CS648 : Randomized Algorithms
CSE, IIT Kanpur

Practice sheet 3

The topics are:

- coupon collector problem
- Partition experiment into stages
- Miscellaneous application of Chernoff bound and Chebyshev's Inequality

1. **Coupon collector problem**
   Recall the coupon collector problem with n different coupons. What is the expected number of coupons to be collected to have $0.99n$ different coupons ?

2. **Random Walk on a Lollipop graph**

   A lollipop graph on n vertices consists of a clique on $n/2$ vertices, and a path on the remaining vertices. There is a vertex u in the clique to which the path is attached; let v denote the other end of the path. What is the expected number of steps to reach v by a random walk starting from u; given that on reaching any vertex x, any of the neighbours of x are equally likely to be visited at the next step ? What is the expected number of steps to reach from v to u ? Conclude that the expected number of steps required to travel from a vertex u to v is not necessarily the same as the expected number of steps required to travel from v to u in a graph.

3. **Revisiting Randomized Quick Sort**

   Recall the randomized quick sort algorithm. Let e_i denote ith smallest element in the array. Let X_{ij} be the random variable which takes value 1 if e_i is compared with e_j, and zero otherwise.

   - Prove that the random variables $\{X_{ij} | 1 \le i < j \le n\}$ are not independent.
     Hint: Conditioned on "$X_{1n} = 1$", what can you say about Prob$[X_{1i} = 1 \text{ OR } X_{in} = 1]$.
   - What about the random variables $\{X_{1j} | 1 < j < n\}$?

4. **A biased random walk**

   Recall the random walk on a line discussed in the class. Suppose, probability of taking a step in the right direction is $3/4$ and the probability of taking a step in the left direction is $1/4$. What is the expected number of steps till the particle reaches n ?

5. **A variant of client server problem**

   Consider a parallel computer consisting of n processors and n memory modules. In the first step, each processor sends a memory request to a memory module selected randomly uniformly. If more than one processor sends request to the same memory module, the memory module discards all of them. If a memory module receives only one request, that request gets satisfied. After the first step, all those processors whose memory request was satisfied, leave the system. The remaining processors follow the same protocol in the following round. What is the expected number of rounds when all the memory request have been satisfied ?

6. **Set Balancing problem.**

   Given an $n \times n$ matrix A all of whose entries are 0 or 1, our aim is to compute a column vector $b \in \{-1, +1\}^n$ minimizing $\|Ab\|_\infty$. Here is a surprisingly simple algorithm: Construct the vector b as follows: each entry of b is independently and equiprobably chosen from $\{-1, +1\}$. Show that with very high probability, $\|Ab\|_\infty$ is going to be $O(\sqrt{n \ln n})$.

   (**Note:** $\|Ab\|_\infty$ evaluates to the element of the vector Ab with largest absolute value.)

7. Suppose that we roll a standard fair dice 100 times. Let X be the sum of the numbers that appear over the 100 rolls. Use Chebyshev's inequality to bound $\mathbf{P}(|X - 350| \geq 50)$.

8. Suppose $X_1, \ldots, X_n$ be n random variables defined over a probability space. Moreover, $\mathbf{E}[X_i \cdot X_j] = \mathbf{E}[X_i] \cdot \mathbf{E}[X_j]$. Prove that

$$\mathbf{Var}[\sum_i X_i] = \sum_i \mathbf{Var}[X_i].$$

9. Suppose X and Y are random variables defined over a probability space. Moreover, they are independent. Prove that $\mathbf{Var}[X - Y] = \mathbf{Var}[X] + \mathbf{Var}[Y]$.

10. Find an example of a random variable with finite expectation and unbounded variance.

11. Let Y be a nonnegative integer-values random variable with positive expectation. Prove that

$$\mathbf{E}[Y] \geq \mathbf{P}[Y \neq 0] \geq \frac{(\mathbf{E}[Y])^2}{\mathbf{E}[Y^2]}$$

12. Recall the ball-bin problem discussed many times in the course. Suppose the number of balls = number of bins = n. Let X be the random variable for the number of empty bins. Calculate $\mathbf{Var}[X]$.

13. Alice and Bob play checkers often. Alice is a better player, so the probability that she wins any game is 0.6, independent of all other games. they decide to play a tournament of n games. Give the best possible bound on the probability that Alice loses the tournament.

14. We have a standard six-sided dice. Let X denote the number of times that a 6 occurs over n throws of the dice. Let p be the probability of the event $X \geq n/4$. Compare the best upper bounds on p using Markov's inequality, Chebyshev's inequality, and Chernoff's bound.

15. **Hunters and rabbits** There are n hunters and n live rabbits. In each round each hunter targets a live rabbit picked uniformly randomly and independent of other hunters and shoots. After each round all the dead rabbits are removed. What is the expected number of rounds till all rabbits are dead ?