COUNTING AND COMBINATORICS

10.1 Why study Counting?

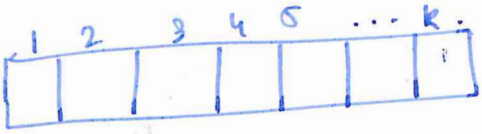
- Lets say if we design an Algorithm, we have to compute the time & space complexity we have to count the no of times a line of code is execut
- In discrete mathematics we count the #relations, #graphs etc
- In Probability theory we use counting to evaluate probabilities
- Many applications in computer science also for a S/w Engineer.

10.2 PERMUTATIONS AND COMBINATIONS : AN INTRODUCTION

Permutations / Arrangements

- If we have n distinct objects and an array with k slots

$a_1 \; a_2 \; a_3 \; a_4 \; \ldots \ldots \; a_n$.



$K \leq n$

No of ways we can place these n elements in k slots such that there are no repetitions (means an item can come only once in the array).

If $K=3$ $n=10$

a_7	a_1	a_2
1	2	3

a_7	a_7	a_2

not valid (a_7 is repeated)

a_7	a_1	a_2

a_1	a_2	a_7

these are two distinct permutations (ordering is important in case of permutations).

<u>Case 1</u>: Permutation without repetitions.

$a_1\ a_2\ a_3\ \dots\ a_n$

1	2	3	4	$\dots$	k
a_i				$\dots$	
can be selected in n ways	$n-1$	$(n-2)$		$\dots$	$(n-k+1)$ ways

Here we can have all elements except a_i

$\therefore$ Total no of arrangements $= n\times(n-1)\times(n-2)\times(n-3)\dots(n-k+1)$

$$= \frac{n!}{(n-k)!} = {}^{n}P_{k}$$

If $k = n$

we have n objects and n slots

$a_1 \; a_2 \; \ldots \; a_n$

We can arrange in $n \times (n-1) \times (n-2) \cdots \times 1$

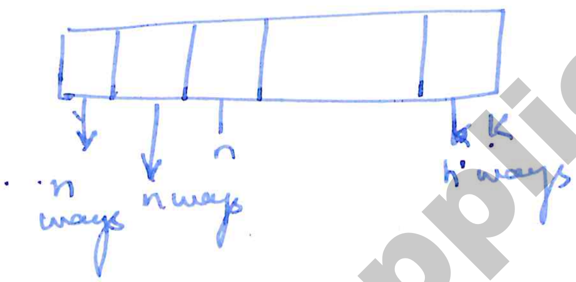
$= n!$ ways.

Also if we use the formula $^nP_n = \frac{n!}{(n-n)!} = n!$

Case 2 :- Unlimited repetitions

$a_1, a_2, a_3 \ldots a_n$

$\boxed{a_7 | a_7 | a_1}$ is valid, any element can repeat any number of times.



# of arrangements $= n \times n \times n - k$ times

$= n^k$

Case 3 : Limited Repetitions Case.

Example Suppose we have 5 balls < 2 red (r_1, r_2) — identical. / 3 black (b_1, b_2, b_3) — identical.

r_1	b_1	b_2	b_3	r_2
1	2	3	4	5

(1) Lets assume they are not identical = 5! of arranging

r_2	b_1	b_2	b_3	r_1

The above 2 arrangments should be counted as 1 arrangment.

∴ Because there are 2 identical red balls to get the correct no of arrangements we should ÷ by 2!

Similarly for 3 black balls 3! such arrangments should be counted as 1 arrangement.

$r_1\ b_1\ b_2\ b_3\ r_2$
$r_1\ b_1\ b_3\ b_2\ r_2$
$r_1\ b_2\ b_1\ b_3\ r_2$
$r_1\ b_2\ b_3\ b_1\ r_2$
$r_1\ b_3\ b_1\ b_2\ r_2$
$r_1\ b_3\ b_2\ b_1\ r_2$

There are the 3! = 6 arrangments which should be counted as 1 because of the identical balls.

∴ Total no of limited repetation = $\dfrac{5!}{2!\,3!}$

Because of 2 Red Balls. (Identical ones)

Because of 3 Identical Black Balls.

→ If we have n balls in which

n_1 balls are identical
n_2 balls are identical
n_3 balls are identical
⋮
n_k balls are identical

# of arrangements in n slots are given by $\dfrac{n!}{n_1! \times n_2! \times n_3! \times \cdots \times n_k!}$

Combinations / Selections / Choosing

→ If we have n items and k slots,

→ In case of permutations ordering is important, but in case of combinations ordering is not important.

$a_1 a_2 a_3$
$a_1 a_3 a_2$ are the same in case of combinations but in permutations they are considered different.

Case 1: No Repetitions and all items are distinct.

In case of permutations $n=10$ $k=3$ the following permutations

a_1, a_3, a_4 $\quad$ a_4, a_3, a_1
a_1, a_4, a_3
a_3, a_4, a_1 $\quad$ are considered as one combination
a_3, a_1, a_4 $\quad$ because
a_4, a_1, a_3

For k items we can have k! possible permutations, in order to get the no of combinations we need to divide the no of permutations by k!

$$\therefore \text{ \# of combinations} = \frac{{}^nP_k}{k!} = \frac{n!}{(n-k)!\,k!} \quad \text{also known as } {}^nC_r$$

$$^nC_k = \frac{n!}{(n-k)!\,k!}$$

→ Alternate way to think about permutations :-

If we choose k elements from n elements, we can do it in ^nC_k ways and these k elements can be arranged/permuted in k! ways.

$$\therefore \text{ No of permutations} = {}^nC_k \times k!$$

$$= \frac{n!}{\cancel{k!}\,(n-k)!}\cancel{k!}$$

$$= \frac{n!}{(n-k)!} = {}^nP_r$$

Case 2 Unlimited Repetitions

We have n distinct elements $a_1, a_2 \ldots a_n$ and we have to choose n items.

We can think about this problem in the following items

$a_1, a_2, a_3, a_4 \ldots a_n$

For each item we can either pick multiple times or move to the next item. Lets denote a pick by p and a "move to the next item as m" then the string is constructed as a string of picks and moves. It will have a total of (n-1) moves and exactly k picks, the total length of the string is

$(n+k-1)$

$$\underbrace{\begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & \ldots & a_n \\ p\,p & m & m & p\,k & \ldots & \end{array}}$$

n-1 moves
k picks.

No of ways we can organize the string = no of ways we can choose k items with infinite repetitions.

Now it is reduced to no of ways we can choose k locations from a total of $(n+k-1)$ locations $= {}^{n+k-1}C_k$.

Case 3 Limited repetitions (covered later in generating functions).

10.3 Solved Problems - 1

(1) If we have 10 people and 4 chairs, What are the no of ways to seat them?

→ If nothing is mentioned in the question we have to assume that the people are distinct and chairs are distinct by default.

#ways $\underset{c_1}{\underline{10}}$ $\underset{c_2}{\underline{9}}$ $\underset{c_3}{\underline{8}}$ $\underset{c_4}{\underline{7}}$

$$^{10}P_4 = 10 \cdot 9 \cdot 8 \cdot 7.$$

another way of thinking

from 10 people we can choose 4 people in $^{10}C_4$ ways

Now we can arrange them in $^{10}C_4 \times 4!$ ways.

$$= \frac{10!}{6!\,4!} \times 4! = \frac{10!}{6!} = 10 \times 9 \times 8 \times 7$$

2) 4 people on 10 chairs

$^nP_r \quad n<r \quad \underline{\underline{0}}$

3) # of 5 letter English words that can be built?

- By using this we can count the passwords / email id's.

a) no repetitions allowed?

$\underline{26} \quad \underline{25} \quad \underline{24} \quad \underline{23} \quad \underline{22}$.

$= {}^{26}P_5 = \frac{26!}{21!} = 26 \times 25 \times 24 \times 23 \times 22$

b) repetation is allowed

$\underline{26} \quad \underline{26} \quad \underline{26} \quad \underline{26} \quad \underline{26}$

$= 26^5$

c) No repetation and starting with a and ending with z

$\underline{a} \quad \underline{24} \quad \underline{23} \quad \underline{22} \quad \underline{z} = {}^{24}P_3 = 24 \cdot 23 \cdot 22.$

24 possibilities

(d) contains atleast one a.

To count the number of words which contain atleast one a

= Total no of words − Words with no 'a's.

\# words with no 'a's = $\underline{25}\ \underline{25}\ \underline{25}\ \underline{25}\ \underline{25}$

$= (25)^5$

$= 26^5 - 25^5$

(4) How many ways we can rearrange the letters in the word

MISSISSIPPI

We Have: M − 1
I − 4
S − 4
P − 2

\# of permutations = $\dfrac{11!}{1!\,4!\,4!\,2!}$

(b) all the S are together

consider the four 'S' as one "SSSS"

Other characters are M-1, I-4, P-2 :- Total = 8
"SSSS"-1

$$= \frac{8!}{1!\,4!\,2!\,1!} = \frac{8!}{2!\,4!}$$

(c) # of permutations which do not start with M.

