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Online Portfolio Selection Strategy Based on Combining Experts' Advice

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Abstract The *weak aggregating algorithm* (WAA) developed from learning and prediction with expert advice makes decisions by considering all the experts' advice, and each expert's weight is updated according to his performance in previous periods. In this paper, we apply the WAA to the online portfolio selection problem. We first consider a simple case in which the expert advice is the strategy for investing in one stock; for this case, we obtain a portfolio selection strategy *WAAS* and prove that the WAAS can identify the best stock. We also discuss a more complicated case in which *constant rebalanced portfolios* are considered as expert advice, and obtain a corresponding portfolio selection strategy *WAAC*. The theoretical result shows that the cumulative gain that WAAC achieves is as large as that of the *best constant rebalanced portfolio*. Numerical analysis shows that the cumulative gains of our proposed strategies are as large as those of the best expert advice.

Keywords Online portfolio selection · Online learning · Expert advice · Weak aggregating algorithm

1 Introduction

The online portfolio selection problem was proposed by Cover (1991a). It emphasizes the online method of choosing a portfolio and independence from statistical assumptions, which makes it different from the traditional portfolio selection problem proposed in the seminal paper by Markowitz (1952). Traditional portfolio selection

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models based on probability theory are useful in theory (Zhang 2007; Huang 2012); however, it is difficult to apply these models in practice. Possible reasons for such difficulties are that the trend of stock prices is hard to predict or a suitable distribution to describe the trend is unavailable. These difficulties become even more obvious when the process that governs stock price behavior changes with time. Cover's *universal portfolio* (UP), which produces portfolios based only on the sequence of past stock prices, avoids the need for statistical assumptions about the behavior of stock prices (Cover 1991a). Thus, it has become a useful complement to traditional research methods. The UP strategy has been generalized and several relevant contributions presented (Cover 1991a, 1994, 1996; Cover and Ordentlich 1996, 1998). Meanwhile, the competitive theory of online portfolios has been studied intensively (Cover 1991b; Ordentlich and Cover 1996; Borodin et al. 2000). However, Cover's UP approach has the disadvantage that the portfolios it produces are hard to compute for a relatively large number of stocks. Helmbold et al. (1998) proposed the *exponentiated gradient* (EG) update, which adopts a multiplicative update rule derived from a framework introduced by Kivinen and Warmuth (1997) to deal with the UP computational issues. EG is easy to implement and competitive with the *Best Constant Rebalanced Portfolio* (BCRP) determined in hindsight.

The method of learning and prediction with expert advice, which is developed in computer science, provides a new approach to the online portfolio selection problem (Cesa-Bianchi and Lugosi 2006). This method aims to develop algorithms that compete with a benchmark set of "experts," who can be free agents or decision strategies. The *weak aggregating algorithm* (WAA) provided by Kalnishkan and Vyugin (2008) is an example of such an algorithm. In this paper, we apply the WAA to the online portfolio selection problem. We first consider the simple case in which the expert advice relates to pure strategies of investing in one fixed stock. For this case, we provide a simple portfolio selection strategy and name it WAAS. We also discuss a more complicated case in which *constant rebalanced portfolios* (CRPs) are considered as expert advice. For this case we obtain a corresponding portfolio selection strategy and name it WAAC. In these two applications, the expert advice is stationary, which means each expert advice is fixed in all trading periods. The duration of expert advice is the investors' investment time. The online portfolio is a sequential decision-making problem and the number of total trading periods is unavailable in advance. Therefore, our application of the WAA to stationary expert advice provides investor strategies in an online manner. More importantly, these strategies can guarantee that investors' cumulative gains are as large as those of the corresponding best expert advice. Our numerical analysis illustrates the competitive performance of our proposed strategies.

The remainder of this paper is organized as follows. In Sect. 2, we introduce the notation used in the paper and present a simple literature review. In Sect. 3, we describe the WAA's setting for the online portfolio selection problem. In Sect. 4, we present the portfolio selection strategy WAAS and demonstrate its competitive performance. In Sect. 5, we present the portfolio selection strategy WAAC and its theoretical result. Numerical analyses are provided in Sect. 6 to demonstrate the effectiveness of the proposed strategies. Section 7 concludes the paper.

2 Notation and Literature Review

In this paper, we assume that stocks are traded in a market without short selling or borrowing. The stock market evolves in discrete time; and each day of discrete time is referred to as a trading day. The stock market comprises N stocks for which prices vary from one trading day to another. The performance of the N stocks is denoted by a vector of price relatives $\mathbf{x} = (x^1, x^2, \cdots, x^N)$, where x^i represents the ratio of the closing to the opening price for the ith stock on the current day. All possible portfolio vectors are denoted by

$$\mathbf{B} = \{\mathbf{b} = (b^1, b^2, \ldots, b^N); b^i \geq 0, \sum_{i=1}^{N} b^i = 1\},$$

where b^i is the proportion of current wealth invested in the ith stock. Given the price relatives $\mathbf{x}$, an investor using portfolio $\mathbf{b}$ to invest will decrease or increase his wealth from one day to the next by a factor of $\mathbf{b} \cdot \mathbf{x} = \sum_{i=1}^{N} b^i x^i$.

At the start of each day t, the investor immediately chooses portfolio $\mathbf{b}_t$ for the day according to the previous price relatives of the stock market $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{t-1}$. At the beginning of the next day (day $t+1$), the price relatives for day t are observed and the investor's wealth increases or decreases by a factor of $\mathbf{b}_t \cdot \mathbf{x}_t$. When T trading days pass, a sequence of daily price relatives $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_T$ is observed and a sequence of portfolios $\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_T$ is selected. Thus, from the beginning of day 1 to the beginning of day $T+1$, the investor's wealth will have increased or decreased by a factor of

$$S_T(\{\mathbf{b}_t\}, \{\mathbf{x}_t\}) = \prod_{t=1}^{T} \mathbf{b}_t \cdot \mathbf{x}_t,$$

where the initial wealth $S_0 = 1$ is normalized to 1.

A CRP is a strategy that uses the same investment vector $\mathbf{b}$ on each trading day and the resulting wealth after T trading days is

$$S_T(\mathbf{b}) = S_T(\mathbf{b}, \{\mathbf{x}_t\}) = \prod_{t=1}^{T} \mathbf{b} \cdot \mathbf{x}_t.$$

A CRP strategy might require vast amounts of trading because at the beginning of each trading day the investment proportions are rebalanced back to $\mathbf{b}$. Its frequent transactions incur transaction costs, which are ignored in this paper. It is possible to calculate an optimal offline portfolio for the CRP strategy as,

$$\mathbf{b}_T^* = \arg\max_{\mathbf{b} \in \mathbf{B}} S_T(\mathbf{b}),$$

which can be efficiently solved. The CRP strategy with $\mathbf{b}_T^*$ is denoted as BCRP.

