CS648 : Randomized Algorithms
CSE, IIT Kanpur

Final practice sheet

The topics are:

- **Routing**
- **Probabilistic methods**
- **Random walk and electric networks**
- **Pure fun ...**

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1. **Randomized algorithms are not designed using random idea**
   Recall the bit fixing protocol we discussed for permutation routing on n-hypercube. We showed that this protocol will take $\Omega(2^n/n)$ rounds for the transpose permutation : $ab \rightarrow ba$, where a and b are any two binary strings of length $n/2$ each. Now suppose some one suggests the following randomized algorithm based on any arbitrary random thought: Instead of fixing the bits from left to right, a packet fixes its bits in a uniformly random order independent of other packets. Unfortunately, this randomized protocol will be useless: It can be shown that it will also take $2^{\Omega(n)}/n$ rounds for the transpose permutation. Make a sincere attempt to establish this fact.
   (*Hint for this problem is given on the last page of this practice sheet.*)

2. **Monochromatic cliques**
   Prove that, for every integer n, there exists a coloring of the edges of the complete graph K_n by two colors so that the total number of monochromatic copies of K_4 is at most $\binom{n}{4}2^{-5}$.

3. **Line intersecting a circle**
   There are several circles of total circumference 10 inside a square of side length 1. Prove that there is a line that intersects at least 4 of the circles.
   (*Hint for this problem is given on the last page of this practice sheet.*)

4. **Large cut**
   In the lecture class, we showed that a graph having m edges has a cut of size at least $m/2$. In fact, this bound can be further improved slightly: If G has $2n+1$ vertices and m edges, then it has a cut of size at least $m(n+1)/(2n+1)$.
   (*Hint for this problem is given on the last page of this practice sheet.*)

5. **An application of alteration**
   Let G be an undirected graph on n vertices, with $nd/2$ edges. A subset of vertices is said to be an independent set of G if no two vertices of the set are neighbors in G. Consider the following probabilistic experiment for finding an independent set in G. Delete each vertex of G independently (together with all its edges) with probability $1 - 1/d$.

(a) Let V' and E' be the set of vertices and edges that remain after the deletion process. Calculate expected size of V' and E'.

(b) Perform some *alteration* step on V' to output an independent set.
Hint for this step is given on the last page of this practice sheet.

(c) What is the expected size of the independent set computed in the step 2 above ?

6. **Commute time of a dense graph**
Let G be an undirected graph on n vertices where degree of each vertex is at least $2n/3$. What is the maximum commute time in this graph ?

7. **Commute time of a lollipop graph**
Let G be a complete graph on n vertices and H be a line graph on n vertices. We join one end of the graph H with any arbitrary vertex of G. This graph is called lollipop graph. What is the maximum commute time of this graph. Calculate it using

   (a) the relation between random walks and electric networks.

   (b) appropriate equations from scratch.

   Compare the two expressions you get.

8. **Cover time of a graph**
Let $G = (V, E)$ be an undirected graph on n vertices and m edges. Let $C(v)$ be the expected number of steps of a uniform random walk originating from v to visit every vertex at least once. Let $C(G) = \max_{v \in V} C(v)$. Show that with very high probability that $C(G) = O(mR \log n)$ where R is the maximum effective resistance between any two nodes in the electric circuit associated with G.
(*Hint for this problem is given on the last page of this practice sheet.*)

9. **Hitting time on a directed graph**
Construct a strongly connected directed graph on n vertices with two vertices u and v in it such that the expected number of steps needed to reach v from u is exponential in n.

10. **Random permutation using few random bits.**
Let n be a prime number and let $S = \{1, 2, ..., n-1\}$. We are given an array A storing a permutation of S. We wish to permute A randomly such that the following condition is satisfied.

$$\mathbf{P}(A[i] = j) = \frac{1}{n-1}$$

    How will you do it using $O(\log n)$ random bits only ?

11. **Last fun problem**
Note: This problem is just for fun and you may skip it for the exam.

    You enter a shopping mall. There is a row of apples and you are standing at one end of this row. The row is so long that you can not see the other end of the row and hence don't know the exact number of apples in the row. You have a fair coin in your pocket and there is a positive integer k. How will you get out of the mall with a uniformly random sample of k apples subject to the following constraints:

    - You have a bag that can accommodate at most k apples at any time.
    - You are allowed to make only a single pass over the row of apples. (Note that once you discard an apple, you can not place it back in the bag).

    What is the expected number of coin tosses that you will have to make ?

Hint for Problem 1:
We shall bound the packets that pass through 0^n. For this we focus on all those packets that originate from address $a0^{n/2}$ where a is a binary string of length $n/2$.

1. Let P be the set of packets with source address $a0^{n/2}$ for any binary string a of length $n/2$. What is $|P|$?

2. Let P_k be the subset of P consisting of those packets whose source address has exactly k 1's. What is $|P_k|$?

3. Consider any packet $p \in P_k$. State the necessary and sufficient condition for the packet p to pass though 0^n while executing the randomized protocol described above.

4. What is the probability that p passes though 0^n ?

5. What is the expected number of packets from the set P_k that will pass though 0^n ?

6. Pick suitable value of $k = \Omega(n)$ and conclude that the expected number of packets passing through 0^n will be $2^{\Omega(n)}$.

(*Note:* make use of some formulas in the formula sheet to simplify calculations.)

Hint for Problem 3:
Choose any one side of the square. Select a random point on this side and draw a line perpendicular to the side. What will be expected number of circles it will intersect ?

Hint for Problem 4:
Use another simple randomized algorithm to partition the vertices. In particular, try to partition the set of vertices randomly uniformly but as evenly as possible.

Hint for Problem 5:

1. For each edge from E', remove any one of its endpoint from V'. Let V'' be the set of vertices left finally.

2. Show that V'' is an independent set of the original graph.

Hint for Problem 8:
Use the formula for commute time along with the tool of partitioning of an experiment into stages suitably.