= # Total. − # start with M (its easy to calculate this)

$$= \left(\frac{11!}{4!\,4!\,2!} - \frac{10!}{4!\,4!\,2!} \right)$$

M _ _ _ _ _ _ _ _ _ _

Remaining characters = 10!

I-4
P-2 $= \frac{10!}{4!\,4!\,2!}$.
S-4

(5) No of ways we can arrange 4 Boys and 5 Girls.
a

# of arrangements = 9! $\quad {}^{n}P_{n} = {}^{9}P_{9} = 9!$

(b) 20 coins

outcomes.

$$\underset{1}{\underline{2}}\ \underset{2}{\underline{2}}\ \underset{3}{\underline{2}}\ \underset{4}{\underline{2}}\ \underset{5}{\underline{2}}\ \underset{6}{\underline{2}}\ \ldots\ \underset{20}{\underline{2}} = 2^{20}$$ outcomes are possible.

→ 8 dice

6 outcomes/die

$\#$ of outcomes $= 6\times6\times6\times6\times6\times6\times6\times6$

$= \underline{6^8 \text{ outcomes}}$

(7) If we have a MCQ question paper with 65 questions each having 4 options (a, b, c, d), # of ways a student can answer the question paper.

For each question: he/she can choose one of the four options or not attempt the question

$\underset{Q_1}{\underline{5 \text{ ways}}}\ \underset{Q_2}{\underline{\quad}}\ \underset{Q_3}{\underline{\quad}}\ \dots\ \underset{Q_{65}}{\underline{\quad}}$

$\#$ of ways $= \underline{\underline{5^{65}}}$

(8) EQUATION

vowels A E I O U

consonents Q T N.

(a) ~~How~~ How many ways can we rearrange the letters words EQUATION such that all vowels are together

(1). $\underbrace{\text{AEIOU}}_{1}$ Remaining are consonents 3 can be arranged in 3! ways

consonents _ _ _

(2) The vowels are togeather they can be arranged among themselves in 5! ways.

(3)

$$\uparrow \ \frac{\text{Consonant}}{\text{letter}} \ \uparrow \ \frac{C}{L} \ \uparrow \ \frac{C}{L} \ \uparrow$$

We can place the vowels in one of the above 4 places.

\# no of arrangemts = $3! \times 5! \times 4$

(b) All vowels and consonents are togeather

Treat all vowels as a single block and all consonents as one block.

$\boxed{\text{Vow}}$ $\boxed{\text{Cons}}$

1) They can be arranged in 2! ways (2 Blocks).

2) Vowels can be arranged in 5! ways.

3) Consonents can be arranged in 3! ways.

$\therefore$ Total # arrangements = $2! \times 5! \times 3!$

(c) Not all vowels are togeather

= Total arrangements − # arrangemts with all vowels being togeather

= $\underline{(8!) - (5!4!)}$

(d) No two consonents are togeather

A E I O U

1) Initially we want to arrange all the vowels we can do it in 5! ways.

2). Now we can place the 3 consonents such in the 6 places.

_ A _ E _ I _ O _ U _

in $^6C_3 \times 3!$ ways.

Choose 3 out of 6 places

Among the 3 consonents we can have 3! arrangements.

$\therefore$ # of arrangements = $\underline{\underline{5! \times {}^6C_3 \times 3!}}$.

(e) No two vowels are togeather

_ Q _ T _ N _

locations = 4
vowels = 5

we cannot place them such that no two vowels are togeather

$\therefore$ No of arrangements = 0.

(9) 5 lettered english words starting with A (or) ending in M.

1. $\underline{A}$ $\underline{26}$ $\underline{26}$ $\underline{26}$ $\underline{26}$

or

2. $\underline{26}$ $\underline{26}$ $\underline{26}$ $\underline{26}$ $\underline{M}$

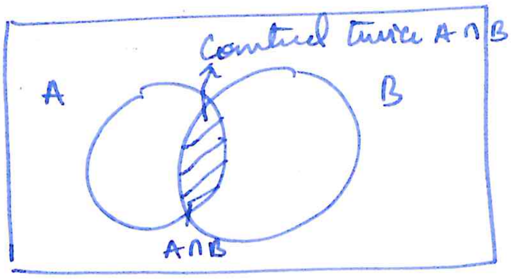
Case 1. We can have 26^4 words.

2. We can have 26^4 words.

But there will be some words which start with A and end with M, these are counted twice we need to subtract them once.

(Principle of inclusion and exclusion).

$|A \cup B| = |A| + |B| - |A \cap B|$



No of words which start with A and end with M

A $\underline{26}$ $\underline{26}$ $\underline{26}$ $\underline{M}$

$= 26^3$

$\therefore$ Total # of words $= 26^4 + 26^4 - 26^3$

$= 2 \times (26)^4 - 26^3$

Q10 (a) # of 8bit strings that start with 00 or end with '111'

_ _ _ _ _ _ _ _
8 bit string

Case 1 Start with 00

0 0 2 2 2 2 2 2.

$= 2^6$.

Case 2. end with 111 — $\underset{2}{\_}\ \underset{2}{\_}\ \underset{2}{\_}\ \underset{2}{\_}\ \underset{2}{\_}$ ways $\underline{1}\ \underline{1}\ \underline{1}$

$= 2^5$

Case 3 Start ~~the~~ with 00 and end with 111

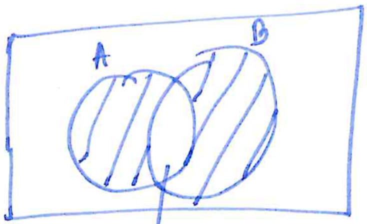
0 0 $\underset{2}{\_}\cdot\underset{2}{\_}\cdot\underset{2}{\_}$ 1 1 1

$= 2^3$

$\therefore$ # of strings $= 2^6 + 2^5 - 2^3$ (Principle of Inclusion & Exclusion).

(b) Strings that start with 00 (or) ~~but~~ end with 111 but not both.

→ We need strings that start with 00 or end with 111 but not string which start with 00 and end with 111



We need $A \cap B$.

$|A| + |B| - 2|A \cap B|$

We do not need $A \cap B$.

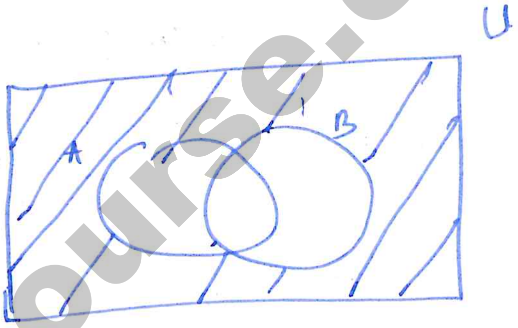
∴ We need to subtract it twice.

$$\# \text{ of strings} = 2^6 + 2^5 - 2 \times 2^3$$

(c) We need strings that neither start with '00' nor end with '111'

= We are interested in -

$U - (A \cup B) = (A \cup B)'$



$$\text{Total no of strings} = 2^8 - |A \cup B|$$
$$= 2^8 - (2^6 + 2^5 - 2^3)$$

SOLVED PROBLEMS - 2

(Q) # of numbers b/w 100 - 999 which do not contain 7?

These are 3 digited numbers.

_ _ _

cannot be 0, 7 — 8 ways. cannot be 7 — 9 ways. cannot be 7 — 9 ways.

$= 8 \times 9 \times 9$.

(Q) 5 Boys and 4 Girls

(a) No two girls are together

__ B __ B __ B __ B __ B __

1. First place all the boys - 5! ways.
2. 4 girls can be placed in 6 places
   we need to choose 4 places $^6C_4$
3. The 4 girls among themselves can be arranged 4! ways

\# no of arrangements $= 5! \times {}^6C_4 \times 4!$

(b) No two boys are together

__ G __ G __ G __ G __

1. First we place all the girls, this can be done in 4! ways.
2. In 5 places we can place 5 boys in 5! ways.

$\therefore$ # of arrangements $= 4! \times 5!$

(c) particular pair of girls always sit together

G_1, G_2 - say we have to treat them as one

We have now 5 - B
2 - G.
1 - girl pair

8.

8 people can be arranged in 8!

Among the girls who want to sit together we can have 2! arrangements.

$\therefore$ Total no of arrangements = $\underline{\underline{8! \times 2!}}$.

(d) a particular pair of girls do not sit together.

= # Total -- # of ways that particular pair of girls sit together

$= 9! - 8! \times 2!$

(Q) MISSISSIPPI

M - 1
I - 4
S - 4
P - 2

11.

Total = 11!

(a) No two S are together

M - 1
I - 4
P - 2

7

$\_ C_1 \_ C_2 \_ C_3 \_ C_4 \_ C_5 \_ C_6 \_ C_7 \_$

8 positions

1. The 7 characters can be placed in $\frac{7!}{4!\,2!}$

2. In 8 places we can place 4 S's in $\frac{{}^8C_4 \times 4!}{4!} = {}^8C_4$

Because the 4 S's are identical

$\#\text{ways} = \frac{7!}{4!\,2!} \times {}^8C_4$.