Note that the BCRP strategy is a hindsight strategy that can only be computed by assuming perfect knowledge of future stock prices. Cover (1991a) showed that the wealth of BCRP exceeds both the wealth of the best stock and the wealth of the arithmetic mean of N stocks. Thus, BCRP has been used as the reference benchmark for the evaluation of online portfolio selection strategies. An online portfolio selection strategy A is said to be universal if it has the same exponential rate of growth as BCRP, i.e., the sequence of portfolios $\{\mathbf{b}_t\}$ produced by the online portfolio selection strategy A should satisfy

$$\lim_{T\to\infty} \max_{\{\mathbf{x}_t\}}[LS^*(\{\mathbf{x}_t\}) - LS(\{\mathbf{b}_t\}, \{\mathbf{x}_t\})] \leq 0 \tag{1}$$

where

$$LS^*(\{\mathbf{x}_t\}) = \frac{1}{T}\ln S_T(\mathbf{b}_T^*), \qquad LS(\{\mathbf{b}_t\}, \{\mathbf{x}_t\}) = \frac{1}{T}\ln S_T(\{\mathbf{b}_t\}, \{\mathbf{x}_t\}).$$

Cover (1991a) proposed a UP strategy and later extended it to cases with side information (Cover and Ordentlich 1996). Blum and Kalai (1997) further extended the strategy by introducing transaction costs. With UP, the current portfolio is the weighted average of all CRPs and the weight is decided by the performance of the corresponding CRP strategy in the previous trading days, i.e.

$$\mathbf{b}_t = \frac{\int_{\mathbf{B}} \mathbf{b} S_{t-1}(\mathbf{b}) d\mathbf{b}}{\int_{\mathbf{B}} S_{t-1}(\mathbf{b}) d\mathbf{b}} \tag{2}$$

The UP presents an elegant formula; however, it relies on multidimensional integration, which is difficult to compute for relatively large numbers of stocks. Using ideas drawn from nonstationary stochastic optimization, the adaptive portfolio selection policies described in Gaivoronski and Stella (2000, 2003) were constructed. When the stock price varies greatly, a UP strategy cannot achieve ideal performance. Singer (1997) proposed a switching method to design online portfolios in this case. Helmbold et al. (1998) proposed the EG online portfolio selection strategy, which uses relative entropy as a distance function for motivating updates and derives the portfolio vector by maximizing an objective function. The EG can be easily computed and is universal. Based on linear function learning, Zhang et al. (2012) generalized the EG to a class of universal portfolios.

Most online portfolio selection strategies take BCRP as the reference benchmark. Indeed, transaction costs affect the return as they exist in adjusting and can be very large. Hence, Kozat and Singer (2011) took the *semi-constant rebalanced portfolio* (S-CRP) as a benchmark and used the method developed in signal processing to provide an online portfolio selection strategy. Taking the best stock as the benchmark, Albeverio et al. (2001) used the "Cross rate" to present another method and further considered transaction costs to generalize it. The methods developed in machine learning, information science and data mining are applied to the design of an online portfolio selection strategy. Li and Hoi (2012) presented a detailed survey about online portfolio selection. The WAA is an online learning method of prediction with expert advice. In the following sections, we apply the WAA to online portfolio and provide investment strategies.

3 The WAA for the Online Portfolio Selection Problem

The online learning method WAA was proposed by Kalnishkan and Vyugin (2008). The application of the WAA to the online portfolio selection problem is as follows. Given a set of experts who provide decisions (portfolio vectors) each day, an investor employs the WAA to combine these decisions in a certain way. He assigns weights to all the experts. These weights are updated every day and a real outcome (a vector of price relatives) is obtained to reflect the change in the investor's level of trust in each of the experts' decisions. The performances of the investor and experts are measured by a gain function. The investor uses the WAA to combine experts' decisions and update the weights such that his cumulative gain is comparable to the cumulative gain of the best expert.

Alternatively, this process can be presented in the form of a sequential game between two players, the investor and the market. The investor's role is to choose portfolio vectors and the market's role is to choose vectors of price relatives. The investor uses the WAA to decide on a portfolio vector each day. The investor's performance improves with time as more vectors of price relatives become available. After the investor decides on his portfolio vector, the market announces the actual vector of price relatives. Then the investor updates the WAA parameters and the game continues.

We denote the index set for experts by Θ and assume that Θ is measurable and each individual expert is denoted by $\theta, \theta \in \Theta$. Different expert advice corresponds to different index set Θ. The market chooses the actual outcome (vector of price relatives) from the outcome set Ω. In this paper $\Omega = [0, 1]^N$ as we can assume that the price relative belongs to [0, 1], which can be seen in Helmbold et al. (1998) and Zhang et al. (2012). Both the experts and the investor choose their decisions from a prediction set $\mathbf{B}$. Let $g = \pi(\mathbf{b}, \mathbf{x})$ be the gain function. On day t, given decision $\mathbf{b}_t \in \mathbf{B}$ and the actual outcome $\mathbf{x}_t \in \Omega$, the investor's gain function is $g_t = \pi(\mathbf{b}_t, \mathbf{x}_t)$. Given expert θ's decision $\mathbf{b}_t^\theta \in \mathbf{B}$, his gain function is $g_t^\theta = \pi(\mathbf{b}_t^\theta, \mathbf{x}_t)$, the cumulative gains for the investor and expert θ during the first t days are $G_t = \sum_{s=1}^{t} g_s$ and $G_t^\theta = \sum_{s=1}^{t} g_s^\theta$.

Let $q(\theta)$ be a prior probability measure of Θ and $p_t(d\theta)$ be the weight the investor assigns to expert θ on day t. The pseudo-code of the WAA for the online portfolio selection problem is as follows.

- The initial cumulative gains for the investor and experts are 0: $G_1 := 0$; $G_1^\theta := 0, \theta \in \Theta$
- In each day $t = 2, 3, \ldots$
  (1) The investor assigns weights to each of the experts:

$$p_t(d\theta) := \frac{\beta_t^{G_{t-1}^\theta} q(d\theta)}{\int_\Theta \beta_t^{G_{t-1}^\theta} q(d\theta)}, \theta \in \Theta,$$

where $\beta_t = \exp\left(\frac{1}{\sqrt{t}}\right)$ is the learning rate parameter used by Kalnishkan and Vyugin (2008);

(2) The experts give their decisions $\mathbf{b}_t^\theta, \theta \in \Theta$;

(3) The investor announces his decision

$$\mathbf{b}_t := \int_\Theta \mathbf{b}_t^\theta p_t(d\theta) \tag{3}$$

(4) The market announces the actual outcome $\mathbf{x}_t$;

(5) The investor updates his cumulative gain $G_t := G_{t-1} + \pi(\mathbf{b}_t, \mathbf{x}_t)$ and the experts' cumulative gains $G_t^\theta := G_{t-1}^\theta + \pi(\mathbf{b}_t^\theta, \mathbf{x}_t), \theta \in \Theta$.

