Linear Algebra (Review)

Vectors

$$V = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$||V|| = (\sum_i v_i^2)^1/2$$

$$||V||_\infty = \max(v_i)$$

$$||V||_1 = \sum_i |v_i|$$

$$< V, U > = \sum_i (v_i * u_i) = V_T U$$

$$< V, U > = ||V||||U||cos(\theta)$$

$$| < V, U > | \leq ||V||||U|$$

- θ is angle between two vectors
- Orthogonal vectors $< V, U > = 0$
- Orthonormal vectors $< V, U > = 0, ||V|| = 1$
    - $U = \sum_n \alpha_i V_i \;\; \alpha_i = < V_i, U >$
    - Does this hold for non orthogonal $\{V_i\}$?
- Properties of norm - non negativity, zero if vector is all zero, scalar multiplication, triangle inequality
- Cauchy Schartz is a equality iff V = a U

Vector Spaces

A vector space over a field F is a set V together with two binary operations that satisfy the eight axioms listed below. In this context, the elements of V are commonly called *vectors*, and the elements of F are called *scalars*.

- The first operation, called *vector addition* or simply *addition* assigns to any two vectors $\mathbf{v}$ and $\mathbf{w}$ in V a third vector in V which is commonly written as $\mathbf{v} + \mathbf{w}$, and called the *sum* of these two vectors.
- The second operation, called *scalar multiplication*, assigns to any scalar a in F and any vector $\mathbf{v}$ in V another vector in V, which is denoted $a\mathbf{v}$.[nb 2]

For having a vector space, the eight following axioms must be satisfied for every $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in V, and a and b in F.[1]

Axiom	Meaning
Associativity of vector addition	$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
Commutativity of vector addition	$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
Identity element of vector addition	There exists an element $\mathbf{0} \in V$, called the *zero vector*, such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$.
Inverse elements of vector addition	For every $\mathbf{v} \in V$, there exists an element $-\mathbf{v} \in V$, called the *additive inverse* of $\mathbf{v}$, such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
Compatibility of scalar multiplication with field multiplication	$a(b\mathbf{v}) = (ab)\mathbf{v}$ [nb 3]
Identity element of scalar multiplication	$1\mathbf{v} = \mathbf{v}$, where 1 denotes the multiplicative identity in F.
Distributivity of scalar multiplication with respect to vector addition	$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
Distributivity of scalar multiplication with respect to field addition	$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$

- Unit vector spaces — x, y, z axis that we all work with
  - Rotated spaces, translated from origin spaces
- Fourier space — frequency domain analysis of signal
- KLT space

Basis

- Given a set of vectors $\{V_i\}$ in space *S*
  - Can any vector *U* in *S* be written as a linear combination $\{V_i\}$?
    $U = \sum_n \alpha_i V_i$
  - Are the vectors linearly independent? That is there is no V_j in $\{V_i\}$ such that
    $V_j = \sum_{i \neq j} \alpha_i V_i$
  - If so this set of vectors is called **basis** for *S*

Linear Algebra

$$\begin{bmatrix} A_{11}x_1 + A_{12}x_2 + \dots A_{1n}x_n = y_1 \\ A_{21}x_1 + A_{22}x_2 + \dots A_{2n}x_n = y_2 \\ \vdots \\ A_{m1}x_1 + A_{m2}x_2 + \dots A_{mn}x_n = y_m \end{bmatrix}$$

$$A_{mxn}X_{nx1} = Y_{mx1}$$

$$\begin{bmatrix} < A_1^T, X > = y_1 \\ < A_2^T, X > = y_2 \\ \vdots \\ < A_m^T X > = y_m \end{bmatrix}$$

- System of linear equations
  - Matrix multipication — each row is inner product of row vector of first matrix with column vector of second matrix
  - Space of solution of a linear system for a vector subspace
  - Simplest linear equation $y = mx + c$

$$\begin{bmatrix} m \cdot x + c \cdot 1 = y \\ 0 \cdot x + 1 \cdot 1 = 1 \end{bmatrix}$$

Scaling, Translation, Rotation

$$S = \begin{bmatrix} s_1,0,0 \\ 0,s_2,0 \\ 0,0,1 \end{bmatrix} \qquad T = \begin{bmatrix} 1,0,t_1 \\ 0,1,t_2 \\ 0,0,1 \end{bmatrix} \qquad R = \begin{bmatrix} cos(\theta), -sin(\theta),0 \\ sin(\theta), cos(\theta),0 \\ 0,0,1 \end{bmatrix}$$

- Rotation matrix derived in class.
  - Basic idea is that any vector can be written in terms of linear combination of its basis vectors
  - In particular for orthonormal basis (original coordinate or rotated) the linear combination is a inner product
  - This can be further written as a matrix equation because each row of linear equation is a inner product

