Deformations of G_2-instantons on nearly G_2 manifolds

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Abstract. We study the deformation theory of G_2-instantons on nearly G_2 manifolds. There is a one-to-one correspondence between nearly parallel G_2 structures and real Killing spinors, thus the deformation theory can be formulated in terms of spinors and Dirac operators. We prove that the space of infinitesimal deformations of an instanton is isomorphic to the kernel of an elliptic operator. Using this formulation we prove that abelian instantons are rigid. Then we apply our results to describe the deformation space of the canonical connection on the four normal homogeneous nearly G_2 manifolds.

Contents

1 Introduction

Nearly parallel G_2 structures on a 7-manifold M are defined by a positive 3-form φ. A positive 3-form induces a metric g_φ and an orientation on M (see §2). When a non-zero real constant τ_0 is such that

$$d\varphi = \tau_0 *_\varphi \varphi,$$

then the 3-form φ defines a nearly parallel G_2 structure. The existence of such a G_2 structure on a manifold was shown to be equivalent to the existence of a spin structure with a real Killing spinor in [BFGK91]. Nearly G_2 manifolds are manifolds with a nearly parallel G_2 structure. Friedrich–Kath–Moroianu–Semmelmann in [FKMS97] showed that excluding the case of the 7-dimensional sphere there

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are three types of nearly parallel G$_2$ structures depending on the dimension of the space $K\$$ of all Killing spinors . The dimension of $K\$$ is bounded above by 3, giving rise to the three different types,

1. $\dim(K\$) = 1$: type 1 or *proper nearly* G$_2$ *manifolds*.
2. $\dim(K\$) = 2$: type 2 or *Sasaki-Einstein manifolds*.
3. $\dim(K\$) = 3$: type 3 or *3-Sasakian manifolds*.

Nearly G$_2$ manifolds were introduced as manifolds with weak holonomy G$_2$ by Gray in [Gra71]. These manifolds are positive Einstein. Some examples of such manifolds are

type 1:	$(S^7, \text{round}), (S^7, \text{squashed})$, the Aloff–Wallach spaces $N_{k,l}, k \neq l$
type 2:	$N_{1,1}$, Stiefel manifold $V_{5,2}$
type 3:	$\frac{\mathrm{SU}(3)\times\mathrm{SU}(2)}{\mathrm{SU}(2)\times\mathrm{U}(1)}$

The cones over these manifolds have holonomy contained in Spin(7), specifically $\mathrm{Spin}(7), \mathrm{SU}(4), \mathrm{Sp}(2)$ when the nearly parallel G$_2$ structure on the link is of type 1, 2, 3 respectively. This property makes these spaces particularly important in the construction and understanding of manifolds with torsion free Spin(7)-structures.

Let M^7 be a manifold with a G$_2$ structure φ and let η be the Killing spinor associated to φ. A connection A on M is a G$_2$*-instanton* if its curvature F_A satisfies the algebraic condition

$$F_A \wedge \varphi = *_\varphi F_A.$$

The above condition is equivalent to $F_A \cdot \eta = 0$ as shown in §3. When the G$_2$ structure is parallel (the case when the constant $\tau_0 = 0$) these instantons clearly solve the Yang–Mills equation $d^*_\nabla F = 0$. The analogous result was proved in the nearly G$_2$ case by Harland–Nölle [HN12]. They showed that the instantons on manifolds with real Killing spinors solve the Yang–Mills equation which makes the study of instantons on nearly G$_2$ manifolds important from the point of view of gauge theory in higher dimensions. However G$_2$-instantons in the parallel case are the minimizers of the Yang–Mills functional which is not necessarily true for the nearly parallel case, as proved by Ball–Oliveira in [BO19]. The first examples of G$_2$-instantons on parallel G$_2$ manifolds were constructed in [Cla14], [Wal16] and [SEW15]. In [BO19] the authors proved the existence of nearly G$_2$-instantons on certain Aloff–Wallach spaces and classified invariant G$_2$-instantons on these spaces with gauge group U(1) and SO(3). Recently, Waldron [Wal20] proved that the pullback of the standard instanton on S^7 obtained from ASD instantons on the 4-sphere via the quaternionic Hopf fibration lies in a smooth, complete, 15-dimensional family of G$_2$-instantons.

In [CH16] Charbonneau–Harland studied the infinitesimal deformation space of irreducible instantons with semi-simple structure group on nearly Kähler 6-manifolds by identifying it with the eigenspace of a Dirac operator. In this article, we investigate the infinitesimal deformation space of G$_2$-instantons on nearly G$_2$ manifolds by applying a similar approach. A significant difference between nearly G$_2$ manifolds and the 6 dimensional nearly Kähler manifolds is that the Killing spinors $\eta, \mathrm{vol}\cdot\eta$ are linearly dependent in the former and independent in the latter case. This prevents us from having a result like [CH16, Proposition 4(iii)] where the authors relate the λ^2-eigenspace of the square of the Dirac operator to the λ-eigenspace of the Dirac which makes the computation of the infinitesimal deformation space much more convenient. In fact we show in §4 that such a relation does not exist in the nearly G$_2$ case by explicitly computing the kernel of the elliptic operator for the homogeneous nearly G$_2$ manifolds.

In §2 we describe a 1-parameter family of connections on the spinor bundle $\$$ over nearly G$_2$ manifolds and the associated Dirac operators. In [DS20] the authors introduced a Dirac type operator and used it to completely describe the cohomology of nearly G$_2$ manifolds and proved the obstructedness of infinitesimal deformations of the nearly G$_2$ structure on the Aloff–Wallach space. We remark that the Dirac type operator introduced there is not associated to any connection in the 1-parameter family.

We prove the following main theorems for a nearly G$_2$-instanton A on a principal bundle $\mathcal{P}$ with curvature F_A. Let EM be a vector bundle associated to $\mathcal{P}$ and the Dirac operator $D^{-1,A}$ is as defined in (3.3).

Theorem 3.2. *The space of infinitesimal deformations of a* G_2*-instanton* A *on a principal bundle* $\mathcal{P}$ *over a nearly* G_2 *manifold is isomorphic to the kernel of the elliptic operator*

$$\left(D^{-1,A} + 2\,\mathrm{Id}\right) \colon \Gamma(\Lambda^1(\not{S}) \otimes EM) \to \Gamma(\Lambda^1(\not{S}) \otimes EM).$$

Theorem 3.7. *Any* G_2*-instanton* A *on a principal* G*-bundle over a compact nearly* G_2 *manifold* M *is rigid if*

(i) *the structure group* G *is abelian, or*

(ii) *all the eigenvalues of the operator*

$$\begin{aligned} L_A \colon \Lambda^1 \otimes Ad_{\mathcal{P}} &\to \Lambda^1 \otimes Ad_{\mathcal{P}} \\ w &\mapsto -2w \lrcorner F_A \end{aligned}$$

are greater than $-\frac{28}{5}$.

A similar result as above has been proved in [BO19, Proposition 8] when the structure group is abelian or the eigenvalues are less than 6. For the proof of the upper bound the authors used the Weitzenböck formula on the connection associated to the Levi-Civita connection and A. In contrast, the proof of the lower bound on the eigenvalue uses the Schrödinger–Lichnerowicz formula for the family of Dirac operators constructed in §3.

We describe the infinitesimal deformation space of the canonical connection on all the homogeneous nearly G_2 manifolds whose nearly G_2 metric is normal. By considering the actions of the Lie groups H and G_2 on G/H we can view the canonical connection as an H-connection or a G_2-connection. We compute its infinitesimal deformation spaces in both of these cases. The results are recorded in Theorem 4.6. It will be interesting to see if these infinitesimal deformations are genuine. As of now, the author is unaware of any known family of nearly G_2-instantons for which the infinitesimal deformations are the ones found in Theorem 4.6.

This article is organized as follows. In §2 we give some preliminaries on nearly parallel G_2 structures to set up conventions and notations. In §3 we introduce the infinitesimal deformation space of nearly parallel G_2-instantons on nearly G_2 manifolds and prove that it is isomorphic to the kernel of a Dirac operator. In §4 we prove Theorem 4.6.

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2 Preliminaries

2.1 Nearly parallel G_2 structures

Let M be a 7-dimensional Riemannian manifold equipped with a positive 3-form $\varphi \in \Omega^3_+(M)$. The 3-form φ induces an orientation and a metric on M and thus a Hodge star operator $*_\varphi$ on the space of differential forms (see [Bry87]). The G_2 structure φ is called a nearly parallel G_2 structure on M if it satisfies the following differential equation for some non-zero $\tau_0 \in \mathbb{R}$,

$$d\varphi = \tau_0 *_\varphi \varphi. \tag{2.1}$$

We denote the 4-form $*_\varphi \varphi$ by ψ in the remainder of this article. The condition $d\varphi = \tau_0\psi$ implies $\mathrm{d}\psi = 0$, thus the nearly parallel G_2 structure φ is co-closed.

Every manifold with a G_2 structure is orientable and spin, and thus admits a spinor bundle $\$$. Let ∇^{LC} be the Levi-Civita connection of the induced metric on M. A spinor $\eta \in \Gamma(\$)$ is a real Killing spinor if for some non-zero $\delta \in \mathbb{R}$,

$$\nabla^{LC}_X \eta = \delta X \cdot \eta \qquad \forall\ X \in \Gamma(TM). \tag{2.2}$$

There is a one-to-one correspondence between nearly parallel G_2 structures and real Killing spinors on M. Given a nearly parallel G_2 structure φ that satisfies (2.1) there exists a real Killing spinor η that satisfies (2.2) with $\delta = -\frac{1}{8}\tau_0$ and vice-versa. See [BFGK91] for more details.

The constant τ_0 can be altered by rescaling the metric and readjusting the orientation. In this article we use $\tau_0 = 4$. With this choice of τ_0 our nearly G_2 structure φ and Killing spinor η satisfies the following equations respectively

$$\begin{aligned} d\varphi &= 4\psi, \\ \nabla^{LC}_X \eta &= -\frac{1}{2} X \cdot \eta. \end{aligned} \tag{2.3}$$

Manifolds with nearly parallel G_2 structures have several nice properties which can be found in detail in [BFGK91]. In particular they are positive Einstein. Let g be the metric induced by φ, then the Ricci curvature $\mathrm{Ric}_g = \frac{3}{8}\tau_0^2 g$ and the scalar curvature $\mathrm{Scal}_g = 7\mathrm{Ric}_g = \frac{21}{8}\tau_0^2$. A G_2 structure on M induces a splitting of the spaces of differential forms on M into irreducible G_2 representations. The space of 2-forms $\Lambda^2(M)$ decomposes as

$$\Lambda^2(M) = \Lambda^2_7(M) \oplus \Lambda^2_{14}(M),$$

where Λ^2_l has pointwise dimension l. More precisely, we have the following description of the space of forms :

$$\begin{aligned} \Lambda^2_7(M) &= \{X \lrcorner \varphi \mid X \in \Gamma(TM)\} = \{\beta \in \Lambda^2(M) \mid *(\varphi \wedge \beta) = -2\beta\}, \\ \Lambda^2_{14}(M) &= \{\beta \in \Lambda^2(M) \mid \beta \wedge \psi = 0\} = \{\beta \in \Lambda^2(M) \mid *(\varphi \wedge \beta) = \beta\}. \end{aligned}$$

Note that we are using the convention of [Kar09] which is opposite to that of [Joy00] and [Bry06].

The space Λ^2_{14} is isomorphic to the Lie algebra of G_2 denoted by $\mathfrak{g}_2$. Since the group G_2 preserves the G_2 structure φ, it preserves the real Killing spinor η induced by φ. The space Λ^2_{14} can be equivalently defined as

$$\Lambda^2_{14} = \{\omega \in \Lambda^2 \mid \omega \cdot \eta = 0\}. \tag{2.4}$$

We make use of this identification when defining the instanton condition on M in §3.

2.2 The spinor bundle

For a 7-dimensional Riemannian manifold M with a nearly parallel G_2 structure φ, the spinor bundle $\$$ is a rank-8 real vector bundle over M and is isomorphic to the bundle $\underline{\mathbb{R}} \oplus TM = \Lambda^0 \oplus \Lambda^1$. At each point $p \in M$, we can identify the fiber of $\$$ with $\mathbb{R} \oplus T_pM \cong \mathbb{R} \oplus \mathbb{R}^7 \cong \mathrm{Re}(\mathbb{O}) \oplus \mathrm{Im}(\mathbb{O}) = \mathbb{O}$. If η is the real Killing spinor on M induced by φ then we have the isomorphism

$$\$ = (\Lambda^0 TM \cdot \eta) \oplus (\Lambda^1 TM \cdot \eta) \cong \Lambda^0 TM \oplus \Lambda^1 TM.$$

Under this isomorphism any spinor $s = (f \cdot \eta, \alpha \cdot \eta) \in \$$ can be written as $s = (f, \alpha) \in \Lambda^0 \oplus \Lambda^1$.

The 3-form φ induces a cross product $\times_\varphi$ on vector fields $X, Y \in \Gamma(TM)$. Throughout this article we use e_i to denote both tangent vectors and 1-forms, identified using the metric. All the computations are done in a local orthonormal frame $\{e_1, \ldots, e_7\}$ and any repeated indices are summed over all possible values. With respect to this local orthonormal frame, we have $(X \times_\varphi Y)_l = X_i Y_j \varphi_{ijl}$. The octonionic product of two octonions (f_1, X_1) and (f_2, X_2) is given by,

$$(f_1, X_1) \cdot (f_2, X_2) = (f_1 f_2 - \langle X_1, X_2 \rangle, f_1 X_2 + f_2 X_1 + X_1 \times X_2).$$

As shown in [Kar10] the Clifford multiplication of a 1-form Y and a spinor (f, Z) is the octonionic product of an imaginary octonion and an octonion and is thus given by

$$Y \cdot (f, Z) = -(-\langle Y, Z\rangle, fY + Y \times Z). \tag{2.5}$$

We define the Clifford multiplication of any p-form $\beta = \beta_{i_1 \ldots i_p} e_{i_1} \wedge e_2 \wedge \cdots \wedge e_{i_p}$ with a spinor by,

$$\beta \cdot (f, X) = \beta_{i_1 \ldots i_p}(e_{i_1} \cdot (e_{i_2} \cdot \ldots \cdot (e_{i_p} \cdot (f, X)) \ldots)).$$

We record an identity for Clifford algebras for later use and refer the reader to [LM89, Proposition 3.8, Ch1] for the proof.

Proposition 2.1. *For $\alpha \in \Lambda^p(M)$*

$$\sum_j e_j \cdot \alpha \cdot e_j = (-1)^{p+1}(n - 2p)\ \alpha.$$

The Clifford multiplication between a p-form α and a 1-form v can be written as [LM89, Proposition 3.9]

$$v \cdot \alpha = v \wedge \alpha - v \lrcorner \alpha.$$

The vector bundle $\$$ is a G_2-representation and since G_2 is the isotropy group of the 3-form φ the map $\mu \mapsto \varphi \cdot \mu$ from the bundle of spinors $\$$ to itself is an isomorphism. The same argument holds for the 4-form ψ.

Lemma 2.2. *The subbundles of $\$$ isomorphic to Λ^0 and Λ^1 are eigenspaces of the operations of Clifford multiplication by φ and ψ. The associated eigenvalues are*

	Λ^0	Λ^1
φ	7	-1
ψ	7	-1.

Proof. The bundle $\$$ is a G_2-representation. The spaces Λ^0, Λ^1 are its irreducible subrepresentations and thus are eigenspaces of the operators defined by the Clifford multiplication by φ, ψ respectively. By Schur's Lemma there exist real constants $\lambda_0, \lambda_1, \mu_0, \mu_1$ such that for all $f \in \Lambda^0, \alpha \in \Lambda^1$

$$\varphi \cdot f = \lambda_0 f, \quad \varphi \cdot \alpha = \lambda_1 \alpha,$$
$$\psi \cdot f = \mu_0 f, \quad \psi \cdot \alpha = \mu_1 \alpha.$$

Proposition 2.1 then implies $\sum_i e_i \cdot \varphi \cdot e_i = \varphi$ and $\sum_i e_i \cdot \psi \cdot e_i = \psi$ thus

$$\lambda_0 f = \varphi \cdot f = \sum_{i=1}^{7} e_i \cdot \varphi \cdot e_i \cdot f,$$

$$\mu_0 f = \psi \cdot f = \sum_{i=1}^{7} e_i \cdot \psi \cdot e_i \cdot f.$$

Using the fact that $e_i \cdot f \in \Lambda^1$ and summing over i we get

$$\lambda_0 + 7\lambda_1 = 0, \tag{R1}$$
$$\mu_0 + 7\mu_1 = 0. \tag{R2}$$

We find the eigenvalues corresponding to Λ^0 by explicit calculations and use relations (R1) and (R2) to show the result for Λ^1. Let $(f, 0) \in \Lambda^0$ be a spinor. In the local orthonormal frame $e_1, \ldots, e_7$, we have $\varphi = \frac{1}{6}\varphi_{ijk} e_i \wedge e_j \wedge e_k$, where φ_{ijk} is skew-symmetric in each pair of indices. Using (2.5) we get that

$$\varphi \cdot (f, 0) = \frac{1}{6}\varphi_{ijk} e_i \cdot (e_j \cdot (e_k \cdot (f, 0))) = -\frac{1}{6}\varphi_{ijk} e_i \cdot (e_j \cdot (0, f e_k))$$

$$= \frac{1}{6}\varphi_{ijk}e_i \cdot (-f\delta_{kj}, f\varphi_{jkt}e_t)$$
$$= -\frac{1}{6}\varphi_{ijk}(-f\varphi_{ijk}, -f\delta_{kj}e_i + f\varphi_{jkt}\varphi_{itp}e_p).$$

By using the skew-symmetry of φ and the contraction identities $\varphi_{ijk}\varphi_{ijl} = 6\delta_{kl}, \varphi_{ijk}\varphi_{ijk} = 42$ (see [Kar09]), we get

$$\varphi \cdot (f, 0) = \frac{1}{6}(42f, -6f\delta_{it}\varphi_{itp}e_p) = (7f, 0).$$

Similarly, in above local orthonormal frame, $\psi = \frac{1}{24}\psi_{ijkl}e_i \wedge e_j \wedge e_k \wedge e_l$ and using (2.5) we get

$$\begin{aligned}
\psi \cdot (f, 0) &= \frac{1}{24}\psi_{ijkl}e_i \cdot (e_j \cdot (e_k \cdot (e_l \cdot (f, 0)))) = -\frac{1}{24}\psi_{ijkl}e_i \cdot (e_j \cdot (e_k \cdot (0, fe_l)))\\
&= \frac{1}{24}\psi_{ijkl}e_i \cdot (e_j \cdot (-f\delta_{kl}, f\varphi_{klp}e_p))\\
&= -\frac{1}{24}\psi_{ijkl}e_i \cdot (-f\varphi_{klp}\delta_{jp}, -f\delta_{kl}e_j + f\varphi_{klp}\varphi_{jpt}e_t)\\
&= \frac{1}{24}\psi_{ijkl}(f\delta_{kl}\delta_{ij} - f\varphi_{klp}\varphi_{jpt}\delta_{it}, f\varphi_{klp}\delta_{jp}e_i - f\delta_{kl}\varphi_{ijs}e_s + f\varphi_{klp}\varphi_{jpt}\varphi_{itr}e_r).
\end{aligned}$$

Here we can use the skew-symmetry of ψ, the contraction identity $\psi_{ijkl}\varphi_{klp} = -4\varphi_{ijp}$ along with the contraction identities of φ mentioned before to obtain

$$\psi \cdot (f, 0) = \frac{1}{24}(24\delta_{il}\delta_{il}f, 0) = \frac{1}{24}(24.7f, 0) = (7f, 0).$$

Substituting $\lambda_0 = 7$ and $\mu_0 = 7$ in Relations (R1), (R2) respectively proves the desired result. □

A common feature between nearly Kähler 6-manifolds and manifolds with nearly parallel G$_2$ structures is the presence of a unique canonical connection ∇^{can} with totally skew-symmetric torsion defined below. The Killing spinor η is parallel with respect to this connection and thus we have $\mathrm{Hol}(\nabla^{can}) \subset \mathrm{G}_2$. It was proved by Cleyton–Swann in [CS04, Theorem 6.3] that a G-irreducible Riemannian manifold (M, g) with an invariant skew-symmetric non-vanishing intrinsic torsion falls in one of the following categories:

1. it is locally isometric to a non-symmetric isotropy irreducible homogeneous space, or,

2. it is a nearly Kähler 6-manifold, or,

3. it admits a nearly parallel G$_2$ structure.

For the nearly G$_2$ manifold (M, φ) we define a 1-parameter family of connections on TM that include the canonical connection ∇^{can}. Let $t \in \mathbb{R}$ and let ∇^t be the 1-parameter family of connections on TM defined for all $X, Y, Z \in \Gamma(TM)$ by

$$g(\nabla^t_X Y, Z) = g(\nabla^{LC}_X Y, Z) + \frac{t}{3}\varphi(X, Y, Z). \tag{2.6}$$

Let T^t be the torsion $(1, 2)$-tensor of ∇^t. Since the connection ∇^{LC} is torsion free

$$\begin{aligned}
g(X, T^t(Y, Z)) &= g(X, \nabla^t_Y Z) - g(X, \nabla^t_Z Y) - g(X, [Y, Z])\\
&= g(\nabla^{LC}_Y Z, X) + \frac{t}{3}\varphi(Y, Z, X) - g(\nabla^{LC}_Z Y, X) - \frac{t}{3}\varphi(Z, Y, X)\\
&\quad - g(X, [Y, Z])\\
&= \frac{2t}{3}\varphi(X, Y, Z).
\end{aligned}$$

Therefore the torsion tensor T^t is given by

$$T^t(X,Y) = \frac{2t}{3}\varphi(X,Y,\cdot) \tag{2.7}$$

which is proportional to φ and is thus totally skew-symmetric.