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Levina et al. applied the WAA to the newsvendor problem and proved a theoretical guarantee on the performance of the WAA in the case of the bounded gain function (Levina et al. 2010, Lemma 2). The result is stated in Lemma 1.

Lemma 1 *Let $\pi \in [-L, 0]$. The WAA guarantees that, for all T,*

$$G_T \geq \sqrt{T}\left(\ln \int_\Theta \exp\left(\frac{G_T^\theta}{\sqrt{T}}\right) q(d\theta) - L^2\right) \tag{4}$$

4 Simple Portfolio Selection Strategy WAAS and Its Competitive Performance

To apply the WAA to the online portfolio selection problem, we let $g = \pi(\mathbf{b}, \mathbf{x}) = \ln(\mathbf{b} \cdot \mathbf{x})$ be the gain function for portfolio vector $\mathbf{b}$ and price relatives vector $\mathbf{x}$. Without generalization, suppose the price relatives satisfy $a \leq x_t^i \leq 1$ for all i and t. Suppose that on the first day the proportion of investment on each stock is identical, i.e., $\mathbf{b}_1 = (1/N, \ldots, 1/N)$. For simplicity, we first consider simple expert advice that invests only in one stock. As there are N stocks, the index set for the expert can be denoted by $\Theta = \{1, 2, \ldots, N\}$, where $\mathbf{e}^\theta \in \mathbf{B}, \theta \in \Theta$ denotes expert advice that always invests in the θth stock. Denote this online portfolio selection strategy by WAAS. The decision-making process for WAAS can be described as follows.

– On the first day, the experts' cumulative gains are denoted by $G_1^\theta = 0, \theta \in \Theta$ and the investor's cumulative gain is denoted by $G_1 = 0$;

– On each day $t = 2, 3, \cdots$

(1) The investor applies the WAA to compute each expert's weight $p_t^\theta, \theta \in \Theta$, i.e.,

$$p_t^\theta = \frac{\beta_t^{G_{t-1}^\theta}}{\sum_{\tilde{\theta} \in \Theta} \beta_t^{G_{t-1}^{\tilde{\theta}}}} \tag{5}$$

(2) The investor applies the WAA to obtain portfolio vector $\mathbf{b}_t$, i.e.,

$$\mathbf{b}_t = \sum_{\theta \in \Theta} \mathbf{e}^\theta p_t^\theta \tag{6}$$

(3) The investor computes his gain $g_t = \ln(\mathbf{b}_t \cdot \mathbf{x}_t)$ and cumulative gain $G_t = G_{t-1} + g_t$, and each expert's gain $g_t^\theta = \ln(\mathbf{e}^\theta \cdot \mathbf{x}_t)$ and cumulative gain $G_t^\theta = G_{t-1}^\theta + g_t^\theta, \theta \in \Theta$.

Based on (5) and (6), the specific decision $\mathbf{b}_t$ can be obtained. From the notations introduced above, we have

$$\mathbf{b}_t = \sum_{\theta \in \Theta} \mathbf{e}^\theta p_t^\theta = \sum_{\theta \in \Theta} \mathbf{e}^\theta \frac{\exp\left(\frac{\sum_{s=1}^{t-1} \ln(\mathbf{e}^\theta \cdot \mathbf{x}_s)}{\sqrt{t}}\right)}{\sum_{\tilde{\theta} \in \Theta} \exp\left(\frac{\sum_{s=1}^{t-1} \ln(\mathbf{e}^{\tilde{\theta}} \cdot \mathbf{x}_s)}{\sqrt{t}}\right)}.$$

By simple computation, we obtain

$$\mathbf{b}_t = \frac{1}{\sum_{\tilde{\theta} \in \Theta} \left(S_{t-1}(\mathbf{e}^{\tilde{\theta}})\right)^{\frac{1}{\sqrt{t}}}} \left(\left(S_{t-1}(\mathbf{e}^1)\right)^{\frac{1}{\sqrt{t}}}, \cdots, \left(S_{t-1}(\mathbf{e}^N)\right)^{\frac{1}{\sqrt{t}}}\right) \tag{7}$$

which shows that the proportion invested in each stock by WAAS is decided by its cumulative gain in previous trading days. The larger the cumulative gain, the greater the corresponding proportion invested.

Expression (7) provides a specific decision for online portfolio selection. However, it is important to prove that WAAS is theoretically optimal compared with an off-line benchmark. Let stock that performs best on all trading days be the off-line benchmark. The corresponding best expert's index on day T is

$$* = \arg\max_{\theta \in \Theta} \ln S_T(\mathbf{e}^\theta) = \arg\max_{\theta \in \Theta} G_T^\theta,$$

which means that the best expert advice is $\mathbf{e}^*$.

Using Lemma 1 and (4), the competitive performance of WAAS is obtained.

Theorem 1 *Let* $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_T$ *be a sequence of price relatives with* $x_t^i \geq a$ *for all* i, t *and* $\max_i x_t^i = 1$ *for all* t. *The natural logarithmic cumulative gain of the sequence of portfolios* $\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_T$ *generated by* WAAS *satisfies*

$$\sum_{t=1}^{T} \ln(\mathbf{b}_t \cdot \mathbf{x}_t) \geq \sum_{t=1}^{T} \ln(\mathbf{e}^* \cdot \mathbf{x}_t) - \left(\ln N + (\ln a)^2\right)\sqrt{T} \tag{8}$$

where (8) holds for all $T = 1, 2, \ldots$.

Proof Based on the assumption $a \le x_t^i \le 1$, we have

$$\ln a \le \mathrm{g} = \ln(\mathbf{b} \cdot \mathbf{x}) \le \ln 1 = 0,$$

which means that $L = -\ln a$. By applying Lemma 1, we obtain

$$\begin{aligned} G_T &\ge \left(\ln \sum_{\theta \in \Theta} \frac{1}{N} \exp\left(\frac{G_T^\theta}{\sqrt{T}} \right) - L^2 \right) \sqrt{T} \\ &\ge \left(\ln \frac{1}{N} \exp\left(\frac{G_T^*}{\sqrt{T}} \right) - (-\ln a)^2 \right) \sqrt{T} \\ &= G_T^* - \left(\ln N + (\ln a)^2 \right) \sqrt{T}, \end{aligned}$$

where $G_T = \sum_{t=1}^{T} \ln(\mathbf{b}_t \cdot \mathbf{x}_t)$ and $G_T^* = \sum_{t=1}^{T} \ln(\mathbf{e}^* \cdot \mathbf{x}_t)$. □

Inequality (8) shows that WAAS's cumulative gain approaches that of the best stock when T is large enough. However, we have to obtain a in advance. Hence, the situation of $a = 0$ should be considered. Using the technique proposed in Helmbold et al. (1998), we prove that WAAS can identify the best stock.

