Derivation of Black Scholes Equation

The price of the stock is assumed to follow a Geometric Brownian Motion as follows

$$dS_t \ = \mu dS_t \ + \ \sigma dS_t dW_t$$

with constant μ and constant σ. The interest rate is also assumed to be constant. So, the differential of the bank account has no stochastic term:

$$dB_t \ = rB_t dt$$

The option price is a function of 5 parameters: V_t = V (T-t, S_t, r, σ , K) ; that is, time to maturity, stock price, interest rate, volatility, and strike price.

However, if the last three parameters are assumed to be constant, the value of the option can be approximated to be a function of the time to maturity and Stock price; V_t ~ V (T-t, S_t)

Derivation [i]

Ito's Lemma for $X_t \ = \ f(Y_t)$ is as follows

$$dX_t \ = f'Y_t \, dY_t \ + \frac{1}{2} f''(Y_t) \, dt$$

Applying Ito's lemma to our option price V, we have the following equation (as V is also a function of time, we are differentiating with respect to t as well).

$$dV \ = \ \frac{\partial V}{\partial t} dt \ + \ \frac{\partial V}{\partial S} dS \ + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} dS^2$$

Now, substituting dS and writing dS^2 in terms of dt, we get:

$$dV \ = \ \frac{\partial V}{\partial t} dt \ + \ \frac{\partial V}{\partial S} (\mu dS \ + \ \sigma \, dS \, dW) \ + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma^2 S^2 dt$$

Rearranging,

$$dV \ = \ (\frac{\partial V}{\partial t} \ + \ \mu S_t \frac{\partial V}{\partial S} \ + \frac{1}{2} \ \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}) \ dt \ + \ \sigma S \frac{\partial V}{\partial S} dW$$

The first part of the equation now is the deterministic part (yellow) and the second part is the stochastic part (blue).

Now, Black-Scholes deals with the stochastic part by delta hedging. Let's create a portfolio with Δ units of stock and α units of Cash (bank account). Δ and α can be either negative or positive.

$$\Pi \ = \Delta \mathrm{S} \ + \ \alpha B$$

The value of this portfolio will change as the stock price moves or as time changes.

$$\mathrm{d}\Pi \ = \Delta \mathrm{dS} \ + \ \alpha dB$$

Substituting dS and dB

$$\mathrm{d}\Pi \ = \Delta(\mu dS \ + \ \sigma \, dS \, dW) \ + \ \alpha \, r \, B \, dt$$

Rearranging,

$$d\Pi = \Delta(\mu S + \alpha\, r\, B)dt + \Delta\, \sigma\, S\, dW$$

So, by combining the Option and the delta hedging portfolio, we get

$$dV + d\Pi = \left(\frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial s} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \sigma S \frac{\partial V}{\partial s} dW + \Delta(\mu S + \alpha\, r\, B)dt + \Delta\, \sigma\, S\, dW$$

Now, we want to find delta Δ which makes the coefficient of the stochastic part = 0

$$\Delta\, \sigma\, S + \sigma S \frac{\partial V}{\partial s} = 0 \quad \Rightarrow \quad \Delta = -\frac{\partial V}{\partial s}$$

If we replace this $\Delta = -\frac{\partial V}{\partial s}$ in our combined portfolio, the stochastic terms become 0, and we are left with the drift terms

$$dV + d\Pi = \left(\frac{\partial V}{\partial t} + \cancel{\mu S \frac{\partial V}{\partial s}} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \cancel{\frac{\partial V}{\partial s}\mu S} + \alpha r B\right)dt$$

$$dV + d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \alpha r B\right)dt$$

Also replacing $\Delta = -\frac{\partial V}{\partial s}$ in our replicating portfolio $\Pi = \Delta S + \alpha B$,we get

$$\Pi = -\frac{\partial V}{\partial S} S + \alpha B$$

Since, we now only have deterministic parts in our equation, it must grow at the risk-free rate r to avoid any arbitrage

$$dV + d\Pi = (V + \Pi)\, r\, dt$$

Substituting for Π on the RHS,

$$dV + d\Pi = \left(V - \frac{\partial V}{\partial s} S + \alpha B\right) r\, dt$$

Substituting $dV + d\Pi$ on the LHS,

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \alpha r B\right)dt = \left(V - \frac{\partial V}{\partial s} S + \alpha B\right) r\, dt$$

Rearranging,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial t} - rV = 0$$

This is the Black Scholes PDE

Mathematical definition of greeks Δ, Γ, and Θ and their meaning

Delta Δ: Mathematically, delta is the first derivative of the option price with respect to the stock price [ii]

- Call $\Delta = e^{-r_f \tau} N(d_1)$
- Put $\Delta = -e^{-r_f \tau} N(-d_1)$

Delta is the change in the price of an option with a change in the price of the stock

Gamma Γ: Mathematically, gamma is the second partial derivative of the price with respect to S. It is the same for both Calls and Puts. [iii]

- Call and Put Γ: $\frac{e^{-r_f\tau}}{S\sigma\sqrt{\tau}}n(d_1)$

Gamma represent the acceleration of the option price with respect to a change in the stock price. It also represents the inherent convexity in options.

Theta Θ: Mathematically, theta is the first derivative of the option price with respect to time [iv]

- Call Θ: $Se^{-r_f\tau}r_f N(d_1) - Ke^{-r_d\tau}r_d N\left(d_1 - \sigma\sqrt{\tau}\right) - Se^{-r_f\tau}\frac{\sigma}{2\sqrt{\tau}}n(d_1)$
- Put Θ:$-Se^{-r_f\tau}r_f N(-d_1) + Ke^{-r_d\tau}r_d N\left(-d_1 + \sigma\sqrt{\tau}\right) - Se^{-r_f\tau}\frac{\sigma}{2\sqrt{\tau}}n(d_1)$

Theta represents the time decay of the option price, the change in price of an option due to passage of time.

Black-Scholes PDE in Greek form

Black Scholes PDE: $\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} = rV$

where $\frac{\partial V}{\partial t}$ = Θ theta ; $\frac{\partial^2 V}{\partial S^2}$ = Γ gamma ; $\frac{\partial V}{\partial t}$ = Δ delta

Therefore, the Black-Scholes PDE in Greeks form is as follows:

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma + rS\Delta = rV$$

[i] “Black Scholes PDE Derivation Using Delta Hedging.” Quantpie.co.uk, 17 June 2019, www.youtube.com/watch?v=-qa2B_sCpZQ.

[ii] “Black Scholes Greeks, Delta.” Quantpie.co.uk, quantpie.co.uk/bsm_formula/bs_delta.php .

[iii] “Black Scholes Greeks, Gamma.” Quantpie.co.uk, quantpie.co.uk/bsm_formula/bs_gamma.php .

[iv] “Black Scholes Greeks, Theta.” Quantpie.co.uk, quantpie.co.uk/bsm_formula/bs_theta.php .