Contents

27. Boundary Conditions for Electric Fields

Under static condition, as all charges lie on the outer surface of the conductor, the $\overrightarrow{E}$ and $\overrightarrow{D}$ are zero within the conductor.

27.1 Boundary conditions at the conductor-dielectric interface

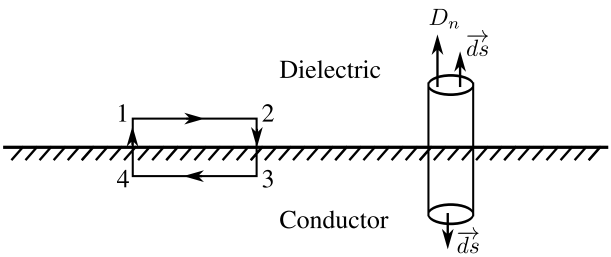


Fig. 27.1: Conductor-dielectric interface

We know that static electric field is a conservative field.

$$\text{i.e.,} \qquad \oint \overrightarrow{E} \bullet \overrightarrow{dL} = 0$$

Consider a closed rectangular path 12341 shown in Fig. ????

$$\oint \overrightarrow{E} \bullet \overrightarrow{dL} = \int_1^2 \overrightarrow{E} \bullet \overrightarrow{dL} + \int_2^3 \overrightarrow{E} \bullet \overrightarrow{dL} + \int_3^4 \overrightarrow{E} \bullet \overrightarrow{dL} + \int_4^5 \overrightarrow{E} \bullet \overrightarrow{dL} = 0$$

In Eqn. (27.1), the 3$^{\text{rd}}$ integral is zero because the path 3 to 4 is within the conductor where $\overrightarrow{E}$ is zero. If the path lengths 2 to 3 and 4 to 1 approaches zero (i.e., across boundary), the 2$^{\text{nd}}$ and 4$^{\text{th}}$ integral, in Eqn. (27.1) are zero.

$$\therefore \quad \text{We get} \quad \int_1^2 \overrightarrow{E} \bullet \overrightarrow{dL} = \int E_t dL = 0$$

Where E_t is the tangential component of $\overrightarrow{E}$ at the surface of the dielectric. From Eqn. , we get

$$E_t = 0 \quad \& \quad D_t = 0$$

Now consider a Gaussian right circular cylinder placed across the boundary as shown in Fig-. From Gauss's law,

$$\oint \overrightarrow{D} \bullet \overrightarrow{ds} = Q_{\text{enclosed}}.$$

$$\int_{\text{top}} \overrightarrow{D} \bullet \overrightarrow{ds} + \int_{\text{bottom}} \overrightarrow{D} \bullet \overrightarrow{ds} + \int_{\text{side}} \overrightarrow{D} \bullet \overrightarrow{ds} = \int_{s} P_s ds$$

In Eqn. (27.1), the 2^{nd} integral is zero because the bottom of the cylinder is within the conductor where $\overrightarrow{D}$ and $\overrightarrow{E}$ are zero. The 3^{rd} integral is zero because $D_t = 0$.

$$\therefore \oint_{\text{top}} \overrightarrow{D} \bullet \overrightarrow{ds} = \int_{\text{top}} D_n ds = \int_{s} \rho_s ds$$

$$\therefore D_n = \rho_s \;\&\; E_n = \frac{\rho_s}{\epsilon}$$

27.2 Boundary conditions at the dielectric-dielectric interface

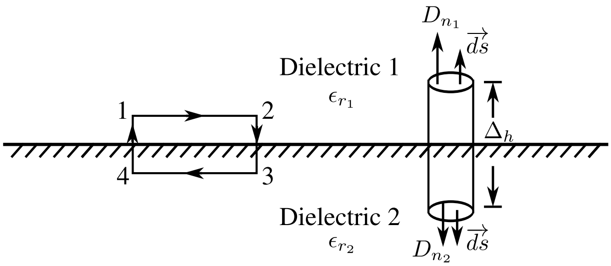


Fig. 27.2: Dielectric-dielectric interface

Consider a dielectric - dielectric interface as shown in Fig. 27.2

Applying Eqn. (27.2) to the closed path 12341,

$$\int_1^2 \overrightarrow{E} \bullet \overrightarrow{dL} + \int_2^3 \overrightarrow{E} \bullet \overrightarrow{dL} + \int_3^4 \overrightarrow{E} \bullet \overrightarrow{dL} + \int_4^1 \overrightarrow{E} \bullet \overrightarrow{dL} = 0$$