Theorem 2 *Let* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_T$ *be a sequence of price relatives with* $x_t^i \ge 0$ *for all* i, t *and* $\max_i x_t^i = 1$ *for all* t*. The online portfolio selection strategy WAAS can identify the best stock in* T *trading days.*

Proof When $0 \le x_t^i \le 1$, we let

$$\tilde{\mathbf{x}}_t = (1 - \alpha/N)\mathbf{x}_t + (\alpha/N)\mathbf{1}, \alpha \in [0, 1],$$

where $\mathbf{1}$ is the all 1's vector. According to WAAS, on day t we use $\tilde{\mathbf{x}}_t$ to obtain the updated portfolio vector, i.e.,

$$\mathbf{b}_t = \frac{1}{\sum_{\theta \in \Theta} (S_{t-1}^\star(\mathbf{e}^\theta))^{\frac{1}{\sqrt{t}}}} \left((S_{t-1}^\star(\mathbf{e}^1))^{\frac{1}{\sqrt{t}}}, \ldots, (S_{t-1}^\star(\mathbf{e}^N))^{\frac{1}{\sqrt{t}}} \right),$$

where $S_{t-1}^\star(\mathbf{e}^\theta)$ denotes the cumulative gain of expert θ who always invests in θth stock when the price vector sequence is $\{\tilde{\mathbf{x}}_j\}_{j=1}^{t-1}$. Meanwhile, the invested portfolio vector becomes $\tilde{\mathbf{b}}_t = (1 - \alpha)\mathbf{b}_t + (\alpha/N)\mathbf{1}, \alpha \in [0, 1]$. We denote this strategy by WAAS*. According to Helmbold et al. (1998) and (8), the cumulative gain of WAAS* satisfies

$$\sum_{t=1}^{T} \ln(\tilde{\mathbf{b}}_t \cdot \mathbf{x}_t) \ge \sum_{t=1}^{T} \ln(\mathbf{e}^* \cdot \mathbf{x}_t) - \left(\ln N + (\ln \frac{\alpha}{N})^2 \right) \sqrt{T} - 2\alpha T \qquad (9)$$

On the one hand, we use inequality $(\ln N)^2 \le N, N \ge 2$ to transfer (9). As $\alpha \in [0,1]$, we have

$$\sum_{t=1}^{T} \ln(\tilde{\mathbf{b}}_t \cdot \mathbf{x}_t) \ge \sum_{t=1}^{T} \ln(\mathbf{e}^* \cdot \mathbf{x}_t) - \left(\ln N + \frac{N}{\alpha}\right)\sqrt{T} - 2\alpha T.$$

Let $\alpha = (N^2/T)^{1/4}$, then $2\alpha T = 2N^{1/2}T^{3/4}$, $N/\alpha = N^{1/2}T^{1/4}$. When $N \ge 2$, we obtain

$$\begin{aligned}\sum_{t=1}^{T} \ln(\tilde{\mathbf{b}}_t \cdot \mathbf{x}_t) &\ge \sum_{t=1}^{T} \ln(\mathbf{e}^* \cdot \mathbf{x}_t) - \left(\ln N + N^{\frac{1}{2}}T^{1/4}\right)T^{\frac{1}{2}} - 2N^{\frac{1}{2}}T^{\frac{3}{4}} \\ &= \sum_{t=1}^{T} \ln(\mathbf{e}^* \cdot \mathbf{x}_t) - T^{1/2}\ln N - 3N^{1/2}T^{3/4} \\ &\ge \sum_{t=1}^{T} \ln(\mathbf{e}^* \cdot \mathbf{x}_t) - T^{1/2}\ln N - (2N^6)^{1/4}T^{3/4}.\end{aligned}$$

On the other hand, we have $\ln N/T^{1/4} \le (2N^6)^{1/4}$, and obtain

$$\begin{aligned}\sum_{t=1}^{T} \ln(\tilde{\mathbf{b}}_t \cdot \mathbf{x}_t) &\ge \sum_{t=1}^{T} \ln(\mathbf{e}^* \cdot \mathbf{x}_t) - T^{1/2}\ln N - (2N^6)^{1/4}T^{3/4} \\ &= \sum_{t=1}^{T} \ln(\mathbf{e}^* \cdot \mathbf{x}_t) - T^{3/4}\ln N/T^{1/4} - (2N^6)^{1/4}T^{3/4} \\ &\ge \sum_{t=1}^{T} \ln(\mathbf{e}^* \cdot \mathbf{x}_t) - T^{3/4}(2N^6)^{1/4} - (2N^6)^{1/4}T^{3/4}.\end{aligned}$$

Thus, we have

$$\sum_{t=1}^{T} \ln(\tilde{\mathbf{b}}_t \cdot \mathbf{x}_t) \ge \sum_{t=1}^{T} \ln(\mathbf{e}^* \cdot \mathbf{x}_t) - 2(2N^6)^{1/4}T^{3/4} \tag{10}$$

Inequality (10) shows that the cumulative gain of WAAS* approaches that of the best stock when T is large enough. However, the setting of α still needs to obtain T in advance. According to the technique proposed in Helmbold et al. (1998) and Zhang et al. (2012), this situation can be dealt with in stages. For large enough T ($T \ge 2N^6$), the stage technique works as follows. The first and second $2N^6$ trading days are numbered stage 0 and 1, respectively. The following 2^iN^6 trading days are numbered stage $i\,(i > 1)$. At the start of each stage the portfolio vector is reinitialized to the uniform portfolio vector and α is set as stated above, using 2^iN^6 in stage $i\,(i \ge 1)$ as the value for T. In every stage i, the WAAS* is carried out to obtain the updated portfolios. This strategy is called staged WAAS*. The upper bound of the last stage number is $[\log_2(T/2N^6)]$ (Zhang et al. 2012). Then we have

$$\sum_{t=1}^{T}\ln(\mathbf{e}^*\cdot\mathbf{x}_t)-\sum_{t=1}^{T}\ln(\tilde{\mathbf{b}}_t\cdot\mathbf{x}_t)\le 6(1+\left(\frac{T}{2N^6}\right)^{3/4})N^6.$$

