ME 543 Assignment 1

Computational Fluid Dynamics: 2D Elliptic problems

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1 Introduction

This assignment aims to solve two computational fluid problems of 2D elliptic nature: 1. To solve for the stream function of a flow as defined in Figure 1 below; 2. To solve for heat conduction inside four walls as shown in Figure 2.

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

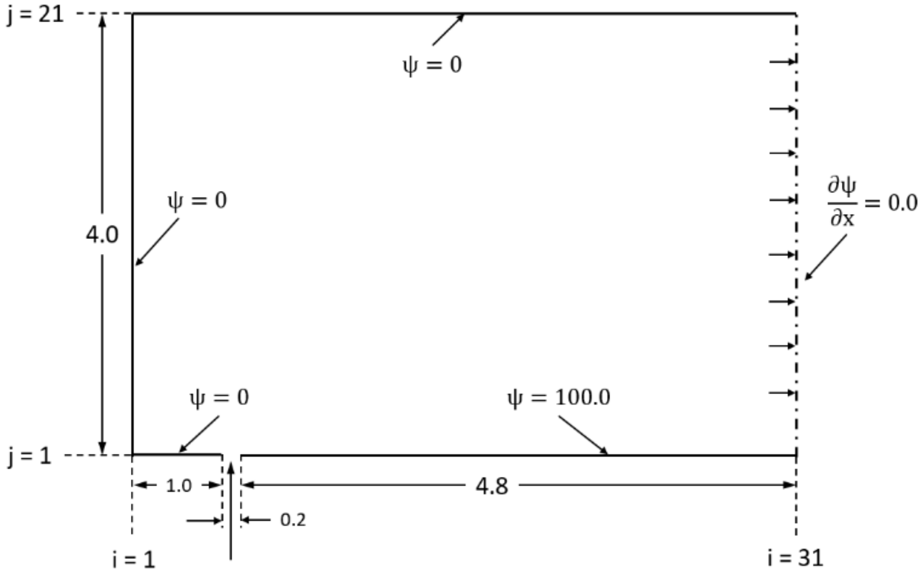
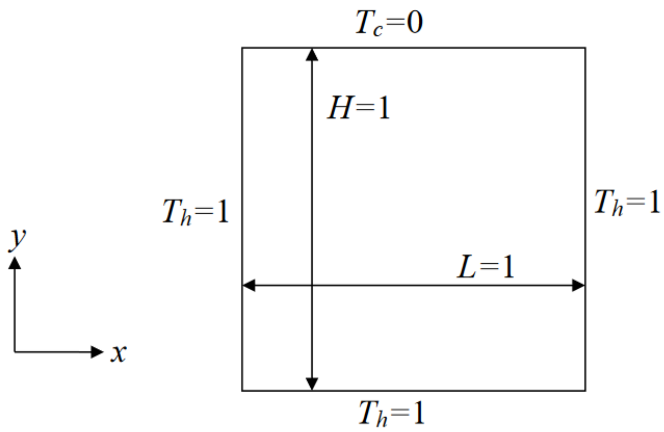


Figure 1: Question 1



$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Figure 2: Question 2

2 Methods

There are several algorithms to solve 2 D elliptic problems through finite difference method. Some algorithms and their formulation are discussed below:

2.1 Jacobi Method

Jacobi is an explicit method that is formulated as:

$$\phi_{i,j}^{k+1} = \frac{1}{2(1+\beta^2)}(\beta^2\phi_{i,j-1}^{k} + \phi_{i-1,j}^{k} + \phi_{i+1,j}^{k} + \beta^2\phi_{i,j+1}^{k}) \tag{1}$$

2.2 Point Gauss Seidel Method

Point Gauss Seidel is a method that uses current iteration values in previous locations (i,j-1 and i-1,j points). It is formulated as:

$$\phi_{i,j}^{k+1} = \frac{1}{2(1+\beta^2)}(\beta^2\phi_{i,j-1}^{k+1} + \phi_{i-1,j}^{k+1} + \phi_{i+1,j}^{k} + \beta^2\phi_{i,j+1}^{k}) \tag{2}$$

2.3 Point successive over relaxation

Point successive over relaxation (PSOR) is a modification of Point Gauss Seidel that introduces a parameter - ω, called relaxation factor. It is formulated as:

$$\phi_{i,j}^{k+1} = (1-\omega)\phi_{i,j}^{k} + \frac{\omega}{2(1+\beta^2)}(\beta^2\phi_{i,j-1}^{k+1} + \phi_{i-1,j}^{k+1} + \phi_{i+1,j}^{k} + \beta^2\phi_{i,j+1}^{k}) \tag{3}$$