In Eqn. (27.2), if the path lengths 2 to 3 and 4 to 1 approaches zero (i.e.., across boundary), the 2^{nd} and 4^{th} integrals are zero.

$$\therefore \quad \int_1^2 \overrightarrow{E} \bullet \overrightarrow{dL} + \int_3^4 \overrightarrow{E} \bullet \overrightarrow{dL} = 0$$

$$\int E_{t_1} dL - \int E_{t_2} dL = 0$$

$$\int (E_{t_1} - E_{t_2})\, dL = 0$$

$$E_{t_1} - E_{t_2} = 0$$

$$\therefore \quad E_{t_1} = E_{t_2} \quad \& \quad \frac{D_{t_1}}{\varepsilon_{r_1}} = \frac{Dt_2}{\varepsilon_{r_2}}$$

Therefore across dielectric-dielectric interface tangential component of $\overrightarrow{E}$ is continuous cylindrical.

Applying Eqn .(**??**), to the Gaussian surface,

$$\int_{\text{top}} \overrightarrow{D} \bullet \overrightarrow{ds} + \int_{\text{bottom}} \overrightarrow{D} \bullet \overrightarrow{ds} + \int_{\text{side}} \overrightarrow{D} \bullet \overrightarrow{ds} = \int_{s} P_s ds$$

In Eqn. (**??**), as Δh approaches zero (i.e., across boundary) the 3$^{\text{rd}}$ integral in Eqn. ???? is zero. The RHS of Eqn. ???? is zero because is dielectric $\rho_s = 0$.

$$\int_{\text{top}} \overrightarrow{D} \bullet \overrightarrow{ds} + \int_{\text{bottom}} \overrightarrow{D} \bullet \overrightarrow{ds} = 0$$

$$\int Dn_1 ds - \int Dn_2 ds = 0$$

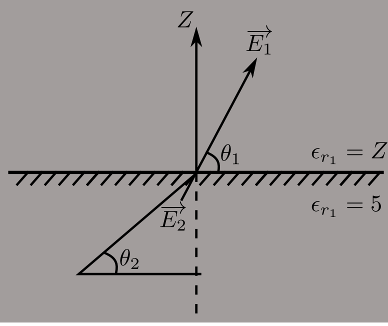
$$\int (D_{n_1} - D_{n_2}) ds = 0$$

$$D_{n_1} - D_{n_2} = 0$$

$$\therefore \quad D_{n_1} = D_{n_2} \quad \& \quad \varepsilon_{r_1} E_{n_1} = \varepsilon_{r_2} E_{n_2}.$$

Therefore across dielectric-dielectric interface normal component of $\overrightarrow{D}$ is continuous.

Problem 27.1 *Given that $\overrightarrow{E}_1 = 2\overrightarrow{a}_x - 3\overrightarrow{a}_y + 5\overrightarrow{a}_z$ across dielectric-dielectric interface shown in Fig.* **??**. *Find $\overrightarrow{E}_2, \overrightarrow{D}_1, \overrightarrow{D}_2$ and the angles θ_1 and θ_2.*



✍ **Solution**. $\overrightarrow{a}_x$ and $\overrightarrow{a}_y$ are tangential & $\overrightarrow{a}_2$ is normal to the interface

Given : $\overrightarrow{E}_1 = 2\overrightarrow{a}_x - 3\overrightarrow{a}_y + 5\overrightarrow{a}_z$

$$\therefore \quad \overrightarrow{D}_1 = \varepsilon_0 \varepsilon_{r_1} = 4\varepsilon_0 \overrightarrow{a}_x - 6\varepsilon_0 \overrightarrow{a}_y + 10\varepsilon_0 \overrightarrow{a}_z$$

From Eqns. ????? & ?????

$$\overrightarrow{E}_2 = 2\overrightarrow{a}_x - 3\overrightarrow{a}_y + \left(\frac{2}{5}\right) 5\overrightarrow{a}_z$$

$$\therefore \quad \overrightarrow{E}_2 = 2\overrightarrow{a}_x - 3\overrightarrow{a}_y + 2\overrightarrow{a}_z$$

$$\vec{D}_2 = \left(\frac{5}{2}\right) + \varepsilon_0 \vec{a}_x - \left(\frac{5}{2}\right) 6\varepsilon_0 \vec{a}_y + 10\varepsilon_0 \vec{a}_z$$

$$\therefore \quad \vec{D}_2 = 10\varepsilon_0 \vec{a}_x - 15\varepsilon_0 \vec{a}_y + 10\varepsilon_0 \vec{a}_z$$

From Fig. ????

