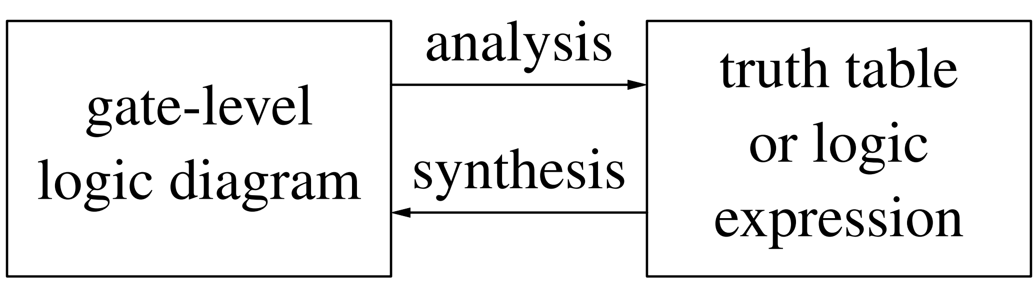
Chapter 3. Switching Algebra and Combinational Logic (Part 1)

Logic expressions describe functions that need to be implemented by digital circuits.

For a *combinational* circuit, the outputs depend only on the current inputs.
There are no feedback loops that create memory.



We will first consider single-output combinational circuits.

Switching algebra is a mathematical tool for analysis and synthesis of logic circuits.
It is a special case of Boolean algebra. We will define switching algebra directly.

A symbolic variable, such as X, is used to represent the condition of a logic signal.
X can be 0 or 1.
For example, 0 can correspond to LOW voltage and 1 to HIGH voltage.

Axioms

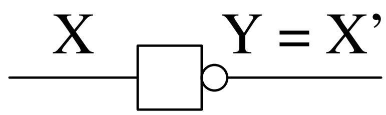
The axioms of a mathematical system are a minimal set of definitions that are assumed to be true.
From the axioms it is possible to derive all the other properties of the system.

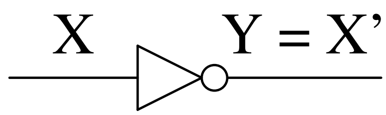
The following axioms state that a variable can only assume one of two values.
(A1) $X = 0$ if $X \neq 1$
(A1D) $X = 1$ if $X \neq 0$

The difference between A1 and A1D is that the symbols 0 and 1 are interchanged.
This is referred to as duality.
Most properties of swithcing algebra have duals.

The function of an inverter is given by the following axioms.
(A2) If $X = 0$ then $X' = 1$
(A2D) If $X = 1$ then $X' = 0$

For an inverter with input X and output Y we will write $Y = X'$.





The following axioms state the definitions of the AND and OR operations by listing the output produced by each operation for every possible input combination.