2.4 Line Gauss Seidel Method

Line Gauss Seidel is an implicit method that is solved line by line and has 3 unknowns. In the equation below, the three values in the horizontal line (i-1,j and i,j and i+1,j points) are unknown at current iteration. It is formulated as

$$\phi_{i-1,j}^{k+1} - 2(1+\beta^2)\phi_{i,j}^{k+1} + \phi_{i+1,j}^{k+1} = -\beta^2(\phi_{i,j-1}^{k+1} + \phi_{i,j+1}^{k}) \tag{4}$$

It is solved using TDMA (Tri-Diagonal Matrix Algorithm).

$$a_i\phi_i + b_i\phi_{i+1} + c_i\phi_{i-1} = d_i$$

2.5 Alternating-Direction Implicit method

Alternating-direction implicit is an implicit method that solves Line Gauss Seidel in two different directions. In the same iteration, Line Gauss Seidel is solved in either x or y direction followed by solving in the other direction. Similar to Line Gauss Seidel, Tri-Diagonal Matrix algorithm is used for ADI method too.

3 Procedure

For question 1, initial value for ψ for interior points of the grid is taken as 50 and the boundary conditions are taken as provided in the problem statement. For question 2, initial temperature is taken as 0.5 in the interior grid points. And, the grid size is taken as 30 x 30 for question 2. Boundary conditions are applied and the previously discussed methods were used to solve the problems. Convergence of solution is achieved based on error. When error goes below 10^{-6}, iterations are halted. Error is calculated as,

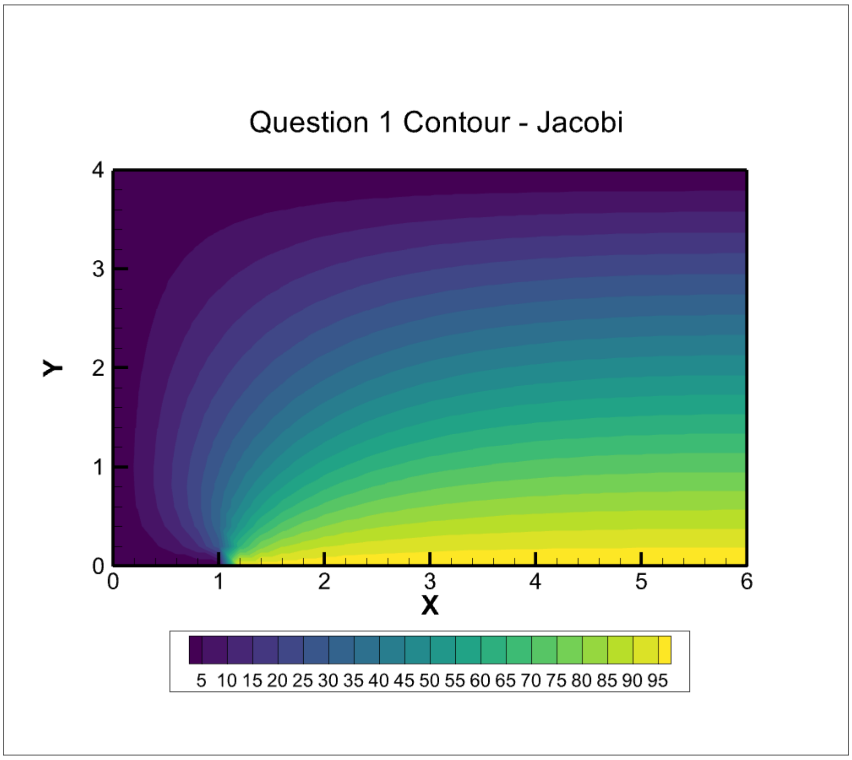
$$\epsilon = \sqrt{\frac{\sum_{i,j}(\phi^{k+1} - \phi^{k})^2}{(M-2)\times(N-2)}} \tag{5}$$

where M and N are the number of horizontal and vertical grid points.