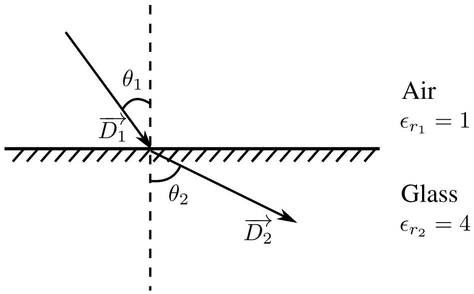
$$E_{z_1} = \vec{E}_1 \bullet \vec{a}_z = E_1 \cos(90 - \theta_1)$$
$$5 = \sqrt{2^2 + 3^2 + 5^2} \sin\theta_1$$
$$\theta_1 = 54.2^\circ$$

$$E_{z_2} = \vec{E}_2 \bullet \vec{a}_z = E_2 \cos(90 - \theta_2)$$
$$2 = \sqrt{2^2 + 3^2 + 2^2} \sin\theta_2$$
$$\theta_2 = 29.0^\circ$$

Problem 27.2 *At the boundary between given ($\varepsilon_r = 4$) and air, the lines of electric field makes an angle of 40° with normal to the boundary. If the electric flux density in air is 0.25μ c/m^2, determine the orientation and magnitude of electric flux density in glass.*

[VTU : Jan 2005, Aug 2001]

✍ **Solution**.



For dielectric-dielectric interface,

$$D_{t_1} = \frac{\varepsilon_{r_1}}{\varepsilon_{r_2}} \bullet D_{t_2}$$
$$D_1 \sin\theta_1 = \frac{1}{4} D_2 \bullet \sin\theta_2$$
$$D_{n_1} = D_{n_2}$$
$$D_1 \cos\theta_1 = D_2 \cos\theta_2$$
$$\therefore \quad \tan\theta_1 = \frac{1}{4} \tan\theta_2$$
$$\tan 40 = \frac{1}{4} \tan\theta_2$$

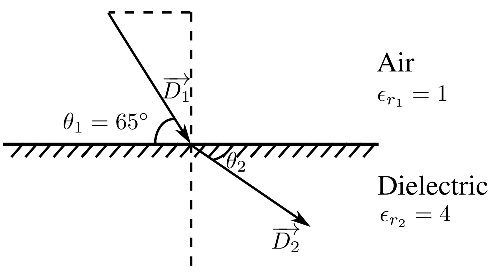
$$\therefore \quad \theta_2 = 73.41^\circ$$

Given that $D_1 = 0.25\mu$ c/m^2,
From: Eqn. ???,

$$0.25 \times 10^{-16} \times \cos 40 = D_2 \cos 73.41$$
$$\therefore \quad D_2 = 0.67\mu \text{c/m}^2$$

Problem 27.3 *An electric field strength 1.2 V/m is entering a dielectric medium of $\epsilon_r = 4$ from air. The orientation of the electric field in air is 65° w.r.t the boundary. Determine the orientation and the strength of the electric field in the dielectric medium.*

✍ **Solution.**



Across dielectric-dielectric boundary,

$$\begin{aligned}
E_{t_1} &= E_{t_2} \\
E_1 \cos\theta_1 &= E_2 \cos\theta_2 \\
\varepsilon_{r_1} E_{n_1} &= \varepsilon_{r_2} E_{n_2} \\
E_{n_1} &= \frac{\varepsilon_{r_2}}{\varepsilon_{r_1}} E_{n_2} \\
E_1 \sin\theta_1 &= \frac{\varepsilon_{r_2}}{\varepsilon_1} E_2 \sin\theta_2 \\
\therefore \quad \tan\theta_1 &= \frac{\varepsilon_{r_2}}{\varepsilon_1} \tan\theta_2 \\
\tan 65 &= \frac{4}{1} \tan\theta_2 \\
\therefore \quad \theta_2 &= 28.2^\circ
\end{aligned}$$

