06\ \mathrm{Re}_D^{2/3}\right) \mathrm{Pr}^{0.4} \left(\frac{\mu}{\mu_s}\right)^{1/4}$$

All properties except μ_s are evaluated at T_∞.

Problem: Sphere

The decorative plastic film on a copper sphere of 10 mm diameter is cured in an oven at 75°C. Upon removal from the oven, the sphere is subjected to an airstream at 1 atm and 23°C having a velocity of 10 m/s. Estimate how long it will take to cool the sphere to 35°C.

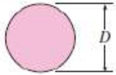
Copper ($T = 55$°C):

$\rho = 8933$ kg/m^3, $k = 399$ W/m K, $C_p = 387$ J/kg

Air ($T_\infty = 39$°C):

$\nu = 17.15 \times 10^{-6}$ m^2/s, Pr = 0.705

Air ($T_s = 55$°C): $\mu = 197.8 \times 10^{-7}$ Ns/m^2

Cross-section of the cylinder	Fluid	Range of Re	Nusselt number
Circle 	Gas or liquid	0.4–4 4–40 40–4000 4000–40,000 40,000–400,000	$Nu = 0.989 Re^{0.330} Pr^{1/3}$ $Nu = 0.911 Re^{0.385} Pr^{1/3}$ $Nu = 0.683 Re^{0.466} Pr^{1/3}$ $Nu = 0.193 Re^{0.618} Pr^{1/3}$ $Nu = 0.027 Re^{0.805} Pr^{1/3}$

All properties are evaluated at T_f

Quiz 4

A flat plate of 0.3 m long is maintained at a uniform surface temperature, $T_s = 230°$C, by using two independently controlled electrical strip heaters. The first heater 0.2 m long and the second one is 0.1 m long. The ambient air temperature at $T_\infty = 25°$C flows over the plate at a velocity of 60 m/s. At what heater is the electrical input a maximum? What is the value of this input.

Air ($T_f = 400$ K):

$\nu = 26.41 \times 10^{-6}$ m^2/s, $k = 0.0338$ W/m K, Pr $= 0.69$

Relations

Laminar boundary layer $\quad \bar{\text{Nu}}_x = \dfrac{\bar{h}_x x}{k} = 0.664\ \text{Re}_x^{1/2}\text{Pr}^{1/3}$

$(\text{Re} < 5 \times 10^5)$

Mixed boundary layer $\quad \bar{\text{Nu}}_L = \dfrac{\bar{h}_L L}{k} = (0.037\ \text{Re}_L^{4/5} - 871)\ \text{Pr}^{1/3}$

$(\text{Re} > 5 \times 10^5)$

Heat and Mass Transfer

Internal Flow

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Internal Flows

External flow

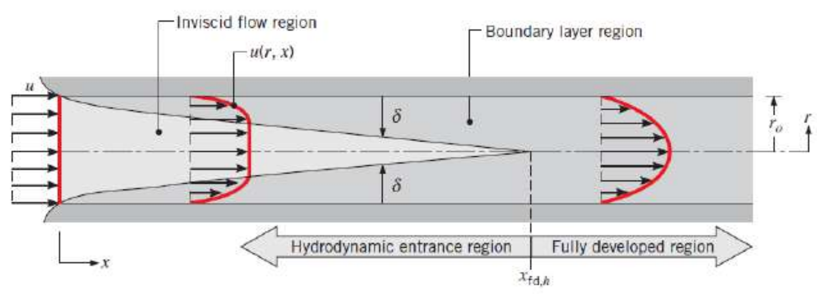
- Fluid has a free surface
- δ is free to grow indefinitely

Internal flow

- Fluid is completely confined by the inner surfaces of the tube
- There is a limit on how much δ can grow

Circular pipes can withstand large pressure difference between inside and outside without distortion. Provide the most heat transfer for the least pressure drop.

Internal Flows

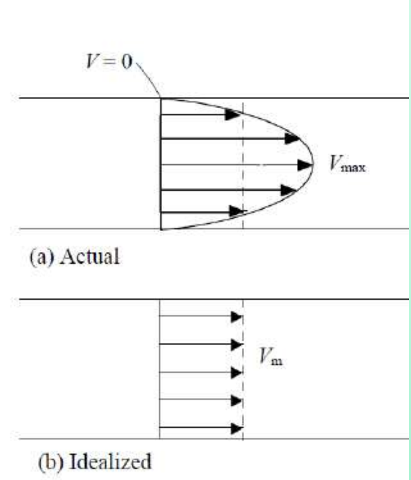


$$\mathrm{Re}_D = \frac{\rho u_m D}{\mu}$$

Critical $\quad \mathrm{Re}_{D,c} \approx 2300$

Laminar $\quad \left(\frac{x_{\mathrm{fd},h}}{D}\right)_{\mathrm{lam}} \approx 0.05\mathrm{Re}_D$

Turbulence $\quad \left(\frac{x_{\mathrm{fd},h}}{D}\right)_{\mathrm{turb}} \approx 10$

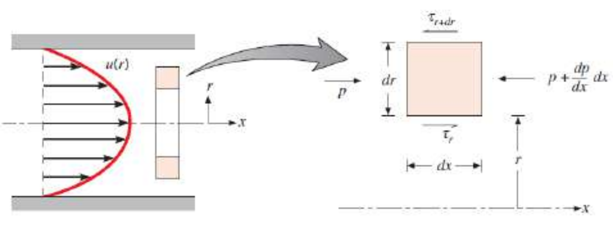


$$\dot{m} = \rho u_m A_c = \int_{A_c} \rho u(r,x) dA_c$$

$$u_m = \frac{2}{r_o^2} \int_0^{r_o} u(r,x) r dr$$

Velocity Profile in Fully Developed Region

Assumptions: Laminar, incompressible, constant property fluid in fully developed region, circular tube.



$$u(r) = -\frac{1}{4\mu}\left(\frac{dp}{dx}\right) r_o^2 \left[1 - \left(\frac{r}{r_o}\right)^2\right]$$

$$u_m = -\frac{r_o^2}{8\mu}\frac{dp}{dx}$$

$$\frac{u(r)}{u_m} = 2\left[1 - \left(\frac{r}{r_o}\right)^2\right]$$

Friction Factor

The Moody (or Darcy) friction factor:

$$f \equiv \frac{-(dp/dx)D}{\rho u_m^2/2}$$

Friction coefficient, Fanning friction factor,

$$C_f = \frac{\tau_s}{\rho u_m^2/2} = \frac{f}{4}$$

For laminar flow:

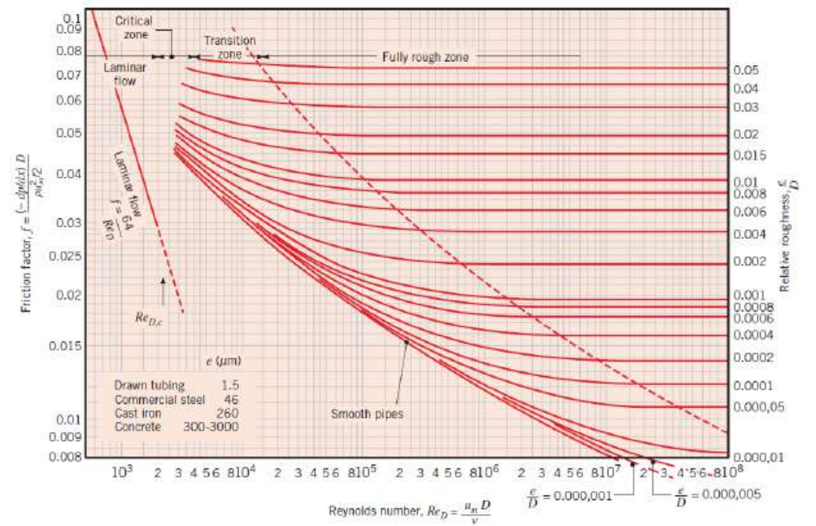
$$f = \frac{64}{\mathsf{Re}}$$

For turbulent flow:

$$f = 0.316\mathsf{Re}_D^{-1/4} \qquad \mathsf{Re}_D \lesssim 2 \times 10^4$$

$$f = 0.184\mathsf{Re}_D^{-1/5} \qquad \mathsf{Re}_D \gtrsim 2 \times 10^4$$

Moody Diagram

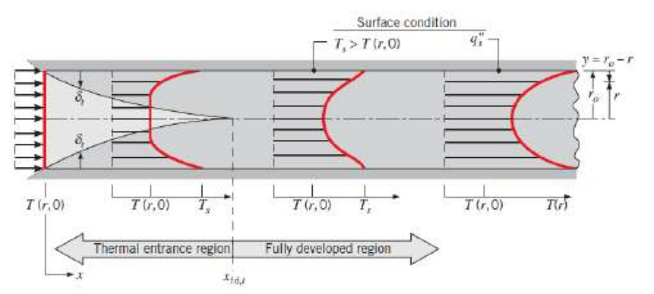


Pump or Fan Power

$$\Delta p = -\int_{p_1}^{p_2} dp = f\frac{\rho u_m^2}{2D}\int_{x_1}^{x_2} dx = f\frac{\rho u_m^2}{2D}(x_1 - x_2)$$

$$\text{Power, } P = \Delta p\frac{\dot{m}}{\rho}$$

Temperature



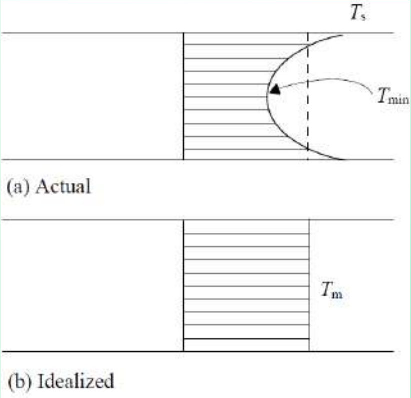
$$\mathrm{Re}_D = \frac{\rho u_m D}{\mu}$$