4 Results and Discussion

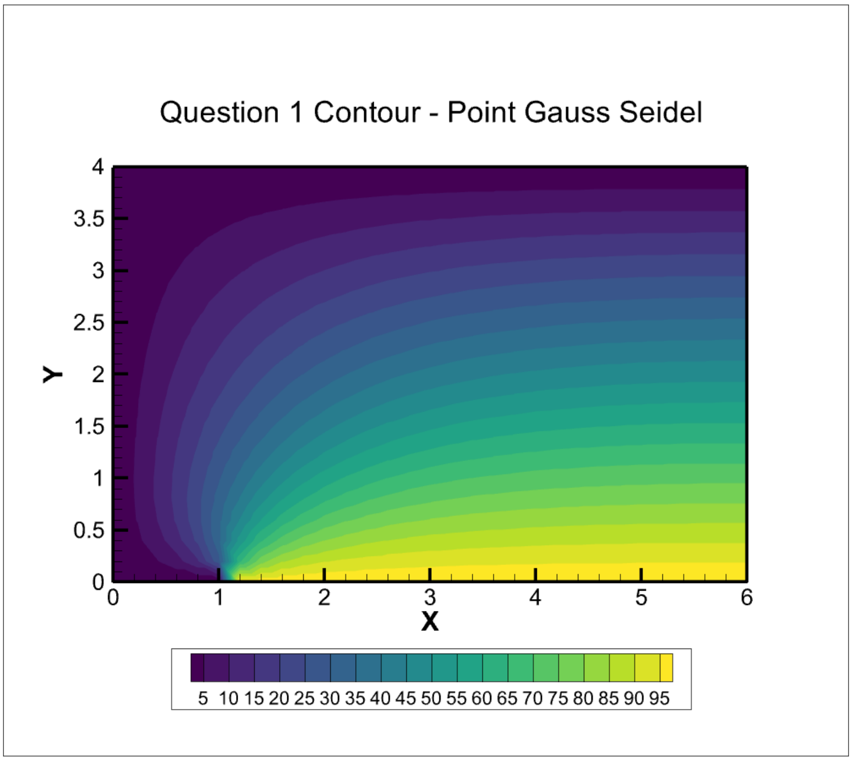
4.1 Question 1

4.1.1 Jacobi



Iterations to converge: 1536

4.1.2 Point Gauss Seidel



Iterations to converge: 849

4.1.3 PSOR

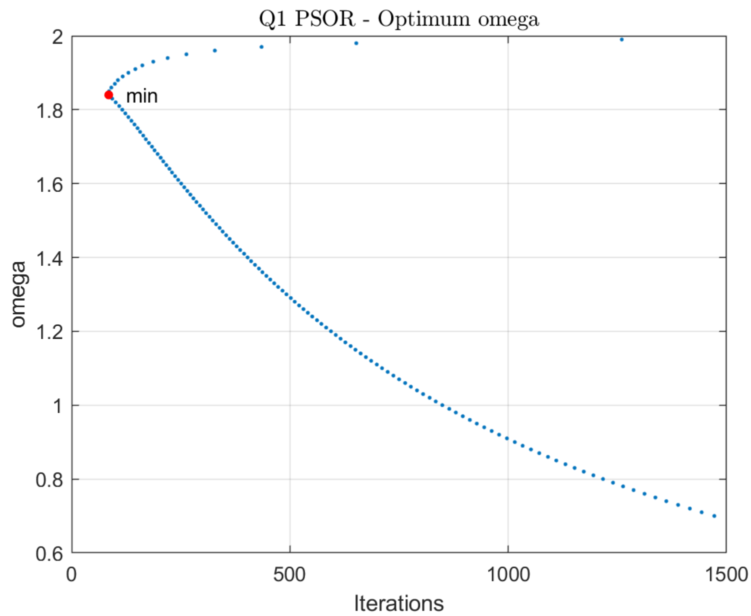
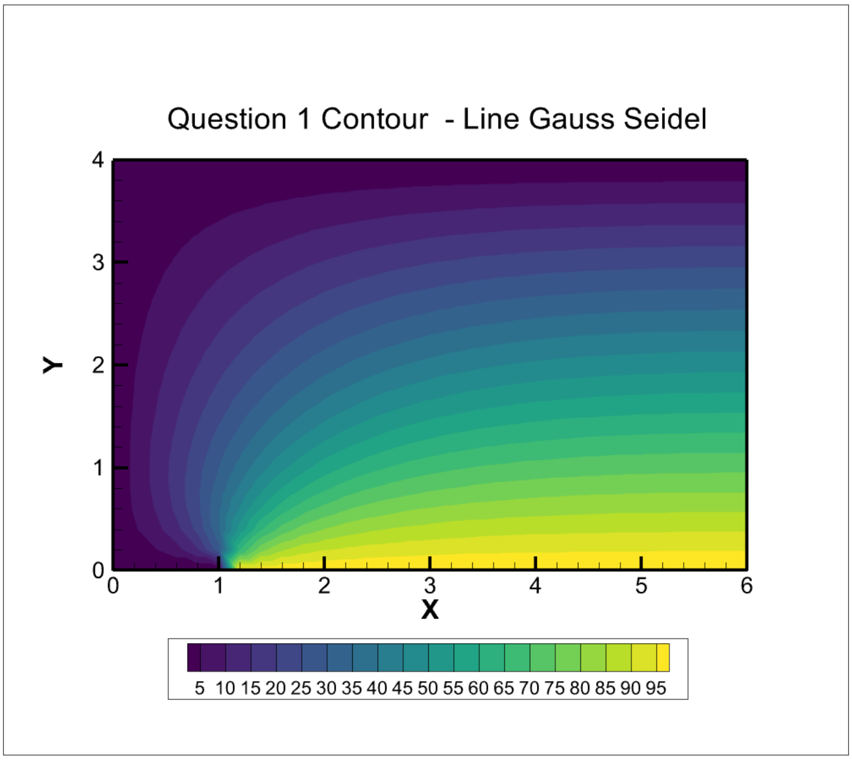