(Q) # of binary strings that has n 0's and n 1's such that 0's and 1's are alternating.

This can be done only in 2 ways
0's in even posn, 1's in odd
AND
1's in even, 0's in odd

eg n = 3

010101

101010.

(Q) 5 B, 6 G arrange them alternatively

_ B _ B _ B _ B _ B _

Boys arrangments $5!$

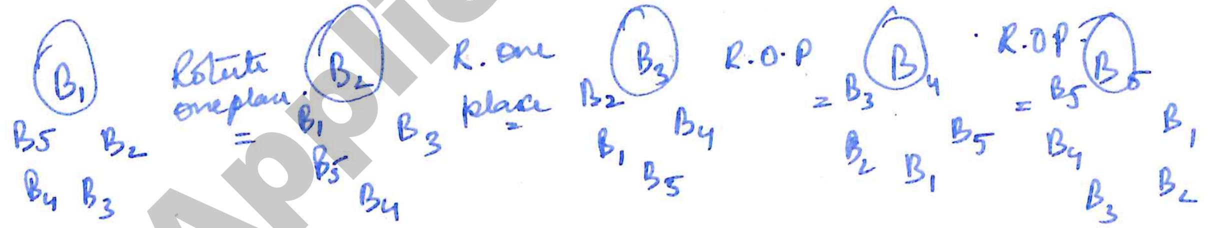
6 spaces for 6 girls $= 6!$

$\#$ arrgmts $= \underline{\underline{5! \times 6!}}$

Circular Arrangements

5 B

$\#$ of linear arrangmts $= 5!$



All the above 5 circular arrangements are considered to be one only.

If we write them down starting from the first circled person

1. $B_1\ B_2\ B_3\ B_4\ B_5$
2. $B_2\ B_3\ B_4\ B_5\ B_1$
3. $B_3\ B_4\ B_5\ B_1\ B_2$
4. $B_4\ B_5\ B_1\ B_2\ B_3$
5. $B_5\ B_1\ B_2\ B_3\ B_4$

All these 5 distinct linear permutations / arrangments are equal to one circular arrangmt.

1. → No of circular arrangements for n people = $(n-1)! = \frac{n!}{n}$

2. No of circular arrangements for 6B, 7G.

$$\frac{13!}{13} = 12!$$

3. 6B, 7G circular (AND) all G's together

6B ↓, 1 unit ↓

$$\frac{7!}{7} \times 7! \text{ - (units can arrange themselves)}$$

$$= \underline{\underline{6! \times 7!}}$$

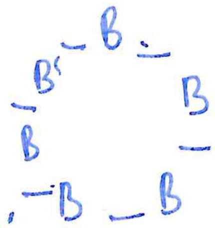
4. 6B, 7G, Circular and 2 girls always want to sit together

6B 7G < 2G together ⇒ 1 unit; 5G → 6G. ; → 6B

$$\Rightarrow \frac{12!}{12} \times 2$$ → ways girls that are together can sit

$$= \underline{\underline{11! \times 2}}$$

5. 6B, 7G, Circular, no two girls are seated together

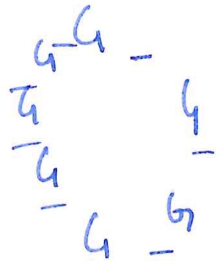


There are 6 spaces and 7G.

its not possible

# of ways — .

(6) 6 B, 7 G.



7 girls in a circular = 6!.

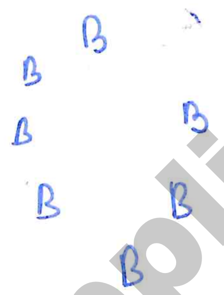
7 positions for 6 Boys

$= {}^7C_6 \times 6!$

# of arrangements $= 6! \times {}^7C_6 \times 6!$

$= 6! \times {}^7P_6$

(7) 7 B, 7 G. Circular, alternatively



7 B - Circular = 6!

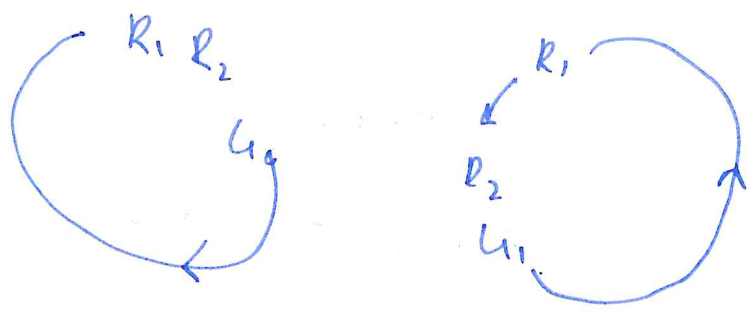
7 position for 7 G = 7! ways.

$= 6! \times 7!$

(8) Necklace 5 R, 6 G, 3 Y   Not identical stone

5 - R
6 - G
3 - Y

14



In case of a necklace the direction does not matter clockwise or counter clock wise we have to ÷ by 2.

# of necklaces $= \frac{14!}{14 \times 2} = \frac{13!}{2}$.

(b) Stones are identical.

# of linear arrangements $= \frac{14!}{5!\,6!\,3!}$

# of circular arrangements $= \frac{14!}{14\,5!\,6!\,3!}$

circular

# of Necklace arrangements $= \frac{14!}{14\,5!\,6!\,3! \times 2}$

Direction

(Q)(a) 4 digit even number

1-9	0-9	0-9	0,2,4,6,8
9	10	10	5

$= 9 \times 10 \times 10 \times 5$

(b) 4 digit odd no with distinct digits

			5
8	8	7	

$= 8 \times 8 \times 7 \times 5$

(c) 4 digit even number with distinct digits.

Case 1 — 2, 4, 6, 8.

8	8	7	4

Case 2

9	8	7	0

$= (8 \times 8 \times 7 \times 4) + (9 \times 8 \times 7 \times 1)$

(9) (a) 5 lettered words with atleast one 'a'

Words of 5 letters − # 5 letter words with no a's.

$= \underline{\underline{26^5 - 25^5}}$

(b) exactly one a

consider 4 lettered words without a's

$\underline{\ 25\ }\ \underline{\ 25\ }\ \underline{\ 25\ }\ \underline{\ 25\ } = 25^4$

Now we can have a's at any one of the 5 positions

in 5 ways.

$\uparrow_a \underline{\quad} \uparrow_a \underline{\quad} \uparrow_a \underline{\quad} \uparrow_a \underline{\quad} \uparrow_a$

No of such words. $= \underline{\underline{5 \times 25^4}}$

(c) Atmost 1 a

$= 0a's + 1a.$

$= 25^5 + (5 \times 25^4)$

└ Previous problem.

(d) Atleast 2 a's.

Words − { 0 a's + 1 a's }

$= 26^6 - \{25^5 + 5 \times 25^4\}.$

(Q) 10 lettered word, each of the character is distinct with 4 vowels and 6 consonents.

From 5 vowels − choose 4 vowels − $^5C_4$

21 consonants − choose 6 consonants − $^{21}C_6$.

Now this 10 characters can be arranged in 10! ways

no of words $= 10! \times {}^5C_4 \times {}^{21}C_6$.

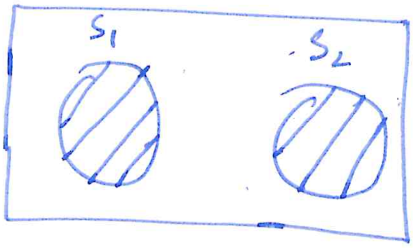
10.5 Recap: Sum Rule, Product Rule, Principle of Inclusion-Exclusion

Sum Rule

Lets consider an example sets $S = \{a, b, c\}$

# of ways we can pick one object $\longrightarrow$ $^3C_1$ ...
# of ways we can pick two objects ——— $^3C_2$
Let S_1 represent the set of all ways we can pick one item from S.

Let S_2 represent the set of all ways we can pick 2 items from S.



Here S_1 and S_2 are disjoint i.e $S_1 \cap S_2 = \phi$

Then the no of ways we can pick 1 item or 2 items

$|S_1 \cup S_2| = |S_1| + |S_2|$.

~~$\neq \{S_1 \cap S_2\}$~~

Sum Rule :- If sets S_1 and S_2 are disjoint then the no of elements

$|S_1 \cup S_2| = |S_1| + |S_2|$.

- Sum Rule is a special case of principle of Inclusion and Exclusion.