Critical $\mathrm{Re}_{D,c} \approx 2300$

Laminar $\left(\frac{x_{\mathrm{fd},t}}{D}\right)_{\mathrm{lam}} \approx 0.05\mathrm{Re}_D\mathrm{Pr}$

Turbulence $\left(\frac{x_{\mathrm{fd},t}}{D}\right)_{\mathrm{turb}} \approx 10$

Advection rate: integrating the product of mass flux (ρu) and the thermal energy (or enthalpy) per unit mass, C_pT, over A_c.



$$\dot{m}C_pT_m = \int_{A_c} \rho u C_p T dA_c$$

For incompressible flow in a circular tube with constant C_p:

$$T_m = \frac{2}{u_m r_o^2}\int_0^{r_o} uTrdr$$

$$q_s'' = h(T_s - T_m)$$

Dimensionless Temperature

$$\frac{\partial}{\partial x}\left[\frac{T_s(x)-T(r,x)}{T_s(x)-T_m(x)}\right]_{\text{fd,t}} = 0$$

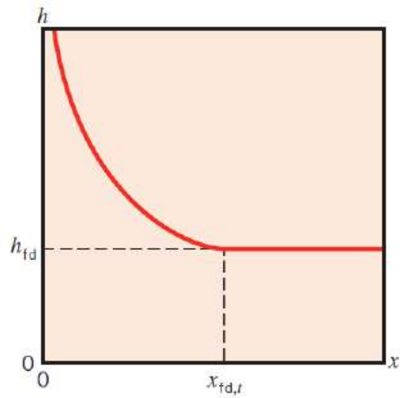
$$\implies \frac{\partial}{\partial x}\left[\frac{T_s(x)-T(r,x)}{T_s(x)-T_m(x)}\right]_{\text{fd,t}} = \frac{-\partial T/\partial r|_{r=r_o}}{T_s - T_m} \neq f(x)$$

$$q_s'' = -k\,\frac{\partial T}{\partial y}\bigg|_{y=0} = -k\,\frac{\partial T}{\partial r}\bigg|_{r=r_o}$$

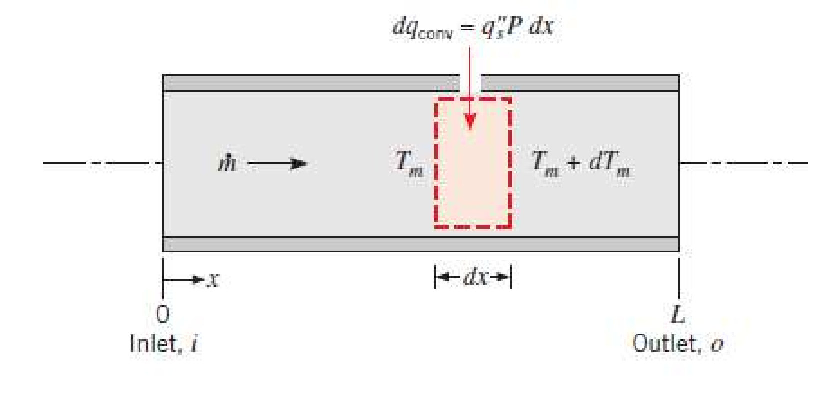
$$\implies \frac{h}{k} \neq f(x)$$

In thermally fully developed flow of a fluid with constant properties, h_{local} is a constant, independent of x.

Heat Transfer Coefficient for Flow in a Tube



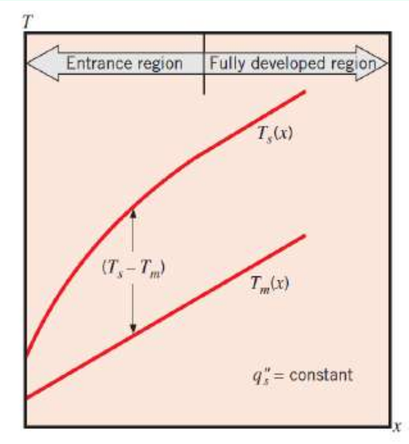
General Considerations



$$dq_{conv} = \dot{m}C_p[(T_m + dT_m) - T_m]$$

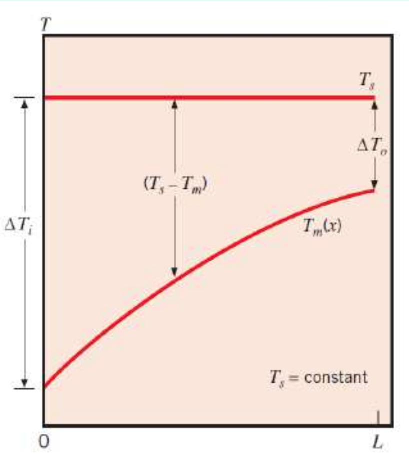
$$dq_{conv} = \dot{m}C_p dT_m$$

General Considerations: Const. q



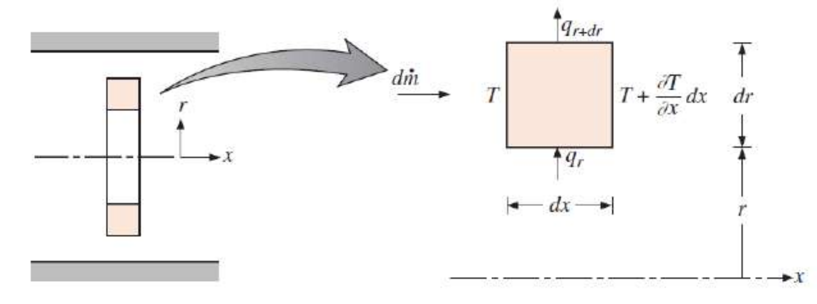
$$T_m(x) = T_{m,i} + \frac{q_s'' P}{\dot{m} C_p} x$$

General Considerations: Const. T



$$\frac{T_s - T_m(x)}{T_s - T_{m,i}} = \exp\left(-\frac{Px}{\dot{m}C_p}\bar{h}\right)$$

Fully Developed Laminar Flow: Const. q



$$u\frac{\partial T}{\partial x} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$$

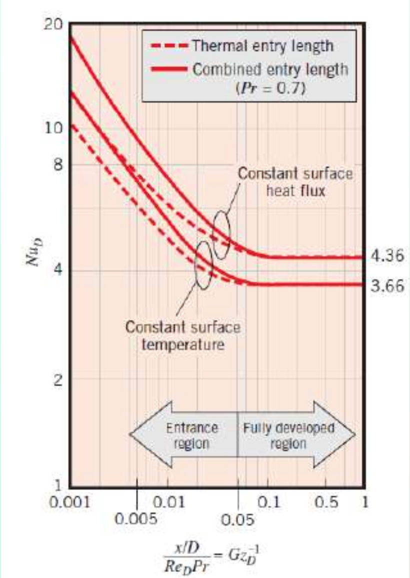
$$T(r,x) \implies T_m(x) - T_s(x) = -\frac{11}{48}\frac{q_s'' D}{k}$$

$$h = \frac{11}{48}\frac{k}{D} \implies \mathrm{Nu}_D = \frac{hD}{k} = 4.36$$

Fully Developed Laminar Flow: Const. T

$$\mathsf{Nu}_D = \frac{hD}{k} = 3.66$$

The Entry Region

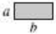
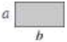
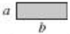


Dittus-Boelter equation

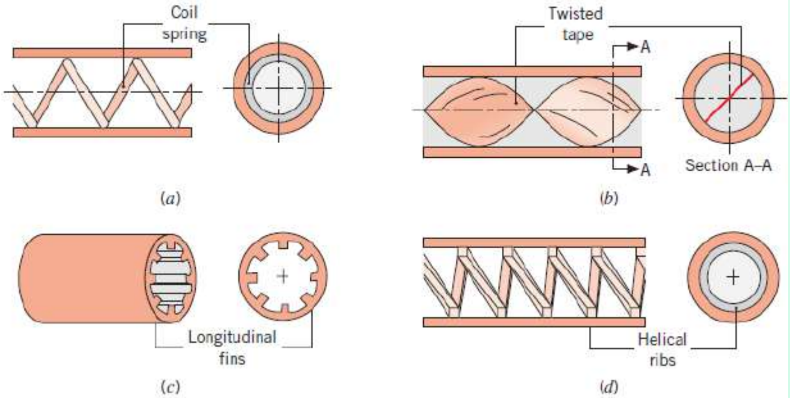
$$\mathsf{Nu}_D = 0.023\mathsf{Re}_D^{4/5}\mathsf{Pr}^n$$

where $n = 0.4$ for heating $(T_s > T_m)$ and 0.3 for cooling $(T_s < T_m)$.