Figure 3: No. of iterations for varying omega

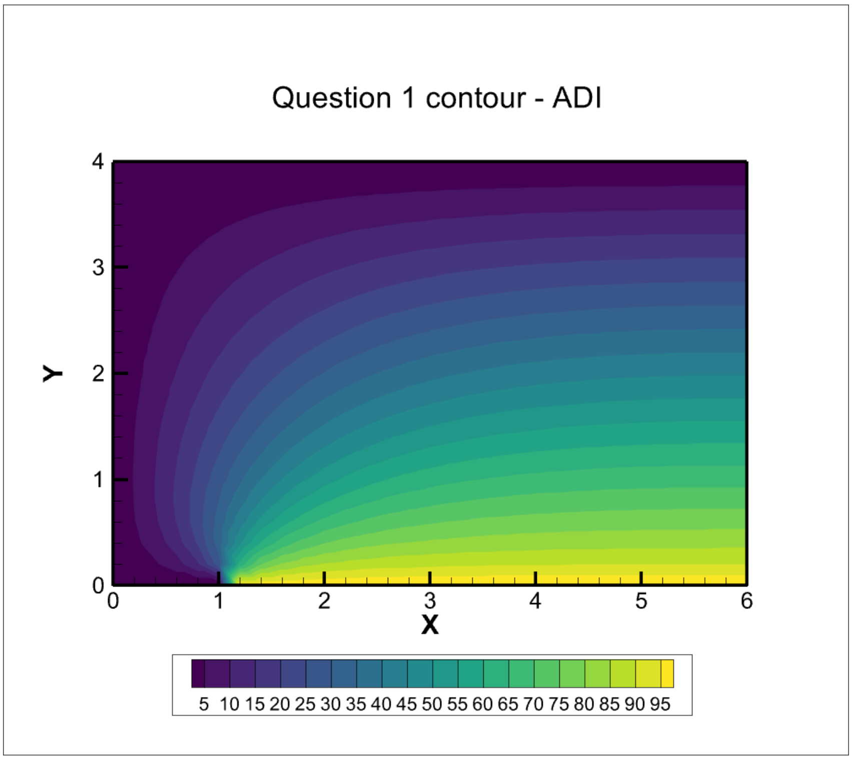
Optimum omega = 1.84
No. of iterations to converge for optimum omega: 85