Principle Of Inclusion - Exclusion

$$|A \cup B \cup C| = |A| + |B| + |C| - (|A \cap B| + |A \cap C| + |B \cap C|) + (|A \cap B \cap C|)$$

For n sets

$$|S_1 \cup S_2 \cup S_3 \cup \dots \cup S_n| = +(|S_1| + |S_2| + |S_3| + \dots |S_n|)$$

$$- \underbrace{\{ |S_1 \cap S_2| + |S_2 \cap S_3| + \dots \}}_{\text{All 2 way}}$$

$$+ \underbrace{\{ |S_1 \cap S_2 \cap S_3| + |\;| \dots \}}_{\text{3 way}}$$

$$- \underbrace{\{ |S_1 \cap S_2 \cap S_3 \cap S_4| + \dots \}}_{\text{4 ways}}$$

$$\pm$$

$$\vdots$$

$$\underbrace{\{ |S_1 \cap S_2 \cap S_3 \dots \cap S_n| \}}_{\text{n - way}}$$

Product rule. If we have two sets S_1 and S_2 # of ordered pairs in $|S_1 \times S_2|$

$$|S_1 \times S_2| = |S_1| \times |S_2|$$

n way product rule: $S_1\ S_2\ S_3 \dots S_n$ sets # of ordered pairs in $|S_1 \times S_2 \dots \times S_n|$

$$= |S_1| \times |S_2| \times |S_3| \times |S_4| \dots \times |S_n|$$

(Q) # of 8 bit numbers with exactly 3 zeros
(a)

$\underline{\quad}$ $\underline{\quad}$ $\underline{\quad}$ $\underline{\quad}$ $\underline{\quad}$ $\underline{\quad}$ $\underline{\quad}$ $\underline{\quad}$
1 2 3 4 5 6 7 8

Here any of the bits can be 0's.

From 8 places we can choose 3 in ${}^8C_3$ ways & remaining places should be filled with 1's in 1 way.

$$\# \text{ of numbers} = \underline{\underline{{}^8C_3 \times 1}}$$

$$\boxed{{}^nC_R = {}^nC_{n-R}}$$

(b) Atleast 3 0's.

= # Total 8 bit nos − [0 0's + 1 0's + 2 0's]

$$= 2^8 - \left[{}^8C_0 + {}^8C_1 + {}^8C_2\right]$$

choose 1 0 from 8 places

choose 2 places for 0's.

Selections from multiple groups.

(4) 8 B and 5 G select 3 B and 1 G.

$^8C_3$ $^5C_1$.

# of ways = $^8C_3 \times {}^5C_1$

(5) Select 3 B or 1 G.

ways to select 3 B = $^8C_3 = S_1$

ways to select 1 G = $^5C_1 = S_2$

Because of or.

$|S_1 \cup S_2| = |S_1| + |S_2|$

$= {}^8C_3 + {}^5C_1$

Note = Because of the "or" term in the question we are performing addition.

Because of the "And" we performed multiplication in the previous problem.

Q) If we have a deck of 52 cards

(a) we need to select 3 kings and 2 Queens.

the deck contains

4 kings

4 Queens

$\therefore$ 3 of 4 kings can be selected in $^4C_3$

of 4 Queens can be selected $= {}^4C_2$ ways.

ways $= {}^4C_3 \times 4_2$

(b) Select 5 cards - Exactly 1. 3 kings

2. and 1 Queen

3. 1 other card. (not King and not Queen).

3 Kings — ${}^4C_3$ $= {}^4C_3$

1 Queen — ${}^4C_1$ $= {}^4C_1$

1 other — ${}^{(52-8)}C_1$ $= {}^{44}C_1$

# of selected ways $= \underline{\underline{{}^4C_3 \times {}^4C_1 \times {}^{44}C_1}}$.

(8) (eg) 8G., 5B

(a) Select 5 people with atleast 3 girls.

3G, 2B → ${}^8C_3 \times {}^5C_2$

or +

4G, 1B → ${}^8C_4 \times {}^5C_1$

or +

5G, 0B → ${}^8C_5 \times {}^5C_0$

→ Because these sets are disjoint.

(b) Select 5 people : atleast one boy atleast 3G.

1B 4G. - ${}^8C_4 \times {}^5C_1$

+

2B, 3G - $\underline{\underline{{}^8C_3 \times {}^5C_2}}$

~~1B, 3G~~ (not possible)

(c) 5 $\longrightarrow$ 3G, 2B. exactly a particular girl is never included.

We have total 8G and 5 boys. if one girl is never included

it is as good as having 7G and 5B.

$= {}^7C_3 \times {}^5C_2$.

(d) 5 $\longrightarrow$ 3G, 2B exactly.

a girl is always included

This is as good as selecting 4 people such that 2G and 2B

from 7G and 5B.

$= {}^7C_3 \times {}^5C_2$

(b) 3G and 2 Boys exactly a particular boy and girl should not be selected.

= Total no of ways − no of ways that pair is selected

$= {}^8C_3 \cdot {}^5C_2 - {}^7C_2 \cdot {}^4C_1$

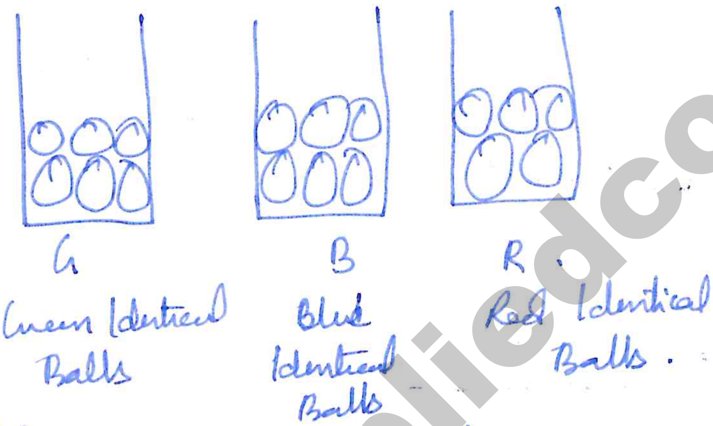
Combinations with Unlimited repetitions

eg $\{a,b,c,d,e,f\}$ Select 3 characters with unlimited repetitions.

$^{n+R-1}C_{R-1}$ Can be done using the pick and move approach.

$^{6+3-1}C_3 = {}^{8}C_3$

eg 2



{Also known as Chocolate picking problem}

- No of ways we can choose 10 balls from these 3 jars

If g green balls are picked. $g \geq 0$

b blue. $b \geq 0$

r Red. $r \geq 0$.

$g + b + r = 10$.

This can be solved using the pick and move.

G B R.

pp...m pppp...m pppp.

The pand m string should be such that it has exactly 10 picks and 2 moves.

12 length string $\left.\begin{array}{l}10 p's\\ 2 m's\end{array}\right\}$ — ${}^{10+2}C_{10}$ or ${}^{10+2}C_2$

$= \underline{{}^{12}C_2}$ ways.

Alternate approach.

(b) $g+b+r=10$ $g\geq 0, b\geq 0, r\geq 0$.

$\underbrace{\_\ |\ \_}_{2}\ |\ \pm\ |\ \underbrace{\_}_{2}\ |\ \pm\ |\ \underbrace{\_|\_|\_|\_|\_|\_}_{6}$

2 + symbols in 11 places ${}^{11}C_2$ X

It cannot handle the case where.

$g=0$

$b=0$

$r=10$.

For that we need to consider 12 blanks.

$\_\ \pm\ |\_|\_|\_|\_|\_|\_|\_|\_|\_|\_|\ \pm$

$g=0$ $b=0$ $r=10$ $g=0$ $b=10$ $r=0$. we can do it in $\underline{{}^{12}C_2}$ ways

→ If we have $x_1 \geq 0$ $x_2 \geq 0$ $x_3 \geq 0$ (non negative integer).

$$x_1 + x_2 + x_3 = n.$$

solutions = ${}^{n+3-1}C_n$.

$\underline{\underline{K \geq 3}}$

(eg) If we want to distribute 10 identical balls in 3 bins How many ways can we do it?

$\sqcup \quad \sqcup \quad \sqcup$

$x_1 \quad x_2 \quad x_3$.

$x_1 + x_2 + x_3 = 10$ $\quad$ $x_1 \geq 0$ $x_2 \geq 0$ $x_3 \geq 0$.

$${}^{10+3-1}C_{10} = {}^{12}C_{10}$$

(eg) 3 dice are thrown, no of ways they can sum to 6

$x_1 + x_2 + x_3 = 6.$ $\quad$ $x_1 \geq 1$ $\quad$ $1 \leq x_1 \leq 6$

$1 \leq x_2 \leq 6$

$1 \leq x_3 \leq 6.$

$- \; 1 \_ 1 \_ 1 \_ 1 \_ 1 \_ 1 \; -$

we can't have blank here. becaus $x_1\, x_2\, x_3$ can't be 0

2 plus in 5 locations $= {}^5C_2$

algebraic way of solving :-

$x_1' = x_1 - 1 \geq 0 \quad x_2' = x_2 - 1 \geq 0 \quad x_3' = x_3 - 1 \geq 0.$

$(x_1'+1) + (x_2'+1) + (x_3'+1) = 6.$

$x_1' + x_2' + x_3' = 3$

$n=3 \quad k=3$

$\therefore$ No of ways ~~3+3-1~~ $^{3+3-1}C_{3-1} = {}^5C_2 = {}^5C_3$

Eg Q) If we have 3 dice and the sum of the outcomes should be =10.