By [LM89, Theorem 4.14] the lift of the connection ∇^t on the spinor bundle which is also denoted by ∇^t acts on sections μ of $\not{S}$ as

$$\nabla^t_X\mu = \nabla^{LC}_X\mu + \frac{t}{6}(i_X\varphi)\cdot\mu. \tag{2.8}$$

The space of real Killing spinors is isomorphic to Λ^0 thus for a Killing spinor η it follows from (2.3) and Lemma 2.2 that for any vector field X since $X\cdot\varphi+\varphi\cdot X = -2\ i_X\varphi$,

$$\begin{aligned}
\nabla^t_X\eta &= \nabla^0_X\eta + \frac{t}{6}(i_X\varphi)\cdot\eta \\
&= -\frac{1}{2}X\cdot\eta - \frac{t}{12}(X\cdot\varphi+\varphi\cdot X)\cdot\eta \\
&= -\frac{1}{2}X\cdot\eta - \frac{t}{12}(7X\cdot\eta - X\cdot\eta) \\
&= -\frac{t+1}{2}X\cdot\eta.
\end{aligned}$$

Therefore η is parallel with respect to the connection ∇^{-1}. The connection ∇^{-1} thus has holonomy group contained in G_2 with totally skew-symmetric torsion and is therefore the canonical connection on the nearly G_2 manifold M described in [CS04].

Proposition 2.3. *The Ricci tensor* Ric^t *of the connection* ∇^t *is given by*

$$\mathrm{Ric}^t = (6 - \frac{2t^2}{3})g.$$

Proof. By using the expression of the Ricci tensor for a connection with a totally skew-symmetric torsion from [FI02], we have

$$\mathrm{Ric}^t(X,Y) = \mathrm{Ric}^0(X,Y) - \frac{t}{3}\mathrm{d}^*\varphi(X,Y) - \frac{2t^2}{9}g(i_X\varphi, i_Y\varphi)$$

The Ricci tensor for the Levi-Civita connection is given by $\mathrm{Ric}^0 = 6g$. Since $d\psi = 0$, φ is co-closed and the second term in the above expression vanishes. The third term can be calculated in a local orthonormal frame $e_1,\ldots,e_7$ using the contraction identity $\varphi_{ijk}\varphi_{ijl} = 6\delta_{kl}$ as follows

$$\begin{aligned}
g(i_X\varphi, i_Y\varphi) &= \frac{1}{4}\sum_{i,j,k,\alpha,\beta,\gamma} X_kY_\gamma\varphi_{ijk}\varphi_{\alpha\beta\gamma}g(e_i\wedge e_j, e_\alpha\wedge e_\beta) \\
&= \frac{1}{4}\sum_{i,j,k,\gamma} X_kY_\gamma(\varphi_{ijk}\varphi_{ij\gamma} - \varphi_{ijk}\varphi_{ji\gamma}) \\
&= 3\sum_{k,\gamma} X_kY_\gamma\delta_{k\gamma} = 3g(X,Y).
\end{aligned}$$

Summing up all the terms together give the desired identity for Ric^t. □

3 Deformation theory of instantons

Let $\mathcal{P}\to M$ be a principal K-bundle. We denote by $\mathrm{Ad}_\mathcal{P}$ the adjoint bundle associated to $\mathcal{P}$. Let A be a connection 1-form on $\mathcal{P}$ and $F_A \in \Gamma(\Lambda^2T^*M\otimes\mathrm{Ad}_\mathcal{P})$ be the curvature 2-form associated to A given by

$$F_A = \mathrm{d}A + \frac{1}{2}[A\wedge A].$$

There are many ways to define the instanton condition on A. If (M, g) is equipped with a G-structure such that $G \subset \mathrm{O}(n)$, there is a subbundle $\mathfrak{g}(T^*M) \subset \Lambda^2 T^*M$ whose fibre is isomorphic to $\mathfrak{g} = \mathrm{Lie}(G)$. The connection A is an instanton if the 2-form part of F_A belongs to $\mathfrak{g}(T^*M)$. In global terms, A is an instanton if

$$F_A \in \Gamma(\mathfrak{g}(T^*M) \otimes \mathrm{Ad}_{\mathcal{P}}) \subset \Gamma(\Lambda^2 T^*M \otimes \mathrm{Ad}_{\mathcal{P}}).$$

Note that in dimension 7 if M is equipped with a G_2 structure then this condition implies that A is an instanton if the 2-form part of $F_A \in \mathfrak{g}_2(T^*M) = \Gamma(\Lambda^2_{14})$.

The second definition of an instanton is a special case of the first when the Lie algebra $\mathfrak{g}$ is simple. Its quadratic Casimir is a G-invariant element of $\mathfrak{g} \otimes \mathfrak{g}$ which may be identified with a section of $\Lambda^2 \otimes \Lambda^2$ and thus to a 4-form Q by taking a wedge product. Since this Q is G-invariant the operator $u \to *(*Q \wedge u)$ acting on 2-forms commutes with the action of G and hence by Schur's Lemma the irreducible representations of G in Λ^2 are eigenspaces of the operator. Then F_A is an instanton if

$$*(*Q \wedge F_A) = \nu F_A.$$

for some $\nu \in \mathbb{R}$. In dimension 7 it turns out that $Q = \psi$ (see [HN12]) and the above condition is equivalent to $F_A \in \Gamma(\Lambda^2_{14})$ when $\nu = 1$.

Furthermore if M is a spin manifold, and the spinor bundle admits one or more non-vanishing spinors η, then A is an instanton if

$$F_A \cdot \eta = 0.$$

When M has a G_2 structure and η is the corresponding spinor, (2.4) implies that a the above condition is satisfied if and only if A is a G_2-instanton. An interested reader can read further on these definitions and their relations in [HN12].

We remark that for an instanton A on a manifold with a G_2 structure φ all the above definitions are equivalent. They all imply that the curvature F_A associated to A lies in $\Gamma(\Lambda^2_{14})$ and thus satisfies *all* of these equivalent conditions:

$$\begin{aligned} F_A \cdot \eta &= 0, \\ F_A \wedge \varphi &= *F, \\ F_A \wedge \psi &= 0, \\ F_A \lrcorner \varphi &= 0. \end{aligned} \tag{3.1}$$

From now on in this article we use these instanton conditions interchangeably according to the context without further specification. Note that the above definitions are valid for any general G_2 structure and not only for nearly parallel ones.

On a manifold with real Killing spinors it was shown in [HN12] that instantons solve the Yang–Mills equation. In the case of a nearly G_2-instanton we can prove this fact by direct computation. For an instanton A, (3.1) and the second Bianchi identity imply

$$\begin{aligned} (d^A)^* F_A &= *d^A * F_A \\ &= *d^A(\varphi \wedge F_A) \\ &= 4 * (\psi \wedge F_A) = 0. \end{aligned}$$

3.1 Infinitesimal deformation of instantons

Let M^7 be a nearly G_2 manifold. We are interested in studying the infinitesimal deformations of nearly G_2-instantons on M. An infinitesimal deformation of a connection A represents an infinitesimal change in A and thus, is a section of $T^*M \otimes Ad_{\mathcal{P}}$. If $\epsilon \in \Gamma(T^*M \otimes Ad_{\mathcal{P}})$ is an infinitesimal deformation of A, the corresponding change in the curvature F_A up to first order is given by $\mathrm{d}^A\epsilon$. A standard gauge fixing condition on this perturbation is given by $(\mathrm{d}^A)^*\epsilon = 0$. So in total the pair of equations whose solutions define an infinitesimal deformation of an instanton A is given by

$$(\mathrm{d}^A\epsilon)\cdot\eta = 0, \quad (\mathrm{d}^A)^*\epsilon = 0. \tag{3.2}$$

On a nearly G_2 manifold we can define a 1-parameter family of Dirac operators

$$D^{t,A} = D^A + \frac{t}{2}\varphi\cdot$$

The 1-parameter family of connections on the spinor bundle $\not{S}$ defined in (2.8) and the connection A on $\mathcal{P}$ can be used to construct a 1-parameter family of connections on the associated vector bundle $\not{S}\otimes Ad_{\mathcal{P}}$. We denote by $\nabla^{t,A}$, the connection induced by ∇^t and A for all $t\in\mathbb{R}$ respectively. We denote by $D^{t,A}$ the Dirac operator associated to $\nabla^{t,A}$. The following proposition associates the solutions to (3.2) to a particular eigenspace of $D^{t,A}$ for each t. The proposition was proved in [Fri12] for $t=0$.

Proposition 3.1. *Let ϵ be a section of $T^*M\otimes Ad_{\mathcal{P}}$, and let $D^{t,A}$ be the Dirac operator constructed from the connections $\nabla^{t,A}$ for $t\in\mathbb{R}$. Then ϵ solves (3.2) if and only if*

$$D^{t,A}(\epsilon\cdot\eta) = -\frac{t+5}{2}\epsilon\cdot\eta. \tag{3.3}$$

Proof. Let $\{e_a, a = 1\dots 7\}$ be a local orthonormal frame for T^*M. Then

$$\begin{aligned} D^{0,A}(\epsilon\cdot\eta) &= e_a\cdot\nabla^0_a(\epsilon\cdot\eta)\\ &= (e_a\cdot\nabla^0_a\epsilon)\cdot\eta + e_a\cdot\epsilon\cdot\nabla^0_a\eta\\ &= (\mathrm{d}^A\epsilon + (\mathrm{d}^A)^*\epsilon)\cdot\eta + e_a\cdot\epsilon\cdot\nabla^0_a\eta. \end{aligned}$$

Applying Proposition 2.1 to the 1-form part of ϵ we get $e_a\cdot\epsilon\cdot e_a\cdot\eta = 5\epsilon\cdot\eta$. So if η is a real Killing spinor then (2.3) together with the above identity imply

$$\begin{aligned} D^{0,A}(\epsilon\cdot\eta) &= (\mathrm{d}^A\epsilon + (\mathrm{d}^A)^*\epsilon)\cdot\eta - \frac{1}{2}e_a\cdot\epsilon\cdot e_a\cdot\eta\\ &= (\mathrm{d}^A\epsilon + (\mathrm{d}^A)^*\epsilon - \frac{5}{2}\epsilon)\cdot\eta. \end{aligned}$$

It follows from (2.8) and the identity $\sum_a e_a\cdot i_a\varphi = 3\varphi$ that

$$D^{t,A} = D^{0,A} + \frac{t}{2}\varphi\cdot$$

Since $\epsilon\cdot\eta\in\Lambda^1\cdot\eta$, by Lemma 2.2 we have

$$D^{t,A}(\epsilon\cdot\eta) = \left(\mathrm{d}^A\epsilon + (\mathrm{d}^A)^*\epsilon + \frac{-t-5}{2}\epsilon\right)\cdot\eta.$$

The equation $D^{t,A}(\epsilon\cdot\eta) = -\frac{t+5}{2}\epsilon\cdot\eta$ is thus equivalent to $(\mathrm{d}^A\epsilon + (\mathrm{d}^A)^*\epsilon)\cdot\eta = 0$, which in turn is equivalent to the pair of equations $(\mathrm{d}^A\epsilon)\cdot\eta = 0, (\mathrm{d}^A)^*\epsilon = 0$ since these two components live in complementary subspaces. □

Since η is parallel with respect to ∇^{-1} we can view $D^{-1,A}$ as an operator on $\Lambda^1\otimes Ad_P$ defined by $D^{-1,A}(\epsilon\cdot\eta) = (D^{-1,A}\epsilon)\cdot\eta$. The following theorem is an immediate consequence of the above proposition.

Theorem 3.2. *The space of infinitesimal deformations of a G_2-instanton A on a principal bundle $\mathcal{P}$ over a nearly G_2 manifold M is isomorphic to the kernel of the operator*

$$\left(D^{-1,A} + 2\ \mathrm{Id}\right) : \Gamma(\Lambda^1\otimes Ad_{\mathcal{P}}) \to \Gamma(\Lambda^1\otimes Ad_{\mathcal{P}}). \tag{3.4}$$

Remark 3.3. By Proposition 3.1, the $-\frac{t+5}{2}$ eigenspace of the operator $D^{t,A}$ on $\Lambda^1 \cdot \eta \otimes Ad_{\mathcal{P}}$ is isomorphic to the infinitesimal deformation space of the instanton A for all $t \in \mathbb{R}$ and all these eigensapces are thus isomorphic to each other. In particular

$$\ker(D^{-1/3,A} + \frac{7}{3}\mathrm{id}) \cong \ker(D^{-1,A} + 2\mathrm{id}). \tag{3.5}$$

We can obtain an expression for the square of the Dirac operators constructed above using the Schrödinger–Lichnerowicz formula in the case of skew-symmetric torsion obtained by Agricola–Friedrich in [AF04]. The proof adapted to our setting is presented to keep the discussion self contained.

Proposition 3.4. *Let EM be a vector bundle associated to $\mathcal{P}$ and $\mu \in \Gamma(\$ \otimes EM)$. Let A be any connection on $\mathcal{P}$. Then for all $t \in \mathbb{R}$,*

$$(D^{t/3,A})^2\mu = (\nabla^{t,A})^*\nabla^{t,A}\mu + \frac{1}{4}\mathrm{Scal}_g\mu + \frac{t}{6}\mathrm{d}\varphi \cdot \mu - \frac{t^2}{18}\|\varphi\|^2\mu + F \cdot \mu. \tag{3.6}$$

Proof. Let $\{e_1, \dots, e_7\}$ be an orthonormal frame for the tangent bundle. As before we obtain

$$D^{t,A}\mu = (D^{0,A} + \frac{t}{2}\varphi\cdot)\mu.$$

Squaring both sides we obtain,

$$\begin{aligned}(D^{t/3,A})^2\mu &= \left(D^{0,A} + \frac{t}{6}\varphi\cdot\right)^2\mu \\ &= (D^{0,A})^2\mu + \frac{t}{6}(D^{0,A}(\varphi\cdot\mu) + \varphi \cdot D^{0,A}\mu) + \frac{t^2}{36}\varphi\cdot\varphi\cdot\mu.\end{aligned}$$

The first term of the above expression is given by the Schrödinger–Lichnerowicz formula

$$(D^{0,A})^2\mu = (\nabla^{0,A})^*\nabla^{0,A}\mu + \frac{1}{4}\mathrm{Scal}_g\mu + F\cdot\mu. \tag{E1}$$

The anti-commutator in the second term is given by

$$\begin{aligned}D^{0,A}(\varphi\cdot\mu) + \varphi\cdot D^{0,A}\mu &= e_a \cdot \nabla^{0,A}_a(\varphi\cdot\mu) + \varphi\cdot e_a\cdot\nabla^{0,A}_a\mu \\ &= (e_a\cdot\nabla^{0,A}_a\varphi)\cdot\mu + (e_a\cdot\varphi + \varphi\cdot e_a)\cdot\nabla^{0,A}_a\mu \\ &= \mathrm{d}\varphi\cdot\mu + \mathrm{d}^*\varphi\cdot\mu - 2(e_a \lrcorner \varphi)\cdot\nabla^{0,A}_a\mu\end{aligned} \tag{E2}$$

but since M is nearly G_2, φ is coclosed, therefore

$$D^{0,A}(\varphi\cdot\mu) + \varphi\cdot D^{0,A}\mu = \mathrm{d}\varphi\cdot\mu - 2(e_a \lrcorner \varphi)\cdot\nabla^{0,A}_a\mu$$

For the 3-form φ, $\varphi\cdot\varphi = \|\varphi\|^2 - (e_a\lrcorner\varphi)\wedge(e_a\lrcorner\varphi)$ and $(e_a\lrcorner\varphi)\cdot(e_a\lrcorner\varphi) = -3\|\varphi\|^2 + (e_a\lrcorner\varphi)\wedge(e_a\lrcorner\varphi)$ which imply

$$\begin{aligned}\varphi\cdot\varphi\cdot\mu &= \|\varphi\|^2\mu - (e_a\lrcorner\varphi)\wedge(e_a\lrcorner\varphi)\cdot\mu, \\ &= \|\varphi\|^2\mu - ((e_a\lrcorner\varphi)\cdot(e_a\lrcorner\varphi) + 3\|\varphi\|^2)\cdot\mu \\ &= -2\|\varphi\|^2\mu - (e_a\lrcorner\varphi)\cdot(e_a\lrcorner\varphi)\cdot\mu.\end{aligned} \tag{E3}$$

At the center of a normal frame,

$$\begin{aligned}(\nabla^{t,A})^*\nabla^{t,A}\mu &= -(\nabla^{0,A}_a + \frac{t}{6}(e_a\lrcorner\varphi))(\nabla^{0,A}_a + \frac{t}{6}(e_a\lrcorner\varphi))\mu \\ &= -\nabla^{0,A}_a\nabla^{0,A}_a\mu - \frac{t}{6}(e_a\lrcorner\varphi)\cdot\nabla^{0,A}_a\mu - \frac{t}{6}\nabla^{0,A}_a((e_a\lrcorner\varphi)\cdot\mu) \\ &\quad - \frac{t^2}{36}(e_a\lrcorner\varphi)\cdot(e_a\lrcorner\varphi)\cdot\mu\end{aligned}$$

$$= (\nabla_a^{0,A})^* \nabla_a^{0,A}\mu - \frac{t}{6}(e_a \lrcorner \varphi)\cdot \nabla_a^{0,A}\mu - \frac{t}{6}(-d^*\varphi\cdot\mu + (e_a \lrcorner \varphi)\cdot\nabla_a^{0,A}\mu) - \frac{t^2}{36}((e_a \lrcorner \varphi)\cdot(e_a \lrcorner \varphi))\cdot\mu.$$

Again using the fact that $d^*\varphi = 0$ we get

$$(\nabla_a^{0,A})^*\nabla_a^{0,A}\mu = (\nabla^{t,A})^*\nabla^{t,A}\mu + \frac{t}{3}(e_a \lrcorner \varphi)\cdot\nabla_a^{0,A}\mu + \frac{t^2}{36}((e_a \lrcorner \varphi)\cdot(e_a \lrcorner \varphi))\cdot\mu. \tag{E4}$$

Substituting the three terms in the expression of $(D^{t/3,A})^2\mu$ using (E1), (E2), (E3), (E4) we get the result. □

When the connection A is an instanton on a nearly G_2 manifold the expression for $(D^{t/3,A})^2$ can be simplified further. For the G_2 structure φ, $\|\varphi\|^2 = 7$ and under our choice of convention $d\varphi = 4\psi$ and $\mathrm{Scal}_g = 42$. Thus we can calculate the action of $(D^{t/3,A})^2$ on spinors in $\Lambda^0\eta$ and $\Lambda^1\cdot\eta$ as follows.

Let $\eta \in \Gamma(\Lambda^0 M \otimes EM)$ be a real Killing spinor then Lemma 2.2 implies $\psi\cdot\eta = 7\eta$ and $F_A\cdot\eta = 0$ by (3.1). Thus by above proposition we obtain,

$$(D^{t/3,A})^2\eta = (\nabla^{t,A})^*\nabla^{t,A}\eta - \frac{7}{18}(t^2 - 12t - 27)\eta. \tag{3.7}$$

Now suppose ϵ is an infinitesimal deformation of A. Then $\epsilon\cdot\eta \in \Gamma(\Lambda^1 M\otimes EM)$. From Lemma 2.2 we know that $\psi\cdot\epsilon\cdot\eta = -\epsilon\cdot\eta$ and since $F\cdot\eta = 0$, $F\cdot\epsilon\cdot\eta = (F\cdot\epsilon + \epsilon\cdot F)\cdot\eta = -2(\epsilon \lrcorner F)\cdot\eta$. Thus by above proposition

$$(D^{t/3,A})^2(\epsilon\cdot\eta) = (\nabla^{t,A})^*\nabla^{t,A}(\epsilon\cdot\eta) - \frac{1}{18}(7t^2 + 12t - 189)\epsilon\cdot\eta - 2(\epsilon \lrcorner F)\cdot\eta. \tag{3.8}$$

In the special case when the bundle EM is equal to $Ad_{\mathcal{P}}$, the holonomy group $H \subset G$ of the connection A acts on the Lie algebra $\mathfrak{g}$ of G. Let us denote by $\mathfrak{g}_0 \subset \mathfrak{g}$ the subspace on which H acts trivially. Let $\mathfrak{g}_1$ be the orthogonal subspace of $\mathfrak{g}_0$ with respect to the Killing form of G. The corresponding splitting of the adjoint bundle is given by $Ad_{\mathcal{P}} = L_0 \oplus L_1$. By Proposition 3.4 $(D^{-1/3,A})^2$ is self adjoint and hence respects the decomposition

$$\$ \otimes Ad_{\mathcal{P}} = (\Lambda^1 M\otimes L_0)\oplus(\Lambda^1 M\otimes L_1)\oplus(\Lambda^0 M\otimes L_0)\oplus(\Lambda^0 M\otimes L_1).$$

We use the shorthand $\Lambda^i L_j$ for $\Lambda^i M\otimes L_j$ where $i,j = 0,1$. For compact M we have the following proposition.

Proposition 3.5. *Let A be a G_2-instanton on a principal G-bundle $\mathcal{P}$ with holonomy group H and suppose $Ad_{\mathcal{P}}$ splits as above. Then*

(i) $\ker((D^{-1/3,A})^2 - \frac{49}{9}\mathrm{id}) = \ker((D^{-1/3,A})^2 - \frac{49}{9}\mathrm{id}) \cap (\Lambda^1 L_1 \oplus \Lambda^0 L_0)$.

(ii) $\ker((D^{-1/3,A})^2 - \frac{49}{9}\mathrm{id}) \cap \Lambda^1 L_1 = \Big(\ker(D^{-1/3,A} + \frac{7}{3}\mathrm{id}) \oplus \ker(D^{-1/3,A} - \frac{7}{3}\mathrm{id})\Big) \cap \Lambda^1 L_1$.

Proof. To prove (i) we need to show that $\ker((D^{-1/3,A})^2 - (\frac{7}{3})^2\mathrm{id}) \cap (\Lambda^0 L_1 \oplus \Lambda^1 L_0)$ is trivial.