(A3) $0 \cdot 0 = 0$

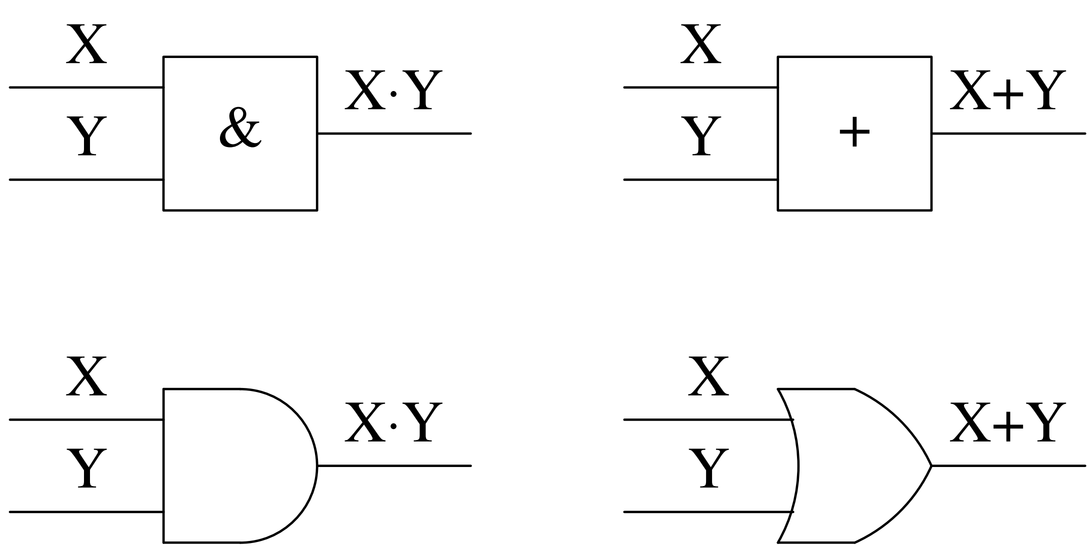
(A4) $1 \cdot 1 = 1$

(A5) $0 \cdot 1 = 1 \cdot 0 = 0$

(A3D) $1 + 1 = 1$

(A4D) $0 + 0 = 0$

(A5D) $1 + 0 = 0 + 1 = 1$



AND has a higher precedence than OR in logical expressions.
$W \cdot Y + Y \cdot Z = (W \cdot Y) + (Y \cdot Z)$.

All the other properties of switching algebra can be derived from the axioms A1-A5 and A1D-A5D.

Properties are stated as theorems that need to be proved.

Once they are proved, they can be used for proving other theorems.

The properties are used for simplifying logic expressions, and digital circuits implementing them.

Single-Variable Theorems

(T1) $X + 0 = X$
(T2) $X + 1 = 1$
(T3) $X + X = X$
(T4) $(X')' = X$
(T5) $X + X' = 1$

(T1D) $X \cdot 1 = X$
(T2D) $X \cdot 0 = 0$
(T3D) $X \cdot X = X$
(T5D) $X \cdot X' = 0$

These theorems can be proved using *perfect induction*.

Since a variable X can only take two values, 0 and 1 (based on A1 and A1D), we will prove a theorem if we can show that it is correct for both $X = 0$ and $X = 1$.

Proof of (T2) $X + 1 = 1$:

If $X = 0$, then $X + 1 = 0 + 1 \underset{A5D}{=} 1$

If $X = 1$, then $X + 1 = 1 + 1 \underset{A3D}{=} 1$

Proof of (T3) X + X = X:

If $X = 0$, then $X + X = 0 + 0 \underset{A4D}{=} 0 = X$

If $X = 1$, then $X + X = 1 + 1 \underset{A3D}{=} 1 = X$

Two- and Three-Variable Theorems

Commutativity:
(T6) $X + Y = Y + X$
(T6D) $X \cdot Y = Y \cdot X$

Associativity:
(T7) $(X + Y) + Z = X + (Y + Z)$
(T7D) $(X \cdot Y) \cdot Z = X \cdot (Y \cdot Z)$

This allows us to write expressions such as $X + Y + Z$ without ambiguity.
Considering circuits, it allows us to form multi-input gates from two-input gates.

Distributivity:
(T8) $X \cdot Y + X \cdot Z = X \cdot (Y + Z)$
(T8D) $(X + Y) \cdot (X + Z) = X + Y \cdot Z$

Covering:
(T9) $X + X \cdot Y = X$
(T9D) $X \cdot (X + Y) = X$

Combining:
(T10) $X \cdot Y + X \cdot Y' = X$
(T10D) $(X + Y) \cdot (X + Y') = X$

Consensus:

(T11) $X \cdot Y + X' \cdot Z + Y \cdot Z = X \cdot Y + X' \cdot Z$

(T11D) $(X + Y) \cdot (X' + Z) \cdot (Y + Z) =$
$(X + Y) \cdot (X' + Z)$

This can be used to simplify expressions or circuits.
It can also be used to add gates that will prevent hazards.

These theorems can also be proved by perfect induction. In this case, we need to consider four (eight) combinations of two (three) variables.

Proof of (T8) $X \cdot Y + X \cdot Z = X \cdot (Y + Z)$:

X	Y	Z	$X \cdot Y$	$X \cdot Z$	$X \cdot Y + X \cdot Z$
0	0	0	0	0	0
0	0	1	0	0	0
0	1	0	0	0	0
0	1	1	0	0	0
1	0	0	0	0	0
1	0	1	0	1	1
1	1	0	1	0	1
1	1	1	1	1	1

X	Y	Z	$Y + Z$	$X \cdot (Y + Z)$
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	1	0
1	0	0	0	0
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

It is also possible to prove some of the theorems using other theorems.

