

Linear Algebra (Review)

Vectors

$$V = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\begin{aligned} ||V|| &= (\sum_i v_i^2)^{1/2} \\ ||V||_\infty &= \max(v_i) \\ ||V||_1 &= \sum_i |v_i| \end{aligned}$$

$$\begin{aligned} \langle V, U \rangle &= \sum_i (v_i * u_i) = V_T U \\ \langle V, U \rangle &= ||V|| ||U|| \cos(\theta) \\ |\langle V, U \rangle| &\leq ||V|| ||U|| \end{aligned}$$

- θ is angle between two vectors
- Orthogonal vectors $\langle V, U \rangle = 0$
- Orthonormal vectors $\langle V, U \rangle = 0, ||V|| = 1$
 - $U = \sum_n \alpha_i V_i \quad \alpha_i = \langle V_i, U \rangle$
 - Does this hold for non orthogonal $\{V_i\}$?
- Properties of norm - non negativity, zero if vector is all zero, scalar multiplication, triangle inequality
- Cauchy Schartz is a equality iff $V = a U$

Vector Spaces

A vector space over a [field \$F\$](#) is a [set \$V\$](#) together with two [binary operations](#) that satisfy the eight axioms listed below. In this context, the elements of V are commonly called *vectors*, and the elements of F are called *scalars*.

- The first operation, called *vector addition* or simply *addition* assigns to any two vectors \mathbf{v} and \mathbf{w} in V a third vector in V which is commonly written as $\mathbf{v} + \mathbf{w}$, and called the *sum* of these two vectors.
- The second operation, called [scalar multiplication](#), assigns to any scalar a in F and any vector \mathbf{v} in V another vector in V , which is denoted $a\mathbf{v}$.^[nb 2]

For having a vector space, the eight following [axioms](#) must be satisfied for every \mathbf{u} , \mathbf{v} and \mathbf{w} in V , and a and b in F .^[1]

Axiom	Meaning
Associativity of vector addition	$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
Commutativity of vector addition	$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
Identity element of vector addition	There exists an element $\mathbf{0} \in V$, called the zero vector , such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$.
Inverse elements of vector addition	For every $\mathbf{v} \in V$, there exists an element $-\mathbf{v} \in V$, called the additive inverse of \mathbf{v} , such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
Compatibility of scalar multiplication with field multiplication	$a(b\mathbf{v}) = (ab)\mathbf{v}$ ^[nb 3]
Identity element of scalar multiplication	$1\mathbf{v} = \mathbf{v}$, where 1 denotes the multiplicative identity in F .
Distributivity of scalar multiplication with respect to vector addition	$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
Distributivity of scalar multiplication with respect to field addition	$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$

- Unit vector spaces — x, y, z axis that we all work with
 - Rotated spaces, translated from origin spaces
- Fourier space — frequency domain analysis of signal
- KLT space

Basis

- Given a set of vectors $\{V_i\}$ in space S

- Can any vector U in S be written as a linear combination $\{V_i\}$?

$$U = \sum_n \alpha_i V_i$$

- Are the vectors linearly independent? That is there is no V_j in $\{V_i\}$ such that

$$V_j = \sum_{i \neq j} \alpha_i V_i$$

- If so this set of vectors is called **basis** for S

Linear Algebra

$$\begin{bmatrix} A_{11}x_1 + A_{12}x_2 + \dots + A_{1n}x_n = y_1 \\ A_{21}x_1 + A_{22}x_2 + \dots + A_{2n}x_n = y_2 \\ \vdots \\ A_{m1}x_1 + A_{m2}x_2 + \dots + A_{mn}x_n = y_m \end{bmatrix}$$

$$A_{mxn}X_{nx1} = Y_{mx1}$$

$$\begin{bmatrix} \langle A_1^T, X \rangle = y_1 \\ \langle A_2^T, X \rangle = y_2 \\ \vdots \\ \langle A_m^T, X \rangle = y_m \end{bmatrix}$$

- System of linear equations
 - Matrix multiplication – each row is inner product of row vector of first matrix with column vector of second matrix
 - Space of solution of a linear system for a vector subspace
 - Simplest linear equation $y = mx + c$

$$\begin{bmatrix} m \cdot x + c \cdot 1 = y \\ 0 \cdot x + 1 \cdot 1 = 1 \end{bmatrix}$$