Nusselt numbers and friction factors for fully developed laminar flow in tubes of differing cross section

Cross Section	$\frac{b}{a}$	$Nu_D = \frac{hD_h}{k}$ (Uniform q_s'')	$Nu_D = \frac{hD_h}{k}$ (Uniform T_s)	$f Re_{D_h}$
	—	4.36	3.66	64
	1.0	3.61	2.98	57
	1.43	3.73	3.08	59
	2.0	4.12	3.39	62
	3.0	4.79	3.96	69
	4.0	5.33	4.44	73
	8.0	6.49	5.60	82
	∞	8.23	7.54	96
	∞	5.39	4.86	96
	—	3.11	2.49	53

The Entry Region



Internal flow heat transfer enhancement schemes: (*a*) longitudinal section and end view of coil-spring wire insert, (*b*) longitudinal section and cross-sectional view of twisted tape insert, (c) cut-away section and end view of longitudinal fins, and (*d*) longitudinal section and end view of helical ribs.

Heat and Mass Transfer

Free Convection

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Physical Considerations

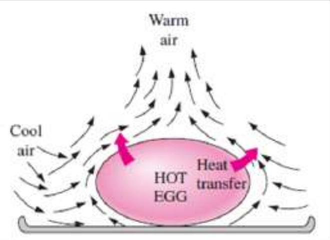
Buoyancy

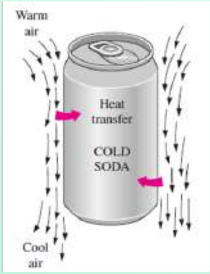
Combined presence of a fluid density gradient and a body force that is proportional to density.

Body Force

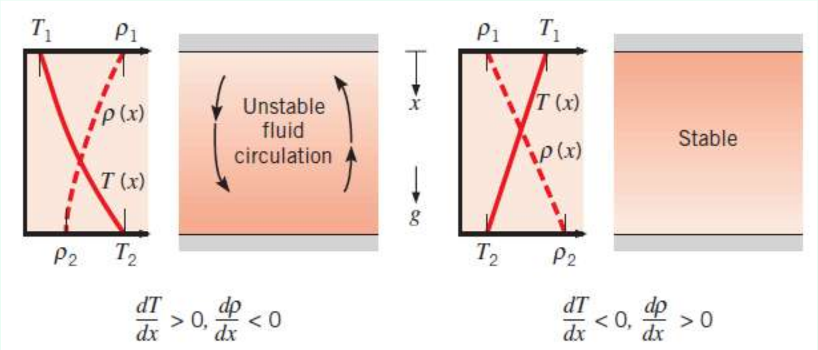
- Gravitational
- Centrifugal force in rotating fluid machinery
- Coriolis force in atmospheric
- Oceanic rotational motions

Free Convection: Examples





Conditions of Stable & Unstable T Gradient

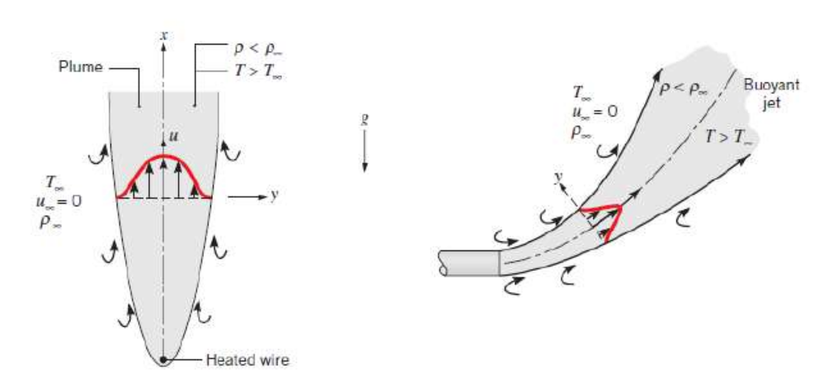


Classification: bounded or unbounded

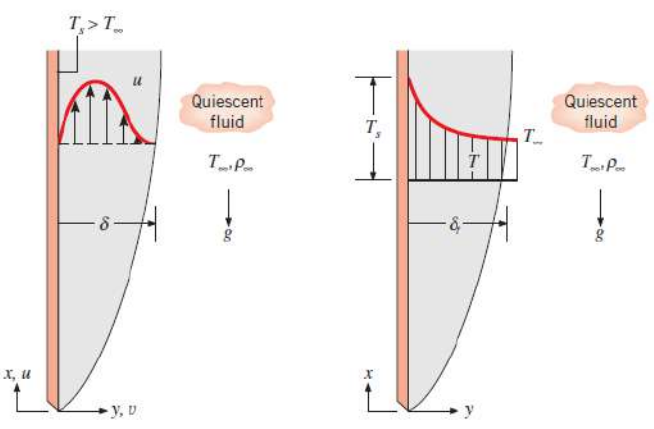
- Free boundary flows
- Bounded by a surface

May occur in the form of a plume or a buoyant jet. plume is associated with fluid rising from a submerged heated object.

A heated wire immersed in an extensive, quiescent fluid.



Heated Vertical Plate



Momentum: $$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{1}{\rho}X + \nu\frac{\partial^2 u}{\partial y^2}$$

Buoyancy Force

Force by gravity per unit volume:

$$X = -\rho g$$

$$\implies u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} - g + \nu\frac{\partial^2 u}{\partial y^2}$$

From y-momentum equation: $(\partial p/\partial y) = 0$

$$\frac{\partial P}{\partial x} = -\rho_\infty g$$

$$\implies u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\frac{\Delta\rho}{\rho} + \nu\frac{\partial^2 u}{\partial y^2}$$

where $\Delta\rho = \rho_\infty - \rho$

Boussinesq Approximation

Thermodynamic property, Volumetric thermal expansion coeff.,

$$\beta = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p$$

Approximate form:

$$\beta \approx -\frac{1}{\rho}\frac{\Delta\rho}{\Delta T} = -\frac{1}{\rho}\frac{\rho_\infty - \rho}{T_\infty - T}$$

$$(\rho_\infty - \rho) \approx \rho\beta(T - T_\infty)$$

$$\boxed{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\beta(T - T_\infty) + \nu\frac{\partial^2 u}{\partial y^2}}$$

The Governing Equations

Steady incompressible, free convection, laminar flow of a fluid with constant properties:

$$\text{Continuity:} \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\text{Momentum:} \qquad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\beta(T - T_\infty) + \nu\frac{\partial^2 u}{\partial y^2}$$

$$\text{Energy:} \qquad u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$$

For an ideal gas ($\rho = p/RT$),

$$\beta = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p = \frac{1}{T}$$

Dimensionless Parameters

$$x^* \equiv \frac{x}{L}, y^* \equiv \frac{y}{L}, u^* \equiv \frac{u}{u_0}, v^* \equiv \frac{v}{u_0}, T^* \equiv \frac{T - T_\infty}{T_s - T_\infty}$$

u_0 is an arbitrary reference velocity.

Momentum: $$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \frac{g\beta(T_s - T_\infty)L}{u_0^2} T^* + \frac{1}{\mathrm{Re}_L} \frac{\partial^2 u^*}{\partial y^{*2}}$$

Energy: $$u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{1}{\mathrm{Re}_L \mathrm{Pr}} \frac{\partial^2 T^*}{\partial y^{*2}}$$

Coefficient of T^* is in terms of unknown reference velocity u_0.
Eliminate it to get a dimensionless parameter:

$$\boxed{\mathrm{Gr}_L \equiv \frac{g\beta(T_s - T_\infty)L}{u_0^2} \left(\frac{u_0 L}{\nu}\right)^2 = \frac{g\beta(T_s - T_\infty)L^3}{\nu^2}}$$

Grashof number

$$\boxed{\mathsf{Gr}_L \equiv \frac{g\beta(T_s - T_\infty)L^3}{\nu^2} = \frac{\text{Buoyancy force}}{\text{Viscous force}}}$$

$$\frac{\mathsf{Gr}_L}{\mathsf{Re}_L^2} \ll 1 : \qquad \text{Free convection is neglected}$$

$$\mathsf{Nu}_L = f(\mathsf{Re}_L,\ \mathsf{Pr})$$

$$\frac{\mathsf{Gr}}{\mathsf{Re}^2} \gg 1 : \qquad \text{Forced convection is neglected}$$

$$\mathsf{Nu}_L = f(\mathsf{Gr}_L,\ \mathsf{Pr})$$

$$\frac{\mathsf{Gr}}{\mathsf{Re}^2} \equiv 1 : \qquad \text{Combination of free and forced}$$

Similarity Solution

Boundary conditions:

$$\text{At } y = 0: \quad u = v = 0, \quad T = T_s$$
$$\text{At } y \to \infty: \quad u \to 0, \quad T \to T_\infty$$

Similarity parameter of the form:

$$\eta \equiv \frac{y}{x}\left(\frac{\mathsf{Gr}_x}{4}\right)^{1/4}$$

and representing the velocity components in terms of a stream function defined as:

$$\psi(x,y) \equiv f(\eta)\left[4\nu\left(\frac{\mathsf{Gr}_x}{4}\right)^{1/4}\right]$$

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{2\nu}{x}\mathsf{Gr}_x^{1/2} f'(\eta)$$

Similarity Solution

$$v = -\frac{\partial \psi}{\partial x} \quad \text{and} \quad T^* \equiv \frac{T - T_\infty}{T_s - T_\infty}$$

The reduced partial differential equations:

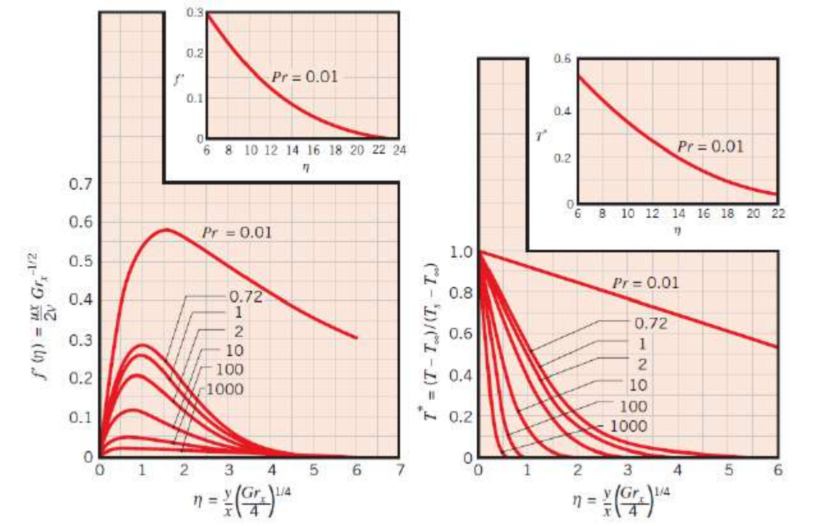
$$f''' + 3ff'' - 2(f')^2 + T^* = 0$$

$$T^{*''} + 3\mathrm{Pr} f T^{*'} = 0$$

Boundary conditions:

$$\text{At } \eta = 0: \quad f = f' = 0, \quad T^* = 1$$
$$\text{At } \eta \to \infty: \quad f' \to 0, \quad T^* \to 0$$

Laminar, Free Convection



Laminar, free convection boundary layer conditions on a isothermal vertical surface.

Laminar, Free Convection

Newton's law of cooling:

$$\mathsf{Nu}_x = \frac{hx}{k} = \frac{[q''_s/(T_s - T_\infty)]x}{k}$$

Fourier's law:

$$q''_s = -k\,\frac{\partial T}{\partial y}\bigg|_{y=0} = -\frac{k}{x}(T_s - T_\infty)\left(\frac{\mathsf{Gr}_x}{4}\right)^{1/4}\,\frac{dT^*}{d\eta}\bigg|_{\eta=0}$$

$$\Longrightarrow \boxed{\mathsf{Nu}_x = \frac{hx}{k} = -\left(\frac{\mathsf{Gr}_x}{4}\right)^{1/4}\,\frac{dT^*}{d\eta}\bigg|_{\eta=0} = \left(\frac{\mathsf{Gr}_x}{4}\right)^{1/4} g(\mathsf{Pr})}$$

$$g(\mathsf{Pr}) = \frac{0.75\mathsf{Pr}^{1/2}}{(0.609 + 1.221\mathsf{Pr}^{1/2} + 1.238\mathsf{Pr})^{1/4}}$$

$$0 \le \mathsf{Pr} \le \infty$$

Laminar, Free Convection

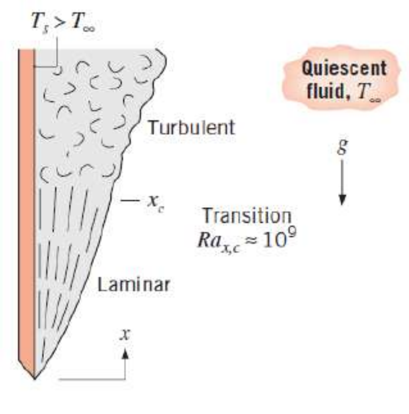
$$\bar{h} = \frac{1}{L}\int_0^L h dx = \frac{k}{L}\left[\frac{g\beta(T_s - T_\infty)}{4\nu^2}\right]^{1/4} g(\mathsf{Pr}) \int_0^L \frac{dx}{x^{1/4}}$$

$$\boxed{\overline{\mathsf{Nu}}_L = \frac{\bar{h}L}{k} = \frac{4}{3}\left(\frac{\mathsf{Gr}_L}{4}\right)^{1/4} g(\mathsf{Pr})}$$

$$\overline{\mathsf{Nu}}_L = \tfrac{4}{3}\,\mathsf{Nu}_L$$

These results apply irrespective of whether $T_s > T_\infty$ or $T_s < T_\infty$.

Turbulence



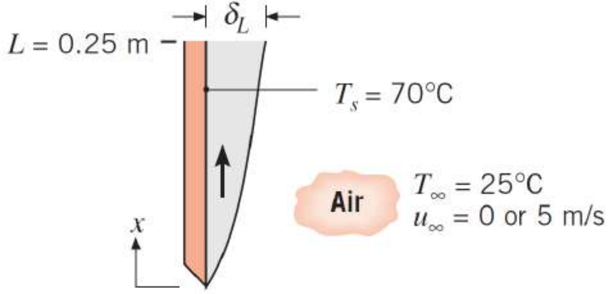
Critical Rayleigh number:

$$\boxed{\mathrm{Ra}_{x,c} = \mathrm{Gr}_{x,c}\,\mathrm{Pr} = \frac{g\beta(T_s - T_\infty)x^3}{\nu\alpha} \approx 10^9}$$

Problem

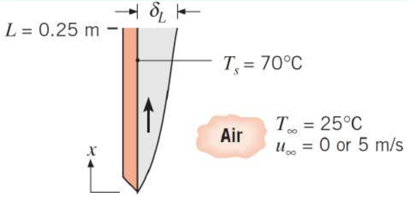
Consider a 0.25 m long vertical plate that is at 70°C. The plate is suspended in air that is at 25°C. Estimate the boundary layer thickness at the trailing edge of the plate if the air is quiescent. How does this thickness compare with that which would exist if the air were flowing over the plate at a free stream velocity of 5 m/s?

Air properties at $T_f = 47.5°$C:
$\nu = 17.95 \times 10^{-6}$ m^2/s, Pr $= 0.7, \beta = \frac{1}{T_f} = 3.12 \times 10^{-3}$ K^{-1}



Solution

Air properties at $T_f = 47.5°\text{C}$:
$\nu = 17.95 \times 10^{-6}$ m^2/s, Pr $= 0.7, \beta = \frac{1}{T_f} = 3.12 \times 10^{-3}$ K^{-1}



$$\text{Gr}_L = \frac{g\beta(T_s - T_\infty)L^3}{\nu^2} = 6.69 \times 10^7$$

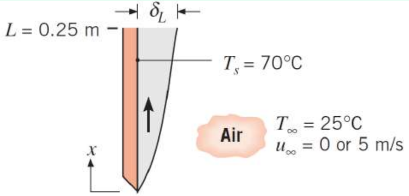
$$\text{Ra}_L = \text{Gr}_L \, \text{Pr} = 4.68 \times 10^7$$

For Pr $= 0.7, \eta \approx 6.0$ at the edge of the boundary layer.

$$\eta \equiv \frac{y}{x}\left(\frac{\text{Gr}_x}{4}\right)^{1/4} = 0.6 \implies \delta_L = 0.37 \text{ m}$$

Solution

Air properties at $T_f = 47.5°\text{C}$:
$\nu = 17.95 \times 10^{-6}$ m^2/s, Pr $= 0.7, \beta = \frac{1}{T_f} = 3.12 \times 10^{-3}$ K^{-1}



For airflow at $u_\infty = 5$ m/s

$$\text{Re}_L = \frac{u_\infty L}{\nu} = 6.97 \times 10^4$$

For laminar boundary layer:

$$\delta = \frac{5x}{\sqrt{\text{Re}_L}} = 0.0047 \text{ m} \quad \frac{\delta}{\delta_t} \approx \text{Pr}^{1/3}$$

Solution

Comments

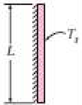
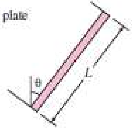
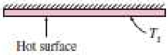
- δ are typically larger for free convection than for forced convection.
- $\mathrm{Gr} / \mathrm{Re}^2 \ll 1$, and the assumption of negligible buoyancy effects for $u_\infty = 5$ m/s is justified.

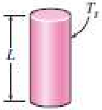
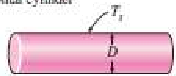
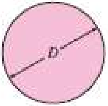
Empirical Correlations

$$\mathsf{Nu} = \frac{h}{L_c}k = C(\mathsf{Gr}_L\,\mathsf{Pr})^n = C\,\mathsf{Ra}_L^n$$

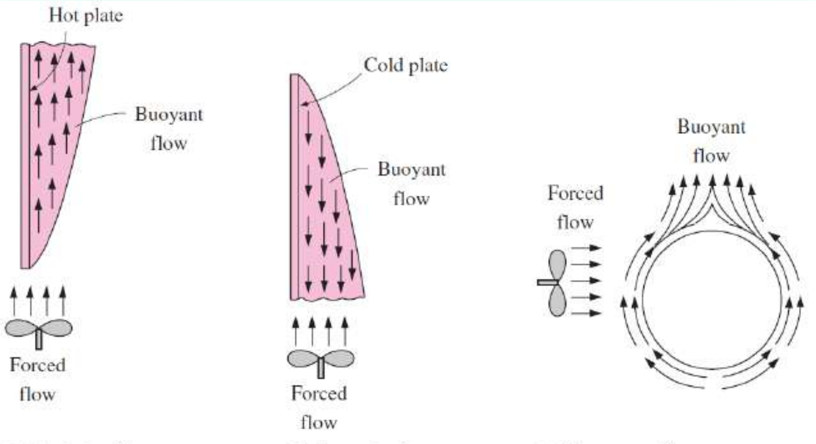
$$\mathsf{Ra}_L = \mathsf{Gr}_L\,\mathsf{Pr} = \frac{g\beta(T_s - T_\infty)L^3}{\nu\alpha}$$

Properties of the fluid are calculated at the mean film temperature, $T_f \equiv (T_s + T_\infty)/2$.