Proof of (T9) $X + X \cdot Y = X$:
We will use:
(T1D) $X \cdot 1 = X$
(T2) $X + 1 = 1$
(T8) $X \cdot Y + X \cdot Z = X \cdot (Y + Z)$

$$
\begin{aligned}
X + X \cdot Y &\underset{T1D}{=} X \cdot 1 + X \cdot Y \\
&\underset{T8}{=} X \cdot (1 + Y) \\
&\underset{T2}{=} X \cdot 1 \\
&\underset{T1D}{=} X
\end{aligned}
$$

Proof of (T10) $X \cdot Y + X \cdot Y' = X$:

We will use:

(T8) $X \cdot Y + X \cdot Z = X \cdot (Y + Z)$

(T5) $X + X' = 1$

(T1D) $X \cdot 1 = X$

$$\begin{aligned} X \cdot Y + X \cdot Y' &\underset{T8}{=} X \cdot (Y + Y') \\ &\underset{T5}{=} X \cdot 1 \\ &\underset{T1D}{=} X \end{aligned}$$

Proof of (T10D) $(X + Y) \cdot (X + Y') = X$:

We will use:

(T8D) $(X + Y) \cdot (X + Z) = X + Y \cdot Z$

(T5D) $X \cdot X' = 0$

(T1) $X + 0 = X$

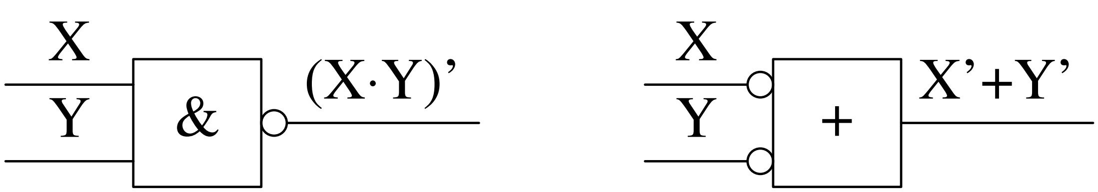
$$
\begin{aligned}
(X + Y) \cdot (X + Y') &\underset{T8D}{=} X + Y \cdot Y' \\
&\underset{T5}{=} X + 0 \\
&\underset{T1}{=} X
\end{aligned}
$$

n-Variable Theorems

(T12) $X + X + \cdots + X = X$
(T12D) $X \cdot X \cdot \cdots \cdot X = X$

DeMorgan's theorems
(T13) $(X_1 \cdot X_2 \cdot \cdots \cdot X_n)' = X_1' + X_2' + \cdots + X_n'$
(T13D) $(X_1 + X_2 + \cdots + X_n)' = X_1' \cdot X_2' \cdot \cdots \cdot X_n'$

T13 shows that a NAND gate is equivalent to an OR gate with complemented inputs.



Complementing both sides, an AND gate is equivalent to a NOR gate with complemented inputs.

T13D shows that a NOR gate is equivalent to an AND gate with complemented inputs. Complementing both sides, an OR gate is equivalent to a NAND gate with complemented inputs.

This is useful when simplifying circuits.

Generalized DeMorgan's theorem
(T14) $[F(X_1, X_2, \cdots, X_n, +, \cdot]' =$
$[F(X_1', X_2', \cdots, X_n', \cdot, +]$

This is useful when computing complemented expressions. The complementation can be removed by interchanging complemented and uncomplemented variables, and the operations $\cdot$ and $+$.

Example:

$F(W,X,Y,Z) = W' \cdot X + X \cdot Y + W \cdot (X' + Z')$

$F'(W,X,Y,Z) =$

$= (W' \cdot X + X \cdot Y + W \cdot (X' + Z'))'$

$= (W + X')(X' + Y')(W' + X \cdot Z)$

Shannon's expansion theorems

(T15) $F(X_1, X_2, \cdots, X_n) =$

$X_1 \cdot F(1, X_2, \cdots, X_n) +$

$X_1' \cdot F(0, X_2, \cdots, X_n)$

(T15D) $F(X_1, X_2, \cdots, X_n) =$

$[X_1 + F(0, X_2, \cdots, X_n)] \cdot$

$[X_1' + F(1, X_2, \cdots, X_n)].$

A theorem can be proved by finite induction:
1. Prove that the theorem is true for $n = 2$ (basis)
2. Prove that if the theorem is true for $n \le i$, then it is true for $n = i + 1$ (the induction step).

Proof of (T13) $(X_1 \cdot X_2 \cdot \cdots \cdot X_n)' = X_1' + X_2' + \cdots + X_n'$:

For $n = 2$ we need to prove that
$(X_1 \cdot X_2)' = X_1' + X_2'$.
This can be done by perfect induction.