Matrices

- Given a basis {t_i} of space S write each basis vector as column vector of a matrix T

$$T = [t_1, t_2, \cdots, t_n]$$

- Then y = Tx is transformation of a vector x to a vector y where y belongs to S
- If t_i is m dimension and there are n vectors in {t_i} T is mxn matrix
- And input space is dimension n
- Output space dimension is m

Matrices

- System of linear equations
  - Solution space is the space spanned by its linearly independent column vector
- Can be one unique solution
- Can be many solutions
- Can be no solution
  - Projection and least square

The four possibilities for linear equations depend on the rank r:

$\boldsymbol{r = m}$	and	$\boldsymbol{r = n}$	*Square and invertible*	$A\boldsymbol{x} = \boldsymbol{b}$	has 1 solution
$\boldsymbol{r = m}$	and	$r < n$	*Short and wide*	$A\boldsymbol{x} = \boldsymbol{b}$	has ∞ solutions
$r < m$	and	$\boldsymbol{r = n}$	*Tall and thin*	$A\boldsymbol{x} = \boldsymbol{b}$	has 0 or 1 solution
$r < m$	and	$r < n$	*Not full rank*	$A\boldsymbol{x} = \boldsymbol{b}$	has 0 or ∞ solutions

The reduced R will fall in the same category as the matrix A. In case the pivot columns happen to come first, we can display these four possibilities for R. For $R\boldsymbol{x} = \boldsymbol{d}$ (and the original $A\boldsymbol{x} = \boldsymbol{b}$) to be solvable, $\boldsymbol{d}$ must end in $m - r$ zeros.

Four types	$R = \begin{bmatrix} I \end{bmatrix}$	$\begin{bmatrix} I & F \end{bmatrix}$	$\begin{bmatrix} I \\ 0 \end{bmatrix}$	$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$
Their ranks	$r = m = n$	$r = m < n$	$r = n < m$	$r < m, r < n$

1. The rank r is the number of pivots. The matrix R has $m - r$ zero rows.
2. $A\boldsymbol{x} = \boldsymbol{b}$ is solvable if and only if the last $m - r$ equations reduce to $0 = 0$.
3. One particular solution $\boldsymbol{x}_p$ has all free variables equal to zero.
4. The pivot variables are determined after the free variables are chosen.
5. Full column rank $r = n$ means no free variables: one solution or none.
6. Full row rank $r = m$ means one solution if $m = n$ or infinitely many if $m < n$.

Dimension

- Dimension of a space is the number of linearly independent vectors that span the space
- For an image with MxN pixels — the dimension of vector is MN
- But the dimension of space << MN
- This is because pixels in an image are highly correlated, knowing a few of MN pixels, we can predict the remaining pixels
- KL Transforms — based on eigenvalues of correlation matrix
- DCT — approximations of KL transforms used in JPEG

Linear time invariant system

- This is primarily a model
- So lets say you have a black box, where you have observed pairs of (input (x), output (y))
  - Lets say $y = f(x)$ where *f()* is an unknown function
  - Lets assume certain properties of *f()*
  - Linearity
    - f(x_1) = y_1, f(x_2)= y_2 ==> f(x_1+x_2) = y_1+y_2
    - f(a x_1) = ay_1
  - Time or Space invariance $\begin{bmatrix} f(x(t)) = y(t) \\ f(x(t+\tau)) = y(t+\tau) \end{bmatrix}$

Examples of LTI systems

- Retina can be modelled as LTSI system
- Image sensors
- Denoising systems
    - Systems that remove noise from signal

Convolution

CONVOLUTION EXAMPLE

Impulse Response $h(t)$

Input $x(t)$

Output $y(t)$

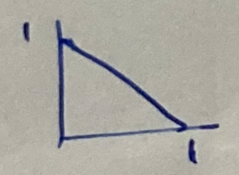
$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

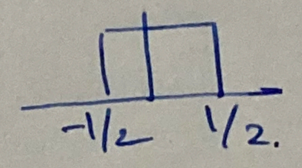
$h(t-\tau) \Rightarrow$ ~~st in~~ $h(-\tau) \Rightarrow h(t-\tau)$