$x_1 + x_2 + x_3 = 10$

$1 <= x_1 <= 6.$

$1 <= x_2 <= 6$

Sticks approach fails here $\quad 1 <= x_3 <= 6$

as

$|\pm| \underbrace{\_|\_|\_|\_|\_|\_|\_|\_|}_{8} \pm|$

we can't have $x_2 = 8 > 6$ ×

Generating Functions is a much more better approach to take.

If in the problem we had values $x_1\ x_2\ x_3 \geq 6$ then

Observation :. to satisfy $x_1+x_2+x_3 = 10$.

Case 1 if $x_1 \geq 6$.

or.

Case 2 : if $x_2 \geq 6$

or

Case 3 if $x_3 \geq 6$.

$x_1' = x_1 - 6 \geq 1$.

$x_2' \geq 1$

$x_3' \geq 1$

$(x_1'+6) + x_2' + x_3' = 10$.

$x_1' + x_2' + x_3 = \cancel{10}\ 6$.

|_|_|_| $^3C_2$ ways.

similarly for. $x_2 \geq 6$ $^3C_2$ ways.

and for $x_3 \geq 6$ $^3C_2$ ways.

$\therefore$ # of total ways $= 3 \times {}^3C_2$

This is because we can observe one of $x_1\ x_2\ x_3 \geq 6$ Otherwise it is a complex problem where we have to consider solving using generating functions.

10.7 DERANGEMENTS

- If we have set of numbers $\{1,2,3,4,5\}$ then an arrangement like

$$a \; \begin{array}{|c|c|c|c|c|} 3 & 1 & 2 & 5 & 4 \\ \end{array}$$
1 2 3 4 5

where $a[i] \neq i$ is known as derangements.

- If we have n numbers. #Derangements $= D_n = !n$.

Derangements = # total − # non-Derangements.

Let
S_i = Set of all arrangements such that ith object is placed in the i th slot.

$S_1 = \{ \ldots \}$ $S_2 = \{ \ldots \}$

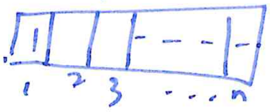
$\begin{array}{|c|c|} 1 & - - - - - \\ \end{array}$ (slots up to n)

$\begin{array}{|c|c|c|} - & 2 & - - - - \\ \end{array}$ (slots 1 2 … n)

No of non derangements $= |S_1 \cup S_2 \cup S_3 \cup \ldots \cup S_n|$

$|S_1 \cup S_2 \cup S_3 \cup S_4 \cup \ldots S_n| = |S_1| + |S_2| + \cdots |S_n|$
$- (|S_1 \cap S_2| + |S_2 \cap S_3| + \cdots)$ 2 way
$+ \cdots$ 3 way
$+$
$\vdots$
$\vdots \; (|S_1 \cap S_2 \cap S_3 \cdots S_n|)$ − n way.

For 1 way.



: 1 element is fixed n-1 elements can be arranged in $(n-1)!$ ways.

$\therefore \sum |S_i| = n \times (n-1)! = 1$

2 way $S_1 \cap S_2$

.. we have 2 fixed $(n-2)$ can be arranged in $(n-2)!$ ways.

we have ${}^nC_2$ such 2 way combinations

$= {}^nC_2 \times (n-2)!$

$= \frac{n(n-1)}{2} \times (n-2)!$

3 way. Similarly we get $(n-3)!$

we have ${}^nC_3$ 3 way intersect $= {}^nC_3 (n-3)!$

4 way $-$ ${}^nC_4 \times (n-4)!$

+

⋮

n way. ${}^nC_n$ $0!$

Total no of non derangements = $= \sum_{i=1}^{n} (-1)^{i+1} \, {}^{n}C_{i} \, (n-i)!$

$!n = D_n = n! \sum_{i=1}^{n} \frac{(-1)^{i}}{i!}$

# of Derangements $= D_n = n! \sum_{i=1}^{n} \left(\frac{(-1)^{i}}{i!} \right)$

eg (a) 5 numbered boxes and 5 numbered balls

$B_1 \; B_2 \; \ldots \; B_5$ $\qquad$ $b_1 \; b_2 \; b_3 \; \ldots \; b_5$.

no of ways we can arrange such that
$b_1 \notin B_1$
$b_2 \notin B_2$
$b_3 \notin B_3$
$b_4 \notin B_4$
$b_5 \notin B_5$.

#ways $= D_5 = 5! \left(\sum_{i=1}^{5} \frac{(-1)^{i}}{i!} \right)$

(b) # of ways s.t b_i is in $B_i \; \forall i$

b_1	b_2	b_3	b_4	b_5
B_1	B_2	B_3	B_4	B_5

= only '1' way.

(c) atleast one ball in the same numbered box.

$$= \text{Total} - D_5$$

$$= 5! - D_5$$

(d) Atleast one ball is in the incorrect box.

$$= \text{Total} - \text{All in the correct box.}$$

$$= n! - 1$$

$$= 5! - 1$$

10.8 Pigeon Hole Principle

If we have 5 pigeons and 4 pigeon holes

5 = n pigeon. 4 = m holes.

As $n > m$ atleast one hole must contain atleast 2 pigeons.

Generalization :- if we have n pigeons and m holes $n > m$ then atleast -

$\left\lfloor \frac{n-1}{m} \right\rfloor + 1$ pigeons will occupy one hole.

let us assume $n = mK + 1$
pigeons holes

K K K K .. K
1 2 3 4 m

at least one of the boxes has $K+1$ pigeons in this case.

If all holes do no contain K pigeons then atleast one hole holds

$\geq \left\lfloor \frac{n-1}{m} \right\rfloor + 1$ pigeons or $\geq K+1$ pigeons.

From the equation $n = mK + 1$

$$mK = n-1$$

$$K = \left\lfloor \frac{n-1}{m} \right\rfloor$$ as K is an integer.

If 50 bikes are painted with 7 colors # bikes with the same color $\geq$?

$C_1 \quad C_2 \quad C_3 \quad \ldots \quad C_7$

If we want each color to be used in min no of bikes we can divide 50 among 7 colors. $\frac{50}{7} = 7.\ldots$

$= 7$ bikes by 1 color but still we are left with one bike to be colored. One color need to be accorded.

$K = \left\lfloor \frac{n-1}{m} \right\rfloor + 1$. $\left\lfloor \frac{49}{7} \right\rfloor = 7$

Atleast $K+1 = 8$ bikes have the same color.

(eg) The min number of cards to be dealt to guarantee that these are 3 cards are of the same suit?

In all we have 4 suits

$m = 4$

$n =$ no of cards.

$K = 3$

$$\left\lfloor \frac{n-1}{4} \right\rfloor + 1 = 3$$

$$\left\lfloor \frac{n-1}{4} \right\rfloor = 2$$

$$\frac{n-1}{4} = 2$$

$n - 1 = 8$

$\underline{\underline{n = 9}}$ min of 9 cards are required.

9 cards

$\boxed{2}$ $\boxed{2}$ $\boxed{2}$ $\boxed{2}$ +1

S1 S2 S3 S4

(eg) If a chit/slip is made for every committe of (3M, 4W) where a committe is formed from a group of (10M, 15W). Now place the chits in 4 hats. One of the hats has ?, — chits?

# of possible committees = ${}^{10}C_3 \times {}^{15}C_4 = n$.

# of hats $m = 4$

each hat contains atleast $\left\lfloor \frac{{}^{10}C_3 \times {}^{15}C_4 - 1}{4} \right\rfloor + 1$ chits

10.9 Bionomial Coefficients - 1

- $(a+b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \dots {}^nC_n a^0 b^n$

Binomial Expansion.

$$= \sum_{i=0}^{n} {}^nC_i a^{n-i} b^i$$

Coefficient of $a^{n-i} b^i$ = Coefficient of $a^i b^{n-i}$

$$\left\{ {}^nC_i = {}^nC_{n-i} \right\}$$

Properties of Bionomial Coeffients

1. Symmetry :- ${}^nC_r = {}^nC_r = \frac{n!}{r!(n-r)!}$

(2) Pascal's Identity $\quad {}^{n}C_{r} = {}^{n-1}C_{r-1} + {}^{n-1}C_{r}$ (can be proved easily by expanding)

(3) Newtons Identity

$$ {}^{n}C_{r}\,{}^{r}C_{k} = {}^{n}C_{k}\,{}^{n-k}C_{r-k}. $$

This can be thought of as

$$ {}^{n}C_{r}\,{}^{r}C_{k}. $$

From n people $\xrightarrow[\text{r people}]{\text{we choose}}$ From r people $\xrightarrow[\text{k leaders}]{\text{we choose}}$ k leaders

This can also be seen as

From n people $\xrightarrow[\text{leaders}]{\text{choose k}}$ From remaining n-k people $\xrightarrow[\text{r-k people}]{\text{we choose}}$ r-k people

$$ {}^{n}C_{k}\,{}^{n-k}C_{r-k}. $$

→ eg # of ways to 8 black and 8 white pawns on a 8x8 chessboard.

$$ {}^{64}C_{16} \times {}^{16}C_{8} \times {}^{8}C_{8} $$

64 total locations

8+8 Black + white pawns

From 16 locations we choose 8 for black pawns

8 locations for 8 white pawns.