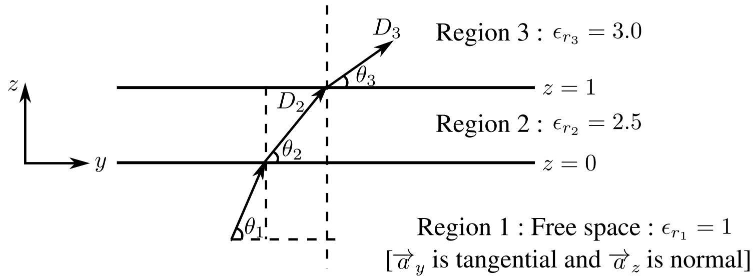
Given that $E_1 = 1.2 V m^{-1}$
From

$$1.2 \cos 65 = E_2 \cos 28.2$$
$$\therefore \quad E_2 = 0.575 V/m$$

Problem 27.4 *In region 1; $(z < 0m)$ (free space) where $D_1 = 5\overrightarrow{a}_y + 7\overrightarrow{a}_z c/m^2$; region 2; $(0 < z < 1m)$ has $\epsilon_{r_2} = 2.5$ and region 3; $(z > 1m)$ has $\epsilon_{r_3} = 3.0$*

Find E_1, D_2, E_2, D_3, E_3 and θ_3

✍ **Solution.**



Given:

$$\overrightarrow{D}_1 = 5\overrightarrow{a}_y + 7\overrightarrow{a}_z$$

$$\overrightarrow{E}_1 = \frac{\overrightarrow{D}_1}{\varepsilon_0} = \frac{1}{\varepsilon_0}[5\overrightarrow{a}_y + 7\overrightarrow{a}_z]$$

$$D_2 = \left(\frac{2.5}{1}\right)5\overrightarrow{a}_y + 7\overrightarrow{a}_z = 12.5\overrightarrow{a}_y + 7\overrightarrow{a}_z$$

$$\overrightarrow{E}_2 = \frac{1}{\varepsilon_0\varepsilon_{r_2}}\overrightarrow{D}_2 = \frac{1}{\varepsilon_0}\left[5\overrightarrow{a}_y + \frac{7}{2.5}\overrightarrow{a}_z\right]$$

$$\overrightarrow{D}_3 = \left(\frac{3}{2.5}\right)12.5\overrightarrow{a}_y + 7\overrightarrow{a}_z = 15\overrightarrow{a}_y + 7\overrightarrow{a}_z$$

$$\overrightarrow{E}_2 = \frac{1}{\varepsilon_0\varepsilon_{r_3}}\overrightarrow{D}_3 = \frac{1}{\varepsilon_0}\left[5\overrightarrow{a}_y + \frac{7}{3}\overrightarrow{a}_z\right]$$

From Fig. :

$$D_{z_3} = \overrightarrow{D}_3 \bullet \overrightarrow{a}_z = D_3\cos(90 - \theta_3)$$

$$7 = \sqrt{15^2 + 7^2}\sin\theta_3$$

$$\therefore \theta_3 = 25.02^\circ$$

(a) Thus current density,

$$\mathbf{J} = \frac{K}{\rho}\,\mathbf{a}_\rho = \frac{31.4658}{\rho}\,\mathbf{a}_\rho A/m^2$$

The electric field intensity in region $2 < \rho < 3$,

$$\mathbf{E}_1 = \frac{K}{100\rho}\,\mathbf{a}_\rho = \frac{31.4658}{100\rho}\,\mathbf{a}_\rho = \frac{0.3147}{\rho}\,\mathbf{a}_\rho V/m$$

For $3 < \rho < 6$, the electric field intensity

$$\mathbf{E}_2 = \frac{K}{25\rho}\,\mathbf{a}_\rho = \frac{31.4658}{25\rho}\,\mathbf{a}_\rho = \frac{1.2586}{\rho}\,\mathbf{a}_\rho V/m$$

(b) At $\rho = 2$ cm, the current crossing the cylindrical surface is

$$I = |\mathbf{J}| \times 2\pi\rho L = \frac{31.3658}{\rho} \times 2\pi\rho L$$
$$= 31.3658 \times 2\pi \times 0.5 = 98.80 \text{ A}$$