1. Let $\mu \in \ker((D^{-1/3,A})^2 - (\frac{7}{3})^2\mathrm{id}) \cap \Lambda^0 L_1$. Thus we have by (3.7) ,

$$\begin{aligned} 0 &= \int_M (\mu, (D^{-1/3,A})^2 - (\frac{7}{3})^2)\mu) \\ &= \int_M (\mu, (\nabla^{-1,A})^*\nabla^{-1,A}\mu + (\frac{49}{9} - \Big(\frac{7}{3}\Big)^2)\mu) \\ &= \int_M \|\nabla^{-1,A}\mu\|^2. \end{aligned}$$

But since the action of the holonomy group of A fixes no non-trivial elements in $\mathfrak{g}_1$ and the holonomy group of ∇^{-1} acts trivially on Λ^0 we get $\mu = 0$.

2. Let $\epsilon \cdot \eta \in \ker((D^{-1/3,A})^2 - (\frac{7}{3})^2\mathrm{id}) \cap \Lambda^1 L_0$. By the definition of L_0 the curvature F_A acts trivially on $\epsilon \cdot \eta$ in (3.8) and we get,

$$\begin{aligned} 0 &= \int_M (\epsilon \cdot \eta, (D^{-1/3,A})^2 - (\frac{7}{3})^2)\epsilon \cdot \eta) \\ &= \int_M (\epsilon \cdot \eta, (\nabla^{-1})^*\nabla^{-1}(\epsilon \cdot \eta) + (\frac{97}{9} - \left(\frac{7}{3}\right)^2)\epsilon \cdot \eta) \\ &= \int_M \|\nabla^{-1}(\epsilon \cdot \eta)\|^2 + \frac{48}{9}\int_M \|\epsilon \cdot \eta\|^2 \end{aligned}$$

hence $\epsilon \cdot \eta = 0$.

For proving (ii) we already know that $\big(\ker((D^{-1/3,A}) + \frac{7}{3}\} \oplus \ker((D^{-1/3,A}) + \frac{7}{3}\}\big) \cap \Lambda^1 L_1 \subset \ker((D^{-1/3,A})^2 - \frac{49}{9}\mathrm{id}) \cap \Lambda^1 L_1$. The reverse inclusion can be seen using the fact that since $D^{-1/3,A}$ and $(D^{-1/3,A})^2$ commute they have the same eigenvectors. Moreover since $D^{-1/3,A}$ is self adjoint, $\epsilon \cdot \mu \in \ker((D^{-1/3,A})^2 - \frac{49}{9}\mathrm{id}) \cap \Lambda^1 L_1$ implies $\|D^{-1/3,A}\epsilon \cdot \mu\| = \frac{7}{3}\|\epsilon \cdot \mu\|$ thus the corresponding eigenvalues of $D^{-1/3,A}$ can only be $\pm\frac{7}{3}$. □

Remark 3.6. Note that part (i) for the above proposition holds only for $D^{-1/3,A}$ and not for any other $D^{t,A}$ where $t \neq -1/3$ since the proof explicitly uses the fact that η is parallel with respect to ∇^{-1}. But since $D^{t,A}$ is self adjoint for all $t \in \mathbb{R}$, for any $\lambda \in \mathbb{R}$ we have the following decomposition

$$\ker\left\{(D^{t,A})^2 - \lambda^2\mathrm{id}\right\} \cap \Lambda^1 Ad_{\mathcal{P}} = \left(\ker\left\{D^{t,A} - \lambda\mathrm{id}\right\} \oplus \ker\left\{D^{t,A} + \lambda\mathrm{id}\right\}\right) \cap \Lambda^1 Ad_{\mathcal{P}}.$$

The above proposition has the following important consequence. If the structure group G is abelian H acts as identity on the whole of $\mathfrak{g}$ which means $\mathfrak{g}_1 = 0$ and L_1 is trivial. Thus by Remark 3.3 the space of infinitesimal deformations of the G_2-instanton A which is isomorphic to $\ker(D^{-1/3,A} + \frac{7}{3}) \cap \Lambda^1 Ad_{\mathcal{P}} = \ker(D^{-1/3,A} + \frac{7}{3}) \cap \Lambda^1 L_1$ is zero dimensional.

In [BO19, Proposition 24] the authors prove that the G_2-instanton A is rigid if all the eigenvalues of the operator

$$\begin{aligned} L_A \colon \Lambda^1 \otimes Ad_{\mathcal{P}} &\to \Lambda^1 \otimes Ad_{\mathcal{P}} \\ w &\mapsto -2w \lrcorner F_A \end{aligned}$$

are smaller than 6. We prove the lower bound for the eigenvalue as follows. Let λ be the smallest eigenvalue of L_A. If $\epsilon \in \Gamma(T^*M \otimes Ad_{\mathcal{P}})$ is an infinitesimal deformation of A then from (3.8) and Theorem 3.2 we know that

$$(\nabla^{t,A})^*\nabla^{t,A}\epsilon \cdot \eta = \left(\frac{5t^2}{12} + \frac{3t}{2} - \frac{17}{4}\right)\epsilon \cdot \eta - L_A(\epsilon) \cdot \eta.$$

Taking the inner product with $\epsilon \cdot \eta$ on both sides we get that if $\lambda > \min\left\{\frac{5t^2+18t-51}{12} \mid t \in \mathbb{R}\right\} = -\frac{28}{5}$ then $\epsilon = 0$ is the only solution. Thus we get the following result.

Theorem 3.7. *Any G_2-instanton A on a principal G-bundle over a compact nearly G_2 manifold M is rigid if*

(i) *the structure group G is abelian, or*

(ii) *the eigenvalues of the operator L_A are either all greater than $-\frac{28}{5}$ or all smaller than 6.*

Some immediate consequences of Theorem 3.7 are that the flat instantons are rigid. Also if all the eigenvalues of L_A are equal then A has to be rigid.

4 Instantons on homogeneous nearly G_2 manifolds

4.1 Classification of homogeneous nearly G_2 manifolds

In [FKMS97] the authors classify all the compact, simply connected homogeneous nearly G_2 manifolds. To exhibit this classification a certain amount of notation must be set. Let $S^1_{k,l} = \{\mathrm{diag}(e^{ik\theta}, e^{il\theta}, e^{-i(k+l)\theta}), \theta \in \mathbb{R}\}$ denote the embedding of $\mathrm{U}(1)$ into $\mathrm{SU}(3)$ needed to define the Aloff–Wallach spaces $N_{k,l}$. Any homogeneous nearly G_2 manifold is one of the six manifolds listed in Table 1. We describe the homogeneous structure on each of these spaces.

$(S^7, g_{round}) = \mathrm{Spin}(7)/\mathrm{G}_2,$	$(S^7, g_{squashed}) = \frac{\mathrm{Sp}(2)\times\mathrm{Sp}(1)}{\mathrm{Sp}(1)\times\mathrm{Sp}(1)},$	$\mathrm{SO}(5)/\mathrm{SO}(3),$
$M(3,2) = \frac{\mathrm{SU}(3)\times\mathrm{SU}(2)}{\mathrm{U}(1)\times\mathrm{SU}(2)},$	$N(k,l) = \mathrm{SU}(3)/S^1_{k,l}\ k,l \in \mathbb{Z},$	$Q(1,1,1) = \mathrm{SU}(2)^3/\mathrm{U}(1)^2.$

Table 1: Homogeneous nearly G_2 manifolds

- In the round S^7 the embedding of G_2 in $\mathrm{Spin}(7)$ is obtained by lifting the standard embedding of G_2 into $\mathrm{SO}(7)$.
- For the squashed metric on S^7 the two copies of $\mathrm{Sp}(1)$ in $\mathrm{Sp}(2)\times\mathrm{Sp}(1)$ denoted by $\mathrm{Sp}(1)_u$ and $\mathrm{Sp}(1)_d$ [AS12] are
$$\mathrm{Sp}(1)_u := \left\{\left(\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}, 1\right) : a \in \mathrm{Sp}(1)\right\}, \quad \mathrm{Sp}(1)_d := \left\{\left(\begin{pmatrix} 1 & 0 \\ 0 & a\end{pmatrix}, a\right) : a \in \mathrm{Sp}(1)\right\}.$$
- In the Steifel manifold $V_{5,2} = \frac{\mathrm{SO}(5)}{\mathrm{SO}(3)}$, the Lie group $\mathrm{SO}(3)$ is embedded into $\mathrm{SO}(5)$ via the 5 dimensional irreducible representation of $\mathrm{SO}(3)$ on $\mathrm{Sym}^2_0(\mathbb{R}^3)$.
- In $\frac{\mathrm{SU}(3)\times\mathrm{SU}(2)}{\mathrm{U}(1)\times\mathrm{SU}(2)}$ the embedding of $\mathrm{SU}(2)$ (denoted by $\mathrm{SU}(2)_d$) and $\mathrm{U}(1)$ in $\mathrm{SU}(2)\times\mathrm{SU}(2)$ is defined as [AS12]
$$\mathrm{SU}(2)_d := \left\{\left(\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}, a\right) : a \in \mathrm{SU}(2)\right\}, \quad \mathrm{U}(1) := \left\{\left(\begin{pmatrix} e^{i}\theta & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & e^{i\theta}\end{pmatrix}, 1\right) : \theta \in \mathbb{R}\right\}$$
- In the Aloff-Wallach spaces $N_{k,l}$ where k,l are coprime positive integers the embedding of $S^1_{k,l} = U(1)_{k,l}$ in $\mathrm{SU}(3)$ is described
$$S^1_{k,l} = \left\{\begin{pmatrix} e^{ik\theta} & 0 & 0 \\ 0 & e^{il\theta} & 0 \\ 0 & 0 & e^{-i(k+l)\theta}\end{pmatrix}, \theta \in \mathbb{R}\right\}$$
- In $Q(1,1,1)$ we denote the two copies of $\mathrm{U}(1)$ inside $\mathrm{SU}(2)^3$ as $\mathrm{U}(1)_u, \mathrm{U}(1)_d$ where their respective embeddings are given by
$$\mathrm{U}(1)_u = \mathrm{Span}\left\{\left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}, \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta}\end{pmatrix}, \mathrm{I}_2\right), \theta \in \mathbb{R}\right\},$$
$$\mathrm{U}(1)_d = \mathrm{Span}\left\{\left(\mathrm{I}_2, \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}, \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta}\end{pmatrix}\right), \theta \in \mathbb{R}\right\}.$$

The first four homogeneous spaces are normal, and for those the nearly G_2 metric g on G/H is related to the Killing form B of G by $g = -\frac{3}{40}B$. The choice of the scalar constant $\frac{3}{40}$ is based on our convention

$\tau_0 = 4$. The general formula for the constant was derived in [AS12, Lemma 7.1]. In the remaining two homogeneous spaces the nearly G_2 metric is not a scalar multiple of the Killing form of G (see [Wil99]).

$$(S^7, g_{round}) \cong \mathrm{Spin}(7)/\mathrm{G}_2, \quad (S^7, g_{squashed}) \cong \tfrac{\mathrm{Sp}(2)\times\mathrm{Sp}(1)}{\mathrm{Sp}(1)\times\mathrm{Sp}(1)},$$

$$\mathrm{SO}(5)/\mathrm{SO}(3), \qquad M(3,2) \cong \tfrac{\mathrm{SU}(3)\times\mathrm{SU}(2)}{\mathrm{U}(1)\times\mathrm{SU}(2)}.$$

Table 2: Normal homogeneous nearly G_2 manifolds

Let $\mathfrak{m}$ be the orthogonal complement of the Lie algebra $\mathfrak{h}$ of H in $\mathfrak{g}$ with respect to g. Then $\mathfrak{m}$ is invariant under the adjoint action of $\mathfrak{h}$ that is, $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ and thus all the homogeneous spaces listed in Table 1 are naturally reductive. The reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ equips the principal H-bundle $G \to G/H$ with a G-invariant connection whose horizontal spaces are the left translates of $\mathfrak{m}$. This connection is known as the characteristic homogeneous connection. On homogeneous nearly G_2 manifolds the characteristic homogeneous connection has holonomy contained in G_2. If we denote by $Z_{\mathfrak{m}}$ the projection of $Z \in \mathfrak{g}$ on $\mathfrak{m}$, the torsion tensor T for any $X, Y \in \mathfrak{m}$ is given by

$$T(X,Y) = -[X,Y]_{\mathfrak{m}},$$

and is totally skew-symmetric. Thus by the uniqueness result in [CS04] it is the canonical connection with respect to the nearly G_2 structure on G/H [HN12]. The canonical connection is a G_2-instanton as proved in [HN12, Proposition 3.1].

The adjoint representation $\mathrm{ad}\colon H \to \mathrm{GL}(\mathfrak{m})$ gives rise to the associated vector bundle $G \times_{\mathrm{ad}} \mathfrak{m}$ on G/H. Similarly since G/H has a nearly G_2 structure we have the adjoint action of G_2 on $\mathfrak{m}$ which we again denote by ad and the isotropy homomorphism $\lambda\colon H \to \mathrm{G}_2$ which we can use to construct the associated vector bundle $G \times_{\mathrm{ad}\circ\lambda} \mathfrak{m}$. The canonical connection is a connection on both $G \times_{\mathrm{ad}} \mathfrak{m}$ and $G \times_{\mathrm{ad}\circ\lambda} \mathfrak{m}$ with structure group H and G_2 respectively. Therefore it is natural to study the infinitesimal deformation space of the canonical connection in both these situations. Since $H \subset \mathrm{G}_2$, the deformation space as an H-connection is a subset of the deformation space as a G_2-connection.

We can completely describe the deformation space when the structure group is H but for structure group G_2 we can only find the deformation space for the normal homogeneous nearly G_2 manifolds listed in Table 2 since our methods do not work for non-normal homogeneous metrics. However since H is abelian in both of the non-normal cases Theorem 3.7 tells us that the canonical connection is rigid as an H-connection. But we cannot say anything about the deformation space for the structure group G_2 in those two cases.

Thus the only cases left to consider are listed in Table 2. The remainder of this article is devoted to computing the infinitesimal deformation space of the canonical connection with the structure group H and G_2 for the homogeneous spaces listed in Table 2.

4.2 Infinitesimal deformations of the canonical connection

Let $M = G/H$ be a homogeneous manifold. Consider the principal H-bundle $G \to M$. If (V, ρ) is an H-representation then the space of smooth sections $\Gamma(G \times_\rho V)$ of the associated vector bundle $G \times_\rho V$ is isomorphic to the space $C^\infty(G,V)_H$ of H-equivariant smooth functions $G \to V$. The space $C^\infty(G,V)_H$ carries the left regular G-representation ρ_L defined by $\rho_L(g)(f) = g.f = f \circ l_{g^{-1}}$ which is also known as the induced G-representation $\mathrm{Ind}_H^G V$.

For any connection A on G the covariant derivative associated to A on any bundle associated to A is denoted by ∇^A. Let $s \in \Gamma(G \times_\rho V)$ and $f_s\colon G \to V$ be the G-equivariant function given by $s(gH) = [g, f_s(g)]$. If we denote by X_h the horizontal lift of $X \in \Gamma(TM)$ via A, then ∇^A acts on s as

$$(\nabla^A_X s)(gH) = (g, X_h(f_s)(g)).$$

For the canonical connection on $G \to M$, $X_h = X$ for every vector field. Thus the covariant derivative ∇^{can} is given by

$$(\nabla^{can}_X s)(gH) = (g, X(f_s)(g)).$$

By the Peter–Weyl Theorem [Hal15] the space of sections can also be formulated as follows. If we denote by G_{irr} the set of equivalence classes of irreducible H-representations then

$$\Gamma(G \times_\rho V) = \overline{\bigoplus_{W \in G_{irr}} \mathrm{Hom}(W, V)_H \otimes W}.$$

The embedding $\mathrm{Hom}(W,V)_H \otimes W$ into $C^\infty(G,V)_H = \Gamma(G \times_\rho V)$ is given by sending (ϕ, w) to the function $f_{(\phi,w)}$ defined by $f_{(\phi,w)}(g) = \phi(\tau(g^{-1})w)$. Thus (ϕ, w) defines a section $s_{(\phi,w)}(gH) = [g, f_{(\phi,w)}(g)]$ which we denote by (ϕ, w) as well.

Claim: The left G-action is given by $g.f_{(\phi,w)} = f_{(\phi,\tau(g)w)}$.

Proof. Let $k \in G$. Then since $g.f = f \circ l_{g^{-1}}$ and $f_{(\phi,w)}(g) = \phi(\tau(g^{-1})w)$ we have

$$\begin{aligned}(g.f_{(\phi,w)})(k) &= f_{(\phi,w)}(g^{-1}k) = \phi(\tau((g^{-1}k)^{-1})w) \\ &= \phi(\tau(k^{-1})\tau(g)w) = f_{(\phi,\tau(g)w)}(k).\end{aligned}$$

The proof of the claim is now complete. □

We can compute the covariant derivative on $s_{(\phi,w)} \in \mathrm{Hom}(W,V)_H \otimes W \subset \Gamma(G \times_\rho V)$ by

$$\begin{aligned}\nabla^{can}_X s_{(\phi,w)}(gH) &= X(f_{(\phi,w)})(g) = \frac{d}{dt}\Big|_{t=0} f(e^{tX}g) \\ &= \frac{d}{dt}\Big|_{t=0} (f_{(\phi,w)} \circ l_{e^{tX}})(g) = \frac{d}{dt}\Big|_{t=0} (e^{-tX}.f)(g) \\ &= \frac{d}{dt}\Big|_{t=0} f_{(\phi,\tau(e^{-tX})w)} = -f_{(\phi,\tau_*(X)w)}(gH).\end{aligned}$$

The above can be written as

$$\nabla^{can}_X (\phi, w) = -(\phi, \tau_*(X)w). \tag{4.1}$$

Thus we get that for the canonical connection the covariant derivative of a section $s \in \Gamma(G \times_\rho V)$ with respect to some $X \in \mathfrak{m}$ translates into the derivative $X(f_s)$, which is minus the differential of the left-regular representation $(\rho_L)_*(X)(f_s)$, see [MS10].

Let $\{a_i,\ i = 1 \ldots n\}$ be an orthonormal basis of $\mathfrak{k}$ with respect to g then the Casimir element $\mathrm{Cas}_{\mathfrak{g}} \in \mathrm{Sym}^2(\mathfrak{g})$ is defined by $\sum_{i=1}^{\dim G} a_i \otimes a_i$. On any $\mathfrak{g}$ representation (V, μ) we can define the Casimir invariant $\mu(\mathrm{Cas}_{\mathfrak{k}}) \in \mathrm{gl}(V)$ by

$$\mu(\mathrm{Cas}_{\mathfrak{k}}) = \sum_{i=1}^{n} \mu(a_i)^2.$$

For the reductive homogeneous spaces G/H let $\{a_i, i = 1 \ldots \dim(H)\}$ and $\{a_i, i = \dim(H) \ldots \dim(G)\}$ be the basis of $\mathfrak{h}$ and $\mathfrak{m}$ respectively. If we define $\mathrm{Cas}_{\mathfrak{h}} = \sum_{i=1}^{\dim(H)} a_i \otimes a_i$ and $\mathrm{Cas}_{\mathfrak{m}} = \sum_{\dim(H)}^{\dim(G)} a_i \otimes a_i$ we can decompose $\mathrm{Cas}_{\mathfrak{g}}$ as

$$\mathrm{Cas}_{\mathfrak{g}} = \mathrm{Cas}_{\mathfrak{h}} + \mathrm{Cas}_{\mathfrak{m}}.$$

Note that $\mathrm{Cas}_{\mathfrak{m}}$ is just used for notational convenience and as $\mathfrak{m}$ may not be a Lie algebra apriori. Also in $\mathrm{Cas}_{\mathfrak{h}}$ the trace is taken over G and not H.

Remark 4.1. If one uses the metric $-cB$ instead of $-B$ then the Casimir operator is divided by the scalar c.

To study the deformation space of the canonical connection ∇^{can} on these homogeneous spaces we rewrite the Schrödinger–Lichnerowicz formula (3.8) in terms of the Casimir operator of $\mathfrak{h}$ and $\mathfrak{g}$ and then use the Frobenius reciprocity formula to compute the deformation space of the canonical connection in each case. Let F be the curvature associated to ∇^{can} then the operator $-2\epsilon\lrcorner F$ can be reformulated in terms of $\mathrm{Cas}_{\mathfrak{h}}$ by doing similar calculations as in [CH16, Lemma 4] which gives

$$-2\epsilon\lrcorner F = (\rho_{\mathfrak{m}^*}(\mathrm{Cas}_{\mathfrak{h}}) \otimes 1_E + 1_{\mathfrak{m}^*} \otimes \rho_E(\mathrm{Cas}_{\mathfrak{h}}) - \rho_{\mathfrak{m}^*\otimes E}(\mathrm{Cas}_{\mathfrak{h}}))\epsilon. \tag{4.2}$$

Let (E, ρ_E) be an H-representation. We denote the tensor product of reprentations on $\mathfrak{m}^*$ and E by $\rho_{m^*\otimes E}$. For every $t \in \mathbb{R}$, $D^{t,A}$ denotes the Dirac operator on $G \times_{\rho_{\mathfrak{m}^*\otimes E}} (\mathfrak{m}^* \otimes E) \otimes \$$ associated to the connection ∇^A and ∇^t on $G \times_{\rho_{\mathfrak{m}^*\otimes E}} (\mathfrak{m}^* \otimes E)$ and $\$$ respectively. For $D^{-1/3,can}$ we record the following proposition. From now on we use the same symbol to denote the Lie group representation and the associated Lie algebra representation wherever there is no confusion.