Scaling, Translation, Rotation

$$S = \begin{bmatrix} s_1, 0, 0 \\ 0, s_2, 0 \\ 0, 0, 1 \end{bmatrix} \quad T = \begin{bmatrix} 1, 0, t_1 \\ 0, 1, t_2 \\ 0, 0, 1 \end{bmatrix} \quad R = \begin{bmatrix} \cos(\theta), -\sin(\theta), 0 \\ \sin(\theta), \cos(\theta), 0 \\ 0, 0, 1 \end{bmatrix}$$

- Rotation matrix derived in class.
 - Basic idea is that any vector can be written in terms of linear combination of its basis vectors
 - In particular for orthonormal basis (original coordinate or rotated) the linear combination is a inner product
 - This can be further written as a matrix equation because each row of linear equation is a inner product

Matrices

- Given a basis $\{t_i\}$ of space S write each basis vector as column vector of a matrix T
- Then $y = Tx$ is transformation of a vector x to a vector y where y belongs to S
- If t_i is m dimension and there are n vectors in $\{t_i\}$ T is mxn matrix
- And input space is dimension n
- Output space dimension is m

$$T = [t_1, t_2, \dots, t_n]$$

Matrices

- System of linear equations
 - Solution space is the space spanned by its linearly independent column vector
- Can be one unique solution
- Can be many solutions
- Can be no solution
 - Projection and least square

The four possibilities for linear equations depend on the rank r :

$r = m$	and	$r = n$	<i>Square and invertible</i>	$Ax = b$	has 1 solution
$r = m$	and	$r < n$	<i>Short and wide</i>	$Ax = b$	has ∞ solutions
$r < m$	and	$r = n$	<i>Tall and thin</i>	$Ax = b$	has 0 or 1 solution
$r < m$	and	$r < n$	<i>Not full rank</i>	$Ax = b$	has 0 or ∞ solutions

The reduced R will fall in the same category as the matrix A . In case the pivot columns happen to come first, we can display these four possibilities for R . For $Rx = d$ (and the original $Ax = b$) to be solvable, d must end in $m - r$ zeros.

Four types	$R = [I]$	$[I \ F]$	$\begin{bmatrix} I \\ 0 \end{bmatrix}$	$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$
Their ranks	$r = m = n$	$r = m < n$	$r = n < m$	$r < m, r < n$

1. The rank r is the number of pivots. The matrix R has $m - r$ zero rows.
2. $Ax = b$ is solvable if and only if the last $m - r$ equations reduce to $0 = 0$.
3. One particular solution x_p has all free variables equal to zero.
4. The pivot variables are determined after the free variables are chosen.
5. Full column rank $r = n$ means no free variables: one solution or none.
6. Full row rank $r = m$ means one solution if $m = n$ or infinitely many if $m < n$.

Dimension

- Dimension of a space is the number of linearly independent vectors that span the space
- For an image with $M \times N$ pixels – the dimension of vector is MN
- But the dimension of space $\ll MN$
- This is because pixels in an image are highly correlated, knowing a few of MN pixels, we can predict the remaining pixels
- KL Transforms – based on eigenvalues of correlation matrix
- DCT – approximations of KL transforms used in JPEG

Linear time invariant system

- This is primarily a model
- So lets say you have a black box, where you have observed pairs of (input (x), output (y))
 - Lets say $y = f(x)$ where $f()$ is an unknown function
 - Lets assume certain properties of $f()$
 - Linearity
 - $f(x_1) = y_1, f(x_2) = y_2 \implies f(x_1+x_2) = y_1+y_2$
 - $f(ax_1) = ay_1$
 - Time or Space invariance $\begin{bmatrix} f(x(t)) = y(t) \\ f(x(t + \tau)) = y(t + \tau) \end{bmatrix}$

Examples of LTI systems

- Retina can be modelled as LTSI system
- Image sensors
- Denoising systems
 - Systems that remove noise from signal