4.1.4 Line Gauss Seidel



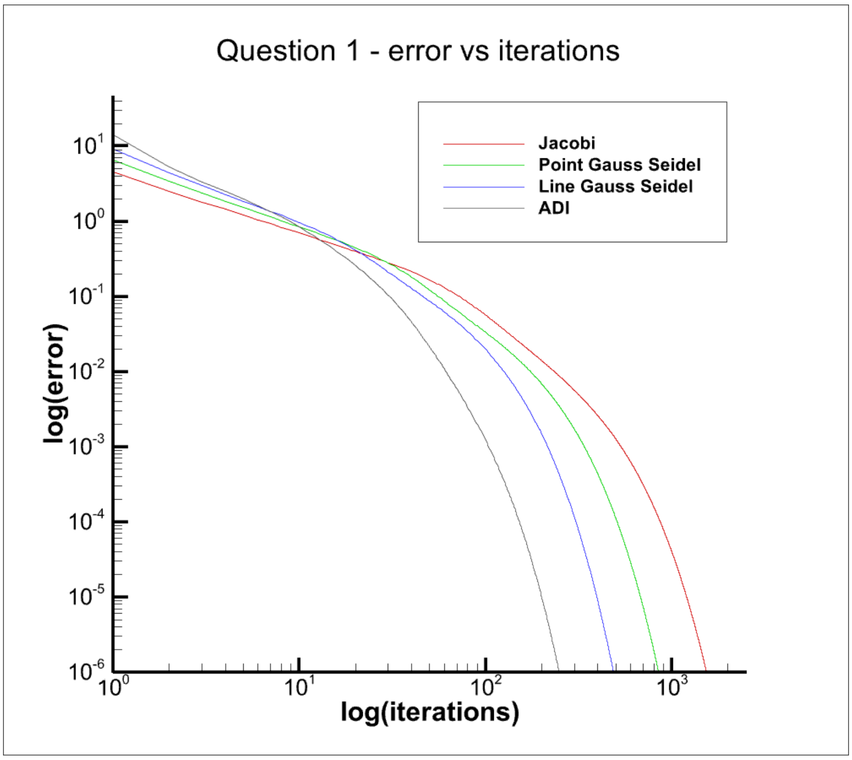
Iterations to converge: 486

4.1.5 ADI



Iterations to converge: 249

4.1.6 Question 1: Error vs Iterations



From the plot showing error vs iterations, it is seen that Jacobi is the least efficient method whereas ADI is the fastest method. Order of rate of convergence is ADI>Line Gauss Seidel>Point Gauss Seidel>Jacobi.