This is same as

$$^{64}C_8 \times {}^{56}C_8.$$

From 64 locations 8 are selected for black pawns.

From remaining 56 8 are selected for white pawns.

it can also be written as.

$$= {}^{64}C_{56} \times {}^{56}C_8 \quad ({}^nC_r = {}^nC_{n-r})$$

Pascals Triangle

Coeff's of $(a+b)^0 \longrightarrow 1 \longrightarrow$ Sum $\Sigma = 1 - 2^0$

Coeff's of $(a+b)^1 \longrightarrow 1 \quad 1 \xrightarrow{\Sigma} 2 - 2^1$

Coeff's of $(a+b)^2 \longrightarrow 1 \quad 2 \quad 1 \longrightarrow 4 - 2^2$

Coeff's of $(a+b)^3 \longrightarrow 1 \quad 3 \quad 3 \quad 1 \longrightarrow 8 - 2^3$

Coeff of $(a+b)^4 \longrightarrow 1 \quad 4 \quad 6 \quad 4 \quad 1 \longrightarrow 16 - 2^4$

Coeff of $(a+b)^5 \rightarrow 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \rightarrow 32 - 2^5$

$^5C_0 \quad {}^5C_1 \quad {}^5C_2 \quad {}^5C_3 \quad {}^5C_4 \quad {}^5C_5$

$$\sum_{i=0}^{n} {}^{n}c_i = 2^n$$ - By Mathematical Induction

We can also proove this combinatorially

If we have a set $\{a_1\ a_2\ a_3\ \ldots a_n\}$

${}^{n}C_0 + {}^{n}C_1 + {}^{n}C_2 + \ldots {}^{n}C_n = 2^n$.

1. No of ways we can construct a set of 0 elements = 1 ways ϕ (${}^{n}C_0$)
2. No of ways we can construct set of 1 element = n ways - ${}^{n}C_1$ ways.

   $\{a_1\}\ \{a_2\}\ \{a_3\} \ldots \{a_n\}$
3. No of ways we can construct a set of 2 elements = ${}^{n}C_2$ ways

   $\{\{a_1, a_2\}\ \{a_2\ a_3\} \ldots \}$.

→ Set containing all the possible subsets of a set is the Power Set

(covered in set theory)

∴ No of elements in power set = $\underline{\underline{2^n}}$

(Q) # of subsets an n element set which have atleast 1 element

(a)

$= 2^n - {}^nC_0 = 2^n - 1.$

ϕ

(b) with atleast 2 elements.

= # total − # of 0 elements − # 1 elements

$= 2^n - (1 + n)$

$= \underline{\underline{2^n - n - 1}}$

(Q) $\sum_{r=0}^{10} {}^{10}C_r \, 2^r \, 3^{10-r} =$

$(a+b)^n$ form.

$n = 10$

$a = 2 \quad b = 3.$

$(2+3)^{10} = 5^{10}.$

(Q) $\sum_{r=0}^{10} \left(\frac{1}{2}\right)^r \left(\frac{3}{5}\right)^{10-r} \, {}^{10}C_r.$

$a = \frac{1}{2} \quad b = \frac{3}{5} \quad n = 10$

$\left(\frac{1}{2} + \frac{3}{5}\right)^{10} \Rightarrow \left(\frac{4}{10}\right)^{10}$

(8) $\sum_{r=0}^{10} 2^r \, {}^{10}C_r = \sum_{r=0}^{10} 2^r (1)^{n-r} \, {}^{10}C_r = (2+10)^{10} = \underline{3^{10}}$

(9) $1 \, {}^{10}C_0 + 2 \, {}^{10}C_1 + 4 \, {}^{10}C_2 + 8 \, {}^{10}C_3 + \cdots + 1024 \, {}^{10}C_{10} = ?$

$2^0 \quad 2^1 \quad 2^2 \quad 2^3 \quad \ldots \quad 2^{10}.$

$2^r \, {}^{10}C_r = 1^{n-r} \cdot 2^r \, {}^{10}C_r.$

$= (1+2)^{10} = 3^{10}$

(10) (a) # of subsets of a 10 element set that always include element 3.

$\{1, 2, 3 \ldots 10\}$

If 3 has to be present in every set we can exclude 3 from that set and calculate # of subsets possible $= 2^9$

Now in each set we can do $\cup \{3\}$ to get the required set. It can be done in 1 way only

$\therefore$ # of sets $= 2^9$

(b) exclude element 3.

For this also we need to remove 3 from original set

# of subsets w/o 3 are $\underline{2^9}$

(c) exclude element 2 and always include element 7

then we construct subset from set excluding elements 2, 7

we can do it in 2^8 ways.

Adding element 7 can be done in 1 way.

# of subsets $= \underline{\underline{2^8}}$

(d). 5 element subset that contain 2, 7.

$\{1, 2, \cancel{3}, 4, 5, 6, \cancel{7}, 8, 9, 10\}$

$\{ \underline{2, 7} \; (\quad) \}$

always present

3 elements remaining need to be chosen from.

8 elements

$\underline{\underline{{}^8C_3}}$ such subsets are possible.

(e). 5 elements subset that exclude 2 & 7.

$\{ \underline{\qquad\qquad} \}$ we can choose from 8 elements

5 elements

$= \underline{\underline{{}^8C_5 \text{ way}}}$ $\underline{\underline{{}^8C_5}}$ subsets are possible.

10.10. Bionomial Coefficiens-2

Alternating Sign Sum.

$$+{}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 - \dots\dots {}^nC_n = \sum_{r=0}^{n} (-1)^r \, {}^nC_r = \boxed{0}$$

Proof n = odd.

$n=5$

$$\cancel{{}^5C_0} - \cancel{{}^5C_1} + \cancel{{}^5C_2} - \cancel{{}^5C_3} + \cancel{{}^5C_4} - \cancel{{}^5C_5}$$

n = even.
$n=4$

$${}^4C_0 - {}^4C_1 + {}^4C_2 - {}^4C_3 + {}^4C_4$$

Pascals Ident:

$${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$$

$$= \cancel{{}^4C_0} - (\cancel{{}^3C_0} + \cancel{{}^3C_1}) + (\cancel{{}^3C_1} + \cancel{{}^3C_2})$$

$$- (\cancel{{}^3C_2} + \cancel{{}^3C_3}) + \cancel{{}^4C_4} = 0.$$

Even-Sum Odd-Sum

$${}^nC_0 + {}^nC_2 + {}^nC_4 + {}^nC_6 + \dots\dots = {}^nC_1 + {}^nC_3 + {}^nC_4 + \dots\dots C_7.$$

$$= \underline{\underline{2^{n-1}}}.$$

# of subsets of n element set that have even no of elemts = ?

$${}^nC_0 + {}^nC_2 + \dots\dots = \underline{2^{n-1}}$$

(eg) $\frac{1!}{0!\,10!} + \frac{1}{2!\,8!} + \frac{1}{4!\,6!} + \frac{1}{6!\,4!} + \frac{1}{8!\,2!} + \dots \frac{1}{10!\,0!}$

$$\frac{\left(\frac{10!}{4!\,6!}\right)}{10!} = \frac{{}^{10}C_4}{10!}$$

$$= \frac{{}^{10}C_0 + {}^{10}C_2 + \dots {}^{10}C_{10}}{10!} = \frac{2^{10-1}}{10!} = \frac{2^9}{10!}$$

Sum of squares of binary coefficients

$${}^nC_0^{\,2} + {}^nC_1^{\,2} + {}^nC_2^{\,2} + \dots {}^nC_n^{\,2} = {}^{2n}C_n.$$

$$(1+x)^n (x+1)^n = (1+x)^{2n}$$

$$\sum {}^nC_r x^r * \sum {}^nC_r x^{n-r} = (1+x)^{2n}$$

In this expansion we are comparing the coeff of x^n.

$$x^n \left(\sum_{r=0}^{n} {}^nC_r\, {}^nC_r \right) = \sum_{r=0}^{n} \left({}^nC_r\right)^2$$

On the R.H.S. we have. Coeff of $x^n = {}^{2n}C_n$

Both should be equal on both the sides.

$$\therefore \sum_{r=0}^{n} ({}^nC_r)^2 = {}^{2n}C_n.$$

→ $${}^nC_r + {}^{r+1}C_r + {}^{r+2}C_r + \cdots {}^nC_r = {}^{n+1}C_{r+1}$$

can be proved using Mathematical Induction.

eg $${}^5C_5 + {}^6C_5 + \cdots {}^{10}C_5 = {}^{11}C_6.$$

Vander Monde's Identity

$$\underbrace{{}^mC_0 \, {}^nC_r}_{r} + \underbrace{{}^mC_1 \, {}^nC_{r-1}}_{r} + {}^mC_2 \, {}^nC_{r-2} + {}^mC_3 \, {}^nC_{r-3} + \cdots$$

$${}^mC_r \, {}^nC_0 = {}^{(m+n)}C_r$$

Sum is r.