Thus, we obtain

$$\lim_{T\to\infty}\frac{1}{T}(\sum_{t=1}^{T}\ln(\mathbf{e}^*\cdot\mathbf{x}_t)-\sum_{t=1}^{T}\ln(\tilde{\mathbf{b}}_t\cdot\mathbf{x}_t)\le\lim_{T\to\infty}\frac{(1+\left(\frac{T}{2N^6}\right)^{3/4})6N^6}{T}=0.$$

Hence, staged WAAS* can identify the best stock when $T\ge 2N^6$. □

5 A More Complicated Portfolio Selection Strategy WAAC

Most existing portfolio selection strategies choose BCRP as the benchmark. Regarding any CRP strategy as expert advice, we can obtain online portfolios by applying the WAA to experts whose strategies are CRPs. The index set for expert is $\Theta=\mathbf{B}$. The expert advice set includes all CRP strategies; each CRP corresponds to an expert and expert advice. Similarly, suppose that the stock price satisfies $a\le x_t^i\le 1$, $1\le i\le N$. Denote this online portfolio selection strategy by WAAC. The decision-making process for WAAC can be described as follows.

- On the first day, the experts' cumulative gains are $G_1^{\mathbf{b}}=0$, $\mathbf{b}\in\mathbf{B}$ and the investor's cumulative gain is $G_1=0$;
- At each day $t=2,3,\cdots$
  (1) The investor computes weight $p_t^{\mathbf{b}}(d\mathbf{b})$, $\mathbf{b}\in\mathbf{B}$, where $q((d\mathbf{b}))=d\mathbf{b}$, i.e.,

$$p_t^{\mathbf{b}}(d\mathbf{b})=\frac{\beta_t^{G_{t-1}^{\mathbf{b}}}d\mathbf{b}}{\int\limits_{\mathbf{B}}\beta_t^{G_{t-1}^{\mathbf{b}}}d\mathbf{b}},\ \mathbf{b}\in\mathbf{B}\tag{11}$$

  (2) The investor applies the WAA to obtain portfolio vector $\mathbf{b}_t$, i.e.,

$$\mathbf{b}_t=\int\limits_{\mathbf{B}}\mathbf{b}p_t^{\mathbf{b}}(d\mathbf{b}),\ \mathbf{b}\in\mathbf{B}\tag{12}$$

  (3) The investor computes his gain $g_t=\ln(\mathbf{b}_t\cdot\mathbf{x}_t)$ and cumulative gain $G_t=G_{t-1}+g_t$, and each expert's gain $g_t^{\mathbf{b}}=\ln(\mathbf{b}\cdot\mathbf{x}_t)$ and cumulative gain $G_t^{\mathbf{b}}=G_{t-1}^{\mathbf{b}}+g_t^{\mathbf{b}}$, $\mathbf{b}\in\mathbf{B}$.

Based on (11) and (12), we obtain

$$\mathbf{b}_t = \frac{\int\limits_{\mathbf{B}} \mathbf{b}\beta_t^{G_{t-1}(\mathbf{b})} d\mathbf{b}}{\int\limits_{\mathbf{B}} \beta_t^{G_{t-1}(\mathbf{b})} d\mathbf{b}} = \frac{\int\limits_{\mathbf{B}} \mathbf{b} \exp\left(\frac{\sum\limits_{s=1}^{t-1} \ln(\mathbf{b}\cdot\mathbf{x}_s)}{\sqrt{t}}\right) d\mathbf{b}}{\int\limits_{\mathbf{B}} \exp\left(\frac{\sum\limits_{s=1}^{t-1} \ln(\mathbf{b}\cdot\mathbf{x}_s)}{\sqrt{t}}\right) d\mathbf{b}} = \frac{\int\limits_{\mathbf{B}} \mathbf{b}\left(S_{t-1}(\mathbf{b})\right)^{\frac{1}{\sqrt{t}}} d\mathbf{b}}{\int\limits_{\mathbf{B}} \left(S_{t-1}(\mathbf{b})\right)^{\frac{1}{\sqrt{t}}} d\mathbf{b}} \tag{13}$$

Comparing (13) and Cover's UP strategy (2), we notice that the two expressions are similar. The only difference between them is that (13) has exponential factor $1/\sqrt{t}$, which enables the weight to update more slowly. Meanwhile, the UP strategy can be seen as a strategy of aggregating experts' strategies. The experts' strategies are CRPs, and each expert's weight is decided by his cumulative gain in the previous $t-1$ days, i.e., the weight is

$$\frac{S_{t-1}(\mathbf{b})}{\int\limits_{B} S_{t-1}(\mathbf{b}) d\mathbf{b}}, \mathbf{b} \in \mathbf{B}.$$

This shows the UP strategy from the other side: the portfolio vectors are updated based on the weighted averages of all CRPs.

The benchmark of WAAC is BCRP. Although the ideas for WAAC and UP are similar, WAAC has its own theoretical basis. In Theorems 3 and 4, we provide a competitive performance analysis of WAAC.

Theorem 3 *Let* $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_T$ *be a sequence of price relatives with* $x_t^i \geq a$ *for all* i, t *and* $\max\limits_i x_t^i = 1$ *for all* t. *The natural logarithmic wealth of the sequence of portfolios* $\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_T$ *generated by* WAAC *satisfies*

$$\sum_{t=1}^{T} \ln(\mathbf{b}_t \cdot \mathbf{x}_t) \geq \sum_{t=1}^{T} \ln(\mathbf{b}_T^* \cdot \mathbf{x}_t) - \frac{N^2}{2a^2} - \left(\frac{N}{2}\ln T + (\ln a)^2\right)\sqrt{T} \tag{14}$$

where (14) holds for all $T = 1, 2, \ldots$.