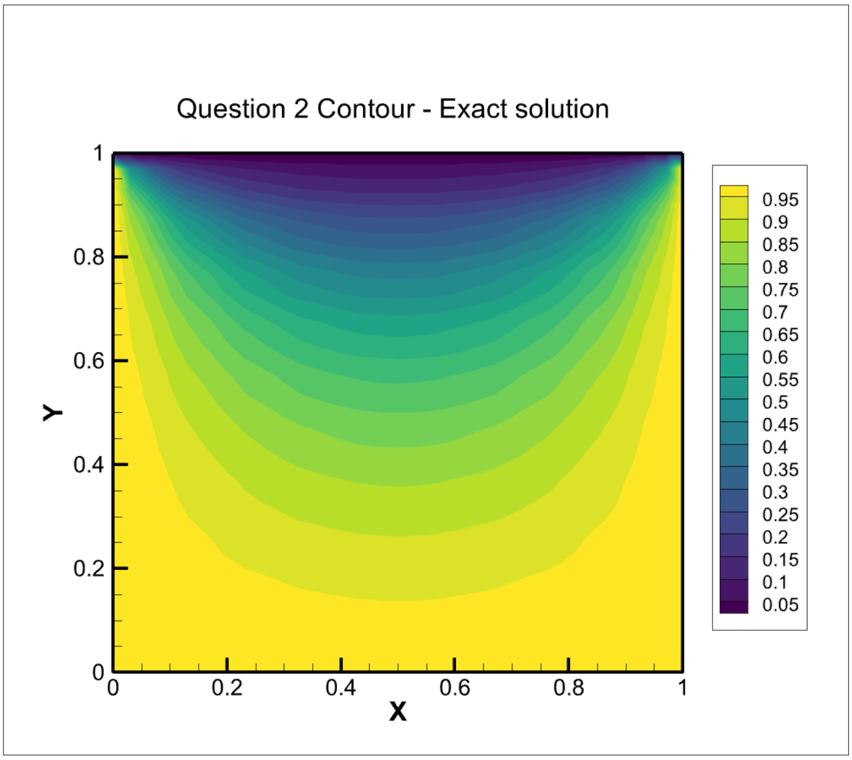
4.2 Question 2

4.2.1 Exact solution

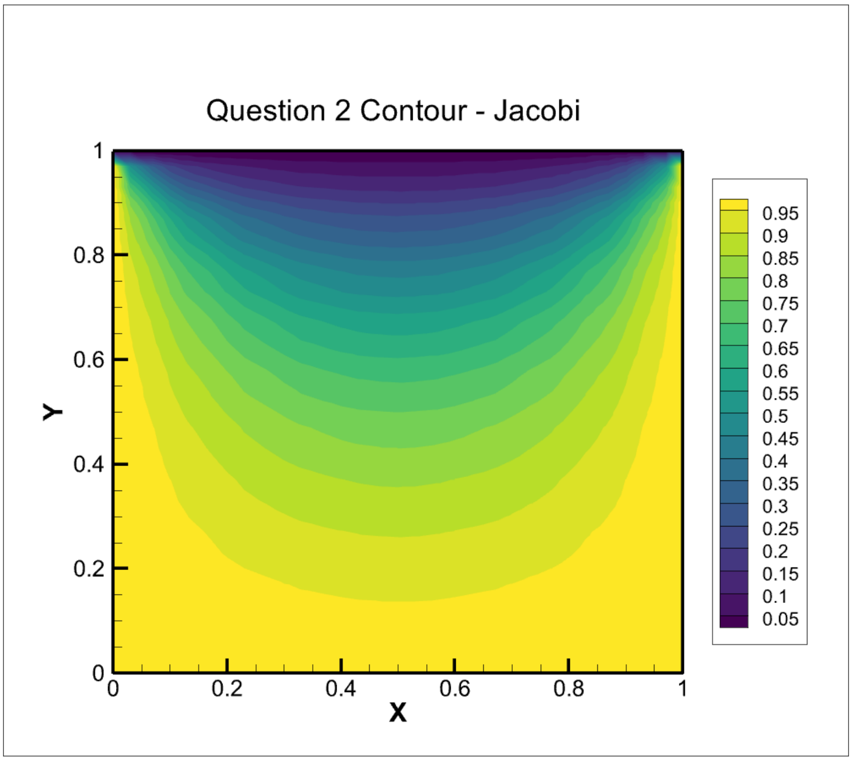
Exact solution of problem defined in Figure 2 is given by the following expression.

$$T(x,y) = T_c + (T_h - T_c)\left[1 - 2\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n\pi}\frac{sinh\left(\dfrac{n\pi y}{L}\right)}{sinh\left(\dfrac{n\pi H}{L}\right)}sin\left(\frac{n\pi x}{L}\right)\right] \tag{6}$$

The above expression is computed at each grid point by approximating the summation to infinity by considering the first 100 terms of the summation. L and H are taken as 1.0 as provided in the problem and x and y are varied from 0 to 1. Boundary conditions are defined as T_c=0 and T_h=1.

