Question 1

Internalizing Min-cut Algorithm

Algorithm 1: Algorithm for $contract(e, G)$ in $O(|V|)$

Input: graph G and an edge $e = (x, y)$
Output: graph $G^{'}$ formed by compressing the edge e in G
$G \leftarrow \{L, n\}$ (L is the linked list defining graph, $n = |V|$)
$L[i] \leftarrow$ Linked list for vertex i
$M[i] \leftarrow$ A map used in merge step (initially set to 0)
$z \leftarrow x$ (the encoding of new node in compressed graph, we need to store this for extracting nodes finally)
$V \leftarrow$ Vertex set of G
Step 1: Merge Linked list of $\mathbf{x}, \mathbf{y}$
$L[z] \leftarrow \phi$
$S \leftarrow \phi$
$p \leftarrow L[x]$
$A \leftarrow \phi$
while $p \neq NULL$ **do**
  $M[p.\text{vertex}] \leftarrow M[p.\text{vertex}] + p.\text{weight}$
  $S \leftarrow S \cup \{p.\text{vertex}\}$
  $A \leftarrow A \cup \{p.\text{address}\}$
  $p \leftarrow p.\text{right}$
end
$p \leftarrow L[y]$
while $p \neq NULL$ **do**
  if $M[p.\textit{vertex}] = 0$ **then**
    $S \leftarrow S \cup \{p.\text{vertex}\}$
  $A \leftarrow A \cup \{p.\text{address}\}$
  $P[p.\text{vertex}] \leftarrow P[p.\text{vertex}] + p.\text{weight}$
  $p \leftarrow p.\text{right}$
end
At this stage M stores the weight of edges for z and S contain unique nodes
Step 2: Modify The linked list of all affected nodes
for $a \in A$ **do**
  delete the node stored at address a. (this denotes either a node from a vertex to x or y)
end
for $s \in S$ **do**
  $n_1 = \text{Create\_Node}(s, M[s], \text{NULL})$
  $n_2 = \text{Create\_Node}(z, M[s], \text{NULL})$
  $n_1.\text{address} = \text{address}(n_2)$
  $n_2.\text{address} = \text{address}(n_1)$
  $L[z] \leftarrow L[z] \cup \{n_1\}$
  $L[s] \leftarrow L[s] \cup \{n_2\}$
end
Step 3: Reset the map M for further use
for $s \in S$ **do**
  $M[s] \leftarrow 0$
end
$G^{'} \leftarrow \{L, n-1\}$ (the updated value of L)
return $G^{'}$

Brief overview: We store the graph in form of adjacency list. Compared to the standard adjacency list, we'll store linked list for each vertex, instead of list. In this linked list we'll store all the vertex which are connected to this vertex, along with weight (representing number of multi-edges), and a pointer. Suppose the linked list of vertex x contain a node of vertex y, then the pointer in this node will store the address of node in linked list of y that contain x. This pointer helps us delete a particular node in $O(1)$ time.
Time Complexity Analysis:
Size of S would be at most n because it contains unique vertices. So each of the steps $1, 2$ and 3 are being done in $O(|V|)$.Thus the time complexity of the overall algorithm would also be $O(|V|)$.

2.

Time Complexity :
Each contract step takes $O(n)$ and we are contracting $\frac{n}{2}$ times so this results in $O(n^2)$ work.
Thus if the time complexity of the algorithm is $T(n)$ then,

$$T(n) = O(n^2) + 2T(\frac{n}{2})$$

$$T(n) = O(n^2) + O(\frac{n^2}{2}) + 2T(\frac{n}{4})$$

$$T(n) = O(n^2) + O(\frac{n^2}{2}) + O(\frac{n^2}{8}) + \dots$$

$$T(n) = O(n^2)$$

Error Analysis :
Let the probability that algorithm preserves the min-cut be $q(n)$.
As shown in class, probability that min-cut is preserved after $\frac{n}{2}$ contractions is at least $\frac{1}{4}$. Thus, we can say that $q(n)$ is at least $\frac{1}{4}$ times the probability that at least one smaller recursive call of size $\frac{n}{2}$ preserves the min-cut. So,

$$q(n) \geq \frac{1}{4}\left(1 - \left(1 - q\left(\frac{n}{2}\right)\right)^2\right)$$

$$q(n) \geq \frac{1}{4}\left(2 - q\left(\frac{n}{2}\right)\right) q\left(\frac{n}{2}\right)$$

$$q(n) \geq \frac{1}{2} q\left(\frac{n}{2}\right) - \frac{1}{4} q\left(\frac{n}{2}\right)^2$$