Convolution

CONVOLUTION EXAMPLE

Impulse Response $h(t)$

Input $x(t)$

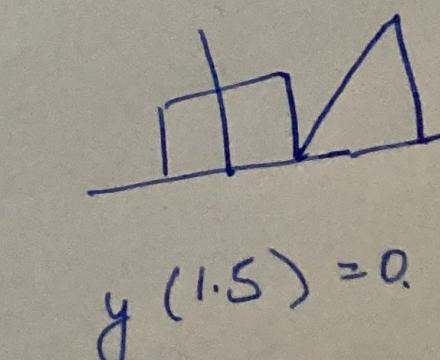
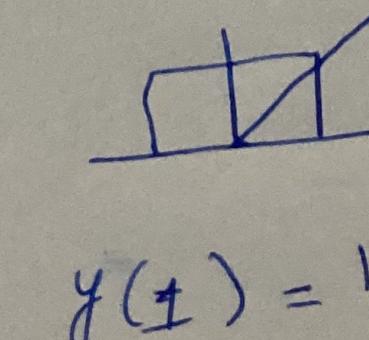
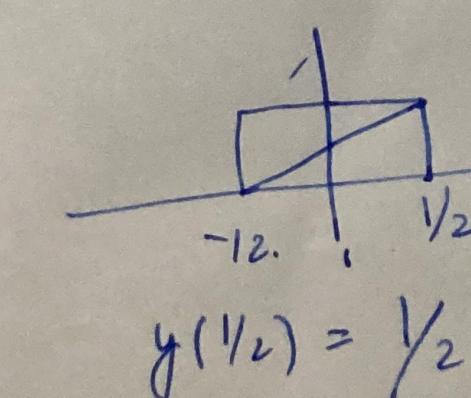
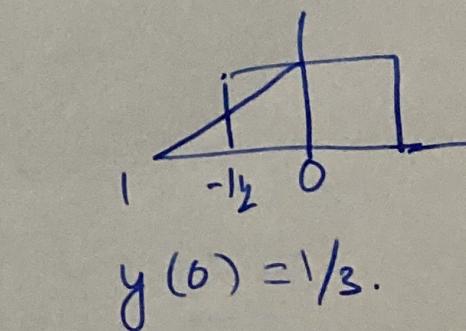
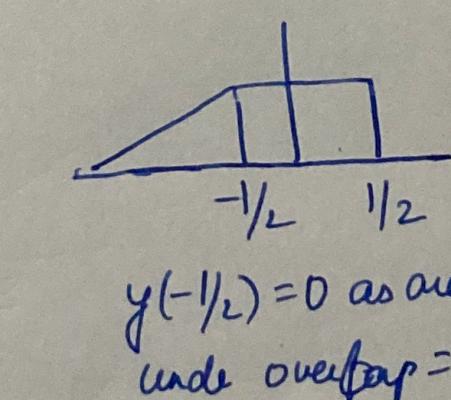
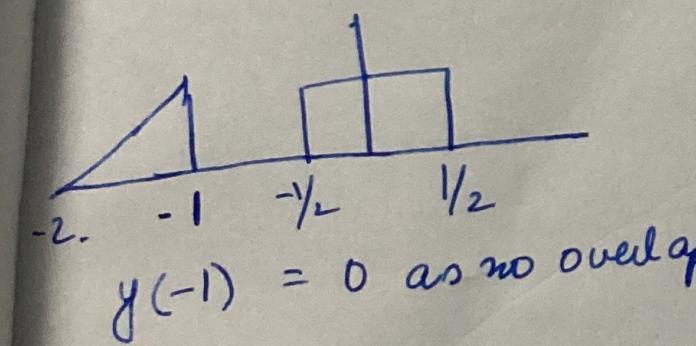
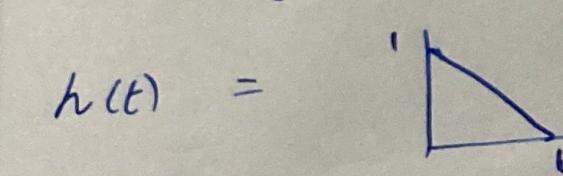
Output $y(t)$

$$y(t) = \int_{-\infty}^{\infty} x(z) h(t-z) dz$$

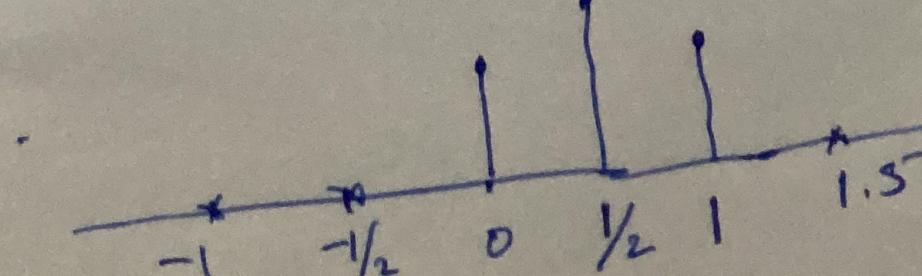
$h(t-z) \Rightarrow$ ~~at t~~ $\Rightarrow h(t-z)$
invert shift

$x(z) h(t-z)$ multiplication/overlap

$\int x(z) h(t-z)$ area of overlap



$y(t)$.