Proposition 4.2. *Let ∇^{can} be the canonical connection on a homogeneous nearly* G$_2$ *manifold $M = G/H$ and η be the Killing spinor. Let (E, ρ_E) be an H-representation and ϵ be a smooth section of $G \times_{\rho_{\mathfrak{m}^*\otimes E}} (\mathfrak{m}^* \otimes E)$. Then*

$$(D^{-1/3,can})^2\epsilon \cdot \eta = (-\rho_L(\mathrm{Cas}_{\mathfrak{g}}) + \rho_E(\mathrm{Cas}_{\mathfrak{h}}))\epsilon + \frac{49}{9}\epsilon) \cdot \eta. \tag{4.3}$$

Proof. We begin by analyzing the rough Laplacian term in the Schrödinger– Lichnerowicz formula for $(D^{-1/3,can})^2\epsilon\cdot\eta$ from (3.8) and then substitute the F-dependent term from (4.2) in the same. We denote by ρ_L the left regular representation of G. From above calculations we know that at the center of a normal orthonormal frame $\{e_i, i = 1\dots7\}$ of $\mathfrak{m}$ with respect to $g = -\frac{3}{40}B$,

$$(\nabla^{-1,can})^*\nabla^{-1,can} = -\nabla^{-1,can}_{e_i}\nabla^{-1,can}_{e_i} = -\rho_L(e_i)^2 = -\rho_L(\mathrm{Cas}_{\mathfrak{m}}).$$

Since $\mathrm{Res}^H_G\, \rho_L = \mathrm{Res}^H_G\, \mathrm{Ind}^G_H(m^*\otimes E) \cong m^*\otimes E$ we have that $\rho_{\mathfrak{m}^*\otimes E}(\mathrm{Cas}_{\mathfrak{h}}) = \rho_L(\mathrm{Cas}_{\mathfrak{h}})$. Also $\rho_{\mathfrak{m}^*}(e_i)^2 = \rho_{\mathfrak{m}^*}(\mathrm{Cas}_{\mathfrak{h}})$ acts as $-\mathrm{Ric}$ of the canonical connection on 1-forms which is equal to $-\frac{16}{3}\mathrm{id}$ from Proposition 2.3. Substituting all the terms in (3.8) for $t = -1$ we get

$$\begin{aligned}(D^{-1/3,can})^2\epsilon\cdot\eta &= (-\rho_L(\mathrm{Cas}_{\mathfrak{m}})\epsilon + \frac{97}{9}\epsilon + (\rho_{\mathfrak{m}^*}(\mathrm{Cas}_{\mathfrak{h}}) \otimes 1_E + 1_{\mathfrak{m}^*} \otimes \rho_E(\mathrm{Cas}_{\mathfrak{h}}) - \rho_{\mathfrak{m}^*\otimes E}(\mathrm{Cas}_{\mathfrak{h}}))\epsilon)\cdot\eta \\ &= (-(\rho_L(\mathrm{Cas}_{\mathfrak{m}}) + \rho_L(\mathrm{Cas}_{\mathfrak{h}}))\epsilon + (\frac{97}{9} - \frac{16}{3})\epsilon + \rho_E(\mathrm{Cas}_{\mathfrak{h}})\epsilon)\cdot\eta \\ &= ((-\rho_L(\mathrm{Cas}_{\mathfrak{g}})\epsilon + \rho_E(\mathrm{Cas}_{\mathfrak{h}})\epsilon + \frac{49}{9}\epsilon)\cdot\eta\end{aligned}$$

which completes the proof. □

Since all the homogeneous spaces considered in Table 2 are naturally reductive and $H \subset \mathrm{G}_2$, there is an adjoint action of H on $\mathfrak{m}, \mathfrak{h}$ and $\mathfrak{g}_2$. Let $\rho_{\mathfrak{m}^*\otimes\mathfrak{h}}, \rho_{\mathfrak{m}^*\otimes\mathfrak{g}_2}$ be the corresponding H-representations. The corresponding Lie algebra representations are denoted similarly. The infinitesimal deformation space of the instanton ∇^{can} is a subspace of $\Gamma(\mathfrak{m}^* \otimes E)$ where E can be either $\mathfrak{h}$ or $\mathfrak{g}_2$.

From Propositions 3.1 and 4.2 it is clear that if ϵ is an infinitesimal deformation of ∇^{can} on the bundle $\mathfrak{m}^* \otimes E$ over G/H then

$$\rho_E(\mathrm{Cas}_{\mathfrak{h}})\epsilon = \rho_L(\mathrm{Cas}_{\mathfrak{g}})\epsilon \tag{4.4}$$

where the trace in both the Casimirs is taken over G.

Using (4.4) we can reformulate the infinitesimal deformation space of the canonical connection. Since the Casimir operator acts as scalar multiple of the identity on irreducible representations we can solve (4.4) for irreducible subrepresentations of L. From Theorem 3.2 the deformations of the canonical connection are the -2 eigenfunctions $\epsilon\cdot\eta$ of $D^{-1,can}$. To explicitly compute the deformation space first we need to find the solutions for (4.4) which by above proposition is identical to the space of $\frac{49}{9}$ eigenfunctions $\epsilon\cdot\eta$ of $(D^{-1/3,can})^2$. For $\alpha \in \Lambda^1\mathrm{Ad}_{\mathcal{P}}$ by Lemma 2.2 $D^{t,A}\alpha\cdot\eta = D^{0,A}\alpha\cdot\eta + \frac{t}{2}\varphi\cdot\alpha\cdot\eta = D^{0,A}\alpha\cdot\eta - \frac{t}{2}\alpha\cdot\eta$.

Therefore the $\pm\frac{7}{3}$ eigenfunctions $\epsilon\cdot\eta$ of $D^{-1/3,can}$ correspond to the -2 and $\frac{8}{3}$ eigenfunction of $D^{-1,A}$ respectively. Thus by Proposition 3.5 we have the following decomposition

$$\ker\left((D^{-1/3,can})^2-\frac{49}{9}\mathrm{id}\right)\cap\Gamma(\mathfrak{m}^*\otimes E)=\left(\ker(D^{-1,can}+2\mathrm{id})\oplus\ker(D^{-1,can}-\frac{8}{3}\mathrm{id})\right)\cap\Gamma(\mathfrak{m}^*\otimes E). \tag{4.5}$$

The first summand on the right hand side is isomorphic to the space of infinitesimal deformations of ∇^{can} by Theorem 3.2. So in the second step we check which of the subspaces in $\ker((D^{-1/3,can})^2-\frac{49}{9}\mathrm{id})\cap(\Gamma(\mathfrak{m}^*\otimes E)\cdot\eta)$ lie in the -2 eigenspace of $D^{-1,can}$.

The Killing spinor η is parallel with respect to ∇^{-1} therefore by the definition of the Dirac operator and Proposition 3.6 we can restrict $D^{-1,can}$ and $(D^{-1/3,can})^2$ to operators from $\Gamma(\mathfrak{m}^*\otimes E)\to\Gamma(\mathfrak{m}^*\otimes E)$. On a homogeneous space we can explicitly compute the canonical connection as we describe below.

Step 1: Calculating $\ker((D^{-1/3,can})^2-\frac{49}{9}\mathrm{id})\cap\Gamma(\mathfrak{m}^*\otimes E)$:

Let $E_{\mathbb{C}}=\oplus_{i=1}^n V_i$ be the decomposition of $E_{\mathbb{C}}$ into complex irreducible H-representations. For each V_i we find all the complex irreducible G-representations $W_{i,j}, j=1\dots n_i$, that satisfy the equation

$$\rho_{V_i}(\mathrm{Cas}_{\mathfrak{h}})=\rho_{W_{i,j}}(\mathrm{Cas}_{\mathfrak{g}}).$$

In order to see whether $W_{i,j}\subset\mathrm{Ind}_H^G(\mathfrak{m}^*\otimes E)_{\mathbb{C}}$ we find the multiplicity $m_{i,j}$ of $W_{i,j}$ in $\mathrm{Ind}_H^G(\mathfrak{m}^*_{\mathbb{C}}\otimes V_i)$. Because of Schur's Lemma this multiplicity is given by $\dim(\mathrm{Hom}(W_{i,j},\mathfrak{m}^*_{\mathbb{C}}\otimes V_i)_H)$. Repeating this process for all the i,j's and summing over all irreducible G-representations $W_{i,j}$ along with their multiplicity we get,

$$\ker((D^{-1/3,can})^2-\frac{49}{9}\mathrm{id})\cap\Gamma(\mathfrak{m}^*\otimes E)_{\mathbb{C}}\cong\bigoplus_{i=1}^{n}\Big(\bigoplus_{j=1}^{n_i}m_{i,j}W_{i,j}\Big). \tag{4.6}$$

Step 2: Calculating $\ker(D^{-1,can}+2\mathrm{id})\cap\Gamma(\mathfrak{m}^*\otimes E)$:

To figure out which of the $W_{i,j}$'s found in Step 1 are in the $\ker(D^{-1,can}+2\mathrm{id})$ we need to calculate the covariant derivative ∇^{can} on $\mathrm{Hom}(W_{i,j},\mathfrak{m}^*_{\mathbb{C}}\otimes V_i)_H\otimes W_{i,j}\subseteq\Gamma(\mathfrak{m}^*\otimes E)_{\mathbb{C}}$.

If (W,τ) is an irreducible G-subrepresentation of $\mathrm{Ind}_H^G(\mathfrak{m}^*\otimes E)$ then $\mathrm{Hom}(W,\mathfrak{m}^*\otimes E)_H$ is non-trivial. By Schur's Lemma the dimension of $\mathrm{Hom}(W,\mathfrak{m}^*\otimes E)_H$ is the number of common irreducible H-subrepresentations in $\mathrm{Res}_G^H W$ and $\mathfrak{m}^*\otimes E$. Let W_α be such a common irreducible H-representation and ϕ_α be a basis element of the 1-dimensional space $\mathrm{Hom}(W|_{W_\alpha},(\mathfrak{m}^*\otimes E)|_{W_\alpha})$. For $X\in\Gamma(TM)$ and $(\phi=\sum c_\alpha\phi_\alpha,w)\in\mathrm{Hom}(W,\mathfrak{m}^*\otimes E)_H\otimes W$, (4.1) implies

$$\nabla^{can}_X(\phi,w)(eH)=-\phi(\tau_*(X)w)\in\mathfrak{m}^*\otimes E$$

where τ_* is the Lie algebra $\mathfrak{g}$ representation associated to the G-representation (W,τ). Using this we can calculate the Dirac operator at eH by

$$D^{-1,can}(\phi_\alpha,w)(eH)=-\sum_{i=1}^{7}e_i\cdot\nabla^{-1,can}_{e_i}(\phi_\alpha,w)(eH)=-\sum_{i=1}^{7}e_i\cdot\phi_\alpha(\tau_*(e_i)w). \tag{4.7}$$

The above method can be extended by linearity to compute the Dirac operator on $\Gamma(\mathfrak{m}^*\otimes E)$. Note that we have omitted the Killing spinor η since it is parallel with respect to η so does not effect the eigenspace.

In the following sections we implement the above procedure on each of the four homogeneous spaces.

Remark 4.3. In a nearly Kähler 6-manifold whose structure is defined by a real Killing spinor η, the spinor $\mathrm{vol}\cdot\eta$ is another independent real Killing spinor. Any Dirac operator $\not{D}$ anti-commutes with the Clifford multiplication by vol that is $\not{D}\mathrm{vol}=-\mathrm{vol}\cdot\not{D}$, hence for all $\lambda\in\mathbb{R}$ we have $\ker(\not{D}-\lambda\mathrm{id})\cong\ker(\not{D}+\lambda\mathrm{id})$. Therefore $\ker(\not{D}^2-\lambda^2\mathrm{id})\cong 2\ker(\not{D}\pm\lambda\mathrm{id})$ and one can compute the λ eigenspace of $\not{D}$ by computing the λ^2 eigenspace of $\not{D}^2$ as done in [CH16, Proposition 4]. In the case of nearly G_2 manifolds $\not{D}$ and the 7-dimensional vol commute and thus we do not have such an isomorphism between the $\pm\lambda$ eigenspaces of the Dirac operator. In fact there is no such automatic relation between $\ker(\not{D}^2-\lambda^2\mathrm{id})$ and $\ker(\not{D}+\lambda\mathrm{id})$ as §4.4 reveals.

Remark 4.4. The Dirac operator is always self-adjoint therefore the above method of finding a particular eigenspace of a Dirac operator D can be used more generally in any bundle associated to the spinor bundle over a homogeneous spin manifold. Often times it is easier to find the eigenspaces of the square of the Dirac operator D^2 similar to the case in hand. Once we know the λ^2-eigenspace of D^2 we can apply D on them to see which of them lie in the λ or $-\lambda$-eigenspace of D.

4.3 Eigenspaces of the square of the Dirac operator

In this section we follow *Step 1* of the above procedure. To see which of the irreducible representations of G satisfy (4.4), we need to compute the Casimir operator on complex irreducible representations. Given any irreducible representation ρ_λ with highest weight λ we use the Freudenthal formula to compute $\rho_\lambda(\mathrm{Cas}_\mathfrak{g})$. We drop the constant $\frac{40}{3}$ in our definition of Casimir operator for this section as it does not play any role in comparing the Casimir operators. Let $\mu = \frac{1}{2}$(sum of the positive roots of $\mathfrak{g}$) then the Freudenthal formula states that

$$\rho_\lambda(\mathrm{Cas}_\mathfrak{g}) = B(\lambda,\lambda) + 2B(\mu,\lambda). \tag{4.8}$$

We compute the deformation space of the canonical connection for $E = \mathfrak{h}$ and $E = \mathfrak{g}_2$ as described earlier. In all the examples listed below Case 1 is for $E = \mathfrak{h}$ and Case 2 is for $E = \mathfrak{g}_2$.

4.3.1 Spin(7)/G$_2$

For this space, $H = \mathrm{G}_2$ so there is only one case to consider.

The adjoint representation $\mathfrak{g}_2$ is the unique 14-dimensional irreducible representation of G$_2$. The complex irreducible representations of G$_2$ are identified with respect to their highest weights of the form $(p,q) \in \mathbb{Z}^2_{\geq 0}$ and are denoted by $V_{(p,q)}$. Here $V_{(1,0)}$ is the 7-dimensional standard G$_2$-representation and $V_{(0,1)}$ is the 14-dimensional adjoint representation. The reductive splitting of the Lie algebra is given by

$$\mathfrak{spin}(7) = \mathfrak{g}_2 \oplus \mathfrak{m}.$$

We have the following isomorphisms of G$_2$ representations,

$$\mathfrak{h}_\mathbb{C} = (\mathfrak{g}_2)_\mathbb{C} \cong V_{(0,1)}$$
$$\mathfrak{m}_\mathbb{C} \cong V_{(1,0)}.$$

The isomorphism $\mathfrak{spin}(7) \cong \mathfrak{so}(7)$ implies that the eigenvalues of their Casimir operators on irreducible representations are equal. For $\mathfrak{so}(7)$, let E_{ij} be the 7×7 skew-symmetric matrix with 1 at the (i,j)th entry and 0 elsewhere. We define $H_1 = E_{45} - E_{23}, H_2 = E_{67} - E_{45}$ and $H_3 = E_{45}$. A Cartan subalgebra for $\mathfrak{so}(7)$ is given by Span$\{H_i, i = 1,2,3\}$. A set of simple roots $\{\alpha_i, i = 1,2,3\}$ is given by

$$\alpha_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

The Cartan matrix C of $\mathfrak{so}(7)$ which is given by

$$C = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 0 \\ 2 & 0 & -2 \end{bmatrix}.$$

Then one can compute the simple co-roots F_is by $\alpha_i(F_j) = C_{ij}$ which give $F_1 = H_2, F_2 = -H_1 + 2H_3$ and $F_3 = -2H_2 - 2H_3$. We denote the fundamental weights in decreasing order by λ_1, λ_2 and λ_3 which are dual to F_3, F_1, F_2 respectively. We can compute easily that

$$\begin{bmatrix} B(H_1,H_1) & B(H_1,H_2) & B(H_1,H_2) \\ B(H_2,H_1) & B(H_2,H_2) & B(H_2,H_3) \\ B(H_3,H_1) & B(H_3,H_2) & B(H_3,H_3) \end{bmatrix} = \begin{bmatrix} -20 & 10 & -10 \\ 10 & -20 & 10 \\ -10 & 10 & -10 \end{bmatrix}$$

which implies,

$$\begin{bmatrix} B(\lambda_1,\lambda_1) & B(\lambda_1,\lambda_2) & B(\lambda_1,\lambda_2) \\ B(\lambda_2,\lambda_1) & B(\lambda_2,\lambda_2) & B(\lambda_2,\lambda_3) \\ B(\lambda_3,\lambda_1) & B(\lambda_3,\lambda_2) & B(\lambda_3,\lambda_3) \end{bmatrix} = \begin{bmatrix} -3/40 & -1/10 & -1/20 \\ -1/10 & -1/5 & -1/10 \\ -1/20 & -1/10 & -1/10 \end{bmatrix}$$

The sum of the positive roots is given by $\lambda_1 + \lambda_2 + \lambda_3$ therefore by (4.8) on an irreducible SO(7)-representation $V_{(m_1,m_2,m_3)}$ with highest weight $m_1\lambda_1 + m_2\lambda_2 + m_3\lambda_3,\ m_1, m_2, m_3 \geq 0$ we have

$$\rho_\lambda(\mathrm{Cas}_{\mathfrak{so}(7)}) = -\frac{1}{40}(3m_1^2 + 8m_2^2 + 4m_3^2 + 8m_1m_2 + 4m_1m_3 + 8m_2m_3 + 18m_1 + 32m_2 + 20m_3).$$

Now we compute the eigenvalues of the Casimir operator for the irreducible representations of $\mathfrak{g}_2 \subset \mathfrak{so}(7)$. A Cartan subalgebra of $\mathfrak{g}_2$ is given by $\mathrm{Span}\{H_1, H_2\}$. Here a pair of simple roots β_1, β_2 is given by

$$\beta_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \beta_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and the Cartan matrix $\tilde{C}$ for $\mathfrak{g}_2$ is given by

$$\tilde{C} = \begin{bmatrix} -2 & 1 \\ 3 & -2 \end{bmatrix}$$

. Let μ_1, μ_2 be the fundamental weights in decreasing order then their duals with respect to B are $-H_1 - 2H_2, H_2$ respectively and one can compute

$$\begin{bmatrix} B(\mu_1,\mu_1) & B(\mu_1,\mu_2) \\ B(\mu_2,\mu_1) & B(\mu_2,\mu_2) \end{bmatrix} = \begin{bmatrix} -1/15 & -1/10 \\ -1/10 & -1/5 \end{bmatrix}.$$

The sum of the positive roots is given by $\mu_1 + \mu_2$. Using these values in the Freudenthal formula for an irreducible G_2-representation $V_{(p,q)}$ with highest weight $p\mu_1 + q\mu_2$ we have

$$\rho_{(p,q)}(\mathrm{Cas}_{\mathfrak{g}_2}) = -\frac{1}{15}(p^2 + 3q^2 + 3pq + 5p + 9q).$$

Case 1: $E = \mathfrak{g}_2$
The adjoint representation $(\mathfrak{g}_2)_{\mathbb{C}} \cong V_{(0,1)}$. From above

$$\rho_{(0,1)}(\mathrm{Cas}_{\mathfrak{g}_2}) = -\frac{4}{5}.$$

Substituting the above found values into (4.4) we get that $V_{(m_1,m_2,m_3)}$ can be an infinitesimal deformation space for the canonical connection if

$$-\frac{1}{40}(3m_1^2 + 8m_2^2 + 4m_3^2 + 8m_1m_2 + 4m_1m_3 + 8m_2m_3 + 18m_1 + 32m_2 + 20m_3) = -\frac{4}{5}.$$

But since there are no positive integral solutions of this equation there are no deformations of the canonical connection on $\mathrm{Spin}(7)/\mathrm{G}_2$.

4.3.2 SO(5)/SO(3)

The complex irreducible SO(5)-representations are characterized by highest weights $(m_1, m_2) \in \mathbb{Z}_{\geq 0}$. The complex irreducible representations of SO(3) are given by $S^k\mathbb{C}^2$ which is a $\binom{2+k-1}{k} = k+1$ dimensional space. The 3-dimensional adjoint representation $\mathfrak{so}(3)_{\mathbb{C}}$ and the 7-dimensional representation $\mathfrak{m}_{\mathbb{C}}$ are irreducible SO(3)-representations therefore

$$\mathfrak{m}_{\mathbb{C}} \cong S^6\mathbb{C}^2,$$
$$\mathfrak{so}(3)_{\mathbb{C}} \cong S^2\mathbb{C}^2.$$

A Cartan subalgebra of $\mathfrak{so}(5)$ is given by $\mathrm{Span}\{H_1, H_2\} = \mathrm{Span}\{E_{12}, E_{34}\}$ where E_{ij} is the 5×5 skew-symmetric matrix with 1 at the (i, j)th position and 0 elsewhere. With respect to the Killing form B on $\mathfrak{so}(5)$, H_1 is orthogonal to H_2 with $B(H_i, H_i) = -6$ for $i = 1, 2$. Let λ_1, λ_2 be the fundamental weights whose duals are $H_1 - H_2, 2H_2$ respectively then the sum of positive roots is given by $\lambda_1 + \lambda_2$. Doing similar computations as above we get

$$\begin{bmatrix} B(\lambda_1, \lambda_1) & B(\lambda_1, \lambda_2) \\ B(\lambda_2, \lambda_1) & B(\lambda_2, \lambda_2) \end{bmatrix} = \begin{bmatrix} -1/6 & -1/12 \\ -1/12 & -1/12 \end{bmatrix}.$$

Using (4.8) for the eigenvalues of the Casimir operator for irreducible representation $V_{(m_1,m_2)}$ of SO(5) with highest weight $m_1\lambda_1 + m_2\lambda_2$ for $m_1, m_2 \geq 0$ we get,

$$\rho_{(m_1,m_2)}(\mathrm{Cas}_{\mathfrak{so}(5)}) = \frac{-1}{12}(2m_1^2 + m_2^2 + 2m_1m_2 + 6m_1 + 4m_2).$$

Under the embedding of $\mathfrak{so}(3)$ in $\mathfrak{so}(5)$ the Cartan subalgebra of $\mathfrak{so}(3)$ is given by $\mathrm{Span}\{2H_1 + H_2\}$. Here the Cartan subalgebra is 1-dimensional and the fundamental weight μ_1 is dual to $4H_1 + 2H_2$. Using $B(H_i, H_i) = -6$ one can compute that $B(4H_1 + 2H_2, 4H_1 + 2H_2) = -120$ the eigenvalue of the Casimir operator on the irreducible representation $S^q\mathbb{C}^2$ of $\mathfrak{so}(3)$ is given by

$$\rho_q(\mathrm{Cas}_{\mathfrak{so}(3)}) = \frac{-1}{120}(q^2 + 2q).$$

Case 1: $E = \mathfrak{so}(3)$
The adjoint representation of $\mathfrak{so}(3)_{\mathbb{C}}$ is an irreducible $\mathfrak{so}(3)$ representation with highest weight 2. Thus

$$\rho_E(\mathrm{Cas}_{\mathfrak{so}(3)}) = \rho_2(\mathrm{Cas}_{\mathfrak{so}(3)}) = \frac{-1}{15}.$$

We need to find irreducible representations $V_{(m_1,m_2)}$ of $\mathfrak{so}(5)$ that satisfy (4.4) which requires

$$\frac{-1}{12}(2m_1^2 + m_2^2 + 2m_1m_2 + 6m_1 + 4m_2) = -\frac{1}{15}.$$

But since there are no integral solutions for the equation the deformation space is trivial in this case.