Now let $p(k) = q(2^k)$. So we have,

$$p(k) \geq \frac{1}{2} p\left(k-1\right) - \frac{1}{4} p\left(k-1\right)^2$$

Call the above inequality to be I_1.
Since $p(k) \leq 1 \; \forall k$ we get

$$p(k) \geq \frac{1}{2} p\left(k-1\right) - \frac{1}{4} p\left(k-1\right)$$

or,

$$p(k) \geq \frac{1}{4} p\left(k-1\right)$$

By the above inequality and I_1 we have,

$$p(k) \geq \frac{1}{2} p\left(k-1\right) - p(k) p\left(k-1\right)$$

Thus,

$$\frac{1}{p(k)} \leq \frac{2}{p(k-1)} + 2$$

Base case - $p(1) = 1$

$$\frac{1}{p(k)} \leq 2\left(\frac{2}{p(k-2)} + 2\right) + 2$$

$$\frac{1}{p(k)} \leq \frac{2^2}{p(k-2)} + 2 + 2^2$$

In general,

$$\frac{1}{p(k)} \leq \frac{2^i}{p(k-i)} + 2 + 2^2 + \dots + 2^i$$

So,

$$\frac{1}{p(k)} \leq \frac{2^{k-1}}{p(1)} + 2 + 2^2 + \dots + 2^{k-1}$$

$$\frac{1}{p(k)} \leq 2^{k-1} + 2(2^{k-1} - 1)$$

$$\frac{1}{p(k)} \leq 2^{k} + 2^{k-1} - 2$$

Thus,

$$\frac{1}{q(n)} \leq n + \frac{n}{2} - 2$$

$$q(n) \geq \frac{2}{3n - 4}$$

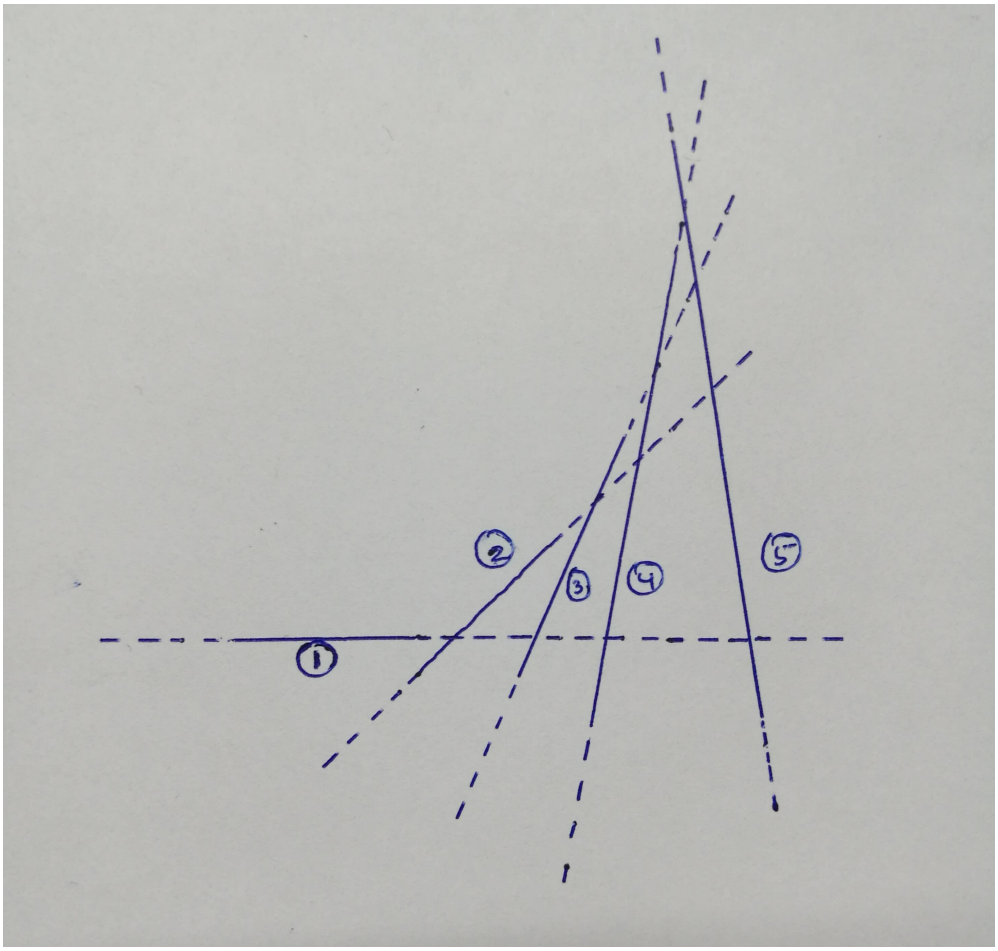
So the probability that the algorithm preserves min-cut is at least $\frac{2}{3n-4}$
To get error probability less than $\frac{1}{n^c}$ repeat our algorithm $cn\log(n)$ times.
Hence, the time complexity of the complete algorithm to get an inverse polynomial error bound is $O(n^3 \log(n))$.