Proof The proof is similar to that for Theorem 1. Using Lemma 1, we have

$$G_T \geq \left(\ln \int\limits_{B} \exp(\frac{G_T^{\mathbf{b}}}{\sqrt{T}}) d\mathbf{b} - L^2\right)\sqrt{T} \geq \left(\ln \int_{Ball(\mathbf{b}_T^*, T^{-1/2})} \exp\left(\frac{G_T^{\mathbf{b}}}{\sqrt{T}}\right) d\mathbf{b} - L^2\right)\sqrt{T}, \tag{15}$$

where $Ball(\mathbf{b}_T^*, T^{-1/2})$ denotes a ball with center $\mathbf{b}_T^*$ and radius $T^{-1/2}$, i.e.,

$$Ball(\mathbf{b}_T^*, T^{-1/2}) = \{\mathbf{b} : ||\mathbf{b} - \mathbf{b}_T^*|| \leq T^{-1/2}\} \tag{16}$$

By Taylor series expansion, we expand $G_T^{\mathbf{b}}$ at $\mathbf{b}_T^*$ with second order, and obtain

$$
\begin{aligned}
G_T^{\mathbf{b}} &= G_T^{\mathbf{b}_T^*} + (\mathbf{b} - \mathbf{b}_T^*)' \Delta G_T^{\mathbf{b}}|_{\mathbf{b}=\mathbf{b}_T^*} \\
&\quad + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial^2 G_T^{\mathbf{b}}}{\partial b^i \partial b^j}|_{\mathbf{b}=\mathbf{b}_T^*} (b^i - b_{i,T}^*)(b^j - b_{j,T}^*) + o(\| \mathbf{b} - \mathbf{b}_T^* \|^2) \\
&\geq G_T^{\mathbf{b}_T^*} + (\mathbf{b} - \mathbf{b}_T^*)' \Delta G_T^{\mathbf{b}}|_{\mathbf{b}=\mathbf{b}_T^*} + \\
&\quad \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial^2 G_T^{\mathbf{b}}}{\partial b^i \partial b^j}|_{\mathbf{b}=\mathbf{b}_T^*} (b^i - b_{i,T}^*)(b^j - b_{j,T}^*) \\
&= G_T^{\mathbf{b}_T^*} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial^2 G_T^{\mathbf{b}}}{\partial b^i \partial b^j}|_{\mathbf{b}=\mathbf{b}_T^*} (b^i - b_{i,T}^*)(b^j - b_{j,T}^*),
\end{aligned}
$$

where $\mathbf{b}_T^* = (b_{1,T}^*, b_{2,T}^*, \cdots, b_{N,T}^*)$ and the second equality is obtained based on the fact that $\mathbf{b}_T^*$ is the maximization point of $G_T^{\mathbf{b}}$. According to $G_T^{\mathbf{b}} = \sum_{t=1}^{T} \ln(\mathbf{b} \cdot \mathbf{x}_t)$ and $a \leq x_t^i \leq 1$, we obtain

$$
\frac{\partial^2 G_T^{\mathbf{b}}}{\partial b^i \partial b^j}|_{\mathbf{b}=\mathbf{b}_T^*} = -\sum_{t=1}^{T} \frac{x_t^i x_t^j}{(\mathbf{b} \cdot \mathbf{x}_t)^2}|_{\mathbf{b}=\mathbf{b}_T^*} \geq \sum_{t=1}^{T} -\frac{1}{a^2} = -\frac{T}{a^2}.
$$

Thus, we have

$$
\begin{aligned}
G_T^{\mathbf{b}} &\geq G_T^{\mathbf{b}_T^*} - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{T}{a^2} (b^i - b_{i,T}^*)(b^j - b_{j,T}^*) && (17) \\
&\geq G_T^{\mathbf{b}_T^*} - \frac{N^2 T}{2a^2} T^{-1/2} T^{-1/2} = G_T^{\mathbf{b}_T^*} - \frac{N^2}{2a^2}, && (18)
\end{aligned}
$$

where (18) is obtained from (16).

Considering (15) and (18), we have

$$
\begin{aligned}
G_T &\geq \left(\ln \int_{Ball(\mathbf{b}_T^*, T^{-1/2})} \exp(\frac{G_T^{\mathbf{b}}}{\sqrt{T}}) d\mathbf{b} - L^2 \right) \sqrt{T} \\
&\geq \left(\ln \int_{Ball(\mathbf{b}_T^*, T^{-1/2})} \exp(\frac{G_T^{\mathbf{b}_T^*} - \frac{N^2}{2a^2}}{\sqrt{T}}) d\mathbf{b} - L^2 \right) \sqrt{T} \\
&= \left(\ln \left(\exp(\frac{G_T^{\mathbf{b}_T^*} - \frac{N^2}{2a^2}}{\sqrt{T}}) \frac{\pi^{N/2}}{\Gamma(N/2 + 1)} T^{-N/2} \right) - L^2 \right) \sqrt{T}
\end{aligned}
$$

$$\geq \left(\ln\Big(\exp(\frac{G_T^{\mathbf{b}_T^*} - \frac{N^2}{2a^2}}{\sqrt{T}})T^{-N/2}\Big) - L^2\right)\sqrt{T}$$

$$= G_T^{\mathbf{b}_T^*} - \frac{N^2}{2a^2} - \Big(\frac{N}{2}\ln T + (\ln a)^2\Big)\sqrt{T},$$

where the first equality holds based on the fact that

$$\int_{Ball(\mathbf{b}_T^*, T^{-1/2})} d\mathbf{b} = \frac{\pi^{N/2}}{\Gamma(N/2+1)} T^{-N/2};$$

the third inequality follows, based on the fact that $\pi^{N/2} \geq \Gamma(N/2+1)$ when $1 \leq N \leq 12$. Using the definition of cumulative gain, we obtain (14). □

Inequality (14) is obtained by assuming that the lower bound on the stock price is $a > 0$. However, the case of $a = 0$ should be considered. Using the techniques provided by Helmbold et al. (1998), we prove WAAC is universal.

Theorem 4 *Let $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_T$ be a sequence of price relatives with $x_t^i \geq 0$ for all i, t and $\max_i x_t^i = 1$ for all t. The online portfolio selection strategy WAAC is universal.*

Proof Let $\tilde{\mathbf{x}}_t = (1 - \alpha/N)\mathbf{x}_t + (\alpha/N)\mathbf{1}, \alpha \in [0, 1]$; the updated portfolio vector $\mathbf{b}_t$ is obtained by using $\tilde{\mathbf{x}}_{t-1}$ rather than $\mathbf{x}_{t-1}$. For the sequence of price relatives $\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \cdots, \tilde{\mathbf{x}}_T$, the natural logarithmic wealth of the sequence of portfolios $\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_T$ and the corresponding BCRP is denoted by $\tilde{G}_T = \sum_{t=1}^{T} \ln(\mathbf{b}_t \cdot \tilde{\mathbf{x}}_t)$ and $\tilde{G}_T^* = \sum_{t=1}^{T} (\tilde{\mathbf{b}}_T^* \cdot \tilde{\mathbf{x}}_t)$, respectively. The modified strategy, which is denoted by $\widetilde{\text{WAAC}}$, uses the portfolio vector

$$\tilde{\mathbf{b}}_t = (1-\alpha)\mathbf{b}_t + (\alpha/N)\mathbf{1}.$$

Therefore, for the sequence of price relatives $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_T$, the natural logarithmic wealth achieved by $\widetilde{\text{WAAC}}$ and the corresponding BCRP is denoted by $G_T = \sum_{t=1}^{T} \ln(\tilde{\mathbf{b}}_t \cdot \mathbf{x}_t)$ and $G_T^* = \sum_{t=1}^{T} (\mathbf{b}_T^* \cdot \mathbf{x}_t)$. Using Theorem 3, we obtain

$$\tilde{G}_T \geq \tilde{G}_T^* - \frac{N^2}{2(\alpha/N)^2} - \left(\frac{N}{2}\ln T + \Big(\ln\frac{\alpha}{N}\Big)^2\right) T^{1/2}.$$