X_1	X_2	$X_1 \cdot X_2$	$(X_1 \cdot X_2)'$
0	0	0	1
0	1	0	1
1	0	0	1
1	1	1	0

X_1	X_2	X_1'	X_2'	$X_1' + X_2'$
0	0	1	1	1
0	1	1	0	1
1	0	0	1	1
1	1	0	0	0

Assume that the theorem is true for $n \le i$.
$(X_1 \cdot X_2 \cdot \cdots \cdot X_{i+1})' =$
$([X_1 \cdot X_2 \cdot \cdots \cdot X_i] \cdot X_{i+1})' =$
(using the theorem with n=2)
$[X_1 \cdot X_2 \cdot \cdots \cdot X_i]' + X'_{i+1} =$
(using the theorem with n=i)
$X'_1 + X'_2 + \cdots + X'_i + X'_{i+1}.$

Duality

The axioms and theorems have two versions that differ in that 0 and 1 are interchanged, and · and + are interchanged.
The principle of duality states that:
A theorem or identity in switching algebra remains true if 0 and 1 are interchanged, and · and + are interchanged throughout.

The principle of duality is true because all the axioms have duals that are true.
A dual of a theorem can be proved by using the duals of the axioms used for proving the theorem.

Remember to use parentheses when replacing $\cdot$ with $+$.
This is important because $\cdot$ has precedence over $+$.

Example:
(T9) $X + X \cdot Y = X$
$X + (X \cdot Y) = X$
After applying the principle of duality:
$X \cdot (X + Y) = X$

Standard Representations of Logic Functions

We will start from the truth table representation of a logic function.
The truth table lists all the possible input combinations, and provides the value of the function for each one.
For a function of n variables, there are 2^n rows in the truth table.
For example:

	X	Y	Z	F
0	0	0	0	$F(0,0,0)$
1	0	0	1	$F(0,0,1)$
2	0	1	0	$F(0,1,0)$
3	0	1	1	$F(0,1,1)$
4	1	0	0	$F(1,0,0)$
5	1	0	1	$F(1,0,1)$
6	1	1	0	$F(1,1,0)$
7	1	1	1	$F(1,1,1)$

The number of logic functions of n variables is 2^{2^n} since there are 2^n rows in the truth table, and there are two options for the output value in each row.

For the analysis and synthesis of logic circuits it is important to be able to represent the information contained in a truth table algebraically.

Definitions:

A *literal* is a variable or the complement of a variable.
For example, X, Y, X', Y' are literals.

A *product term* is a single literal or a product of two or more literals.
For example, X', $X \cdot Y$, $X \cdot Y' \cdot Z'$.

A *sum-of-products expression* is a sum of product terms.
For example, $X' + X \cdot Y + X \cdot Y' \cdot Z'$.

A *sum term* is a single literal or a sum of two or more literals.
For example, X', $X + Y$, $X + Y' + Z'$.

A *product-of-sums expression* is a product of sum terms.
For example, $X' \cdot (X + Y) \cdot (X + Y' + Z')$.

A *normal term* is a product or sum term in which no variable appears more than once.
Non-normal terms can always be simplified.
Examples:
$X \cdot Y \cdot Y \cdot Z = X \cdot Y \cdot Z$.
$X + Y + Y' + Z = 1$.

An n-variable *minterm* is a normal product term with n literals.
There are 2^n such terms.
For example, with $n = 3$:
$X' \cdot Y' \cdot Z'$,
$X' \cdot Y' \cdot Z$,
$X' \cdot Y \cdot Z'$, and so on.

An n-variable *maxterm* is a normal sum term with n literals.
There are 2^n such terms.
For example, with $n = 3$:
$X' + Y' + Z'$,
$X' + Y' + Z$,
$X' + Y + Z'$, and so on.

A minterm is a product term that is 1 in exactly one row of the truth table.
A maxterm is a sum term that is 0 in exactly one row of the truth table.

For example:

	X	Y	Z	F	minterm	maxterm
0	0	0	0	$F(0,0,0)$	$X' \cdot Y' \cdot Z'$	$X + Y + Z$
1	0	0	1	$F(0,0,1)$	$X' \cdot Y' \cdot Z$	$X + Y + Z'$
2	0	1	0	$F(0,1,0)$	$X' \cdot Y \cdot Z'$	$X + Y' + Z$
3	0	1	1	$F(0,1,1)$	$X' \cdot Y \cdot Z$	$X + Y' + Z'$
4	1	0	0	$F(1,0,0)$	$X \cdot Y' \cdot Z'$	$X' + Y + Z$
5	1	0	1	$F(1,0,1)$	$X \cdot Y' \cdot Z$	$X' + Y + Z'$
6	1	1	0	$F(1,1,0)$	$X \cdot Y \cdot Z'$	$X' + Y' + Z$
7	1	1	1	$F(1,1,1)$	$X \cdot Y \cdot Z$	$X' + Y' + Z'$