Question 2

A surprising problem from computational geometry

1.

We will construct the line segments in order they are to be selected in the permutation. Let's say the first line segment is a horizontal line segment having its' left end at origin. Construct the second line segment such that it intersects the extended first line segment on the right side, and makes a small positive angle with the horizontal.
Construct the third line segment such that it will intersect both the previous extended line segments, this line segment makes an angle with the horizontal which is just greater than the angle made by the previous line segment with the horizontal.
Now consider that we have constructed $i-1$ line segments in this manner, construct the i^{th} line segment intersecting all previous extended line segments and making an angle just greater than the angle made by the $(i-1)^{th}$ line segment with the horizontal.
We can always construct such a line segment by chosing a suitable length and suitable angle with the horizontal for the i^{th} line segment. Do this till we get n line segments.
Note - We are considering angle with the horizontal in the anticlockwise sense.
This procedure will result in $\binom{n}{2}$ points of intersection as every pair of line segments will have a distinct point of intersection if constructed this way.
Thus, there exist a set of n segments and a permutation for which the above algorithm will cause $\binom{n}{2}$ points of intersections.
Here is an example construction for $n = 5$. The dots represent the extended part and the order in which the lines are picked is their label.



2.

Take any line segment, call it L. We will look at all points of intersection which we get after extending this segment towards right, analysis for left side is similar.
Label the line segments which intersect the right extended part of L to be $1, 2, ..., i, ...$ in order from left to right. We are considering the intersections after the procedure is completed.

Observation 1: If line segment L has a point of intersection with line labelled i then L should have been extended before all the line segments labelled 1 to $i-1$.

Using Observation 1 and the fact that initially line segments are permuted randomly uniformly(so the probability that any segment is picked first among a subset of size k is $\frac{1}{k}$), the probability that L has a point of intersection with line labelled i is at most $\frac{1}{i}$.
Consider random variable X_i taking value 1 if line segment i intersects with L otherwise 0.
So,

$$E[X_i] = P(X_i = 1) * 1 + P(X_i = 0) * 0 \implies E[X_i] \leq \frac{1}{i}$$

Let X be the number of points of intersection caused by line segment L when extended towards right.
By Linearity of Expectation,

$$E[X] = \sum_{i=1}^{n} E[X_i]$$

$$E[X] \leq \sum_{i=1}^{n} \frac{1}{i}$$

$$E[X] = O(\log n)$$

Similarly, we can get the same bound on the expected number of points of intersection caused by line segment L when extended towards left.
Thus expected number of points of intersection caused by the extension of line segment L is $O(\log n)$.

Observation 2: No two line segments intersected with each other initially, so all points of intersection will be due to extension of line segments.