Proof combinatorically: If we want to choose r items from set of (m+n) items where m items are of one type and n items are of another type, in the L.H.S we have all the particular ways.

in which we can divide r items into two sets (0,r) (1,r-1) . . . . (r,0).

and we have product of these items chosen from sets of m and n item

$${}^{m}C_0\,{}^{n}C_r + {}^{m}C_1\,{}^{n}C_{r-1} + \cdots\cdots {}^{m}C_r\,{}^{n}C_0$$

which is equal to no of ways we can select r items from two sets of m and n items $= {}^{(m+n)}C_r$.

eg 25 → 10 boys, 5 girls — no of ways to pick 5 people.

$${}^{25}C_5$$

$$\sum_{r=0}^{n} r.\,{}^{n}C_r = n\,2^{n-1}$$

$$= 0\,{}^{n}C_0 + 1\,{}^{n}C_1 + 2\,{}^{n}C_2 + \cdots n.\,{}^{n}C_n = n\,2^{n-1}$$

$$= \sum_{r=0}^{n} {}^{r}C_1\,{}^{n}C_r = \sum_{r=0}^{n} \frac{\cancel{r!}}{1!\,(r-1)!} \cdot \frac{n!}{(n-r)!\,\cancel{r!}}$$

$$= n \sum_{r=0}^{n} \frac{(n-1)!}{(n-r)!\,(r-1)!}$$

$$= n \sum {}^{n-1}C_{r-1}$$

$$= n\left(2^{n-1}\right)$$

(eg) coeff of x^{12} in $\left(x^2+\frac{1}{x^3}\right)^{10}$

$$= \sum_{r=0}^{n=10} {}^{10}C_r (x^2)^r \left(\frac{1}{x^3}\right)^{10-r}.$$

$$= {}^{10}C_r \, x^{2r+3r-30}$$

$${}^{10}C_r \, x^{5r-30}.$$

$5r-30=12$

$5r=42$

r = is non integer $= 0$ x^{12} is non present.

(b) coeff of x^{15}

$5r-30=15$

$r=9$

$= {}^{10}C_9 = {}^{10}C_1 = \underline{\underline{10}}$

(q) $(x+y+z)^{10}$ coeff of $x^2y^3z^5$

Trinomial. $(x+y+z)^n = \sum_{n_1+n_2+n_3=n}^{n'} \frac{n!}{n_1!\,n_2!\,n_3!} x^{n_1} y^{n_2} z^{n_3}$

coeff $= \frac{10!}{2!\,3!\,5!}$

(b) coeff $x^3y^3z^7 = 0$ because $3+3+7 \neq 10$.

→ # of terms in a trinomial expansion

No. of ways we can have sum n by using 3 n_1, n_2 and n_3

$n_1 \geq 0$ $n_2 \geq 0$ $n_3 \geq 0$.

↳ we did similar one in combinatorics.

$$^{10+3-1}C_{10} = {}^{12}C_{10}$$

10.11 RECURRENCE RELATIONS:

AN INTRODUCTION

- Application in DS and Algorithms to analyze time and space complexity

eg the time complexity of Merge Sort $T(n) = 2T(n/2) + O(n)$.

- Any recursive algorithm can be analyzed using recurrence relations

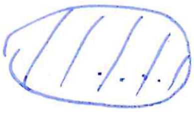
By 1. Recursion tree method

2. Master theorem.

3. Substitution Method.

- Also its used in counting and cryptography.

(eg) Bacteria growth

time = 0 $a_0 = 5$ — base case. 

At every second they double.

$a_n = 2a_{n-1}$ — recursive definition

# of cell at $t = 10$

$a_{10} = 2a_9 = 2^2 a_8 = \cdots = 2^{10} a_0$

$= 2^{10} \times 5 = \underline{1024 \times 5}$ cells.

(eg) Compound interest

$a_0 = 1000$.

$a_n = a_{n-1} + 0.1 a_{n-1}$

$n = 10$ $a_n = (1.1) a_{n-1}$

$= (1.1)^2 a_{n-2}$

$= (1.1)^3 a_{n-3}$

$= \underline{\underline{(1.1)^n a_0}}$.

$= (1.1)^n 1000$

(eg) Fibonacci Numbers 1, 1, 2, 3, 5, 8, 13, 21.

$f_n = f_{n-1} + f_{n-2}$ → recursion

$\left.\begin{array}{l} f_1 = 1 \\ f_2 = 1 \end{array}\right\}$ – Base case.

(eg) Towers of Hanoi (Recursion) covered in data Structure & Algo

$a_n = 2a_{n-1} + 1$

Counting Examples :-

(eg) # of bit strings of length n that do not contain two consecutive 0's.

→ If we have a valid string of length n-1 if we append 1 we get another valid string of length n.

① $\underbrace{\qquad\qquad\qquad}_{(n-1)}\boxed{1}$ → valid n bit string

→ If we have a valid string of length(n-2) if we append 10 we get another valid string of length n.

② $\underbrace{\qquad\qquad\qquad}_{n-2}\boxed{10}$ — valid n bit string

The above two combinations produce all the possible strings of length n.

Recurrence relation = $a_n = a_{n-1} + a_{n-2}$

Base Case $a_1 = 2$ 0 and 1

$a_2 = 3$ 10, 01, 11

For n_3 = 3 bit strings

101
011
111
010
110

} 5

001 x
100 x
000 x

(g) Strings of decimals, valid if it contains even no of 0's

a_n = # valid strings of length n

$\xleftrightarrow{(n-1)}$ [ ] — 9 ways (any+0) → a_{n-1} valid strings (n-1) * 9 ways (1-9)

valid

$\xleftrightarrow{(n-1)}$ [0] ← 1 → invalid string of size (n-1) * 1 way. (00)

invalid

$= 10^{n-1} - a_{n-1}$

$\therefore a_n = a_{n-1} \times 9 + (10^{n-1} - a_{n-1}) \Rightarrow \underline{8a_{n-1} + 10^{n-1}}$

$a_1 = 9$. $a_n = 8a_{n-1} + 10^{n-1}$

eg. → $x_0\ x_1\ x_2\ x_3 \cdots x_n$ $(n+1)$ terms.

C_n - # of ways to parenthesize the expression of $(n+1)$ terms

$C_3 = 5$

$= ((x_0\ x_1)\ x_2)\ x_3$

$= (x_0\ (x_1\ x_2))\ x_3$

$= (x_0 x_1) \cdot (x_2\ x_3)$

$= x_0\ ((x_1 x_2)\ x_3)$

$= x_0\ (x_1\ (x_2\ x_3))$

Also known as Catalan number.

$(x_0\ x_1\ x_2 \cdots x_i) \cdot (x_{i+1} \cdots\cdots x_n)$

↑ The outer dot is always outside the bracket.

The dot can be placed from in between x_0 and x_1 to x_{n-1}, and x_n.

The no of ways we can parenthesize = # of ways we can parenthesize the left part of the dot + # of ways we can parenthesize the right part of the dot.

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-i+1}$$

$$= C_0 C_{n-1} + C_1 C_{n-2} + C_2 C_{n-3} + \cdots C_{n-1} C_0 .$$

$$= \frac{1}{(n+1)} {}^{2n}C_n .$$

10.12 Solving Recurrence Relations-I

Types

Determinate Order.

$a_n = a_{n-1} + 6a_{n-2}$ ← linear

order = $n-(n-2) = \underline{\underline{2}}$ (highest - lowest).

Indeterminate Order.

$a_n = a_{n/2} - 6a_{n-5}$

order = $n - n/2 = n/2$

$n = 100$

$n/2 = 50$.

$n-5 = 95$

Non Linear

$$a_n = a_{n-1} \cdot a_{n-2}$$

$$a_n = a_{n-1}^2 + 6.$$

$$a_n = \sqrt{a_{n-1}} + 6.$$

$$a_n = \sin(a_{n-1}) + n.$$

Homogeneous Recurrence Relation

RHS $= 0$.

$a_n - 3a_{n-1} = 0$.

In homogenous Recurrence Relations

$a_n - 3a_{n-1} = 4$

$a_n - 5a_{n-1} = n$ RHS is non zero.