	X	Y	Z	$X' \cdot Y' \cdot Z'$	$X' \cdot Y' \cdot Z$	$X' \cdot Y \cdot Z'$
0	0	0	0	1	0	0
1	0	0	1	0	1	0
2	0	1	0	0	0	1
3	0	1	1	0	0	0
4	1	0	0	0	0	0
5	1	0	1	0	0	0
6	1	1	0	0	0	0
7	1	1	1	0	0	0

	X	Y	Z	$X + Y + Z$	$X + Y + Z'$	$X + Y' + Z$
0	0	0	0	0	1	1
1	0	0	1	1	0	1
2	0	1	0	1	1	0
3	0	1	1	1	1	1
4	1	0	0	1	1	1
5	1	0	1	1	1	1
6	1	1	0	1	1	1
7	1	1	1	1	1	1

It is possible to create a representation of a logic function from its truth table using minterms or maxterms.

The *canonical sum* of a function is the sum of the minterms corresponding to the truth-table rows where the function is 1.

For example:

	X	Y	Z	F	minterm
0	0	0	0	1	$X' \cdot Y' \cdot Z'$
1	0	0	1	0	$X' \cdot Y' \cdot Z$
2	0	1	0	0	$X' \cdot Y \cdot Z'$
3	0	1	1	1	$X' \cdot Y \cdot Z$
4	1	0	0	1	$X \cdot Y' \cdot Z'$
5	1	0	1	1	$X \cdot Y' \cdot Z$
6	1	1	0	0	$X \cdot Y \cdot Z'$
7	1	1	1	1	$X \cdot Y \cdot Z$

$$F = X' \cdot Y' \cdot Z' + X' \cdot Y \cdot Z +$$
$$+X \cdot Y' \cdot Z' + X \cdot Y' \cdot Z + X \cdot Y \cdot Z.$$

	X	Y	Z	F	$X' \cdot Y' \cdot Z'$	$X' \cdot Y \cdot Z$	$\cdots$
0	0	0	0	1	1	0	
1	0	0	1	0	0	0	
2	0	1	0	0	0	0	
3	0	1	1	1	0	1	
4	1	0	0	1	0	0	
5	1	0	1	1	0	0	
6	1	1	0	0	0	0	
7	1	1	1	1	0	0	

	X	Y	Z	F	0	3	4	5	7
0	0	0	0	1	1	0	0	0	0
1	0	0	1	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0
3	0	1	1	1	0	1	0	0	0
4	1	0	0	1	0	0	1	0	0
5	1	0	1	1	0	0	0	1	0
6	1	1	0	0	0	0	0	0	0
7	1	1	1	1	0	0	0	0	1

$$F = X' \cdot Y' \cdot Z' + X' \cdot Y \cdot Z +$$
$$+X \cdot Y' \cdot Z' + X \cdot Y' \cdot Z + X \cdot Y \cdot Z.$$

The canonical sum can also be represented using *minterm numbers*.
Minterm i is the minterm corresponding to row i of the truth table.
$F = \sum_{X,Y,Z}(0, 3, 4, 5, 7)$

0,3,4,5,7 is the *minterm list* for F. It is also called the $on-set$ of F.

The *canonical product* of a function is the product of the maxterms corresponding to the truth-table rows where the function is 0.
For example:

	X	Y	Z	F	maxterm
0	0	0	0	1	$X+Y+Z$
1	0	0	1	0	$X+Y+Z'$
2	0	1	0	0	$X+Y'+Z$
3	0	1	1	1	$X+Y'+Z'$
4	1	0	0	1	$X'+Y+Z$
5	1	0	1	1	$X'+Y+Z'$
6	1	1	0	0	$X'+Y'+Z$
7	1	1	1	1	$X'+Y'+Z'$

$F = \Pi_{X,Y,Z}(1,2,6) =$
$(X+Y+Z')\cdot(X+Y'+Z)\cdot(X'+Y'+Z)$

We obtain an expression for the function:
In rows 1, 2 and 6, one of the maxterms is 0, and the product is 0.
If all the other rows, all the maxterms are 1, and the product is 1.