Geometry	Characteristic length L_c	Range of Ra	Nu
Vertical plate 	L	10^4–10^9 10^9–10^{13} Entire range	$Nu = 0.59Ra_L^{1/4}$ $Nu = 0.1Ra_L^{1/3}$ $Nu = \left\{0.825 + \frac{0.387Ra_L^{1/6}}{[1 + (0.492/Pr)^{9/16}]^{8/27}}\right\}^2$ (complex but more accurate)
Inclined plate 	L		Use vertical plate equations for the upper surface of a cold plate and the lower surface of a hot plate Replace g by $g\cos\theta$ for $Ra < 10^9$
Horizontal plate (Surface area A and perimeter p) (*a*) Upper surface of a hot plate (or lower surface of a cold plate)  (*b*) Lower surface of a hot plate (or upper surface of a cold plate) 	A_s/p	10^4–10^7 10^7–10^{11} 10^5–10^{11}	$Nu = 0.54Ra_L^{1/4}$ $Nu = 0.15Ra_L^{1/3}$ $Nu = 0.27Ra_L^{1/4}$

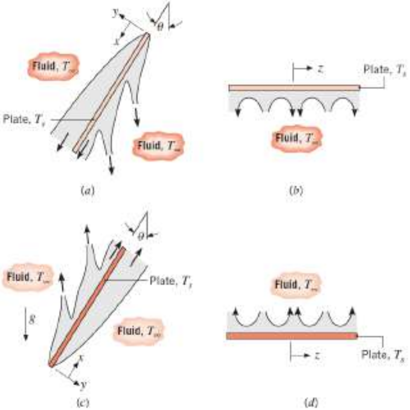
Geometry	Characteristic length L_c	Range of Ra	Nu
Vertical cylinder 	L		A vertical cylinder can be treated as a vertical plate when $D \geq \frac{35L}{Gr_L^{1/4}}$
Horizontal cylinder 	D	$Ra_D \leq 10^{12}$	$Nu = \left\{0.6 + \frac{0.387 Ra_D^{1/6}}{[1 + (0.559/Pr)^{9/16}]^{8/27}}\right\}^2$
Sphere 	D	$Ra_D \leq 10^{11}$ ($Pr \geq 0.7$)	$Nu = 2 + \frac{0.589 Ra_D^{1/4}}{[1 + (0.469/Pr)^{9/16}]^{4/9}}$

Combined Free and Forced Convection

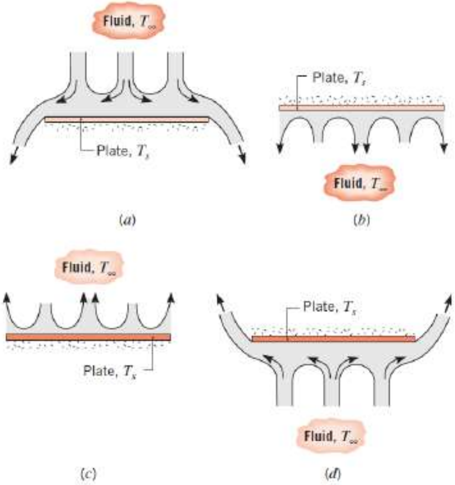


(a) Assisting flow (b) Opposing flow (c) Transverse flow

$$\mathrm{Nu}^n_{\text{combined}} = (\mathrm{Nu}^n_{\text{Forced}} \pm \mathrm{Nu}^n_{\text{Natural}})$$



Buoyancy-driven flows on an inclined plate: (*a*) Side view of flows at top and bottom surfaces of a cold plate ($T_s < T_\infty$). (*b*) End view of flow at bottom surface of cold plate. (*c*) Side view of flows at top and bottom surfaces of a hot plate ($T_s > T_\infty$). (*d*) End view of flow at top surface of hot plate.



Buoyancy-driven flows on horizontal cold ($T_s < T_\infty$) and hot ($T_s > T_\infty$) plates: (*a*) Top surface of cold plate. (*b*) Bottom surface of cold plate. (*c*) Top surface of hot plate. (*d*) Bottom surface of hot plate.

Heat and Mass Transfer

Boiling and Condensation

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Physical Considerations

Free and Forced convection depends on

$\rho, C_p, \mu, k_{fluid}$

Boiling/Condensation Heat Transfer depends on

- $\rho, C_p, \mu, k_{fluid}$
- $\Delta T = |T_s - T_{sat}|$
- Latent heat of vaporization, h_{fg}
- Surface tension at the liquid-vapor interface, σ
- body force arising from the liquid-vapor density difference, $g(\rho_l - \rho_v)$

$$h = h[\Delta T, g(\rho_l - \rho_v), h_{fg}, \sigma, L, \rho, C_p, k, \mu]$$

10 variables in 5 dimensions $\implies 5$ pi-groups.

Dimensionless Parameters

$$\frac{hL}{k} = f\left[\frac{\rho g(\rho_l - \rho_v)L^3}{\mu^2}, \frac{C_p \Delta T}{h_{fg}}, \frac{\mu C_p}{k}, \frac{g(\rho_l - \rho_v)L^2}{\sigma}\right]$$

$$\mathrm{Nu}_L = f\left[\frac{\rho g(\rho_l - \rho_v)L^3}{\mu^2}, \mathrm{Ja}, \mathrm{Pr}, \mathrm{Bo}\right]$$

Jakob number

Ratio of max sensible energy absorbed by liquid (vapor) to latent energy absorbed by liquid (vapor) during condensation (boiling).

Bond number

Ratio of the buoyancy force to the surface tension force.

Unnamed parameter

Represents the effect of buoyancy-induced fluid motion on heat transfer.

Boiling and Evaporation

Boiling

- The process of addition of heat to a liquid such a way that generation of vapor occurs.
- Solid-liquid interface
- Characterized by the rapid formation of vapor bubbles

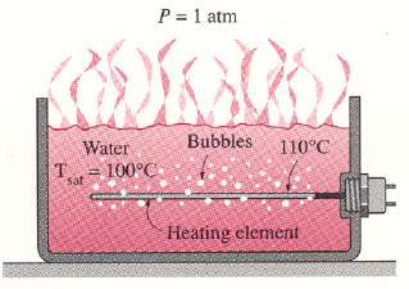
Evaporation

- Liquid-vapor interface
- $P_v < P_{sat}$ of the liquid at a given temp
- No bubble formation or bubble motion

Boiling

Boiling occurs

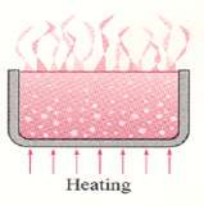
- Solid-liquid interface
- when a liquid is brought into contact with a surface at a temperature above the saturation temperature of the liquid

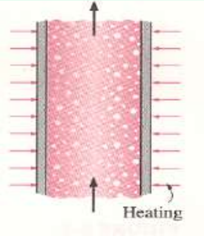


Bubble

- The boiling processes in practice do not occur under equilibrium conditions.
- Bubbles exist because of the surface tension at the liquid vapor interface due to the attraction force on molecules at the interface toward the liquid phase.
- The temperature and pressure of the vapor in a bubble are usually different than those of the liquid.
- Surface tension $\downarrow \quad \uparrow$ Temperature
- Surface tension $= 0$ at critical temperature
- No bubbles at supercritical pressures and temperatures

Boiling - Classification





Pool boiling

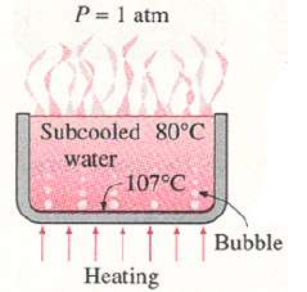
- Fluid is stationary
- Fluid motion is due to natural convection currents
- Motion of bubbles under the influence of buoyancy

Flow boiling

- Fluid is forced to move in a heated pipe or surface by external means such as pump
- Always accompanied by other convection effects