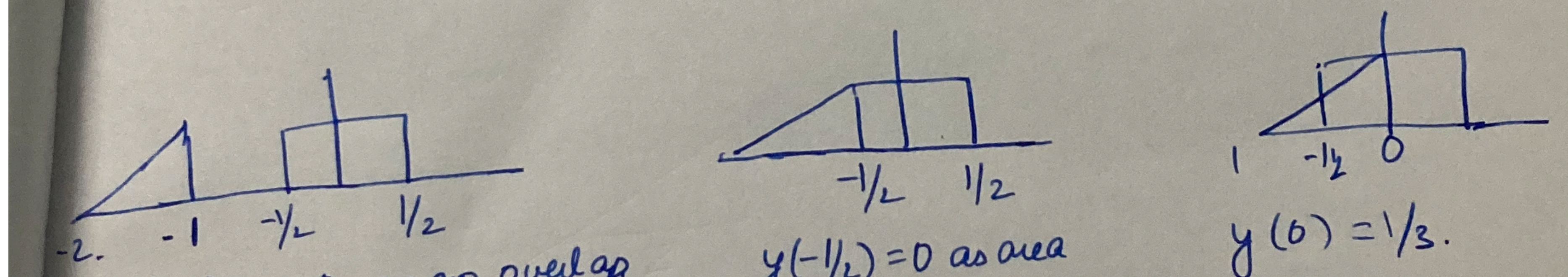
invert shift

$x(\tau)\, h(t-\tau)$ multipication / overlap

$\int x(t)\, h(t-\tau)$ area of overlap

$h(t) =$ 

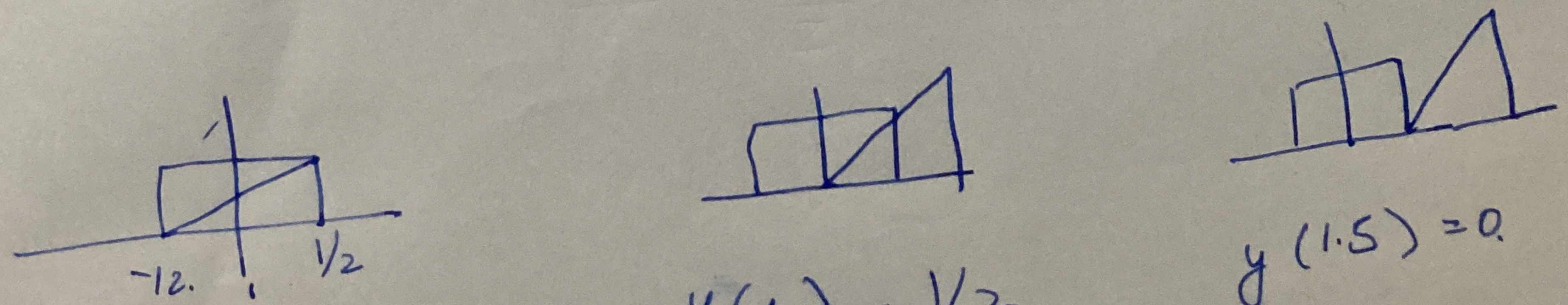
$x(t) =$ 



$y(-1) = 0$ as no overlap

$y(-1/2) = 0$ as area under overlap $= 0$

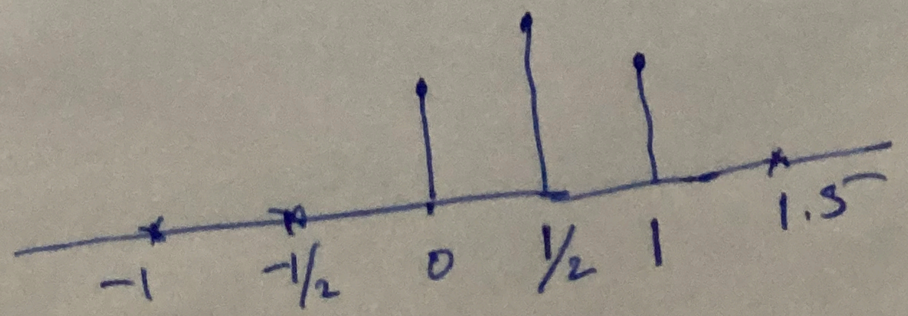
$y(0) = 1/3$.



$y(1/2) = 1/2$.

$y(1) = 1/3$

$y(1.5) = 0$

$y(t)$. 

Convolution

- Discrete — Sum not integrals

$$y[k] = \sum_m x[m]h[k-m]$$

$$y[k+m] = \sum_m x[2m]h[k]$$

- Finite sequences
  - Padding
    - Zero
    - Mirror values
    - Constant
- How long is the output sequence?