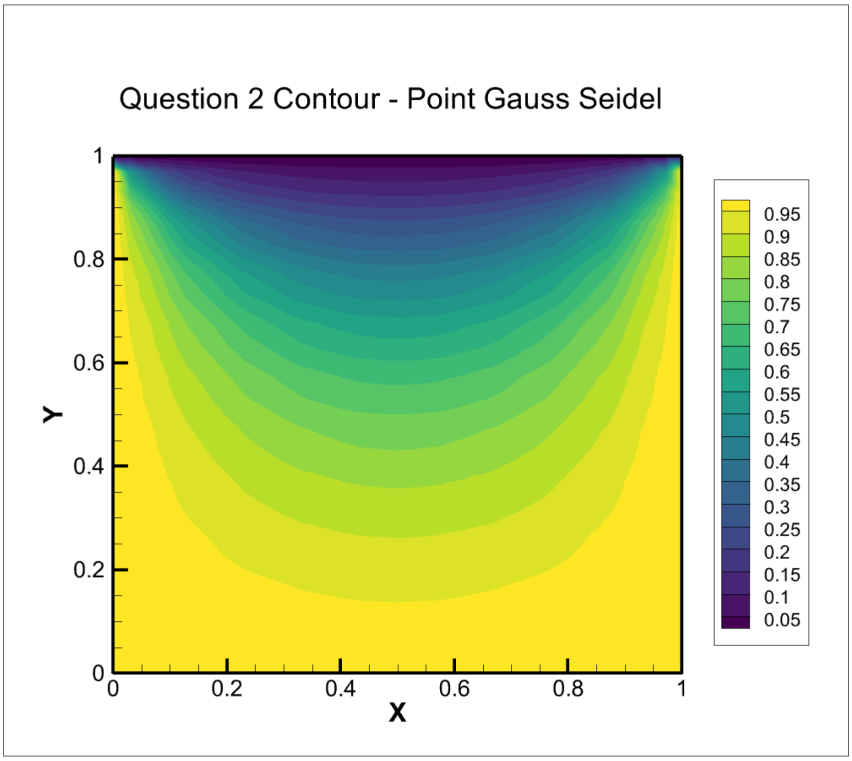


4.2.2 Jacobi



Iterations to converge: 1278

4.2.3 Point Gauss Seidel



Iterations to converge: 699

4.2.4 PSOR

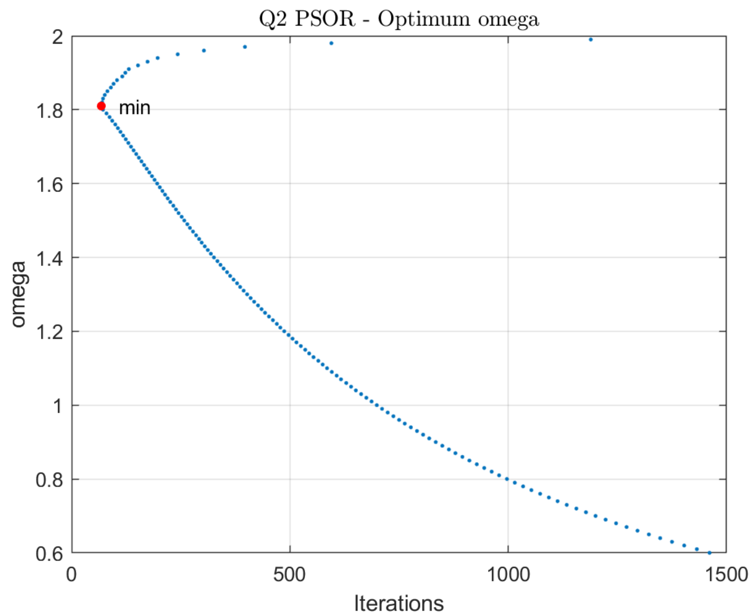
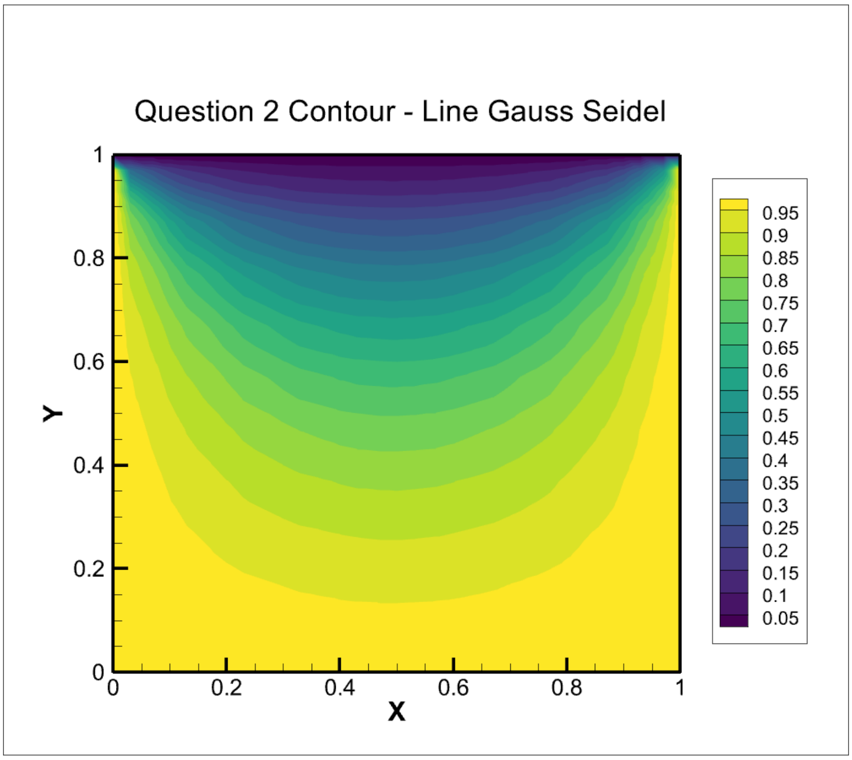