Case 2: $E = \mathfrak{g}_2$
The adjoint representation of $(\mathfrak{g}_2)_{\mathbb{C}}$ splits as an $\mathfrak{so}(3)$ representation into $S^2\mathbb{C}^2 \oplus S^{10}\mathbb{C}^2$. The first component in the splitting has already been studied in case 1 and hence has no contribution to the deformation space. For the second component

$$\rho_{10}(\mathrm{Cas}_{\mathfrak{so}(3)}) = -1.$$

Thus we need to find $\mathfrak{so}(5)$ representations $V_{(m_1,m_2)}$ such that

$$\frac{-1}{12}(2m_1^2 + m_2^2 + 2m_1m_2 + 6m_1 + 4m_2) = -1,$$

which has one integral solution namely $m_1 = 0, m_2 = 2$. Thus $V_{(0,2)} \cong \mathfrak{so}(5)_{\mathbb{C}}$ is the only SO(5)-representation for which $\mathrm{Cas}_{\mathfrak{g}}$ has eigenvalue -1. As $\mathfrak{so}(3)$ representations

$$V_{(0,2)} \cong S^2\mathbb{C}^2 \oplus S^6\mathbb{C}^2,$$

$$\mathfrak{m}^*_{\mathbb{C}} \otimes S^{10}\mathbb{C}^2 \cong \bigoplus_{k=2}^{8} S^{2k}\mathbb{C}^2.$$

Thus $V_{(0,2)}$ and $\mathfrak{m}^*_{\mathbb{C}} \otimes S^{10}\mathbb{C}^2$ have 1 common irreducible $\mathfrak{so}(3)$ representation namely $S^6\mathbb{C}^2$. Thus $V_{(0,2)}$ occurs in $\mathrm{Ind}_H^G(\mathfrak{m}^*_{\mathbb{C}} \otimes S^{10}\mathbb{C}^2)$ with multiplicity 1. Therefore in this case $(\ker((D^{-1/3,can})^2 - \frac{49}{9}\mathrm{id}) \cap \Gamma(\mathfrak{m}^* \otimes \mathfrak{g}_2))_{\mathbb{C}} \cong V_{(0,2)}$.

4.3.3 $\frac{\mathrm{Sp}(2)\times\mathrm{Sp}(1)}{\mathrm{Sp}(1)\times\mathrm{Sp}(1)}$

The Lie algebra $\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)$ decomposes as

$$\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)=\mathfrak{sp}(1)_u\oplus\mathfrak{sp}(1)_d\oplus\mathfrak{m}$$

and the embeddings $\mathfrak{sp}(1)_u,\mathfrak{sp}(1)_d$ are given by

$$\mathfrak{sp}(1)_u=\left\{\left(\begin{pmatrix}a&0\\0&0\end{pmatrix},0\right):a\in\mathfrak{sp}(1)\right\},\quad \mathfrak{sp}(1)_d=\left\{\left(\begin{pmatrix}0&0\\0&a\end{pmatrix},a\right):a\in\mathfrak{sp}(1)\right\}$$

where we follow the notations used in [AS12]. A Cartan subalgebra of $\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)$ is given by $\mathrm{Span}\{H_1,H_2,H_3\}=\mathrm{Span}\{(E_1,0),(E_2,0),(0,E_3)\}$ where

$$E_1=\begin{pmatrix}i&0&0&0\\0&0&0&0\\0&0&-i&0\\0&0&0&0\end{pmatrix},E_2=\begin{pmatrix}0&0&0&0\\0&i&0&0\\0&0&0&0\\0&0&0&-i\end{pmatrix},E_3=\begin{pmatrix}i&0\\0&-i\end{pmatrix}.$$

If B denote the Killing form of $\mathrm{Sp}(2)\times\mathrm{Sp}(1)$ we can compute that H_is are orthogonal with respect to B and $B(H_i,H_i)=-12$ for $i=1,2$ and $B(H_3,H_3)=-8$. The fundamental weights $\lambda_1,\lambda_2,\lambda_3$ are dual to H_1-H_2,H_1,H_3 respectively and the sum of positive roots is given by $\lambda_1+\lambda_2+\lambda_3$. By identical calculations as in other cases we get

$$\begin{bmatrix}B(\lambda_1,\lambda_1)&B(\lambda_1,\lambda_2)&B(\lambda_1,\lambda_3)\\B(\lambda_2,\lambda_1)&B(\lambda_2,\lambda_2)&B(\lambda_2,\lambda_3)\\B(\lambda_3,\lambda_1)&B(\lambda_3,\lambda_2)&B(\lambda_3,\lambda_3)\end{bmatrix}=\begin{bmatrix}-1/12&-1/12&0\\-1/12&-1/6&0\\0&0&-1/8\end{bmatrix}.$$

Applying the Freudenthal formula (4.8) we get that the Casimir operator of $\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)$ acts on the irreducible representations $V_{(m_1,m_2,l)}$ with highest weight $m_1\lambda_1+m_2\lambda_2+l\lambda_3,m_1,m_2,l\geq 0$ with the eigenvalue

$$\rho_{(m_1,m_2,l)}(\mathrm{Cas}_{\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)})=-\frac{1}{12}(m_1^2+2m_2^2+2m_1m_2+4m_1+6m_2)-\frac{1}{8}(l^2+2l).$$

Under the embedding given above a Cartan subalgebra of $\mathfrak{sp}(1)_u,\mathfrak{sp}(1)_d$ is given by $\mathrm{Span}\{H_1\}$ and $\mathrm{Span}\{(E_2,E_3)\}$ respectively. Let P,Q be the standard 2-dimensional representation of $\mathfrak{sp}(1)_u,\mathfrak{sp}(1)_d$ respectively. Then the unique $(n+1)-$dimensional irreducible $\mathfrak{sp}(1)_u$ (respectively $\mathfrak{sp}(1)_d$) representation is given by S^nP (respectively S^nQ). From previous calculations we have $B(H_1,H_1)=-12$ thus the eigenvalue of $\mathrm{Cas}_{\mathfrak{sp}(1)_u}$ on S^nP is given by

$$\rho_n(\mathrm{Cas}_{\mathfrak{sp}(1)_u})=-\frac{1}{12}(n^2+2n).$$

Similarly with the help of previous work one can calculate $B((E_2,E_3),(E_2,E_3))=-20$. Thus $\mathrm{Cas}_{\mathfrak{sp}(1)_d}$ acts on S^pQ as the scalar multiple of

$$\rho_n(\mathrm{Cas}_{\mathfrak{sp}(1)_d})=-\frac{1}{20}(n^2+2n).$$

The adjoint representation $\mathfrak{sp}(1)$ is an irreducible 3-dimensional $\mathfrak{sp}(1)$ representation and hence we have the following decompositions into $\mathrm{Sp}(1)_u\times\mathrm{Sp}(1)_d$ representations

$$(\mathfrak{sp}(1)_u)_{\mathbb{C}}\cong S^2P,\quad(\mathfrak{sp}(1)_d)_{\mathbb{C}}\cong S^2Q,\quad\mathfrak{m}_{\mathbb{C}}\cong S^2Q\oplus PQ$$

where PQ denotes the tensor product of P and Q and we omitted the tensor product sign for clarity and will continue to do so.

Case 1: $E=\mathfrak{sp}(1)_u\oplus\mathfrak{sp}(1)_d$

We need to find the irreducible $\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)$ representations $V_{(m_1,m_2,l)}$ that satisfy (4.4) for each irreducible component of $\mathfrak{h}_{\mathbb{C}}$ that is $(\mathfrak{sp}(1)_u)_{\mathbb{C}}$ and $(\mathfrak{sp}(1)_d)_{\mathbb{C}}$. For $\mathfrak{sp}(1)_u$ this equation takes the form

$$-\frac{1}{12}(m_1^2+2m_2^2+2m_1m_2+4m_1+6m_2)-\frac{1}{8}(l^2+2l)=-\frac{8}{12}.$$

The integral solution (m_1,m_2,l) for this equation is $(0,1,0)$. Thus the only irreducible $\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)$ representations for which $\mathrm{Cas}_{\mathfrak{g}}$ has eigenvalue $-\frac{2}{3}$ is $V_{(0,1,0)}$. As $\mathfrak{sp}(1)_u\oplus\mathfrak{sp}(1)_d$-representations we have the following decomposition

$$\begin{aligned}V_{(0,1,0)} &\cong PQ\oplus\mathbb{C},\\ (\mathfrak{sp}(1)_u\otimes\mathfrak{m})_{\mathbb{C}} &\cong S^2PS^2Q\oplus S^3PQ\oplus PQ.\end{aligned}$$

The irreducible $\mathrm{Sp}(1)\times\mathrm{Sp}(1)$ representation in $(\mathfrak{sp}(1)_u\otimes\mathfrak{m})_{\mathbb{C}}$ common with $V_{(0,1,0)}$ is PQ with multiplicity 1. Thus $V_{(0,1,0)}$ occurs in $\mathrm{Ind}_H^G(\mathfrak{m}^*\otimes\mathfrak{sp}(1)_u)_{\mathbb{C}}$ with multiplicity 1. Therefore the solutions to (4.4) in $\Gamma(\mathfrak{m}^*\otimes\mathfrak{sp}(1)_u)_{\mathbb{C}}$ is the 5-dimensional complex $\mathrm{Sp}(2)\times\mathrm{Sp}(1)$ representation $V_{(0,1,0)}$.

For the next irreducible $\mathfrak{h}_{\mathbb{C}}$ component $(\mathfrak{sp}(1)_d)_{\mathbb{C}}$ (4.4) for $V_{(m_1,m_2,l)}$ becomes

$$-\frac{1}{12}(m_1^2+2m_2^2+2m_1m_2+4m_1+6m_2)-\frac{1}{8}(l^2+2l)=-\frac{8}{20},$$

which has no integral solutions and thus it has no contribution to the deformation space.

Thus from Proposition 4.2 we conclude that $(\ker((D^{-1/3.can})^2-\frac{49}{9}\mathrm{id})\cap\Gamma(\mathfrak{m}^*\otimes\mathfrak{sp}(1)_u\oplus\mathfrak{sp}(1)_d)_{\mathbb{C}}\cong(V_{(0,1,0)})$ when the structure group is $\mathrm{Sp}(1)_u\times\mathrm{Sp}(1)_d$.

Case 2: $E=(\mathfrak{g}_2)_{\mathbb{C}}$
The adjoint representation of $\mathfrak{g}_2$ decomposes into irreducible $\mathfrak{sp}(1)_u\oplus\mathfrak{sp}(1)_d$ as follows:

$$(\mathfrak{g}_2)_{\mathbb{C}}=S^2P\oplus S^2Q\oplus PS^3Q.$$

We have already seen the contribution of the first two irreducible components in the summation. For the third component

$$\rho_{1,3}(\mathrm{Cas}_{\mathfrak{sp}(1)_u\oplus\mathfrak{sp}(1)_d})=-1,$$

so here we need to find the $\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)$ representations $V_{(m_1,m_2,l)}$ such that

$$-\frac{1}{12}(m_1^2+2m_2^2+2m_1m_2+4m_1+6m_2)-\frac{1}{8}(l^2+2l)=-1.$$

The $\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)$-representations that satisfy (4.4) are $V_{(2,0,0)}$ and $V_{(0,0,2)}$, which decompose into $\mathfrak{sp}(1)_u\oplus\mathfrak{sp}(1)_d$ representations as

$$V_{(2,0,0)}\cong\mathfrak{sp}(2)_{\mathbb{C}}\cong S^2P\oplus S^2Q\oplus PQ,\quad V_{(0,0,2)}\cong(\mathfrak{sp}(1)_d)_{\mathbb{C}}\cong S^2Q.$$

Moreover

$$PS^3Q\otimes\mathfrak{m}_{\mathbb{C}}^*\cong S^2PS^4Q\oplus S^2PS^2Q\oplus P(S^5Q\oplus S^3Q\oplus Q)\oplus S^4Q\oplus S^2Q.$$

Thus $V_{(2,0,0)}$ and $PS^3Q\otimes\mathfrak{m}_{\mathbb{C}}^*$ have two common irreducible representations PQ, S^2Q and $V_{(0,0,2)}$ and $PS^3Q\otimes\mathfrak{m}_{\mathbb{C}}^*$ have one common irreducible representation S^2Q. So by Frobenius reciprocity $V_{(2,0,0)}$ and $V_{(0,0,2)}$ lie in $\mathrm{Ind}_H^G(\mathfrak{m}_{\mathbb{C}}^*\otimes PS^3Q)$ with multiplicity $2,1$ respectively. Thus the solution of (4.4) in $\Gamma(\mathfrak{m}^*\otimes\mathfrak{g}_2)_{\mathbb{C}}$ is the 28 dimensional $\mathrm{Sp}(2)\times\mathrm{Sp}(1)$ complex representation $2V_{(2,0,0)}\oplus V_{(0,1,0)}\oplus V_{(0,0,2)}$. So again by Proposition 4.2 we conclude that $\ker((D^{-1/3.can})^2-\frac{49}{9}\mathrm{id})\cap\Gamma(\mathfrak{m}^*\otimes\mathfrak{g}_2)_{\mathbb{C}}\cong 2V_{(2,0,0)}\oplus V_{(0,1,0)}\oplus V_{(0,0,2)}$ when the structure group is G_2.

4.3.4 $\frac{\mathrm{SU}(3)\times\mathrm{SU}(2)}{\mathrm{SU}(2)\times\mathrm{U}(1)}$

The embeddings of $\mathfrak{su}(2)$ and $\mathfrak{u}(1)$ in $\mathfrak{su}(3)\times\mathfrak{su}(2)$ which we denote by $\mathfrak{su}(2)_d$ and $\mathfrak{u}(1)$ following [AS12] in $\mathfrak{su}(3)\oplus\mathfrak{su}(2)$ are given by

$$\mathfrak{su}(2)_d=\left\{\left(\begin{pmatrix}a & 0\\ 0 & 0\end{pmatrix},a\right):a\in\mathfrak{su}(2)\right\},\quad \mathfrak{u}(1)=\mathrm{span}\left\{\left(\begin{pmatrix}i & 0 & 0\\ 0 & i & 0\\ 0 & 0 & -2i\end{pmatrix},0\right)\right\}.$$

A Cartan subalgebra of $\mathfrak{su}(3)\oplus\mathfrak{su}(2)$ is given by $\mathrm{span}\{H_1,H_2,H_3\}=\mathrm{Span}\{(E_1,0),(E_2,0),(0,E_3)\}$ where

$$E_1=\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\quad E_2=\begin{pmatrix}i & 0 & 0\\ 0 & i & 0\\ 0 & 0 & -2i\end{pmatrix},\quad E_3=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

We can check that the H_is are orthogonal with respect to the Killing form B on $\mathrm{SU}(3)\times\mathrm{SU}(2)$. As earlier we denote by $\lambda_1,\lambda_2,\lambda_3$ the fundamental weights which are dual to $\frac{1}{2}(H_1-H_2),\frac{1}{2}(H_1+H_2),H_3$ respectively. By direct computations we get

$$\begin{bmatrix}B(H_1,H_1) & B(H_1,H_2) & B(H_1,H_3)\\ B(H_2,H_1) & B(H_2,H_2) & B(H_2,H_3)\\ B(H_3,H_1) & B(H_3,H_2) & B(H_3,H_3)\end{bmatrix}=\begin{bmatrix}-12 & 0 & 0\\ 0 & -36 & 0\\ 0 & 0 & -8\end{bmatrix},$$

therefore

$$\begin{bmatrix}B(\lambda_1,\lambda_1) & B(\lambda_1,\lambda_2) & B(\lambda_1,\lambda_3)\\ B(\lambda_2,\lambda_1) & B(\lambda_2,\lambda_2) & B(\lambda_2,\lambda_3)\\ B(\lambda_3,\lambda_1) & B(\lambda_3,\lambda_2) & B(\lambda_3,\lambda_3)\end{bmatrix}=\begin{bmatrix}-1/9 & -1/18 & 0\\ -1/18 & -1/9 & 0\\ 0 & 0 & -1/8\end{bmatrix}.$$

The sum of the positive roots is $\lambda_1+\lambda_2+\lambda_3$ and thus by Freudenthal formula (4.8) for a $\mathfrak{su}(3)\oplus\mathfrak{su}(2)$ representation $V_{(m_1,m_2,l)}$ with highest weight $m_1\lambda_1+m_2\lambda_2+l\lambda_3$ where $m_1,m_2,l\geq 0$

$$\rho_{m_1,m_2,l}(\mathrm{Cas}_{\mathfrak{su}(3)\oplus\mathfrak{su}(2)})=-\frac{1}{9}(m_1^2+m_2^2+m_1m_2+3m_1+3m_2)-\frac{1}{8}(l^2+2l).$$

Using the embeddings of $\mathfrak{su}(2)$ and $\mathfrak{u}(1)$ given above we see that Cartan subalgebras of $\mathfrak{su}(2)$ and $\mathfrak{u}(1)$ in $\mathfrak{su}(3)\oplus\mathfrak{su}(2)$ are given by $\mathrm{span}\{(E_1,E_3)\}$ and $\mathrm{span}\{H_2\}$ respectively. By calculations completely analogous to the previous case we then get that if we represent the irreducible $(n+1)$-dimensional $\mathfrak{su}(2)_d$ representations by S^nW where W is the standard $\mathfrak{su}(2)_d$ representation and the 1-dimensional $\mathfrak{u}(1)$ representation with highest weight k by $F(k)$ we get by the Freudenthal formula (4.8)

$$\rho_n(\mathrm{Cas}_{\mathfrak{su}(2)_d})=-\frac{1}{20}(n^2+2n),$$
$$\rho_k(\mathrm{Cas}_{\mathfrak{u}(1)})=-\frac{1}{36}k^2.$$

As $\mathfrak{su}(2)_d\oplus\mathfrak{u}(1)$ representations the 7-dimensional space $\mathfrak{m}_{\mathbb{C}}$ decomposes as

$$\mathfrak{m}_{\mathbb{C}}\cong S^2W\oplus WF(3)\oplus WF(-3),$$

whereas the 3-dimensional adjoint representation of $(\mathfrak{su}(2)_d)_{\mathbb{C}}$ is irreducible and hence is isomorphic to S^2W.

Case 1: $E=\mathfrak{su}(2)_d\oplus\mathfrak{u}(1)$
The adjoint representation $\mathfrak{su}(2)_d\oplus\mathfrak{u}(1)$ splits as irreducible $\mathfrak{su}(2)_d\oplus\mathfrak{u}(1)$ representations as follows:

$$(\mathfrak{su}(2)_d\oplus\mathfrak{u}(1))_{\mathbb{C}}\cong S^2W\oplus\mathbb{C}.$$

Since $U(1)$ is abelian we know by Theorem 3.7 that the component $\mathfrak{u}(1)$ is abelian and thus gives rise to no deformations of the canonical connection. Therefore we only need to check for deformations corresponding to S^2W. For that we need to look for representations $V_{(m_1,m_2,l)}$ such that

$$-\frac{1}{9}(m_1^2+m_2^2+m_1m_2+3m_1+3m_2)-\frac{1}{8}(l^2+2l)=-\frac{8}{20},$$

which as seen before has no integral solutions.

Hence the canonical connection admits no deformations in this case.