	X	Y	Z	F	$X+Y+Z'$	$X+Y'+Z$	$X'+Y'+Z$
0	0	0	0	1	1	1	1
1	0	0	1	0	0	1	1
2	0	1	0	0	1	0	1
3	0	1	1	1	1	1	1
4	1	0	0	1	1	1	1
5	1	0	1	1	1	1	1
6	1	1	0	0	1	1	0
7	1	1	1	1	1	1	1

$$F = \Pi_{X,Y,Z}(1,2,6) =$$
$$(X+Y+Z') \cdot (X+Y'+Z) \cdot (X'+Y'+Z)$$

The *maxterm list* 1,2,6 is also known as the *off – set* of the function.

The minterm list and maxterm list add up to the list of all 2^n rows of a truth table.

Examples:
$\Sigma_{X,Y}(3) = \Pi_{X,Y}(0, 1, 2)$

$\Sigma_{W,X,Y,Z}(0, 1, 3, 5, 7, 8, 11, 12, 14, 15) =$
$\Pi_{W,X,Y,Z}(2, 4, 6, 9, 10, 13)$

The function $F(X, Y, Z) =$
$= \Sigma_{X,Y,Z}(0, 3, 4, 6, 7) = \Pi_{X,Y,Z}(1, 2, 5)$ in Verilog:

```
case {(X,Y,Z)}
    0,3,4,6,7: F = 1;
    default: F = 0;
endcase
```

```
case {(X,Y,Z)}
    1,2,5: F = 0;
    default: F = 1;
endcase
```

Shannon's expansion theorem

(T15) $F(X_1, X_2, \cdots, X_n) =$

$X_1 \cdot F(1, X_2, \cdots, X_n) +$

$X_1' \cdot F(0, X_2, \cdots, X_n)$

is based on a partition of the truth table into two parts corresponding to $X_1 = 0$ and $X_1 = 1$.

X_1	$X_2 \cdots X_n$	F	
0	$\cdots$	0	
0	$\cdots$	1	$F(0, X_2, \cdots, X_n)$
1	$\cdots$	0	
1	$\cdots$	1	$F(1, X_2, \cdots, X_n)$

Example:

	X	Y	Z	F	minterm
0	0	0	0	1	$X' \cdot Y' \cdot Z'$
1	0	0	1	0	
2	0	1	0	0	
3	0	1	1	1	$X' \cdot Y \cdot Z$
4	1	0	0	1	$X \cdot Y' \cdot Z'$
5	1	0	1	1	$X \cdot Y' \cdot Z$
6	1	1	0	0	
7	1	1	1	1	$X \cdot Y \cdot Z$

$$F = X' \cdot (Y' \cdot Z' + Y \cdot Z) +$$
$$+X \cdot (Y' \cdot Z' + Y' \cdot Z + Y \cdot Z)$$

To obtain the dual form of the theorem, note that the minterms where

$$F(1, X_2, \cdots, X_n) = F(0, X_2, \cdots, X_n)$$

can be added to the expression for F without ANDing with X.

It is also possible to add $X_1 \cdot X_1'$ without changing F.

$$
\begin{aligned}
&F(X_1, X_2, \cdots, X_n) = \\
&\qquad X_1 \cdot F(1, X_2, \cdots, X_n) + \\
&\qquad X_1' \cdot F(0, X_2, \cdots, X_n) = \\
&\\
&\qquad X_1 \cdot F(1, X_2, \cdots, X_n) + \\
&\qquad X_1' \cdot F(0, X_2, \cdots, X_n) + X_1 \cdot X_1' \\
&\qquad F(1, X_2, \cdots, X_n) \cdot F(0, X_2, \cdots, X_n) =
\end{aligned}
$$

$$X_1 \cdot F(1, X_2, \cdots, X_n) +$$
$$X_1' \cdot F(0, X_2, \cdots, X_n) + X_1 \cdot X_1'$$
$$F(1, X_2, \cdots, X_n) \cdot F(0, X_2, \cdots, X_n) =$$

$$X_1 \cdot F(1, X_2, \cdots, X_n) + X_1 \cdot X_1' +$$
$$X_1' \cdot F(0, X_2, \cdots, X_n) +$$
$$F(1, X_2, \cdots, X_n) \cdot F(0, X_2, \cdots, X_n) =$$

$$X_1 \cdot [X_1' + F(1, X_2, \cdots, X_n)] +$$
$$F(0, X_2, \cdots, X_n) \cdot [X_1' + F(1, X_2, \cdots, X_n)] =$$

$$[X_1 + F(0, X_2, \cdots, X_n)] \cdot$$
$$[X_1' + F(1, X_2, \cdots, X_n)].$$