According to the result obtained by Helmbold et al. (1998), we have

$$G_T \geq \tilde{G}_T - 2\alpha T.$$

Hence, we obtain

$$
\begin{aligned}
G_T &\geq \tilde{G}_T^* - \frac{N^2}{2(\alpha/N)^2} - \left(\frac{N}{2}\ln T + \left(\ln\frac{\alpha}{N}\right)^2\right) T^{1/2} - 2\alpha T \\
&\geq G_T^* - \frac{N^2}{2(\alpha/N)^2} - \left(\frac{N}{2}\ln T + \left(\ln\frac{\alpha}{N}\right)^2\right) T^{1/2} - 2\alpha T,
\end{aligned}
$$

where $\tilde{G}_T^* \geq G_T^*$ holds. In fact, we have

$$
\mathbf{b}_T^* \cdot \tilde{\mathbf{x}}_t = \mathbf{b}_T^* \cdot \Big((1-\alpha/N)\mathbf{x}_t + (\alpha/N)\mathbf{1}\Big) \geq \mathbf{b}_T^* \cdot \mathbf{x}_t(1-\alpha/N) + \mathbf{b}_T^* \cdot \mathbf{x}_t \frac{\alpha}{N} = \mathbf{b}_T^* \cdot \mathbf{x}_t.
$$

Thus, we obtain

$$
\tilde{G}_T^* = \sum_{t=1}^{T} (\tilde{\mathbf{b}}_T^* \cdot \tilde{\mathbf{x}}_t) \geq \sum_{t=1}^{T} (\mathbf{b}_T^* \cdot \tilde{\mathbf{x}}_t) \geq \sum_{t=1}^{T} (\mathbf{b}_T^* \cdot \mathbf{x}_t) = G_T^*.
$$

Let $\alpha = (\frac{N^2}{T})^{\frac{1}{4}}$, then $2\alpha T = 2N^{\frac{1}{2}}T^{\frac{3}{4}}$. Based on the fact that $\ln T \leq T^{\frac{1}{4}}$, we obtain

$$
G_T \geq G_T^* - \frac{N^3}{2}T^{\frac{1}{2}} - (\frac{N}{2} + 3N^{\frac{1}{2}})T^{\frac{3}{4}}.
$$

When $N \geq 4$, $\frac{N}{2} + 3N^{\frac{1}{2}} \leq 2N \leq (2N^6)^{\frac{1}{4}}$ holds; and when $T \geq \frac{N^6}{32}$, $\frac{N^3}{2}T^{\frac{1}{2}} \leq (2N^6)^{\frac{1}{4}}T^{\frac{3}{4}}$ holds. Therefore, we obtain

$$
G_T \geq G_T^* - 2(2N^6)^{\frac{1}{4}}T^{\frac{3}{4}}. \tag{19}
$$

The following steps are based on inequality (19) and the stage technique provided in Helmbold et al. (1998). Therefore, the universality of WAAC is proved when $T (T \geq 2N^6)$ is large enough. □

6 Numerical Analysis

In this section, we provide numerical analysis to illustrate the feasibility of the WAAS and WAAC strategies. The data used in this paper are obtained from the NYSE. They include 6400 trading days (from 1981 to 2007). Table 1 provides the abbreviation names (Ab. Name) and the increase factors (In. Factor) of the stocks. The portfolios used in this paper consist of two stocks or three stocks.

For portfolios consisting of two stocks, the cumulative gains of BCRP, UP, EG, BEST, WAAS and WAAC, are presented in Table 2, where BEST denotes the best stock in all trading days. Table 2 shows that differences between the cumulative gains of

Table 1 Stock abbreviation names and their increase factors

Ab. name	In. factor	Ab. name	In. factor	Ab. name	In. factor
AA	1.02	MCD	53.63	AXP	1.09
MRK	29.40	GE	8.31	XOM	47.78
WMT	8.21	PG	206.30	BA	8.22
KO	123.92	DD	9.37		

Table 2 Cumulative gains of portfolios consisting of two stocks achieved by BCRP, UP, EG, BEST, WAAS and WAAC

Strategy	BCRP	UP	EG	BEST	WAAS	WAAC
GE-WMT	**12.16**	10.73	12.11	8.31	12.12	**12.14**
BA-DD	**11.52**	10.51	11.44	9.37	11.42	**11.45**
XOM-MRK	**52.62**	45.92	49.86	47.78	49.95	**49.97**
AA-AXP	**1.50**	1.33	1.49	1.09	1.49	**1.49**
MCD-KO	**129.28**	104.41	112.73	123.92	112.48	**112.79**
MCD-XOM	**69.20**	62.42	68.91	53.60	68.91	**68.97**
KO-PG	**208.06**	179.49	188.30	206.30	188.21	**188.44**

The cumulative gains of WAAC and BCRP are presented in bold

Table 3 Cumulative gains of portfolios consisting of three stocks achieved by BCRP, UP, EG, BEST, WAAS and WAAC

Strategy	BCRP	UP	EG	BEST	WAAS	WAAC
MCD-KO-PG	**208.06**	109.70	158.51	206.30	158.87	**160.08**
XOM-MRK-MCD	**69.33**	44.34	63.63	53.63	63.87	**64.11**
GE-WMT-BA	**13.65**	9.95	13.31	8.31	11.57	**13.64**
WMT-BA-DD	**13.51**	11.36	13.18	9.37	11.65	**13.47**

The cumulative gains of WAAC and BCRP are presented in bold

WAAC, WAAS and EG are not obvious. However, the cumulative gains they achieved are larger than those achieved by UP. WAAC performs a little better than EG. The cumulative gains of WAAC, WAAS, EG, and UP approach those of BCRP. Meanwhile, we conclude that the cumulative gains of WAAS approach, or even exceed, those of BEST.