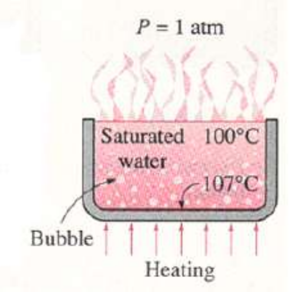
Boiling - Classification

Subcooled boiling



$$T_{\text{bulk of liquid}} < T_{sat}$$

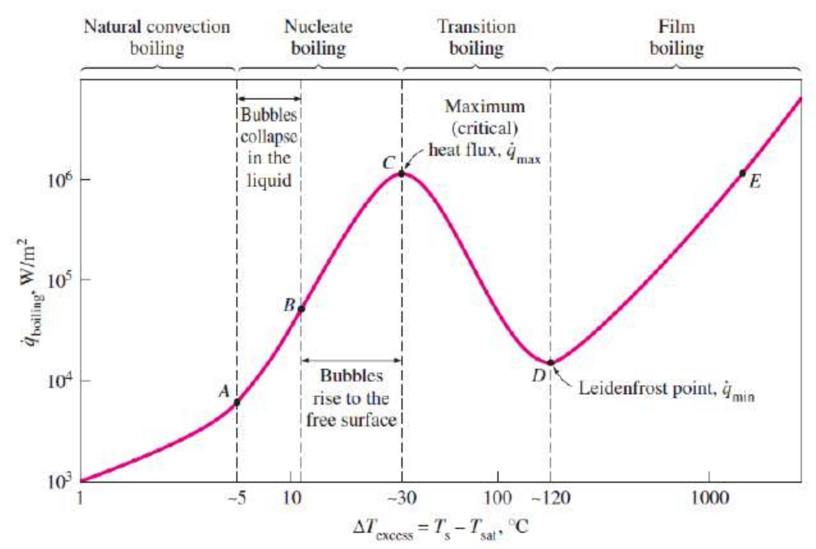
Saturated boiling



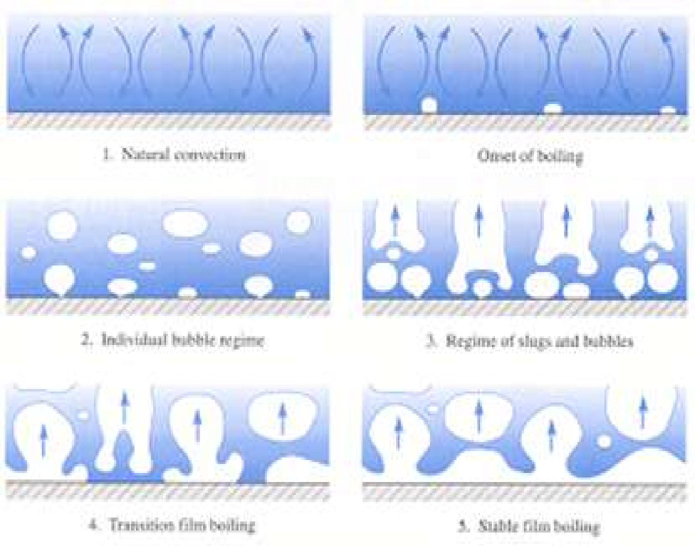
$$T_{\text{bulk of liquid}} = T_{sat}$$

Boiling Regimes - Nukiyama, 1934

Boiling curve for saturated water at atmospheric pressure



Boiling Regimes - Nukiyama, 1934

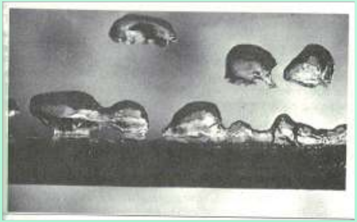


Boiling Regimes

Methanol on horizontal 1 cm steam-heated copper tube

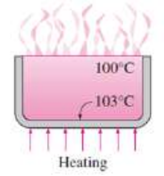
Nucleate boiling

Transition boiling



Film boiling

Boiling Regimes

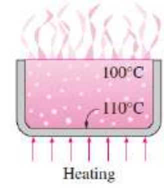


Natural convection

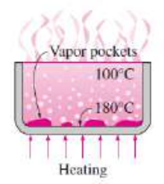
- Governed by natural convection currents
- Heat transfer from the heating surface to the fluid is by natural convection

Nucleate boiling

- Stirring and agitation caused by the entrainment of the liquid to the heater surface increases h, q''
- High heat transfer rates are achieved



Boiling Regimes



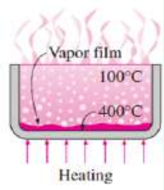
Transition boiling

Unstable film boiling

- Governed by natural convection currents
- Heat transfer from the heating surface to the fluid is by natural convection

Film boiling

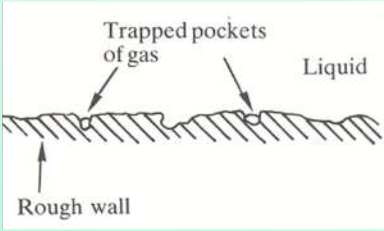
- Presence of vapor film is responsible for the low heat transfer rates
- Heat transfer rate increases with increasing ΔT_e as a result of heat transfer from the heated surface to the liquid through the vapor film by radiation.



Boiling Nucleation - Boiling Inception

The process of bubble formation is called Nucleation

Enlarged view of a boiling surface



- The cracks and crevices do not constitute nucleation sites for the bubbles. Must contain pockets of gas/air trapped
- It is from these pockets of trapped air that the vapor bubbles begin to grow during nucleate boiling
- These cavities are the sites at which bubble nucleation occurs

Heat Transfer in Nucleate Boiling

Rohsenow postulated:

- Heat flows from the surface first to the adjacent liquid, as in any single-phase convection process
- High h is a result of local agitation due to liquid flowing behind the wake of departing bubbles

$$q_s'' = \mu_l h_{fg} \left[\frac{g(\rho_l - \rho_v)}{\sigma} \right]^{1/2} \left(\frac{C_{p,l} \Delta T_e}{C_{s,f} h_{fg} \, \mathrm{Pr}_l^n} \right)^3$$

Nucleate boiling

When used to estimate q'', errors can amount to $\pm 100\%$. The errors for estimating ΔT_e reduce by a factor of 3 $\because \Delta T_e \propto (q_s'')^{1/3}$

Heat Transfer in Nucleate Boiling

Surface–Fluid Combination	$C_{s,f}$	n
Water–copper		
Scored	0.0068	1.0
Polished	0.0128	1.0
Water–stainless steel		
Chemically etched	0.0133	1.0
Mechanically polished	0.0132	1.0
Ground and polished	0.0080	1.0
Water–brass	0.0060	1.0
Water–nickel	0.006	1.0
Water–platinum	0.0130	1.0
n-Pentane–copper		
Polished	0.0154	1.7
Lapped	0.0049	1.7
Benzene–chromium	0.0101	1.7
Ethyl alcohol–chromium	0.0027	1.7

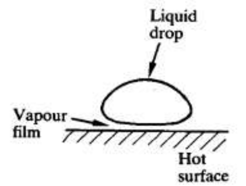
Critical Heat Flux

Zuber's correlations for flat horizontal plate:

$$q''_{\mathsf{max}} = 0.149 h_{fg} \rho_v \left[\frac{g(\rho_l - \rho_v)}{\sigma} \right]^{1/4}$$

Critical heat flux

Rewetting of hot surfaces: Liquid does not we hot surface.



$$q''_{\mathsf{min}} = C h_{fg} \rho_v \left[\frac{g\sigma(\rho_l - \rho_v)}{(\rho_l + \rho_v)^2} \right]^{1/4}$$

C is a non-dimensional constant which lies between 0.09 and 0.18.

$C = 0.09$ provides a better fit.

$C = 0.13$ is sometimes taken as an intermediate value

Prof. Dr. Johann G. Leidenfrost (1715-1794)

- Father - Minister
- Started off with Theological studies
- PhD thesis, ''On the Harmonious Relationship of Movements in the Human Body"
- Professor at University of Duisburg
- Areas of influences:
  - Theologian
  - Physician (Private Medical practice)
  - As a Prof. taught:
    Medicine, Physics, and Chemistry

Film Boiling

$$\overline{Nu}_D = \frac{\bar{h}_{\text{conv}} D}{k_v} = C \left[\frac{g(\rho_l - \rho_v) h'_{fg} D^3}{\nu_v k_v (T_s - T_{sat})} \right]^{1/4}$$

$C = 0.62$ for horizontal cylinders

$C = 0.67$ for spheres

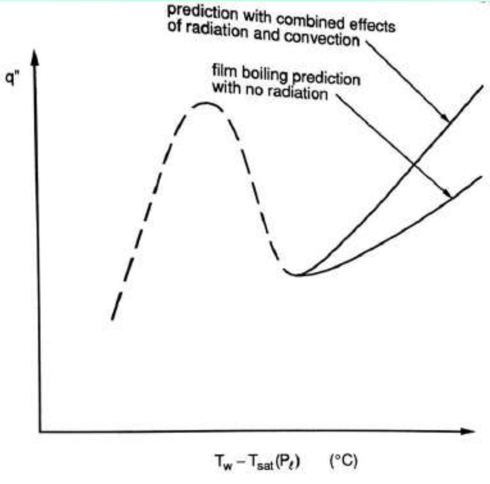
The effective latent heat of vaporization allows for the inclusion of sensible heating effects in the vapor film.

$$h'_{fg} = h_{fg} + 0.5 C_{p,v} (T_s - T_{sat})$$

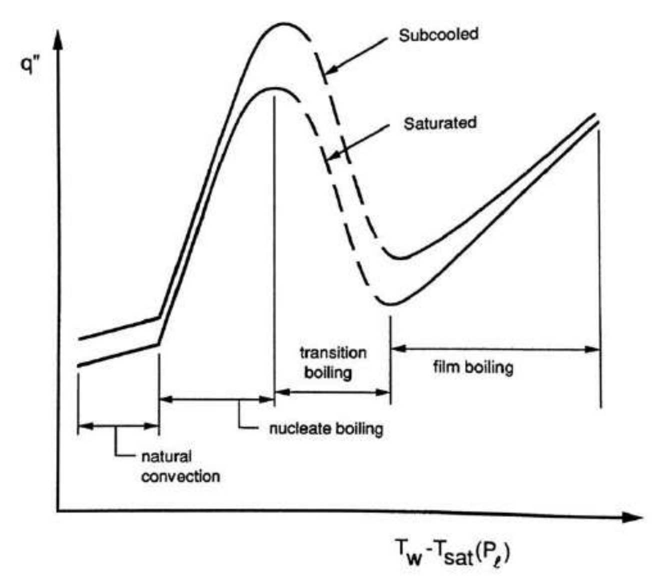
Vapor properties are evaluated at the film temperature, $T_f = (T_s + T_{sat})/2$.