Case 2: $E = \mathfrak{g}_2$
The adjoint representation $(\mathfrak{g}_2)_{\mathbb{C}}$ splits as $\mathfrak{su}(2)_d \oplus \mathfrak{u}(1)$ representation as follows:

$$(\mathfrak{g}_2)_{\mathbb{C}} = S^3WF(3) \oplus S^3WF(-3) \oplus S^2W \oplus F(6) \oplus F(-6) \oplus \mathbb{C}.$$

We need to follow the same procedure as above for each of the components. For each component we need to find the $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$ representation $V_{(m_1,m_2,l)}$ that satisfies (4.4). We have already solved this for $S^2W \oplus \mathbb{C}$ so we just need to compute it for the rest.
From above calculations $\rho_{S^3WF(3)}(\mathrm{Cas}_{\mathfrak{h}}) = -1$ therefore $V_{(m_1,m_2,l)}$ should satisfy

$$-\frac{1}{9}(m_1^2 + m_2^2 + m_1m_2 + 3m_1 + 3m_2) - \frac{1}{8}(l^2 + 2l) = -1.$$

The only possible solutions are $V_{(0,0,2)}, V_{(1,1,0)}$. As $\mathfrak{su}(2) \otimes \mathfrak{u}(1)$ representations $V_{(0,0,2)} \cong S^2W$ and $V_{(1,1,0)} \cong \mathfrak{su}(3)_{\mathbb{C}}$. Further one can compute

$$\begin{aligned} V_{(0,0,2)} &\cong \mathfrak{su}(2)_{\mathbb{C}} \cong S^2W, \\ V_{(1,1,0)} &\cong \mathfrak{su}(3)_{\mathbb{C}} \cong S^2W \oplus WF(3) \oplus WF(-3) \oplus \mathbb{C}, \\ S^3WF(3) \otimes \mathfrak{m}^*_{\mathbb{C}} &\cong (S^5W \oplus S^3W \oplus W)F(3) \oplus (S^4W \oplus S^2W)F(6) \oplus S^4W \oplus S^2W. \end{aligned}$$

Thus $V_{(0,0,2)}$ and $S^3WF(3) \otimes \mathfrak{m}^*_{\mathbb{C}}$ has one common component S^2W with multiplicity 1 and $V_{(1,1,0)}$ and $S^3WF(3) \otimes \mathfrak{m}^*_{\mathbb{C}}$ has two common components $S^2W, WF(3)$ both with multiplicity 1 each. So by Frobenius reciprocity $\mathrm{Ind}_H^G(\mathfrak{m}^*_{\mathbb{C}} \otimes S^3WF(3))$ contains a copy of $V_{(0,0,2)} \oplus 2V_{(1,1,0)}$.

The representation $S^3WF(-3)$ is the dual of the representation $S^3WF(3)$ and since $\mathrm{SU}(2) \otimes \mathrm{U}(1)$ representations are isomorphic to their duals the result for this case is same as the above and $\mathrm{Ind}_H^G(\mathfrak{m}^*_{\mathbb{C}} \otimes S^3WF(-3))$ also contains a copy of $V_{(0,0,2)} \oplus 2V_{(1,1,0)}$.

For the $\mathfrak{u}(1)$ representation $F(6)$, $\rho_6(\mathrm{Cas}_{\mathfrak{u}(1)}) = -1$. Thus again the only solutions are $V_{(0,0,2)}, V_{(1,1,0)}$ by the previous case. The $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ representation $F(6) \otimes \mathfrak{m}^*_{\mathbb{C}}$ has the following decompostion

$$F(6) \otimes \mathfrak{m}^*_{\mathbb{C}} \cong S^2WF(6) \oplus WF(9) \oplus WF(3),$$

thus $V_{(0,0,2)}$ is not contained in $\mathrm{Ind}_H^G(\mathfrak{m}^*_{\mathbb{C}} \otimes F(6))$ but $V_{(1,1,0)}$ is with multiplicity 1. Since $F(-6) \cong F(6)^*$ this case is similar to the above case.
Summing up all the parts together we get that $\ker((D^{-1/3,can})^2 - \frac{49}{9}\} \cap \Gamma(\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}} \cong 2(V_{(0,0,2)} \oplus 3V_{(1,1,0)})$ when the structure group is G_2.

Table 3 lists the $\ker((D^{-1/3,can})^2 - \frac{49}{9}\mathrm{id}) \cap \Gamma(\mathfrak{m}^* \otimes E)$ when $E = \mathfrak{h}$ and $E = \mathfrak{g}_2$ for all the homogeneous spaces listed in Table 2. Note that for the remaining two homogeneous spaces $N_{k,l}, k \neq l$ and $\mathrm{SU}(2)^3/\mathrm{U}(1)^2$ our methods does not apply when $E = \mathfrak{g}_2$ although since H is abelian for both of them there are no deformations for the $E = \mathfrak{h}$ case. The space $V^{(0,1)}$ listed in Table 3 denotes the unique irreducible 5-dimensional complex representation of $\mathfrak{sp}(2)$.

4.4 Eigenspaces of the Dirac operator

All the G-representations listed in Table 3 lie in $\ker((D^{-1/3,can})^2 - \frac{49}{9}\mathrm{id}) \cap \Gamma(\mathfrak{m}^* \otimes E)$ which by (4.5) is equal to $(\ker(D^{-1,can} + 2\mathrm{id}) \oplus \ker(D^{-1,can} - \frac{8}{3}\mathrm{id})) \cap \cap\Gamma(\mathfrak{m}^* \otimes E)$. Since the canonical connection is translation invariant it takes an irreducible G-representation to itself. Hence the irreducible subspaces found in Table 3 lie in either $\ker(D^{-1,can} - \frac{8}{3}\mathrm{id})$ or $\ker(D^{-1,can} + 2\mathrm{id})$ where the subspaces in the latter space constitute the infinitesimal deformations of the canonical connection by Theorem 3.2. Thus now it remains to identify which of the subspaces in Table 3 lies in $\ker(D^{-1,can} + 2\mathrm{id})$ for each of the homogeneous spaces. for all the homogeneous spaces G/H in Table 2 the metric corresponding to the nearly G_2 structure φ is given by $-\frac{3}{40}B$ where B is the Killing form of G. For 1-forms X, Y the Clifford product between X and $Y \cdot \eta$ is given by

$$X \cdot Y \cdot \eta = \langle X, Y \rangle \eta - \varphi(X, Y, .) \cdot \eta. \tag{4.9}$$

Thus we have all the ingredients in (4.7) to calculate the action of the Dirac operator $D^{-1,can}$ on each irreducible subspace in Table 3.

Homogeneous space	$\mathfrak{h}$	$\mathfrak{g}_2$
Spin(7)/G$_2$	0	0
SO(5)/SO(3)	0	$\mathfrak{so}(5)$
$\frac{\mathrm{Sp}(2)\times \mathrm{Sp}(1)}{\mathrm{Sp}(1)\times \mathrm{Sp}(1)}$	$V_{\mathbb{R}}^{(0,1)}$	$2\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)\oplus V_{\mathbb{R}}^{(0,1)}$
$\frac{\mathrm{SU}(3)\times \mathrm{SU}(2)}{\mathrm{SU}(2)\times \mathrm{U}(1)}$	0	$2\mathfrak{su}(2)\oplus 6\mathfrak{su}(3)$
$\mathrm{N}_{k,l}$	0	unknown
$\mathrm{SU}(2)^3/\mathrm{U}(1)^2$	0	unknown

Table 3: $\ker((D^{-1/3,can})^2-\frac{49}{9}\mathrm{id})\cap\Gamma(\mathfrak{m}^*\otimes E)$

4.4.1 SO(5)/SO(3)

From the previous section we know that there are no deformation of the canonical connection when the structure group is SO(3). For the structure group G$_2$ we calculated that the smooth sections of $G\times_{\rho_{\mathfrak{m}^*\otimes\mathfrak{g}_2}}(\mathfrak{m}^*\otimes\mathfrak{g}_2)$ in $\ker((D^{-1/3.can})^2-\frac{49}{9}\mathrm{id})\cong V_{(0,2)}\cong\mathfrak{so}(5)_{\mathbb{C}}$. If we denote by E_{ij} the skew-symmetric matrix with 1 at (i,j), -1 at (j,i) and 0 elsewhere and define

$$
\begin{aligned}
e_1 &:= \frac{2}{3}(E_{12}-2E_{34}), \qquad e_2 := \frac{2}{3}(\sqrt{2}E_{45}-\frac{\sqrt{3}}{\sqrt{2}}(E_{23}-E_{14})),\\
e_3 &:= \frac{2\sqrt{5}}{3}E_{25}, \qquad e_4 := \frac{2}{3}(\sqrt{2}E_{35}-\frac{\sqrt{3}}{\sqrt{2}}(E_{13}+E_{24})),\\
e_5 &:= \frac{\sqrt{10}}{3}(E_{24}-E_{13}), \quad e_6 := -\frac{\sqrt{10}}{3}(E_{23}+E_{14}), \quad e_7 := \frac{2\sqrt{5}}{3}E_{15},
\end{aligned}
$$

then $\{e_i, i=1\dots7\}$ defines a basis of $\mathfrak{m}^*$ which is orthonormal with respect to the metric $-\frac{3}{40}B$. With respect to this basis the nearly G_2 structure φ is given by

$$\varphi = e_{124}+e_{137}+e_{156}+e_{235}+e_{267}+e_{346}+e_{457}.$$

We have seen that for SO(5)/SO(3) the canonical connection has no deformation as an SO(3) connection. Now we need to check whether the SO(5)-representation $V_{(0,2)}$ lies in the $\ker(D^{-1,can}-\frac{8}{3}\mathrm{id})\cap\Gamma(\mathfrak{m}^*\otimes\mathfrak{g}_2)_{\mathbb{C}}$ or $\ker(D^{-1,can}+2\mathrm{id})\cap\Gamma(\mathfrak{m}^*\otimes\mathfrak{g}_2)_{\mathbb{C}}$. As seen before the common irreducible $\mathfrak{so}(3)$ representation in $V_{(0,2)}|_{\mathfrak{so}(3)}$ and $(\mathfrak{m}^*\otimes\mathfrak{g}_2)_{\mathbb{C}}$ is $S^6\mathbb{C}^2\cong\mathfrak{m}^*_{\mathbb{C}}$. We denote the 1-dimensional space $\mathrm{Hom}(V_{(0,2)},(\mathfrak{m}^*\otimes\mathfrak{g}_2)_{\mathbb{C}})=\mathrm{Span}(\alpha)$. Let $\mu_i, i=1\dots11$ be a basis of the 11-dimensional subspace of $(\mathfrak{g}_2)_{\mathbb{C}}$ isomorphic to the $\mathfrak{so}(3)$ representation $S^{10}\mathbb{C}^2$. Then the subspace of $\mathfrak{m}^*_{\mathbb{C}}\otimes S^{10}\mathbb{C}^2\subset(\mathfrak{m}^*\otimes\mathfrak{g}_2)_{\mathbb{C}}$ isomorphic to $S^6\mathbb{C}^2$ is given by $\mathrm{Span}\{v_i, i=1\dots7\}$ where

$$
\begin{aligned}
v_1 =& -\frac{e_2}{14}\otimes(5(\mu_1-\mu_7)+3\sqrt{15}\mu_9)+e_3\otimes(\mu_5+\mu_{11})-\frac{e_4}{14}\otimes(5\mu_2+3\sqrt{15}(\mu_3+\mu_4))\\
&+e_5\otimes(\mu_3-\mu_4)+e_6\otimes\mu_9+e_7\otimes(\mu_6-\mu_{10}),\\
v_2 =& e_1\otimes\mu_9+e_2\otimes(-2\mu_5+\mu_4)-\frac{e_3}{28}\otimes(47\mu_1+37\mu_7+3\sqrt{5}\mu_9)-e_4\otimes(\mu_6+2\mu_{10})\\
&-\frac{e_5}{14}\otimes\mu_8+\frac{e_7}{28}\otimes(-37\mu_2+3\sqrt{15}(\mu_3+\mu_4)),\\
v_3 =& -\frac{e_1}{2}\otimes(\mu_3-\mu_4)+\frac{e_2}{2}\otimes(2\mu_6+\mu_{10})+\frac{e_3}{56}\otimes(47\mu_2+3\sqrt{5}(\mu_3+\mu_4))-\frac{e_4}{2}\otimes(\mu_5-2\mu_{11})\\
&-\frac{e_6}{28}\otimes\mu_8+\frac{e_7}{56}(-37\mu_1+6\sqrt{15}\mu_9),\\
v_4 =& -\frac{e_1}{28}\otimes(5\mu_2+3\sqrt{15}(\mu_3+\mu_4))+\frac{5e_2}{28}\otimes\mu_8-\frac{e_3}{56}\otimes(3\sqrt{15}\mu_2+41\mu_3+13\mu_4)\\
&-\frac{e_5}{2}\otimes(\mu_5-2\mu_{11})+\frac{e_6}{2}\otimes(\mu_6+2\mu_{10})+\frac{e_7}{56}\otimes(3\sqrt{15}(\mu_1-\mu_7)+41\mu_9),
\end{aligned}
$$

$$\begin{aligned}
v_5 =& e_1\otimes(\mu_5+\mu_{11})-\frac{e_2}{28}\otimes(3\sqrt{15}(\mu_1-\mu_7)+13\mu_9)+\frac{e_4}{28}\otimes(3\sqrt{15}\mu_2+41\mu_3+13\mu_4)\\
&+\frac{e_5}{28}\otimes(47\mu_2+3\sqrt{15}(\mu_3+\mu_4))+\frac{e_6}{28}\otimes(47\mu_1+37\mu_7+3\sqrt{15}\mu_9)+\frac{2e_7}{28}\otimes\mu_8,\\
v_6 =& e_1\otimes(-\mu_6+\mu_{10})+\frac{e_2}{28}\otimes(3\sqrt{15}\mu_2+13\mu_3+41\mu_4)+\frac{2e_3}{7}\otimes\mu_8+\frac{e_4}{28}\otimes(3\sqrt{15}(\mu_1-\mu_7)+41\mu_9)\\
&+\frac{e_5}{28}\otimes(37\mu_1+47\mu_7-3\sqrt{15}\mu_9)+\frac{e_6}{28}\otimes(-37\mu_2+3\sqrt{15}(\mu_3+\mu_4)),\\
v_7 =& \frac{e_1}{14}\otimes(5(\mu_1-\mu_7)+3\sqrt{15}\mu_9)-\frac{e_3}{28}\otimes(3\sqrt{15}(\mu_1-\mu_7)+13\mu_9)+\frac{5e_4}{14}\otimes\mu_8-2e_5\otimes(\mu_6+\mu_{10})\\
&+e_6\otimes(-2\mu_5+\mu_{11})-\frac{e_7}{28}\otimes(3\sqrt{15}\mu_2+13\mu_3+41\mu_4).
\end{aligned}$$

The subspace of $V_{(0,2)}$ isomorphic to $S^6\mathbb{C}^2$ is $\mathrm{Span}_{\mathbb{C}}\{e_i, i=1\dots 7\}$ and the SO(3) equivariant homomorphism α between $V_{(0,2)}$ and $(\mathfrak{m}^*\otimes\mathfrak{g}_2)_{\mathbb{C}}$ is given by

$$\begin{aligned}
&\alpha(e_1)=v_1,\quad \alpha(e_2)=v_7,\quad \alpha(e_3)=-v_5,\\
&\alpha(e_4)=-2v_4,\quad \alpha(e_5)=2v_3,\quad \alpha(e_6)=-v_2,\quad \alpha(e_7)=v_6.
\end{aligned}$$

Any section of the bundle associated to $\mathfrak{m}^*\otimes\mathfrak{g}_2$ in $\ker((D^{-1/3.can})^2-\frac{49}{9}\mathrm{id})$ can be represented by (α,v) for some $v\in V_{(0,2)}|_{S^6\mathbb{C}^2}\cong\mathfrak{m}^*_{\mathbb{C}}$. The action of the canonical connection on such a section is then given by $\nabla^{-1,can}_X(\alpha,v)(eH)=-\alpha([X,v])$ where the Lie bracket is in $\mathfrak{so}(5)$. We can now calculate the action of the Dirac operator, $D^{-1,can}$ on $(\alpha,e_1)\cdot\eta$ at the point eH as follows. We omit the $\cdot\eta$ from the computations to reduce notational clutter and will continue to do so in every case.

$$\begin{aligned}
D^{-1,can}(\alpha,e_1)(eH) &= \sum_{i=1}^{7} e_i\cdot\nabla^{-1,can}_{e_i}(\alpha,e_1)(eH)\\
&=\frac{-2}{3}(e_2\cdot\alpha(e_4)+e_3\cdot\alpha(e_7)+e_4\cdot\alpha(-e_2)+e_5\cdot\alpha(e_6)+e_6\cdot\alpha(-e_5)+e_7\cdot\alpha(-e_3))\\
&=\frac{2}{3}(2e_2\cdot v_4-e_3\cdot v_6+e_4\cdot v_7+e_5\cdot v_2+2e_6\cdot v_3-e_7\cdot v_5)\\
&=\frac{2}{3}(-3v_1)\cdot\eta=-2\alpha(e_1).
\end{aligned}$$

Thus by the translation invariance of the canonical connection $V_{(0,2)}\subseteq\ker(D^{-1,can}+2\mathrm{id})\cap\Gamma(\mathfrak{m}^*\otimes\mathfrak{g}_2)_{\mathbb{C}}$.

4.4.2 $\frac{\mathrm{Sp}(2)\times\mathrm{Sp}(1)}{\mathrm{Sp}(1)\times\mathrm{Sp}(1)}$

From the previous section we know that for $E=\mathfrak{sp}(1)\oplus\mathfrak{sp}(1)$ the $\ker((D^{-1/3.can})^2-\frac{49}{9}\mathrm{id})\cap\Gamma(\mathfrak{m}^*\otimes E)_{\mathbb{C}}\cong V_{(0,1,0)}$. Let $\{e_i, i=1\dots 7\}$ be an orthonormal basis of $\mathfrak{m}^*$ with respect to the metric $-\frac{3}{40}B$ given by

$$\begin{aligned}
&e_1:=\frac{1}{3}\left(\begin{pmatrix}0&0\\0&2i\end{pmatrix},-3i\right),\quad e_2:=\frac{1}{3}\left(\begin{pmatrix}0&0\\0&2j\end{pmatrix},-3j\right),\quad e_3:=\frac{1}{3}\left(\begin{pmatrix}0&0\\0&2k\end{pmatrix},-3k\right),\\
&e_4:=\frac{\sqrt{5}}{3}\left(\begin{pmatrix}0&1\\-1&0\end{pmatrix},0\right),\quad e_5:=\frac{\sqrt{5}}{3}\left(\begin{pmatrix}0&i\\i&0\end{pmatrix},0\right),\quad e_6:=\frac{\sqrt{5}}{3}\left(\begin{pmatrix}0&j\\j&0\end{pmatrix},0\right),\quad e_7:=\frac{\sqrt{5}}{3}\left(\begin{pmatrix}0&k\\k&0\end{pmatrix},0\right).
\end{aligned}$$

With respect to this basis the nearly G_2 form is given by

$$\varphi=e_{123}-e_{145}-e_{167}-e_{246}+e_{257}-e_{347}-e_{356},$$

From Table 3 we know that as an $\mathrm{Sp}(1)\times\mathrm{Sp}(1)$ connection the deformation space of the canonical connection is an irreducible subrepresentation of $V_{(0,1,0)}$ and is thus trivial or $(V_{(0,1,0)})_{\mathbb{R}}$. We need to check whether this space lies in the -2 eigenspace of $D^{-1,A}$

The $\mathrm{Sp}(2)\times\mathrm{Sp}(1)$-representation $V_{(0,1,0)}$ is 5 dimensional. We need to find the space $\mathrm{Hom}(V_{(0,1,0)},(\mathfrak{m}^*\otimes(\mathfrak{sp}(1)_u\oplus\mathfrak{sp}(1)_d))_{\mathbb{C}})_{\mathrm{Sp}(1)\times\mathrm{Sp}(1)}$. The common irreducible $\mathrm{Sp}(1)\times\mathrm{Sp}(1)$ representations in $V_{(0,1,0)}$ and

$(\mathfrak{m}^*\otimes\mathfrak{sp}(1)_u)_{\mathbb{C}}$ is PQ. Let $S^2P = Span\{I, J, K\}$ then the subspace of $(\mathfrak{m}^*\otimes\mathfrak{sp}(1)_u)_{\mathbb{C}}$ isomorphic to the space PQ is given by $\mathrm{Span}_{\mathbb{C}}\{v_1, v_2, v_3, v_4\}$ where

$$v_1 = e_5\otimes I + e_6\otimes J + e_7\otimes K, \quad v_2 = -e_4\otimes I + e_7\otimes J - e_6\otimes K,$$
$$v_3 = -e_7\otimes I - e_4\otimes J + e_5\otimes K, \quad v_4 = e_6\otimes I - e_5\otimes J - e_4\otimes K.$$

Let the subspace of $V_{(0,1,0)}$ isomorphic to PQ be given by $\mathrm{Span}\{w_1, w_2, w_3, w_4\}$ and the homomorphism space $\mathrm{Hom}(V_{(0,1,0)}, (\mathfrak{m}^*\otimes\mathfrak{sp}(1)_u)_{\mathbb{C}}) = \mathrm{Span}(\beta)$ where β is defined by

$$w_1\mapsto v_3 + iv_4, \quad w_2\mapsto v_1 - iv_2,$$
$$w_3\mapsto v_1 + iv_2, \quad w_4\mapsto v_3 - iv_4.$$

Using this isomprhism one can compute that the only non-trivial $\mathfrak{gl}(V_{(0,1,0)}|_{PQ})$ elements with respect to the basis $\{w_1, w_2, w_3, w_4\}$ are

$$\tau_*(e_1) = \frac{2}{3}\begin{bmatrix} i & 0 & 0 & 0\\ 0 & -i & 0 & 0\\ 0 & 0 & i & 0\\ 0 & 0 & 0 & -i\end{bmatrix}, \quad \tau_*(e_2) = \frac{2}{3}\begin{bmatrix} 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{bmatrix}, \quad \tau_*(e_3) = \frac{2}{3}\begin{bmatrix} 0 & i & 0 & 0\\ i & 0 & 0 & 0\\ 0 & 0 & 0 & -i\\ 0 & 0 & -i & 0\end{bmatrix}.$$

Also by the definition of the canonical connection, $\nabla^{-1,can}_X(\beta, w)(eH) = -\beta(\tau_*(X)w)$. Thus we can calculate

$$\begin{aligned}(D^{-1,can}(\beta, w_1))(eH) &= \sum_{i=1}^{7} e_i\cdot\nabla^{-1,can}_{e_i}(\beta, w_1)(eH) = -\sum_{i=1}^{7} e_i\cdot\beta((\tau_*(e_i)w_1)|_{PQ})\\ &= -(e_1\cdot\beta(\frac{2}{3}iw_1) + e_2\cdot\beta(\frac{2}{3}w_2) + e_3\cdot\beta(\frac{2}{3}iw_2))\\ &= -\frac{2}{3}(ie_1\cdot(v_3 + iv_4) + e_2\cdot(v_1 - iv_2) + ie_3\cdot(v_1 - iv_2))\\ &= -\frac{2}{3}(3(v_3 + iv_4)) = -2\beta(w_1).\end{aligned}$$

Thus we have shown that $V_{(0,1,0)}$ lies in the $\ker(D^{-1,can} + 2\mathrm{id})$.