Table 3 presents a comparison of cumulative gains achieved by BCRP, UP, EG, BEST, WAAS and WAAC for portfolios consisting of three stocks. For all the portfolios, WAAC is the best-performing of the presented online portfolio selection strategies. Interestingly, WAAC achieves much greater cumulative gains than the UP strategy although they have a similar approach to updating portfolios. The results also show that the cumulative gains of WAAC are as large as those of BCRP. Thus, WAAC is universal.

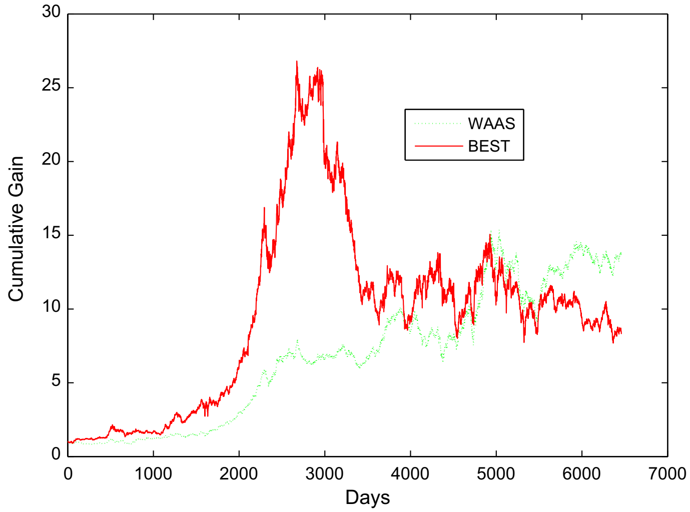

Fig. 1 Comparison of the daily cumulative gain of portfolio GE-WMT-BA achieved by BEST and WAAS

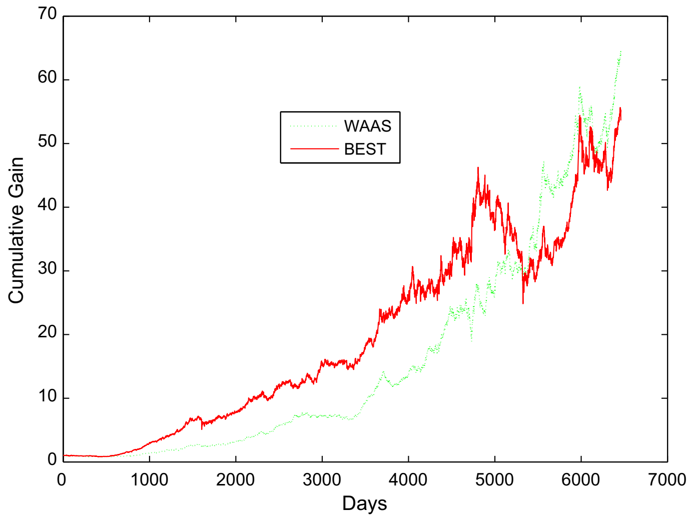

Fig. 2 Comparison of the daily cumulative gain of portfolio XOM-MRK-MCD achieved by BEST and WAAS

Figures 1 and 2 compare the daily cumulative gains of portfolios GE-WMT-BA and XOM-MRK-MCD achieved by BEST and WAAS. In the two figures, the green line representing WAAS's cumulative gain is above the red line representing BEST's cumulative gain on the last trading day, illustrating that WAAS can pursue the best stock.

Figures 3 and 4 further present the daily cumulative gains of the two portfolios achieved by BCRP, WAAC and UP. On the last trading day, the red line representing WAAC's cumulative gain approaches the black line representing BCRP's cumulative gain, and is above the green line representing UP's cumulative gain. These two

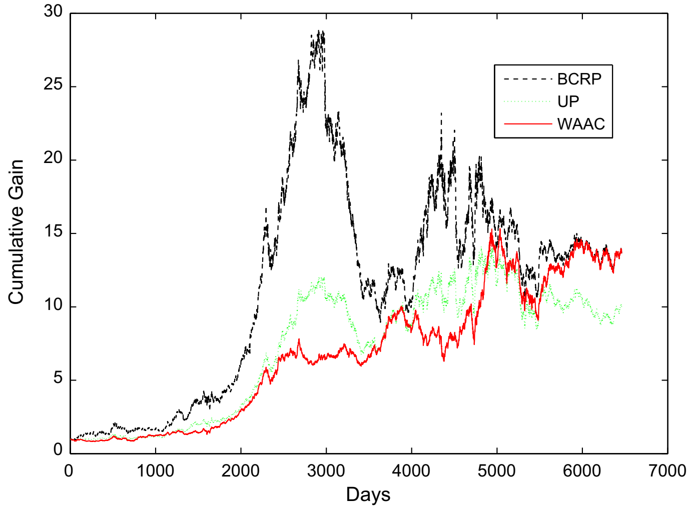

Fig. 3 Comparison of the daily cumulative gain of portfolio GE-WMT-BA achieved by BCRP, UP and WAAC

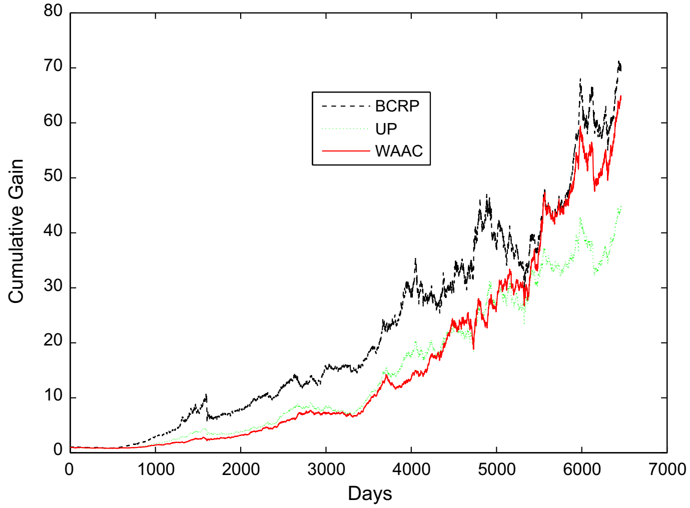

Fig. 4 Comparison of the daily cumulative gain of portfolio XOM-MRK-MCD achieved by BCRP, UP and WAAC

figures show that WAAC performs as well as BCRP, and displays good competitive performance.

7 Conclusions

Many decision-making methods developed in computer science provide new approaches to financial decision-making problems. This paper applies the WAA to the online portfolio selection problem. Using a strategy that invests in a single stock as expert advice, the simple strategy WAAS is obtained. Taking any CRP as expert advice, the more complicated strategy WAAC is obtained. The theoretical and numerical analyses both show that the cumulative gain of WAAS is as large as that of BEST; and the cumulative gain of WAAC is as large as that of BCRP.

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