Let the total number of points of intersection is Y.
By Observation 2 and linearity of expectation,

$$E[Y] = n * O(\log n)$$

$$E[Y] = O(n \log n)$$

Thus, for any arbitrary set of n line segments, if we permute them randomly uniformly and then execute the given procedure, the expected number of points of intersection will be $O(n \log n)$.

Question 3

Internalizing Backward Analysis and Randomized Incremental Construction (RIC)

A point x_i, y_i is said to dominate another point x_j, y_j if $x_i > x_j$ and $y_i > y_j$.
Non dominant point: A point is said to be non dominant if there is no point in P which dominates it.
So we need to find an expected $O(n \log n)$ algorithm to compute all non-dominanted point in P. We'll solve this using *Randomized Incremental Construction*, using conflict graphs.

Some notations used in analysis are G_i : denotes the conflict graph after i^{th} step, p_i : i^{th} point in our incremental construction, I_i : i^{th} interval on X axis.

Conflict Graph Details

Our conflict graph to solve this problem is based on interval division. Formally at i^{th} stage, conflict graph C_i will store 2-way mapping between set of points and set of intervals (division of X axis).

If $I_1 = x_0 = x_{min} \rightarrow x_1, I_2 = x_1 \rightarrow x_2, I_k \ldots x_{k-1} \rightarrow x_k = x_{max}$ denote the intervals at i^{th} stage (x_k being the point in P having largest x coordinate), then our conflict graph will store which interval, point p belongs to $\forall p \in P$. Also we store a list of all the points belonging to a particular interval. Lastly we'll also store the points which define this interval, ie $x_{i-1}, x_i \in P$ for $I_i, \forall i \in [1, k]$, and the interval height (defined by H_i for I_i interval, $H_i =$ maximum value of y coordinate among all points belonging to this interval). For a particular interval, we store the interval which is just left to it, in other words we can get the left neighbor of a interval in $O(1)$.
Note that these interval heights may change when we proceed incrementally.

Some operations and their complexity analysis are:

- **Locating interval of** $p_i \in P$: Well store explicit mapping of interval for each point, so this opearation will take $O(1)$ time.
- **Determining whether** G_{i+1} **will be different from** G_i : Note that G_{i+1} will change only when we change the intervals in our graph. Interval will change if y-coordinate of p_{i+1} is greater than H_t, where H_t is the height of interval at i^{th} stage to which p_{i+1} belong. We can get the interval to which p_{i+1} belongs in $O(1)$, also its height in $O(1)$, we determining whether G_{i+1} will be different will take $O(1)$ time.
- **Updating** G_{i+1}**:** We will update G_{i+1} only if p_{i+1} has height greater than interval height to which p_{i+1} belong to (let this interval be I_t). In that case we'll remove all the intervals to the left of I_t having height less than $y_{p_{i+1}}$ (including I_t), and introduce 2 new interval I_t^1, I_t^2. Lets assume that $I_1^{'}, I_2^{'}, \ldots I_m^{'}$ are the interval to the left of I_t that are removed, then $I_t^1 = (x_0 \rightarrow x_{p_{i+1}}), I_t^2 = (x_{p_{i+1}} \rightarrow x_t)$ (where x_0 is the left end of $I_1^{'}$ interval). Height of I_t^1 is equal to y-coordinate of p_{i+1} and height of I_t^2 is equal to height of I_t. We also need to update the mapping of all points belonging to the interval which are affected. Since we store the left neighbor of each interval, this operation can be done efficiently, however it may take $O(n)$ in worst case.

One last thing before we proceed further is that we'll also store the point p which is on the top right corner of a interval. We have seen above that this point p caused creation of this interval.
Note that after update we do not keep i^{th} point in the set. So after i^{th} round, in the set we have only $n - i$ points.

So using these details of implementation as background our algorithm to find non dominant points is:

Algorithm 2: RIC based algorithm for finding Non dominant points.