Figure 4: No. of iterations for varying omega

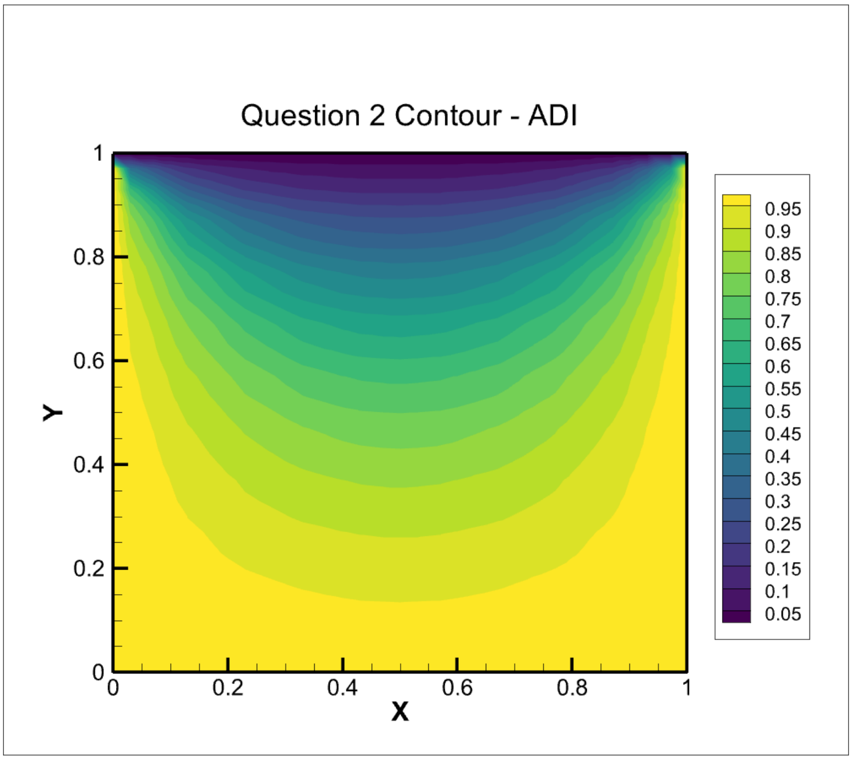
Optimum omega = 1.81
No. of iterations to converge for optimum omega: 68

4.2.5 Line Gauss Seidel



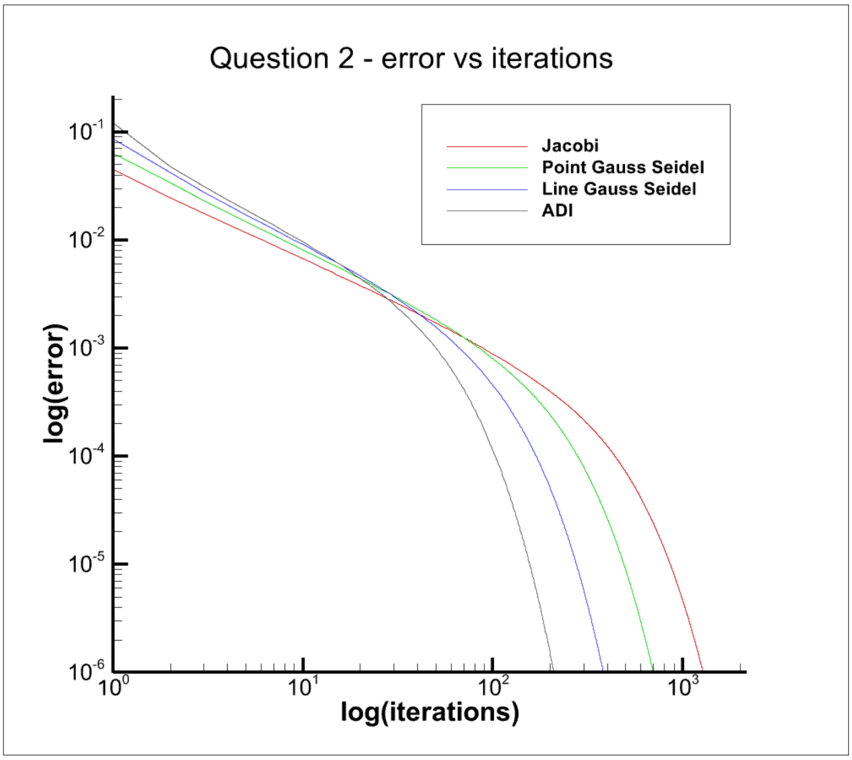
Iterations to converge: 382

4.2.6 ADI



Iterations to converge: 210

4.2.7 Question 2: Error vs Iterations



From the plot showing error vs iterations, it is seen that Jacobi is the least efficient method whereas ADI is the fastest method. Order of rate of convergence is ADI>Line Gauss Seidel>Point Gauss Seidel>Jacobi.

Morever, the contours for temperature obtained from different iterative methods conform to the exact solution plotted at the beginning of the section.