Convolution

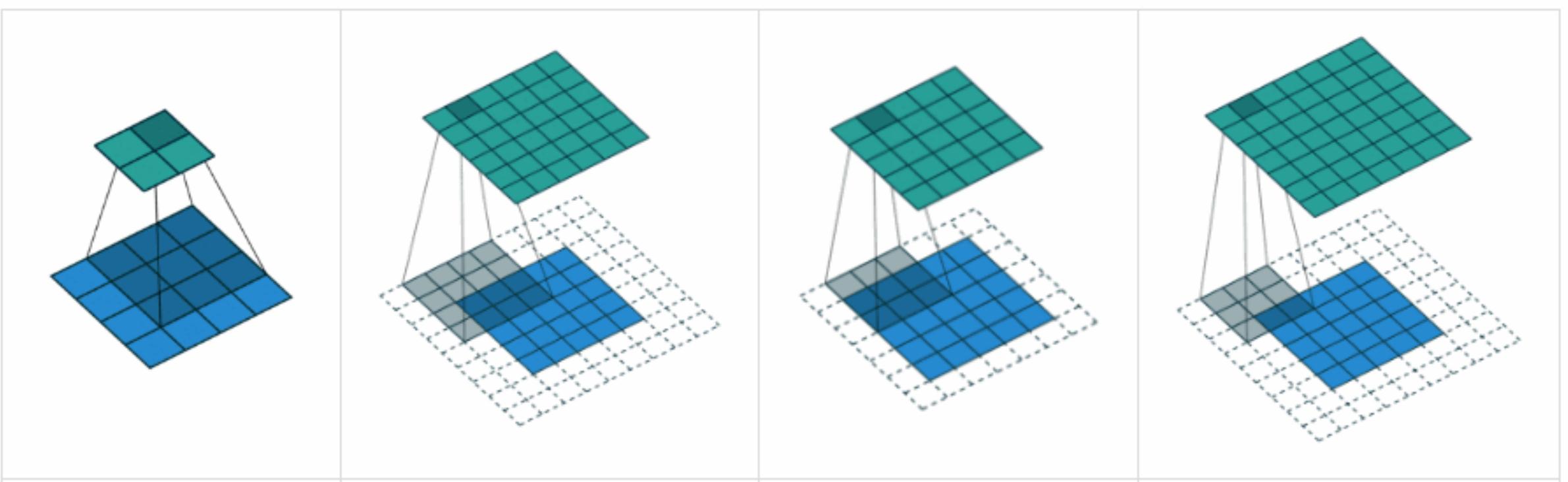
- Discrete – Sum not integrals
- Finite sequences
 - Padding
 - Zero
 - Mirror values
 - Constant
- How long is the output sequence?

Input	Impulse Response	Invert	Shift	
x_0, x_1, x_2, x_3, x_4	h_{-1}, h_0, h_1	h_1, h_0, h_{-1}	x_0, x_1, x_2, x_3, x_4	$y_0 = h_{-1} \cdot x_0$
			h_1, h_0, h_{-1}	$y_1 = h_0 \cdot x_0 + h_{-1} \cdot x_1$
				$y_2 = h_1 \cdot x_0 + h_0 \cdot x_1 + h_{-1} \cdot x_2$

Convolution

- Output sequence
 - If input sequence of length m , impulse response of length n ,
 - Full convolution
 - All positions where there is even a single overlap
 - Length of output is $m+n-1$
 - Valid convolution
 - Positions where impulse response is fully overlapped
 - $\max(m, n) - \min(m, n) + 1$
 - Same convolution
 - Pad minimally to ensure that length is
 - Length is $\max(m, n)$

- $y[k] = \sum_m x[m] h[k-m]$
- $y[m+n] = \sum_m x[m] h[n]$
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$$C[t] = \sum A[\tau]B[t-\tau]$$

$$C[t+\tau] = \sum A[\tau]B[t]$$

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