Radiation Effects in Film Boiling

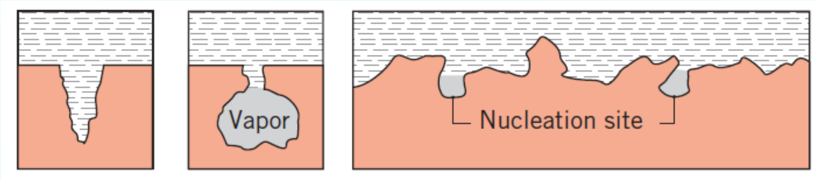
$$h_{\text{total}} = h_{\text{film conv}} + \frac{3}{4} h_{\text{rad}} \qquad h_{\text{rad}} = \frac{\varepsilon_s \sigma (T_s^4 - T_{sat}^4)}{T_s - T_{sat}}$$



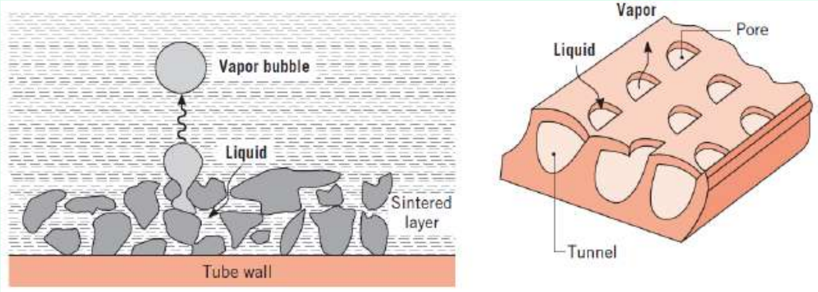
Effect of Liquid Subcooling



Enhancement of Heat Transfer



- Roughening or structuring or coating of the heating surface
- Production of artificial nucleation sites by sintering and
- Addition of gases or liquids or solids



Problem

The bottom of a copper pan, 0.3 m in diameter, is maintained at 118°C by an electric heater. Estimate the power required to boil water in this pan. What is the evaporation rate? Estimate the critical heat flux.

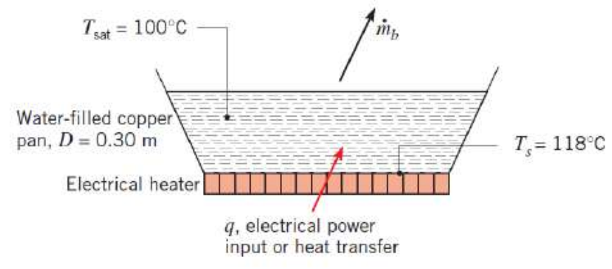
Saturated water, liquid at 100°C:
$\rho_l = 1/v_f = 957.9$ kg/m^3, $C_{p,l} = C_{p,g} = 4.217$ kJ/kg K,
$\mu_l = \mu_f = 279 \times 10^{-6}$ N s/m^2, Pr$_l$ = Pr$_f = 1.76$,
$h_{fg} = 2257$ kJ/kg, $\sigma = 58.9 \times 10^{-3}$

Saturated water, vapor at 100°C:
$\rho_v = 1/v_g = 0.5955$ kg/m^3

Solution

Saturated water, liquid at $100°$C:
$\rho_l = 1/v_f = 957.9$ kg/m^3, $C_{p,l} = C_{p,g} = 4.217$ kJ/kg K,
$\mu_l = \mu_f = 279 \times 10^{-6}$ N s/m^2, $\text{Pr}_l = \text{Pr}_f = 1.76$,
$h_{fg} = 2257$ kJ/kg, $\sigma = 58.9 \times 10^{-3}$

Saturated water, vapor at $100°$C:
$\rho_v = 1/v_g = 0.5955$ kg/m^3



Solution

Saturated water, liquid at 100°C:
$\rho_l = 1/v_f = 957.9$ kg/m^3, $C_{p,l} = C_{p,g} = 4.217$ kJ/kg K,
$\mu_l = \mu_f = 279 \times 10^{-6}$ N s/m^2, $\mathrm{Pr}_l = \mathrm{Pr}_f = 1.76$,
$h_{fg} = 2257$ kJ/kg, $\sigma = 58.9 \times 10^{-3}$

Saturated water, vapor at 100°C:
$\rho_v = 1/v_g = 0.5955$ kg/m^3

$$q_s'' = \mu_l h_{fg} \left[\frac{g(\rho_l - \rho_v)}{\sigma}\right]^{1/2} \left(\frac{C_{p,l}\Delta T_e}{C_{s,f} h_{fg} \mathrm{Pr}_l^n}\right)^3$$

$$q_s'' = 55.8 \text{ kW}$$

$$\dot{m}_{\text{evap}} = \frac{q_s}{h_{fg}} 89 \text{ kg/h}$$

Solution

Saturated water, liquid at $100°$C:
$\rho_l = 1/v_f = 957.9$ kg/m^3, $C_{p,l} = C_{p,g} = 4.217$ kJ/kg K,
$\mu_l = \mu_f = 279 \times 10^{-6}$ N s/m^2, $\text{Pr}_l = \text{Pr}_f = 1.76$,
$h_{fg} = 2257$ kJ/kg, $\sigma = 58.9 \times 10^{-3}$

Saturated water, vapor at $100°$C:
$\rho_v = 1/v_g = 0.5955$ kg/m^3

$$q''_{\text{max}} = 0.149 h_{fg} \rho_v \left[\frac{g(\rho_l - \rho_v)}{\sigma} \right]^{1/4} = 1.26 \text{ MW/m}^2$$

$$q''_{\text{min}} = C h_{fg} \rho_v \left[\frac{g\sigma(\rho_l - \rho_v)}{(\rho_l + \rho_v)^2} \right]^{1/4}$$

Condensation

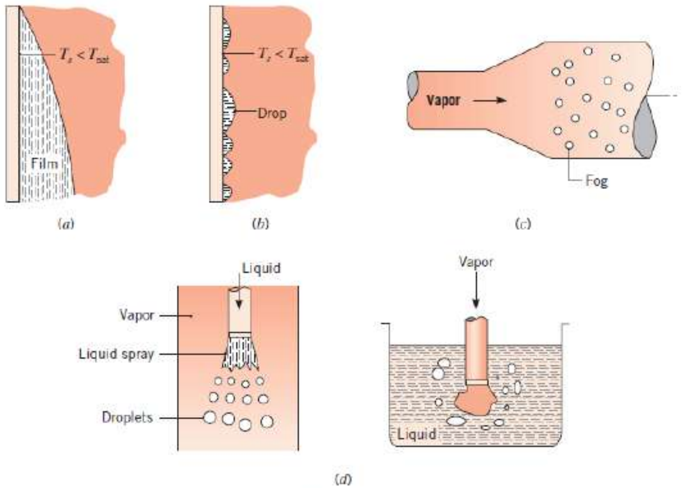
- Condensation is a process in which the removal of heat from a system causes a vapor to convert into liquid.
- Important role in nature:
  - Crucial component of the water cycle
  - Industry
- The spectrum of flow processes associated with condensation on a solid surface is almost a mirror image of those involved in boiling.
- Can also occur on a free surface of a liquid or even in a gas
- Condensation processes are numerous, taking place in a multitude of situations.

Condensation: Classification

1. Mode of condensation: homogeneous, dropwise, film or direct contact.
2. Conditions of the vapor: single-component, multicomponent with all components condensable, multicomponent including non-condensable component(s), etc.
3. System geometry: plane surface, external, internal, etc.

There are overlaps among different classification methods.Classification based on mode of condensation is the most useful.

Condensation: Classification



Modes of condensation. (*a*) Film. (*b*) Dropwise condensation on a surface. (*c*) Homogeneous condensation or fog formation resulting from increased pressure due to expansion. (*d*) Direct contact condensation.

Homogeneous Condensation

- Can happen when vapor is sufficiently cooled $< T_{sat}$ to induce droplet nucleation.
- It may be caused by:
  - Mixing of two vapor streams at different temperatures
  - Radiative cooling of vapor-noncondensable mixtures
    - Fog formation
  - Sudden depressurization of a vapor
    - Cloud formation - adiabatic expansion of warm, humid air masses that rise and cool
    - Cloud - water or ice? -30°C
- Although homogeneous nucleation in pure vapors is possible, in practice dust, other particles act as droplet nucleation embryos

Heterogeneous Condensation

- Droplets form and grow on solid surfaces
- Significant sub-cooling of vapor is required for condensation to start when the surface is smooth and dry.
- The rate of generation of embryo droplets in heterogeneous condensation can be modeled by using kinetic theory
- Heterogeneous condensation leads to:
    - film condensation
    - dropwise condensation

Film vs Dropwise

Film condensation

- The surface is blanketed by a liquid film of increasing thickness.
- ''Liquid wall'' offers resistance.
- Characteristic of clean, uncontaminated surfaces

Dropwise condensation

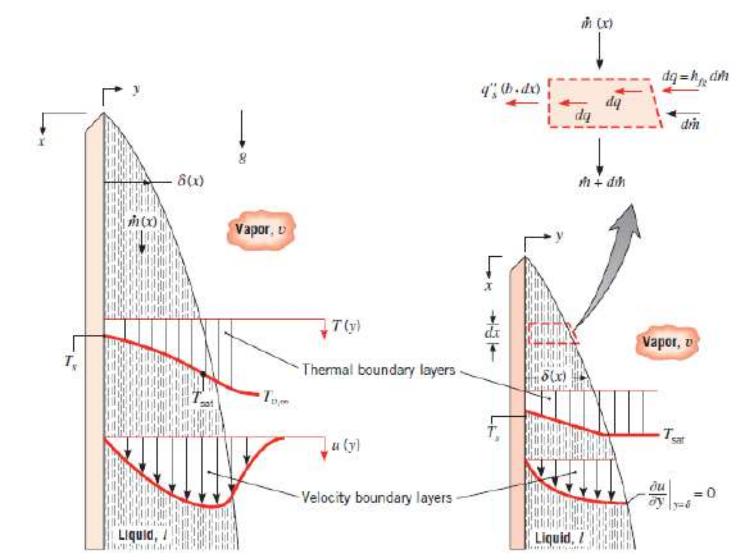
- Surface is coated with a substance that inhibits wetting
- Drops form in cracks, pits, and cavities.
- Typically, $> 90\%$ of the surface is drops.
- Droplets slide down at a certain size, clearing & exposing surface.
- No resistance to heat transfer in dropwise. h is 10 times higher than in film.