Input x_0, x_1, x_2, x_3, x_4

Impulse Response $h_{-1}, h_0, h_1 \quad h_0, h_1, h_2$

Invert h_1, h_0, h_{-1}

0, 0 x_0, x_1, x_2, x_3, x_4

h_1, h_0, h_{-1}

Shift

$$y_0 = h_{-1} \cdot x_0$$

h_1, h_0, h_{-1}

$$y_1 = h_0 \cdot x_0 + h_{-1} \cdot x_1$$

h_1, h_0, h_{-1}

$$y_2 = h_1 \cdot x_0 + h_0 \cdot x_1 + h_{-1} \cdot x_2$$

$$x_0, x_1, x_2, x_3, x_4$$

$$h_{-1}, h_0, h_1$$

$$y_0 = h_{-1} \cdot x_0$$

$$y_1 = h_0 \cdot x_0 + h_{-1} \cdot x_1$$

$$\boxed{y_2 = h_1 \cdot x_0 + h_0 \cdot x_1 + h_{-1} \cdot x_2}$$

$$\boxed{y_3 = h_1 \cdot x_1 + h_0 \cdot x_2 + h_{-1} \cdot x_3}$$

$$\boxed{y_4 = h_1 \cdot x_2 + h_0 \cdot x_3 + h_{-1} \cdot x_4}$$

$$y_5 = h_1 \cdot x_3 + h_0 \cdot x_4$$

$$y_6 = h_1 \cdot x_4$$

$$H = \begin{bmatrix} h_{-1} & ,0 & ,0 & ,0 & ,0 & \\ h_0 & ,h_1 & ,0 & ,0 & ,0 & \\ h_1 & ,h_0 & ,h_{-1} & ,0 & ,0 & \\ 0 & ,h_1 & ,h_0 & ,h_{-1} & ,0 & , \\ 0 & ,0 & ,h_1 & ,h_0 & ,h_{-1} & , \\ 0 & ,0 & ,0 & ,h_1 & ,h_0 & \\ 0 & ,0 & ,0 & ,0 & ,h_1 & \end{bmatrix} \quad \begin{bmatrix} x_0, \\ x_1, \\ x_2, \\ x_3, \\ x_4 \end{bmatrix}$$

Convolution

- Output sequence
  - If input sequence of length m, impulse response of length n,
  - Full convolution
    - All positions where there is even a single overlap
    - Length of output is m+n-1
  - Valid convolution
    - Positions where impulse response is fully overlapped
    - max(m, n) - min(m, n) + 1
  - Same convolution
    - Pad minimally to ensure that length is
    - Length is max(m, n)

- y[m+n] = \sum_m x[m] h[n]

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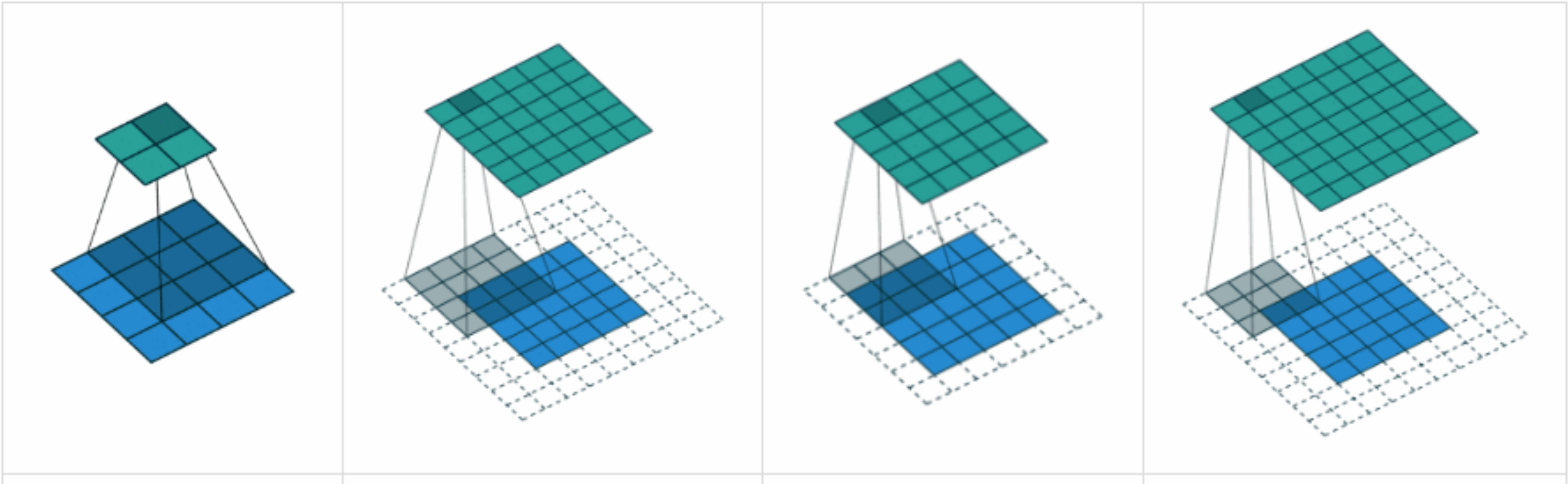

$$C[t] = \sum A[\tau]B[t-\tau]$$

$$C[t+\tau] = \sum A[\tau]B[t]$$

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