For $E = \mathfrak{g}_2$ the subspace of $\Gamma(\mathfrak{m}^*\otimes\mathfrak{g}_2)$ in $\ker((D^{-1/3.can})^2 - \frac{49}{9}\mathrm{id})$ is isomorphic to the $\mathrm{Sp}(1)\times\mathrm{Sp}(1)$ representation $2V_{(2,0,0)}\oplus V_{(0,1,0)}\oplus V_{(0,0,2)}$. We have already dealt with the space $V_{(0,1,0)}$. The remaining spaces are $2V_{(2,0,0)}\cong 2\mathfrak{sp}(2)$ and $V_{(0,0,2)}\cong\mathfrak{sp}(1)$. The two copies of $V_{(2,0,0)}$ arise from $\mathrm{Hom}(V_{(2,0,0)}, \mathfrak{m}^*_{\mathbb{C}}\otimes PS^3Q)_{\mathrm{Sp}(1)\times\mathrm{Sp}(1)}$ and the one copy of $V_{(0,0,2)}$ arises from $\mathrm{Hom}(V_{(0,0,2)}, \mathfrak{m}^*_{\mathbb{C}}\otimes PS^3Q)_{\mathrm{Sp}(1)\times\mathrm{Sp}(1)}$. Thus we have two cases:

Case: 1-$\mathrm{Hom}(V_{(0,0,2)}, \mathfrak{m}^*_{\mathbb{C}}\otimes PS^3Q)_{\mathrm{Sp}(1)\times\mathrm{Sp}(1)}\otimes V_{(0,0,2)}$

Let $\{w_1, w_2, w_3\}$ be the standard basis of $V_{(0,0,2)}\cong\mathfrak{sp}(1)_{\mathbb{C}}$ then the non-trivial actions of $\mathfrak{m}$ on $\mathfrak{sp}(1)_{\mathbb{C}}$ are given by

$$[e_1, .] = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & -2\\ 0 & 2 & 0\end{bmatrix}, \quad [e_2, .] = \begin{bmatrix} 0 & 0 & 2\\ 0 & 0 & 0\\ -2 & 0 & 0\end{bmatrix}, \quad [e_3, .] = \begin{bmatrix} 0 & 2 & 0\\ 2 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

Let $\{\mu_i, i = 1\ldots 8\}$ be a basis of the $\mathrm{Sp}(1)_u\times\mathrm{Sp}(1)_d$ subrepresentation of $(\mathfrak{g}_2)_{\mathbb{C}}$ isomorphic to PS^3Q. The 1-dimensional space $\mathrm{Hom}(V_{(0,0,2)}, (\mathfrak{m}^*\otimes\mathfrak{g}_2)_{\mathbb{C}}) = \mathrm{Span}\{\phi\}$ where ϕ maps

$$w_1\mapsto e_4\otimes(\mu_5 - \mu_2) + e_5\otimes(\mu_1 + \mu_6) + e_6\otimes(\mu_4 - \mu_7) - e_7\otimes(\mu_3 + \mu_8),$$
$$w_2\mapsto e_4\otimes(\mu_3 - 2\mu_8) - e_5\otimes(\mu_4 + 2\mu_7) + e_6\otimes(\mu_1 - 2\mu_6) - e_7\otimes(\mu_2 + 2\mu_5),$$
$$w_3\mapsto -e_4\otimes(2\mu_4 + \mu_7) + e_5\otimes(\mu_8 - 2\mu_3) - e_6\otimes(2\mu_2 + \mu_5) + e_7\otimes(\mu_6 - 2\mu_1).$$

The connection $\nabla^{-1,can}_X(\phi, w) = -\phi([X, w])$ for $w\in\mathfrak{sp}(1)$ where the Lie bracket is in the Lie algebra $\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)$. Thus we can calculate

$$D^{-1,can}(\phi, w_1)(eH) = \sum_{i=1}^{7} e_i\cdot\nabla^{-1,can}_{e_i}(\phi, w_1)(eH) = -\sum_{i=1}^{7} e_i\cdot\phi([e_i, w_1])$$

$$
\begin{aligned}
&= -(e_2 \cdot \phi(-2w_3) + e_3 \cdot \phi(2w_2)) \\
&= -2(e_4 \otimes (\mu_5 - \mu_2) + e_5 \otimes (\mu_1 + \mu_6) + e_6 \otimes (\mu_4 - \mu_7) - e_7 \otimes (\mu_3 + \mu_8)) \\
&= -2\phi(w_1).
\end{aligned}
$$

Hence again by translation invariance of $\nabla^{-1,can}$, $V_{(0,0,2)} \subseteq \ker(D^{-1,can} + 2\mathrm{id}) \cap \Gamma(\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$.

Case: 2-$\mathrm{Hom}(V_{(2,0,0)}, \mathfrak{m}^*_{\mathbb{C}} \otimes PS^3Q)_{\mathrm{Sp}(1)\times\mathrm{Sp}(1)} \otimes V_{(2,0,0)}$

The $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$-representation $V_{(2,0,0)} \cong \mathfrak{sp}(2)_{\mathbb{C}} \cong S^2P \oplus S^2Q \oplus PQ$. The subspace of $(\mathfrak{sp}(2))_{\mathbb{C}}$ isomorphic to S^2Q, PQ is given by $\mathrm{Span}_{\mathbb{C}}\{e_1, e_2, e_3\}, \mathrm{Span}_{\mathbb{C}}\{e_4, e_5, e_6, e_7\}$ respectively. As before the basis of $PS^3Q \subset (\mathfrak{g}_2)_{\mathbb{C}}$ is denoted by $\{\mu_1, \mu_2, \ldots, \mu_8\}$ and the subspace of $(\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$ isomorphic to S^2Q is given by $\mathrm{Span}\{\phi(w_1), \phi(w_2), \phi(w_3)\}$ defined above. The subspace of $(\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$ isomorphic to PQ is given by $\mathrm{Span}\{v_1, v_2, v_3, v_4\}$ where

$$
\begin{aligned}
v_1 &= e_1 \otimes (\mu_1 + \mu_6) - e_2 \otimes (\mu_4 + 2\mu_7) - e_3 \otimes (2\mu_3 - \mu_8), \\
v_2 &= e_1 \otimes (\mu_2 - \mu_5) - e_2 \otimes (\mu_3 - 2\mu_8) + e_3 \otimes (2\mu_4 + \mu_7), \\
v_3 &= -e_1 \otimes (\mu_3 + \mu_8) - e_2 \otimes (\mu_2 + 2\mu_5) - e_3 \otimes (2\mu_1 - \mu_6), \\
v_4 &= -e_1 \otimes (\mu_4 - \mu_7) - e_2 \otimes (\mu_1 - 2\mu_6) + e_3 \otimes (2\mu_2 + \mu_5).
\end{aligned}
$$

Let $\{A_1, A_2\}$ be a basis of the 2-dimensional space $\mathrm{Hom}(V_{(2,0,0)}, (\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}})_{\mathrm{Sp}(1)_u \times \mathrm{Sp}(1)_d}$ and let $A = c_1A_1 + c_2A_2$ for some real constants c_1, c_2 then we have that

$$
\begin{aligned}
&A(e_1) = c_1w_1, \quad A(e_2) = c_1w_2, \quad A(e_3) = c_1w_3 \\
&A(e_4) = -c_2v_2, \quad A(e_5) = c_2v_1, \quad A(e_6) = -c_2v_4, \quad A(e_7) = c_2v_3
\end{aligned}
$$

and A_1, A_2 acts trivially on S^2P.

Let $s_{(A,w)} \in \Gamma(\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$ be the section corresponding to $(A, w) \in \mathrm{Hom}(V_{(2,0,0)}, (\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}})_{\mathrm{Sp}(1)\times\mathrm{Sp}(1)} \otimes \mathfrak{sp}(2)$ then $\nabla^{-1,can}_X(A, w) = -A(\mathrm{ad}(X)w) = A([X, w]|)$ where the Lie bracket is in the Lie algebra $\mathfrak{sp}(2)$. Using this action of $\nabla^{-1,can}$ we can calculate

$$
\begin{aligned}
(D^{-1,can}(A, e_1))(eH) &= \sum_{i=1}^{7} e_i \cdot \nabla^{-1,can}_{e_i}(A, e_1)(eH) = -\sum_{i=1}^{7} e_i \cdot A([e_i, e_1]|) \\
&= -\frac{2}{3}(-e_2 \cdot A(e_3) + e_3 \cdot A(e_2) + e_4 \cdot A(e_5) - e_5 \cdot A(e_4) + e_6 \cdot A(e_7) - e_7 \cdot A(6)) \\
&= -\frac{2}{3}(c_1(-e_2 \cdot w_3 + e_3 \cdot w_2) + c_2(e_4 \cdot v_1 - e_5 \cdot (-v_2) + e_6 \cdot v_3 - e_7 \cdot (-v_4))) \\
&= \frac{4c_1 - 6c_2}{3} w_1 = \frac{4c_1 - 6c_2}{3} A_1(e_1).
\end{aligned}
$$

By doing similar computations we get that

$$
\begin{aligned}
&(D^{-1,can}(A, f_i))(eH) = 0, \quad i = 1, 2, 3, \\
&(D^{-1,can}(A, e_i))(eH) = \frac{4c_1 - 6c_2}{3} A_1(e_i), \quad i = 1, 2, 3, \\
&(D^{-1,can}(A, e_i))(eH) = -\frac{20c_1 + 6c_2}{9} A_2(e_i), \quad i = 4, 5, 6, 7.
\end{aligned}
$$

Therefore the subspace of $\mathrm{Hom}(V_{(2,0,0)}, (\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}})_{\mathrm{Sp}(1)\times\mathrm{Sp}(1)}$ in the $\ker(D^{-1,can} + 2\mathrm{id})$ is given by the condition $c_2 = \frac{5}{3}c_1$ and is thus 1-dimensional. Therefore $V_{(2,0,0)}$ occurs in the $\ker(D^{-1,can} + 2\mathrm{id}) \cap \Gamma(\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$ with multiplicity 1.

Remark 4.5. We can immediately see from above that the only other possible eigenvalue for which $\mathfrak{sp}(2)$ is an eigenspace of $D^{-1,can}$ is $-\frac{8}{3}$ for $c_2 = -\frac{2}{3}c_1$. This shows that not all spaces in $\ker((D^{-1/3,can})^2 - \frac{49}{9}\mathrm{id})$ are in $\ker(D^{-1,can} + 2\mathrm{id})$.

4.4.3 $\frac{\mathrm{SU}(3)\times\mathrm{SU}(2)}{\mathrm{SU}(2)\times\mathrm{U}(1)}$

As before let $\{e_i, i = 1\dots7\}$ be an orthonormal basis of $\mathfrak{m}^*$ with respect to g. If we define $I = \begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix}, J = \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}, K = \begin{pmatrix} 0 & i \\ i & 0\end{pmatrix}$ we have

$$e_1 := \frac{1}{3}\left(\begin{pmatrix} 2I & 0 \\ 0 & 0\end{pmatrix}, -3I\right), \quad e_2 := \frac{1}{3}\left(\begin{pmatrix} 2J & 0 \\ 0 & 0\end{pmatrix}, -3J\right), \quad e_3 := \frac{1}{3}\left(\begin{pmatrix} 2K & 0 \\ 0 & 0\end{pmatrix}, -3K\right),$$

$$e_4 := \frac{\sqrt{5}}{3}\left(\begin{pmatrix} 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \\ -\sqrt{2} & 0 & 0\end{pmatrix}, 0\right), \quad e_5 := \frac{\sqrt{5}}{3}\left(\begin{pmatrix} 0 & 0 & \sqrt{2}i \\ 0 & 0 & 0 \\ \sqrt{2}i & 0 & 0\end{pmatrix}, 0\right),$$
$$e_6 := \frac{\sqrt{5}}{3}\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0\end{pmatrix}, 0\right), \quad e_7 := \frac{\sqrt{5}}{3}\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sqrt{2}i \\ 0 & \sqrt{2}i & 0\end{pmatrix}, 0\right).$$

With respect to this basis the nearly G_2 structure φ is given by

$$\varphi = e_{123} + e_{145} - e_{167} + e_{246} + e_{257} + e_{347} - e_{356}.$$

As an $\mathrm{SU}(2)\times\mathrm{U}(1)$ representation, $\mathfrak{m}^*_{\mathbb{C}} \cong S^2W \oplus WF(3) \oplus WF(-3)$ where

$$S^2W = \mathrm{Span}\{e_1, e_2, e_3\}, \quad WF(3) = \mathrm{Span}\{e_4 - ie_5, e_6 - ie_7\}, \quad WF(-3) = \mathrm{Span}\{e_4 + ie_5, e_6 + ie_7\}.$$

From our previous work we know that the canonical connection has no deformations as an $\mathrm{SU}(2)\times\mathrm{U}(1)$ connection so we only have to consider the case $E = \mathfrak{g}_2$.

As an $\mathrm{SU}(2)\times\mathrm{U}(1)$ representation, $(\mathfrak{g}_2)_{\mathbb{C}} \cong S^3W(F(3)\oplus F(-3)) \oplus S^2W \oplus F(6) \oplus F(-6)$. We have already seen that S^2W gives rise to no deformations. From previous calculations we know that $\ker((D^{-1/3,can})^2 - \frac{49}{9}\mathrm{id}) \cap \Gamma(\mathfrak{m}^*_{\mathbb{C}} \otimes S^3WF(\pm3)) \cong V_{(0,0,2)} \oplus 2V_{(1,1,0)} \cong (\mathfrak{su}(2))_{\mathbb{C}} \oplus 2(\mathfrak{su}(3))_{\mathbb{C}}$ and $\Gamma(\mathfrak{m}^*_{\mathbb{C}} \otimes F(\pm6)) \cap \ker((D^{-1/3,can})^2 - \frac{49}{9}\mathrm{id}) \cong V_{(1,1,0)}$ respectively. Therefore there are 6 subspaces of $\Gamma(\mathfrak{m}^*\otimes\mathfrak{g}_2)$ to consider here.

Case: 1-$\mathrm{Hom}(V_{(0,0,2)}, \mathfrak{m}^*_{\mathbb{C}} \otimes S^3WF(3))_{\mathrm{SU}(2)\times\mathrm{U}(1)} \otimes V_{(0,0,2)}$

We denote by $\{\mu_i, i = 1\dots4\}$ a basis of $S^3WF(3)$. Let $f_i, i = 1\dots3$ be the standard basis of $\mathfrak{su}(2)$ such that $[f_1, f_2] = -2f_3, [f_1, f_3] = 2f_2, [f_2, f_3] = -2f_1$. Then the subspace of $WF(-3)\otimes S^3WF(3) \subset (\mathfrak{m}^*\otimes\mathfrak{g}_2)_{\mathbb{C}}$ isomorphic to $(\mathfrak{su}(2))_{\mathbb{C}}$ is given by $\mathrm{Span}\{v_1, v_2, v_3\}$ where

$$\begin{aligned} v_1 &= \frac{3i}{4}(e_4 + ie_5)\otimes\mu_1 + (e_6 + ie_7)\otimes(\frac{5i}{4}\mu_2 + \mu_4), \\ v_2 &= (e_4 + ie_5)\otimes(-i\mu_2 + \mu_4) + (e_6 + ie_7)\otimes(-i\mu_1 - \mu_3), \\ v_3 &= (e_4 + ie_5)\otimes(-\frac{5i}{4}\mu_1 + \mu_3) - \frac{3i}{4}(e_6 + ie_7)\otimes\mu_2 \end{aligned}$$

and the space $\mathrm{Hom}(V_{(0,0,2)}, (\mathfrak{m}^*\otimes\mathfrak{g}_2)_{\mathbb{C}}) = \mathrm{Span}\{\gamma^A\}$ where γ^A is defined by

$$\gamma^A(f_1) = v_2, \quad \gamma^A(f_2) = i(v_1 - v_3), \quad \gamma^A(f_3) = -2(v_1 + v_3).$$

For $i = 1, 2, 3$, since $e_i = (\frac{2}{3}f_i, -f_i)$ we have $[e_i, v] = -[f_i, v]$ for all $v \in \mathfrak{su}(2)$. The action is trivial for $i = 4\dots7$ since $[e_i, f_j] \notin \mathrm{Span}\{f_1, f_2, f_3\}$. We can thus calculate

$$\begin{aligned} D^{-1,can}(\gamma^A, f_1)(eH) &= \sum_{i=1}^{7} e_i \cdot \nabla^{-1,can}_{e_i}(\gamma^A, f_1)(eH) \\ &= e_2\cdot\gamma^A(2f_3) - e_3\cdot\gamma^A(2f_2) \\ &= -(4e_2\cdot(v_1 + v_3) + 2ie_3\cdot(v_1 - v_3)) \end{aligned}$$

$$= -2v_2 = -2\,\gamma^A(f_1).$$

Hence $\mathrm{Hom}(V_{(0,0,2)}, \mathfrak{m}^*_{\mathbb{C}} \otimes S^3WF(3))|_{\mathrm{Sp}(1)\times\mathrm{Sp}(1)} \otimes V_{(0,0,2)} \subseteq \ker(D^{-1,can} + 2\mathrm{id})$.

Case: 2-$\mathrm{Hom}(V_{(1,1,0)}, \mathfrak{m}^*_{\mathbb{C}} \otimes S^3WF(3))_{\mathrm{SU}(2)\times\mathrm{U}(1)} \otimes V_{(1,1,0)}$

Let a basis of the subspace of $V_{(1,1,0)} \cong (\mathfrak{su}(3))_{\mathbb{C}}$ isomorphic to $S^2W \cong (\mathfrak{su}(2))_{\mathbb{C}}$ be given by

$$p_1 := \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad p_2 := \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}, \quad p_3 := \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}.$$

where I, J, K are defined previously. Then $[p_1, p_2] = -2p_3, [p_1, p_3] = 2p_2, [p_2, p_3] = -2p_1$. The basis of $\mathfrak{m}^*_{\mathbb{C}} \otimes S^3WF(3) \subset \mathfrak{m}^*_{\mathbb{C}} \otimes \mathfrak{g}_2$ isomorphic to S^2W is given by $\mathrm{Span}\{w_1, w_2, w_3\}$ where

$$\begin{aligned} w_1 &= (e_4 + ie_5) \otimes \frac{\mu_2 + i\mu_3}{2} + (e_6 + ie_7) \otimes \frac{\mu_1 - i\mu_4}{2}, \\ w_2 &= (e_4 + ie_5) \otimes \frac{\mu_4 - 2i\mu_1}{2} + (e_6 + ie_7) \otimes \frac{\mu_3 - 2i\mu_2}{2}, \\ w_3 &= -(e_4 + ie_5) \otimes \frac{\mu_1 + 2i\mu_4}{2} + (e_6 + ie_7) \otimes \frac{\mu_2 - 2i\mu_3}{2}. \end{aligned}$$

Since $(\mathfrak{su}(3))_{\mathbb{C}} = \mathfrak{m}_{\mathbb{C}} \oplus \mathbb{C}$, the subspace of $(\mathfrak{su}(3))_{\mathbb{C}}$ isomorphic to $WF(3)$ is given by $\mathrm{Span}_{\mathbb{C}}\{e_4 - ie_5, e_6 - ie_7\}$. The subspace of $S^2W \otimes S^3WF(3) \subset (\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$ isomorphic to $WF(3)$ is given by $\mathrm{Span}\{u_1, u_2\}$ where

$$\begin{aligned} u_1 &= ie_1 \otimes \frac{\mu_2 + i\mu_3}{2} + e_2 \otimes \frac{2\mu_1 + i\mu_4}{2} - ie_3 \otimes \frac{\mu_1 + 2i\mu_4}{2}, \\ u_2 &= ie_1 \otimes \frac{\mu_1 - i\mu_4}{2} + e_2 \otimes \frac{2\mu_2 - i\mu_3}{2} + ie_3 \otimes \frac{\mu_2 - 2i\mu_3}{2} \end{aligned}$$

If we denote the space $\mathrm{Hom}(V^{(1,1,0)}, \mathfrak{m}^*_{\mathbb{C}} \otimes S^3WF(3))$ and $\mathrm{Hom}(V^{(1,1,0)}, \mathfrak{m}^*_{\mathbb{C}} \otimes S^3WF(3))$ by $\mathrm{Span}\{A_1\}$, $\mathrm{Span}\{A_2\}$ respectively then

$$\begin{aligned} A_1(p_i) &= w_i, \quad i = 1, 2, 3, \\ A_2(e_4 - ie_5) &= u_1, \quad A_2(e_6 - ie_7) = u_2. \end{aligned}$$

Define $A = c_1A_1 + c_2A_2$ for some constants c_1, c_2. We need to find the conditions on c_1, c_2 such that $(A, w) \in \Gamma(\mathfrak{m}^* \otimes S^3WF(3)) \cap \ker(D^{-1,can} + 2\mathrm{id})$ for all $w \in \mathfrak{su}(3)$.