Dropwise Condensation

- Providing and maintaining the non-wetting surface characteristics can be difficult.
- The condensate liquid often gradually removes the promoters.
- Furthermore, the accumulation of droplets on a surface can eventually lead to the formation of a liquid film.

It has been postulated that heat transfer occurs at the smaller droplets due to higher thermal resistance in larger drops.

$$\begin{aligned} \bar{h}_{dc} &= 51,104 + 2044 T_{sat} && 22^\circ\text{C} < T_{sat} < 100^\circ\text{C} \\ &= 255,510 && 100^\circ\text{C} < T_{sat} \end{aligned}$$



$\dot{m}(x)$
$q''_s(b \cdot dx)$
$dq = h_{fg}\, d\dot{m}$
dq
dq
$d\dot{m}$
$\dot{m} + d\dot{m}$
y
x
g
$\delta(x)$
$\dot{m}(x)$
Vapor, v
$T(y)$
T_s
Thermal boundary layers
T_{sat}
$T_{v,\infty}$
$u(y)$
Velocity boundary layers
Liquid, l
dx
Vapor, v
$\delta(x)$
T_s
T_{sat}
$\frac{\partial u}{\partial y}\Big|_{y=\delta} = 0$
Liquid, l

Nusselt Integral Analysis

Assumptions:

1. Laminar flow and constant properties
2. Gas is assumed to be pure vapor and at an uniform T_{sat}.
    - With no temperature gradient in the vapor,
    - Heat transfer to the liquid-vapor interface can occur only by condensation at the interface and not by conduction from the vapor
3. Shear stress at the liquid-vapor interface is negligible $\partial u/\partial y|_{y=\delta} = 0$
4. Momentum and energy transfer by advection in the condensate film are assumed to be negligible.

Nusselt Integral Analysis

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu_l}\frac{dp}{dx} - \frac{X}{\mu_l}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{g}{\mu_l}(\rho_l - \rho_v)$$

$$u(0) = 0, \partial u/\partial y|_{y=\delta} = 0$$

$$u(y) = \frac{g(\rho_l - \rho_v)\delta^2}{\mu_l}\left[\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^2\right]$$

Mass flow rate per unit width,

$$\Gamma(x) = \frac{\dot{m}(x)}{b} = \int_0^{\delta(x)} \rho_l u(y) dy = \frac{g\rho_l(\rho_l - \rho_v)\delta^3}{3\mu_l}$$

Nusselt Integral Analysis

$$dq = h_{fg} d\dot{m}$$

$$dq = q_s''(bdx)$$

$$q_s'' = \frac{k_l(T_{sat} - T_s)}{\delta}$$

$$\frac{d\Gamma}{dx} = \frac{k_l(T_{sat} - T_s)}{h_{fg}}$$

$$\delta(x) = \left[\frac{4k_l\mu_l(T_{sat} - T_S)x}{g\rho_l(\rho_l - \rho_g)h_{fg}}\right]^{1/4}$$

Nusselt Integral Analysis

$$\delta(x) = \left[\frac{4k_l\mu_l(T_{sat} - T_S)x}{g\rho_l(\rho_l - \rho_g)h_{fg}}\right]^{1/4}$$

The thermal advection effects may be accounted by:

$$h'_{fg} = h_{fg}(1 + 0.68\text{Ja}) \qquad \text{Ja} = \frac{C_p\Delta T}{h_{fg}}$$

$$q''_x = h_x(T_{sat} - T_s)$$

$$h_x = \frac{k_l}{\delta} = \left[\frac{g\rho_l(\rho_l - rho_v)k_l^3 h'_{fg}}{4\mu_l(T_{sat} - T_s)x}\right]^{1/4}$$

$\because h_x \propto x^{-1/4}$

$$\bar{h}_L = \frac{1}{L}\int_0^L h_x dx = \frac{4}{3}h_L = 0.943\left[\frac{g\rho_l(\rho_l - rho_v)k_l^3 h'_{fg}}{4\mu_l(T_{sat} - T_s)L}\right]^{1/4}$$

Nusselt Integral Analysis

$$\overline{\mathsf{Nu}}_L = \frac{\bar{h}_L L}{k_l} = \left[\frac{g\rho_l(\rho_l - rho_v)L^3 h'_{fg}}{4\mu_l k_l (T_{sat} - T_s)}\right]^{1/4}$$

The total heat transfer to the surface may be obtaned by:

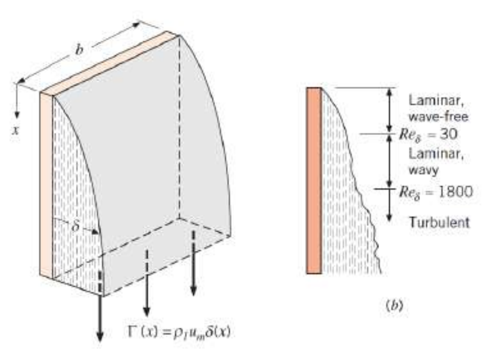
$$q = h_L A(T_{sat} - T_s)$$

$$\dot{m} = \frac{q}{h'_{fg}} = \frac{\bar{h}_L A(T_{sat} - T_s)}{h'_{fg}}$$

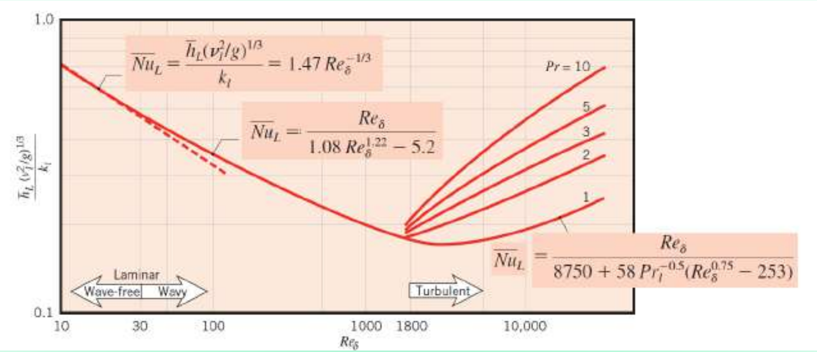
$$\text{Re}_\delta \equiv \frac{4\Gamma}{\mu_l}$$

Condensate mass flow rate, $\dot{m} = \rho_l u_m b\delta$,

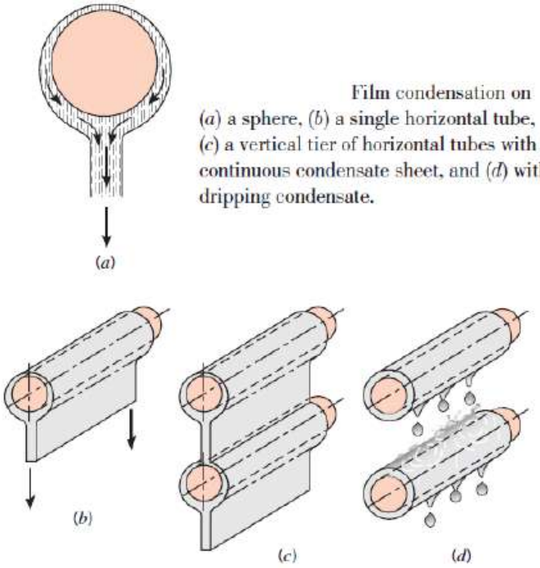
$$\text{Re}_\delta \equiv \frac{4\dot{m}}{\mu_l b} = \frac{4\rho_l u_m \delta}{\mu_l}$$



Turbulent Film Condensation



Film Condensation on Radial Systems



Film condensation on (a) a sphere, (b) a single horizontal tube, (c) a vertical tier of horizontal tubes with a continuous condensate sheet, and (d) with dripping condensate.

Film Condensation on Radial Systems

$$\overline{\mathrm{Nu}}_D = \frac{\bar{h}_D D}{k_l} = C \left[\frac{\rho_l g(\rho_l - \rho_v) h'_{fg} D^3}{\mu_l k_l (T_{sat} - T_s} \right]^{1/4}$$

$C = 0.826$ for sphere and 0.729 for the tube.

For N horizontal unfineed tubes, the average coeff.:

$$\bar{h}_{D,N} = \bar{h}_D N^n$$

$$n = -\frac{1}{4}$$