The total current is independent to radius of cylinder ρ. The potential between the cylinders is 1 V. Thus the resistance offered by the cylindrical resistance is

$$R = \frac{V}{I} = \frac{1}{98.80} = 0.01012\Omega = 10.12 \text{ m}\Omega$$

(c) The potential at $\rho = 3$ cm is

$$V = -\int_6^3 E_2.d\rho = -\int_6^3 \frac{31.4658}{25\rho}.d\rho$$
$$= \frac{31.4658}{25} ln\ 2 = 0.8724 \text{ V}$$

Problem 27.16 *The cross-section of the transmission line shown in Fig. 6.24 is drawn on a sheet of conducting paper with metallic paint. The sheet resistance is* 2500Ω *per square and the dimension a is* 1 *in. (a) Assuming a result for Prob. 6b of* $120pF/m$*, what total resistance would be measured between the metallic conductors drawn on the conducting paper? (b) What would the total resistance be if* $a = 0.6$ *cm?*

Given the metallic paint on the sheet of paper shown in Fig. 6.24. The resistance of sheet is 2500 per square, dimension $a = 1$ inch.

(a) The permittivity is $\epsilon_R = 1.6$. The calculated capacitance is 120 pF/m. Let the thickness of paper be t. The total capacitance offered is $C = 120t$ pF. The current is flowing from inner rim to outer rim in a non-uniform fashion.

The product of resistance and capacitance of a similar geometry is

$$RC = \frac{d}{\sigma S} \times \frac{\epsilon S}{d} = \frac{\epsilon_0 \epsilon_R}{\sigma}$$

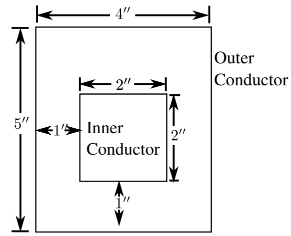


Fig. 27.3: Metallic paint on the sheet of paper-conduction from inner conductor to outer conductor

This gives

$$R = \frac{1.6 \times 8.85}{120t\sigma}$$

But the resistance of sheet is 2500 per square dimension $a = 1$ inch.

$$R_S = \frac{1}{\sigma \times t} = 2500\Omega \Rightarrow \sigma \times t = \frac{1}{2500}$$

Thus the resistance is

$$R = \frac{1.6 \times 8.85}{120t\sigma} = \frac{1.6 \times 8.85 \times 2500}{120} = 295\Omega$$

(b) If $a = 0.6$ cm, the resistance of sheet per square inch does not vary by changing the square size, because

$$R_S = \frac{1}{\sigma \times t} = 2500\Omega \Rightarrow \sigma \times t = \frac{1}{2500}$$

Thus the resistance is again

$$R = \frac{1.6 \times 8.85}{120t\sigma} = \frac{1.6 \times 8.85 \times 2500}{120} = 295\Omega$$

Problem 27.17 *The cross-section of a coaxial transmission line is drawn on a sheet of conducting paper using the dimensions $a = 0.5$ in and $b = 3$ in. Silver paint is used for the two conducting surfaces. If a resistance of 1000Ω is measured between conductors: (a) What is the resistance per square of the paper? (b) If the conducting layer on the paper is 0.008 in thick, what is the conductivity of the resistive coating?*

Dimension of coaxial line $a = 0.5$ in and $b = 3$ in. A silver paint is used for the two conducting surfaces.

(a) Here resistance between the conductor is 1000Ω.

Let the thickness of paint be t. The capacitance is

$$C = \frac{2\pi\epsilon_0}{ln(b/a)} \times t = \frac{2 \times \pi \times 8.85}{ln(3/0.5)} \times t = 31.09 \times t \text{ pF}$$

But $\quad RC = \frac{\epsilon_0}{\sigma}$

The resistance is

$$R = \frac{\epsilon_0}{\sigma C} = \frac{8.85 ln(6)}{2\pi \times 8.85\sigma t} = 1000$$

$$\Rightarrow \quad \sigma t = \frac{ln(6)}{2\pi \times 1000} = 2.83 \times 10^{-4}$$

But the resistance of sheet per square of the paper is

$$R_S = \frac{1}{\sigma \times t} = \frac{1}{2.83 \times 10^{-1}} = 3504.94\Omega$$

(b) The conducting layer of the coating is 0.008 in $= 0.008 \times 0.0254$ m thick. The conductivity of resistive coating is

$$\frac{1}{\sigma \times t} = 3504.94\Omega$$

or $\quad \sigma = \frac{1}{3504.94 \times t} = \frac{1}{3504.94 \times 0.008 \times 0.0254} = 1.404\mho/m$