Let $s_{(A,w)}$ be the section corresponding to (A, w). Then for any vector field X, $\nabla^{-1,can}_X(A, w) = -A(\mathrm{ad}(X)w) = A([X, w]|)$ where the Lie bracket is in the Lie algebra $\mathfrak{su}(3)$. Using this action of $\nabla^{-1,can}$ we can calculate

$$\begin{aligned} D^{-1,can}(A, p_1)(eH) &= \sum_{i=1}^{7} e_i \cdot \nabla^{-1,can}_{e_i}(A, p_1)(eH) \\ &= -(\frac{2}{3}(-e_2 \cdot A(2p_3) + e_3 \cdot A(2p_2))e_4 \cdot A(-e_5) + e_5 \cdot A(e_4) + e_6 \cdot A(e_7) + e_7 \cdot A(e_6)) \\ &= -\frac{2c_1}{3}(-e_2 \cdot w_1 + e_3 \cdot w_2) - c_2(-e_4 \cdot i\frac{u_1}{2} + e_5 \cdot \frac{u_1}{2} + e_6 \cdot i\frac{u_2}{2} - e_7 \cdot \frac{u_2}{2}) \\ &= \frac{4c_1 + 3ic_2}{3} w_1 = \frac{4c_1 + 3ic_2}{3} A_1(e_1). \end{aligned}$$

The operator $D^{-1,can}$ acts trivially on the subspaces of $(\mathfrak{su}(3))_{\mathbb{C}}$ isomorphic to $\mathbb{C}$ and $WF(-3)$. On the remaining subspaces we can compute the action of the Dirac operator as

$$\begin{aligned} D^{-1,can}(A, p_1)(eH) &= \frac{4c_1 + 3ic_2}{3} A_1(e_i), \quad i = 1, 2, 3, \\ D^{-1,can}(A, e_4 - ie_5)(eH) &= \frac{20c_1 - 3ic_2}{9} A_2(e_4 - ie_5), \end{aligned}$$

$$D^{-1,can}(A, e_6 - ie_7)(eH) = \frac{20c_1 - 3ic_2}{9} A_2(e_6 - ie_7).$$

Thus for any $w \in (\mathfrak{su}(3))_{\mathbb{C}}$, $(A, w) \in \ker(D^{-1,can} + 2\mathrm{id})$ if and only if $c_2 = \frac{10i}{3} c_1$. Thus only one copy of $\mathfrak{su}(3)$ lies in $\ker(D^{-1,can} + 2\mathrm{id})$.

Note that similarly to Remark 4.5 here also for $c_2 = -\frac{4i}{3} c_1$, $(A, w) \in \ker(D^{-1,can} - \frac{8}{3}\mathrm{id})$.

Case: 3-$\mathrm{Hom}(V_{(0,0,2)}, \mathfrak{m}^*_{\mathbb{C}} \otimes S^3WF(-3))_{\mathrm{Sp}(1)\times\mathrm{Sp}(1)} \otimes V_{(0,0,2)}$

Let $f_i, i = 1\dots 3$ be as before and denote by $\{\nu_i, i = 1\dots 4\}$ a basis of $S^3WF(-3)$. Then the subspace of $WF(3) \otimes S^3WF(-3)$ isomorphic to S^2W is given by $\mathrm{Span}\{w_1, w_2, w_3\}$ where

$$\begin{aligned}
w_1 &= (e_4 - ie_5) \otimes (\frac{-3i}{4}\nu_1) + (e_6 - ie_7) \otimes (\frac{-5i}{4}\nu_2 + \nu_4),\\
w_2 &= (e_4 - ie_5) \otimes (i\nu_2 + \nu_4) + (e_6 - ie_7) \otimes (i\nu_1 - \nu_3),\\
w_3 &= (e_4 - ie_5) \otimes (\frac{5i}{4}\nu_1 + \nu_3) + (e_6 - ie_7) \otimes (\frac{3i}{4}\nu_2)
\end{aligned}$$

and the space $\mathrm{Hom}(V_{(0,0,2)}, (\mathfrak{m}^*_C \otimes S^3WF(-3))) = \mathrm{Span}\{\gamma^B\}$ where γ^B is defined by

$$\gamma^B(f_1) = \frac{i}{2} w_2, \quad \gamma^B(f_2) = \frac{1}{2}(w_1 - w_3), \quad \gamma^B(f_3) = -i(w_1 + w_3).$$

The action of $e_i, i = 1\dots 7$ on $f_j, j = 1\dots 3$ is the same as Case 1 and thus we can calculate $D^{-1,can}(\gamma^B, f_1)$ as

$$\begin{aligned}
D^{-1,can}(\gamma^B, f_1)(eH) &= \sum_{i=1}^{7} e_i \cdot \nabla^{-1,can}_{e_i}(\gamma^B, f_1)(eH)\\
&= e_2 \cdot \gamma^B(2f_3) - e_3 \cdot \gamma^B(2f_2)\\
&= -2ie_2 \cdot (w_1 + w_3) - e_3 \cdot (w_1 - w_3)\\
&= -iw_2 = -2\,\gamma^B(f_1).
\end{aligned}$$

This implies $V_{(0,0,2)} \subseteq \ker(D^{-1,can} + 2\mathrm{id}) \cap \Gamma(\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$.

Case: 4-$\mathrm{Hom}(V_{(1,1,0)}, \mathfrak{m}^*_{\mathbb{C}} \otimes S^3WF(-3))_{\mathrm{SU}(2)\times\mathrm{U}(1)} \otimes V_{(1,1,0)}$

As above in Case 2, let a basis of the subspace of $(\mathfrak{su}(3))_{\mathbb{C}}$ isomorphic to $S^2W \cong \mathfrak{su}(2)$ be given by $\mathrm{Span}\{p_1, p_2, p_3\}$. The basis of $\mathfrak{m}^*_{\mathbb{C}} \otimes S^3WF(-3) \subset (\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$ isomorphic to S^2W is given by $\mathrm{Span}\{w_1, w_2, w_3\}$ where

$$\begin{aligned}
w_1 &= (e_4 - ie_5) \otimes \frac{\nu_2 - i\nu_3}{2} + (e_6 - ie_7) \otimes \frac{\nu_1 + i\nu_4}{2},\\
w_2 &= (e_4 - ie_5) \otimes \frac{\nu_4 + 2i\nu_1}{2} + (e_6 - ie_7) \otimes \frac{\nu_3 + 2i\nu_2}{2},\\
w_3 &= -(e_4 - ie_5) \otimes \frac{\nu_1 - 2i\nu_4}{2} + (e_6 - ie_7) \otimes \frac{\nu_2 + 2i\nu_3}{2}.
\end{aligned}$$

The subspace of $(\mathfrak{su}(3))_{\mathbb{C}}$ isomorphic to $WF(-3)$ is given by $\mathrm{Span}\{e_4 + ie_5, e_6 + ie_7\}$. The subspace of $S^2W \otimes S^3WF(-3) \subset (\mathfrak{m}^* \otimes \mathfrak{g}_2)_{\mathbb{C}}$ isomorphic to $WF(-3)$ is given by $\mathrm{Span}_{\mathbb{C}}\{u_1, u_2\}$ where

$$\begin{aligned}
u_1 &= -ie_1 \otimes \frac{\nu_2 - i\nu_3}{2} + e_2 \otimes \frac{2\nu_1 - i\nu_4}{2} + ie_3 \otimes \frac{\nu_1 - 2i\nu_4}{2},\\
u_2 &= -ie_1 \otimes \frac{\nu_1 + i\nu_4}{2} + e_2 \otimes \frac{2\nu_2 + i\nu_3}{2} - ie_3 \otimes \frac{\nu_2 + 2i\nu_3}{2}.
\end{aligned}$$

Again if we denote the spaces $\mathrm{Hom}(V(1,1,0), \mathfrak{m}^*_{\mathbb{C}} \otimes S^3WF(-3))$ and $\mathrm{Hom}(V^{(1,1,0)}, \mathfrak{m}^*_{\mathbb{C}} \otimes S^3WF(-3))$ by $\mathrm{Span}\{B_1\}$, $\mathrm{Span}\{B_2\}$ respectively then

$$B_1(p_i) = w_i, \quad i = 1, 2, 3,$$

$$B_2(e_4+ie_5)=u_1,\quad B_2(e_6+ie_7)=u_2.$$

Again as before we need to find the conditions on c_1,c_2 such that $(B=c_1B_1+c_2B_2,w)\in\ker(D^{-1,can}+2\mathrm{id})$ for all $w\in(\mathfrak{su}(3))_{\mathbb{C}}$. By similar computations as Case 2, we can calculate,

$$\begin{aligned}
D^{-1,can}(B,p_1)(eH) &= \sum_{i=1}^{7} e_i\cdot\nabla^{-1,can}_{e_i}(B,p_1)(eH)\\
&= -(\frac{2}{3}(-e_2\cdot B(2p_3)+e_3\cdot B(2p_2))+e_4\cdot B(-e_5)+e_5\cdot B(e_4)+e_6\cdot B(e_7)+e_7\cdot B(e_6))\\
&= -\frac{2c_1}{3}(-e_2\cdot w_1+e_3\cdot w_2)-c_2(-e_4\cdot i\frac{u_1}{2}+e_5\cdot\frac{u_1}{2}+e_6\cdot i\frac{u_2}{2}-e_7\cdot\frac{u_2}{2})\\
&= \frac{4c_1-3ic_2}{3}w_1=\frac{4c_1-3ic_2}{3}B_1(e_1).
\end{aligned}$$

Once can check that $D^{-1,can}$ acts trivially on the subspaces of $(\mathfrak{su}(3))_{\mathbb{C}}$ isomorphic to $\mathbb{C}, WF(3)$ and

$$\begin{aligned}
D^{-1,can}(A,p_1)(eH) &= \frac{4c_1-3ic_2}{3}B_1(e_i),\quad i=1,2,3,\\
D^{-1,can}(A,e_4+ie_5)(eH) &= \frac{20c_1+3ic_2}{9}B_2(e_4+ie_5),\\
D^{-1,can}(A,e_6+ie_7)(eH) &= \frac{20c_1+3ic_2}{9}B_2(e_6+ie_7).
\end{aligned}$$

Thus for all $w\in(\mathfrak{su}(3))_{\mathbb{C}}$, $(B,w)\in\ker(D^{-1,can}+2\mathrm{id})$ if and only if $c_2=-\frac{10i}{3}c_1$ which proves that only one copy of $\mathfrak{su}(3)$ lies in $\ker(D^{-1,can}+2\mathrm{id})$ in this case as well. It immediately follows from the given action that for $c_2=\frac{4i}{3}c_1$, $(B,w)\in\ker(D^{-1,can}-\frac{8}{3}\mathrm{id})$.

Case: 5-$\mathrm{Hom}(V_{(1,1,0)},\mathfrak{m}^*_{\mathbb{C}}\otimes F(6))_{\mathrm{SU}(2)\times\mathrm{U}(1)}\otimes V_{(1,1,0)}$

From before we know that the subspace of $(\mathfrak{su}(3))_{\mathbb{C}}$ isomorphic to $WF(3)$ is given by $\mathrm{Span}\{e_4-ie_5,e_6-ie_7\}$. if we denote by μ a basis vector for the 1-dimensional representation $F(6)$, the subspace of $\mathfrak{m}^*_{\mathbb{C}}\otimes F(6)$ isomorphic to $WF(3)$ is given by $\mathrm{Span}_{\mathbb{C}}\{(e_4+ie_5)\otimes\mu,(e_6+ie_7)\otimes\mu\}$. Let $\mathrm{Hom}(V_{(1,1,0)},\mathfrak{m}^*_{\mathbb{C}}\otimes F(6))=\mathrm{Span}\{\alpha\}$. We can define α as follows,

$$\alpha(e_4-ie_5)=(e_6+ie_7)\otimes\mu,\quad \alpha(e_6-ie_7)=-(e_4+ie_5)\otimes\mu.$$

Since $V_{(1,1,0)}$ is isomorphic to the adjoint representation $(\mathfrak{su}(3))_{\mathbb{C}}$, $\nabla^{-1,can}_X(\alpha,v)(eH)=-\alpha([X,v])$ where $X\in\mathfrak{m}$, $v\in WF(3)\subset\mathfrak{su}(3)$. Thus we can compute

$$\begin{aligned}
D^{-1,can}(\alpha,e_4-ie_5)(eH) &= \sum_{i=1}^{7} e_i\cdot\nabla^{-1,can}_{e_i}(\alpha,e_4-ie_5)(eH)\\
&= -(e_1\cdot\alpha(\frac{2i}{3}(e_4-ie_5))+e_2\cdot\alpha(\frac{2}{3}(e_6-ie_7))+e_3\cdot\alpha(\frac{2i}{3}(e_6-ie_7)))\\
&= -\frac{2}{3}(ie_1\cdot(e_6+ie_7)\otimes\mu-e_2\cdot(e_4+ie_5)\otimes\mu-ie_3\cdot(e_4+ie_5)\otimes\mu)\\
&= -2(e_6+ie_7)\otimes\mu=-2\alpha(e_4-ie_5).
\end{aligned}$$

Therefore $\mathrm{Hom}(V_{(1,1,0)},\mathfrak{m}^*_{\mathbb{C}}\otimes F(6))_{\mathrm{SU}(2)\times\mathrm{U}(1)}\otimes V_{(1,1,0)}\subset\ker(D^{-1,can}+2\mathrm{id})$ and thus lies in the deformation space.

Case: 6-$\mathrm{Hom}(V_{(1,1,0)},\mathfrak{m}^*_{\mathbb{C}}\otimes F(-6))_{\mathrm{SU}(2)\times\mathrm{U}(1)}\otimes V_{(1,1,0)}$

The subspace of $(\mathfrak{su}(3))_{\mathbb{C}}$ isomorphic to $WF(-3)$ is given by $\mathrm{Span}_{\mathbb{C}}\{e_4+ie_5,e_6+ie_7\}$. We denote $F(-6)=\mathrm{Span}\{\nu\}$. Then $\mathfrak{m}^*_{\mathbb{C}}\otimes F(-6)$ isomorphic to $WF(-3)$ is given by $\mathrm{Span}\{(e_4-ie_5)\otimes\nu,(e_6-ie_7)\otimes\nu\}$. Let $\mathrm{Hom}(V_{(1,1,0)},\mathfrak{m}^*_{\mathbb{C}}\otimes F(-6))=\mathrm{Span}\{\beta\}$ then

$$\beta(e_4+ie_5)=-(e_6-ie_7)\otimes\nu,\quad \beta(e_6+ie_7)=(e_4-ie_5)\otimes\nu.$$

Since $V_{(1,1,0)} \cong (\mathfrak{su}(3))_{\mathbb{C}}$, $\nabla_X^{-1,can}(\beta, v)(eH) = -\beta([X, v])$ where $X \in \mathfrak{m}$, $v \in WF(-3) \subset (\mathfrak{su}(3))_{\mathbb{C}}$. Thus we can compute

$$\begin{aligned} D^{-1,can}(\beta, e_4 + ie_5)(eH) &= \sum_{i=1}^{7} e_i \cdot \nabla_{e_i}^{-1,can}(\beta, e_4 + ie_5)(eH) \\ &= -(e_1 \cdot \beta(\frac{-2i}{3}(e_4 + ie_5)) + e_2 \cdot \beta(\frac{2}{3}(e_6 + ie_7)) + e_3 \cdot \alpha^A(\frac{-2i}{3}(e_6 + ie_7))) \\ &= -\frac{2}{3}(ie_1 \cdot (e_6 - ie_7) \otimes \nu + e_2 \cdot (e_4 - ie_5) \otimes \nu - ie_3 \cdot (e_4 - ie_5) \otimes \nu) \\ &= 2(e_6 - ie_7) \otimes \nu = -2\beta(e_4 + ie_5), \end{aligned}$$

which by translation invariance of $D^{-1,can}$ shows that $\mathrm{Hom}(V_{(1,1,0)}, \mathfrak{m}_{\mathbb{C}}^* \otimes F(-6))_{\mathrm{SU}(2)\times\mathrm{U}(1)} \otimes V_{(1,1,0)} \subset \ker(D^{-1,can} + 2\mathrm{id})$.

4.5 Summary

For three out of the four considered normal homogeneous spaces the canonical connection is rigid as an H-connection. As a G_2-connection the canonical connection has a non-trivial infinitesimal deformation space except for the round S^7. Summing up all the results found above we get the following theorem.

Theorem 4.6. *The infinitesimal deformation space for the canonical connection on the four normal homogeneous nearly* G_2 *spaces* G/H *when the structure group is* H *or* G_2 *is isomorphic to*

G/H	Structure group	
	H	G_2
$\mathrm{Spin}(7)/\mathrm{G}_2$	0	0
$\mathrm{SO}(5)/\mathrm{SO}(3)$	0	$\mathfrak{so}(5)$
$\dfrac{\mathrm{Sp}(2)\times\mathrm{Sp}(1)}{\mathrm{Sp}(1)\times\mathrm{Sp}(1)}$	$V_{\mathbb{R}}^{(0,1)}$	$\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)\oplus V_{\mathbb{R}}^{(0,1)}$
$\dfrac{\mathrm{SU}(3)\times\mathrm{SU}(2)}{\mathrm{SU}(2)\times\mathrm{U}(1)}$	0	$2\mathfrak{su}(2)\oplus 4\mathfrak{su}(3)$

where $V^{(0,1)}$ *is the unique* 5*-dimensional complex irreducible* $\mathrm{Sp}(2)$*-representation.*

References

[AF04] Ilka Agricola and Thomas Friedrich, *On the holonomy of connections with skew-symmetric torsion*, Mathematische Annalen **328** (2004Apr), no. 4, 711–748. ↑10

[AS12] Bogdan Alexandrov and Uwe Semmelmann, *Deformations of nearly parallel* G_2*-structures*, Asian J. Math. **16** (2012), no. 4, 713–744. MR3004283 ↑13, 14, 21, 23

[BFGK91] Helga Baum, Thomas Friedrich, Ralf Grunewald, and Ines Kath, *Twistor and killing spinors on riemannian manifolds*, Vol. 108, 1991. ↑1, 4

[BO19] Gavin Ball and Goncalo Oliveira, *Gauge theory on Aloff-Wallach spaces*, Geom. Topol. **23** (2019), no. 2, 685–743. MR3939051 ↑2, 3, 12

[Bry06] Robert L. Bryant, *Some remarks on* G_2*-structures*, Proceedings of Gökova Geometry-Topology Conference 2005, 2006, pp. 75–109. MR2282011 ↑4

[Bry87] ———, *Metrics with exceptional holonomy*, Annals of Mathematics **126** (1987), no. 3, 525–576. ↑3

[CH16] Benoit Charbonneau and Derek Harland, *Deformations of nearly kähler instantons*, Communications in Mathematical Physics **348** (2016Dec), no. 3, 959–990. ↑2, 16, 17

[Cla14] Andrew Clarke, *Instantons on the exceptional holonomy manifolds of Bryant and Salamon*, J. Geom. Phys. **82** (2014), 84–97. MR3206642 ↑2

[CS04] Richard Cleyton and Andrew Swann, *Einstein metrics via intrinsic or parallel torsion*, Mathematische Zeitschrift **247** (2004Jul), no. 3, 513–528. ↑6, 7, 14

[DS20] Shubham Dwivedi and Ragini Singhal, *Deformation theory of nearly* G_2 *manifolds*, arXiv e-prints (July 2020), arXiv:2007.02497, available at `2007.02497`. ↑2

[FI02] Thomas Friedrich and Stefan Ivanov, *Parallel spinors and connections with skew-symmetric torsion in string theory*, Asian J. Math. **6** (2002), no. 2, 303–335. MR1928632 ↑7

[FKMS97] Thomas Friedrich, Ines Kath, Andrei Moroianu, and Uwe Semmelmann, *On nearly parallel g2-structures*, Journal of Geometry and Physics **23** (1997), no. 3, 259 –286. ↑1, 13

[Fri12] Thomas Friedrich, *The second Dirac eigenvalue of a nearly parallel* G_2*-manifold*, Adv. Appl. Clifford Algebr. **22** (2012), no. 2, 301–311. MR2930696 ↑9

[Gra71] Alfred Gray, *Weak holonomy groups*, Mathematische Zeitschrift **123** (1971Dec), no. 4, 290–300. ↑2

[Hal15] Brian Hall, *Lie groups, Lie algebras, and representations*, Second, Graduate Texts in Mathematics, vol. 222, Springer, Cham, 2015. An elementary introduction. MR3331229 ↑15

[HN12] Derek Harland and Christoph Nölle, *Instantons and Killing spinors*, Journal of High Energy Physics **2012** (2012Mar), no. 3, 82. ↑2, 8, 14

[Joy00] Dominic D. Joyce, *Compact manifolds with special holonomy*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000. MR1787733 ↑4

[Kar09] Spiro Karigiannis, *Flows of* G_2*-structures. I*, Q. J. Math. **60** (2009), no. 4, 487–522. MR2559631 ↑4, 6

[Kar10] ———, *Some notes on* G_2 *and* $\mathrm{Spin}(7)$ *geometry* **11** (2010), 129–146. MR2648941 ↑5

[LM89] Harvey B. Lawson and Marie-Louise Michelson, *Spin geometry (pms-38)*, Princeton University Press, 1989. ↑5, 7

[MS10] Andrei Moroianu and Uwe Semmelmann, *The Hermitian Laplace operator on nearly Kähler manifolds*, Comm. Math. Phys. **294** (2010), no. 1, 251–272. MR2575483 ↑15

[SEW15] Henrique N. Sá Earp and Thomas Walpuski, G_2*-instantons over twisted connected sums*, Geom. Topol. **19** (2015), no. 3, 1263–1285. MR3352236 ↑2

[Wal16] Thomas Walpuski, G_2*-instantons over twisted connected sums: an example*, Math. Res. Lett. **23** (2016), no. 2, 529–544. MR3512897 ↑2

[Wal20] Alex Waldron, G_2*-instantons on the 7-sphere*, arXiv e-prints (February 2020), arXiv:2002.02386, available at `2002.02386`. ↑2

[Wil99] Burkhard Wilking, *The normal homogeneous space* $(\mathrm{SU}(3)\times\mathrm{SO}(3))/\mathrm{U}^{\bullet}(2)$ *has positive sectional curvature*, Proc. Amer. Math. Soc. **127** (1999), no. 4, 1191–1194. MR1469441 ↑14