Input: $P = p_1, p_2, \ldots p_n$: set of n points in $2 - D$ plane
Output: Set of non dominant points in P
$G_1 = \{I = \{I_1\}, M, S\}$ (initial conflict graph, here $I_1 = (0 \rightarrow x_{mx})$, where x_{mx} is maximum X coordinate of all points in P)
/*
In subsequent step M will denote the map from set of points to set of intervals
I will denote the set of intervals
I_k will denote k^{th} interval, this will store data for height and interval limit(on X axis).
$H[I_k]$ will denote height of k^{th} interval.
I_k is defined by $x_{k-1} \rightarrow x_k$
$D[I_k]$ will denote the generating point for interval I_k
$S[I_k]$ will denote the set of all points belonging to interval I_k
*/
$S[I_1] = \phi$
for $i \leftarrow 1$ **to** n **do**
  $M[p_i] \leftarrow I_1$
  $S[I_1] \leftarrow S[I_1] \cup p_i$
end
$I_1 \leftarrow (x_{min} \rightarrow x_{max})$
$H[I_1] \leftarrow 0$
$D[I_k] \leftarrow p_{x_{mx}}$ // defined by point having maximum value of X coordinate. **for** $i \leftarrow 1$ **to** n **do**
  I_k = $M[p_i]$
  if $y_{p_i} > H[I_k]$ **then**
    // have to update G
    $I_{new} \leftarrow \phi$
    $I_{temp} \leftarrow I_k$
    while $H[I_{temp}] \leq y_{p_i}$ **do**
      $I_{temp} \leftarrow$ left neighbor of I_{temp}
      $I_{new} \leftarrow I_{new} \cup I_{temp}$
    end
    $I_k^1, I_k^2 \leftarrow$ new intervals
    update the left, right range and neighbor of I_k^1, I_k^2
    $H[I_k^1] \leftarrow y_{p_{i+1}}$
    $H[I_k^2] \leftarrow H[I_k]$
    for $I_t \in I_{new}$ **do**
      for $p \in S[I_t]$ **do**
        change mapping in M depending on y coordinate.
      end
    end
    $D[I_k^2] \leftarrow D[I_k]$
    $D[I_k^1] \leftarrow p_{i+1}$
    Update S value for these 2 interval, but be aware, not to include i^{th} point, because that will give a logarithmic factor compared to linear.
end
$T \leftarrow \phi$
for $Q \in I$ **do**
  $T = T \cup \{D[Q]\}$ // include defining point of this interval
end
return T

Time Complexity Analysis

Running time of i^{th} iteration is of the order of

- Number of intervals that are destroyed.
- Number of new intervals that are created.
- Number of points in the 2 new interval that get created.

Since every intervals destroyed was once created, so the total number of intervals destroyed $<$ total number of intervals created. Hence the total number of intervals created and destroyed $= O(n)$ (since only 2 intervals get created in each stage).

To compute number of points updated in each step we'll use backward analysis.
Let X_i be the random variable denoting the number of points for which the map is updated in i^{th} step.
Backward analysis of this problem is exactly same as that of convex hull.
a : a subset of i point of P
ϵ_a : first i points of P are some permutation of a.

$$\begin{aligned} E[X_i|\epsilon_a] = n_1 + n_2 \ldots (\frac{1}{i}) \\ n_2 + n_3 \ldots (\frac{1}{i}) \\ \ldots \\ n_k \ldots (\frac{1}{i}) \end{aligned}$$

Note that in the last one we have only n_k, because for the rightmost interval, we only have created one in the step.
Using the above analysis we get $E[X_i|\epsilon_a] \leq \frac{2(n-i)}{i}$ Let S_i be the set of all subsets of P of size i.
$E[X_i] = \sum_{a \in S_i} E[X_i|\epsilon_a].P(\epsilon_a) \leq \sum_{a \in S_i} \frac{2(n-i)}{i}.P(\epsilon_a) = \frac{2(n-i)}{i}$

Expected runtime of i^{th} iteration $= E[X_i] + O(1)$ ($O(1)$ is due to creation and deletion of intervals).

$$\begin{aligned} &= E[X_i] + O(1) \\ &= O(\frac{n-i}{i}) \end{aligned}$$

Expected running time of the algorithm $= \mathbf{O(n \log n)}$.