$a_n - 2a_{n-1} = n^2$

Linear Homogenous Recurrence relations

– Easiest ones to solve.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \dots + c_k a_{n-k}$$

$a_n - c_1 a_{n-1} - c_2 a_{n-2} - c_3 a_{n-3} - \dots - c_k a_{n-k} = 0$. $c_1 c_2 \dots c_k \in \mathbb{R}$.

degree k if $c_k \neq 0$.

eg $a_n = 5a_{n-1} - 6a_{n-2}$ $\quad a_0 = 5 \quad a_1 = 3$

Idea ~~try~~ by $a_n = r^n$.

$$r^n - c_1 r^{n-1} - c_2 r^{n-2} - \dots - c_k r^{n-k} = 0$$

divide by r^{n-k}.

$$r^k - c_1 r^{k-1} - c_2 r^{n-2} - \dots - c_k r^{n-k} = 0$$

This is a polynomial in r and its roots will help with the solution

Also known as Characteristic Polynomial.

$$a_n = 5a_{n-1} - 6a_{n-2}$$

$$a_n - 5a_{(n-1)} - 6a_{(n-2)} = 0.$$

$$r^n - 5r^{n-1} - 6r^{n-2} = 0$$

$$r^2 - 5r - 6 = 0.$$

$$r = 2, 3.$$

$$a_n = c_1 \alpha^n + c_2 \beta^n$$

$$a_n = c_1 2^n + c_2 3^n$$

$a_0 = 5 \quad a_0 = c_1 + c_2 = 5$.

$a_1 = 3 \quad 2c_1 + 3c_2 = 3 \quad c_1 = 12, \; c_2 = -7$.

$$a_n = 12(2^n) - 7(3^n)$$

Cases

① If one of the roots is 1

$$a_n = c_1 \cdot 1^n + c_2\beta^n$$

If the roots are distinct.

$$a_n = c_1\alpha^n + c_2\beta^n.$$

② If repeated roots.

$$r^2 - 4r + 4 = 0$$

$$(r-2)(r-2) = 0. \quad \alpha = \beta = 2.$$

$$a_n = c_1\alpha^n + c_2 n\cdot\beta^n.$$

$$a_n = c_1\alpha^n + c_2 n\cdot\alpha^n.$$

③ 3 roots α, β, γ

$$a_n = c_1\alpha^n + c_2\beta^n + c_3\gamma^n$$

④ 3 roots $\alpha = \beta = \gamma$

$$a_n = c_1\alpha^n + c_2 n\alpha^n + c_3 n^2\alpha^n.$$

Inhomogenous Linear Recurrence Relation

- Not all Inhomogenous recurrence relations can be solved, some can be solved.

eg

$$a_n = 5a_{n-1} - 6a_{n-2} + 3$$

constant.

$a_0 = 6$

$a_1 = 3.$

$$a_n - 5a_{n-1} + 6a_{n-2} = 3.$$

2 parts

(1) L.H.S. equate to 0.

$a_n - 5a_{n-1} + 6a_{n-2} = 0$.

$a_n = r^n$.

$a_n^h = C_1 2^n + C_2 3^n$:- homogeneous part of the solution.

$$\boxed{a_n = a_n^h + a_n^p}$$

Homogeneous part ; Particular part.

(2) RHS

$a_n - 5a_{n-1} + 6a_{n-2} = 3 \leftarrow$ const.

Take $a_n = d$

$d - 5d + 6d = 3$

$d = 3/2$

$a_n^p = d = 3/2$

$$\boxed{a_n = C_1 2^n + C_2 \cdot 3^n + 3/2}$$

C_1 and C_2 we can get by using base cases.

$$5 = c_1 + c_2 + 3/2$$

$$3 = 2c_1 + 5c_2 + 3/2.$$

on solving we get $c_1 = 4$ $c_2 = -1/2$

$$\boxed{a_n = 4 \cdot 2^n - \frac{1}{2} 3^n + \frac{3}{2}}$$

10·13 Solving Recurrence Relations - 2

- In inhomogeneous recurrence relation in the RHS if we have other than constant c then we can have the following cases.

Particular part of the solution

RHS	a_n^p
constant (c)	Take $a_n = d$ & solve.
(polynomial) $c_0 + c_1 n + c_2 n^2$	Take $a_n = d_0 + d_1 n + d_2 n^2$ & solve.
exponential c^n	Take $a_n = dc^n$ & solve.
$(c_0 + c_1 n) c^n$	Take $a_n = (d_0 + d_1 n) c^n$ and solve

Ph: +91 844-844-0102

Eg $a_n - 5a_{n-1} + 6a_{n-2} = 3n$

(1) $a_n^h \longrightarrow a_n^h = c_1 2^n + c_2 3^n$ (homogenous part).

(2) a_n^p :— $a_n = d_0 + d_1 n$.

$\Rightarrow (d_0 + d_1 n) - 5(d_0 + d_1(n-1)) + 6(d_0 + d_1(n-2)) = 3n.$

$\Rightarrow 2d_0 + 2d_1 n - 7d_1 = 3n.$

comparing coeff on both sides.

$2d_0 - 7d_1 = 0 \qquad d_1 = 3/2$

$d_0 = \frac{21}{4}$

Eg $a_n = 5a_{n-1} - 6a_{n-2} + 3^n$

(1) $a_n^h = c_1 2^n + c_2 3^n$ (same as previous problem)

(2) $a_n^p =$

$a_n - 5a_{n-1} + 6a_{n-2} = 3^n.$

$a_n = d\,3^n$

$d3^n - 5\,3^{n-1} + 6\,3^{n-2} = 3^n$

gatecse@appliedcourse.com

$$\Rightarrow \quad 3^n \left\{ d - \frac{5}{3}d + \frac{6d}{9} \right\} = 3^n$$

$$d - \frac{5}{3}d + \frac{6d}{9} = 1$$

$$\frac{9d + 15d + 6d}{9} = 1$$

×

fails

In such cases we ~~x~~should use.

$$a_n = d n . 3^n .$$

$$a_n - 5a_{n-1} + 6a_{n-2} = 3^n$$

$$\Rightarrow d . n 3^n - 5d(n-1) 3^{n-1} + 6d(n-2)(3^{n-2}) = 3^n$$

$$\Rightarrow 9nd - 15nd + 15d + 6nd - 12d = 9$$

$$d = 3$$

$$a_n^p = 3 . n . 3^n$$

$$\underline{a_n^p = n . 3^{n+1}} .$$

* $a_n^h = c_1 2^n + c_2 3^n$

$a_n^p = d.3^n$ Same form.

In such case we should take $a_n^p = d\,3^n.n$.

* $a_n^h = c_1 + c_2 n$.

$a_n^p = d$ Same form

Same form

$a_n^p = dn$

These are also called as collisions in some books.

(eg) $a_n - 5a_{n-1} + 6a_{n-2} = n5^n$.

(1) $a_n^h = c_1 2^n + c_2 3^n$

(2) $a_n^p = a_n = (d_0 + d_1 n)5^n$

$\Rightarrow (d_0 + d_1 n)5^n - 5(d_0 + d_1(n-1))\,5^{n-1} + 6(d_0 + d_1(n-2))5^{n-2} = n5^n$

$\Rightarrow (d_0 + d_1 n)25 - 25(d_0 + d_1 n - d_1) + 6(d_0 + d_1 n - 2d_1) = 25n$

On solving we get $d_1 = 25/6$.

we can also compute d_0 using d_1 value.

and $a_n = a_n^h + a_n^p$.

(eg) $na_n = 5na_{n-1} - 5a_{n-1} + 2 \quad a_1 = 5$

Let $b_n = na_n$.

$na_n = 5(n-1)a_{n-1} + 2$.

$b_n = 5b_{n-1} + 2$

$b_n - 5b_{n-1} = 2$.

$\boxed{b_n^h = c_1 5^n}$

$b_n^p = d - 5d = 2$

$d = -1/2$

$b_n = b_n^h + b_n^p$

$b_n = c_1 5^n + \left(\frac{-1}{2}\right)$

$b_n = na_n$.

$na_n = c_1 5^n - \frac{1}{2}$

$a_n = \frac{1}{n}\left[c_1 5^n - \frac{1}{2}\right]$

on solving for c_1 using a_1 we get $c_1 = 11/5$

(g) $a_n^2 = a_{n-1}^2 + 2 \quad (a_1 = 5)$.

$b_n = a_n^2 \qquad a_n = \sqrt{b_n}$

$b_n - b_{n-1} = 2$.

$b_n^h = r^n - r^{n-1} = 0$

$r - 1 = 0$

$r = 1$.

$b_n^h = c_1 1^n$.

$b_n^p =$

$b_n = d$ (same form) as $\underline{b_n^h = c_1}$.

$b_n = nd$,

$nd - (n-1)d = 2$.

$d = 2$.

$b_n^p = 2n$.

$b_n = \overset{b^h + b^p}{c_1 + 2n}$.

$\sqrt{a_n} = \sqrt{c_1 + 2n}$.

$a_1 = 5$

$a_1 = \sqrt{c_1 + 2} = 5 \quad c_1 = 23$

$a_n = \sqrt{23 + 2n}$.

(eg) $a_n^2 = 8a_{n-1}$ $a_1 = 5$

log on both sides.

$2\log a_n = 3 + \log(a_{n-1})$

$b_n = \log a_n$

$2b_n = 3 + b_{n-1} \Rightarrow 2b_n - b_{n-1} = 3$

$c = (\frac{1}{2})^n$

$b_n^h = 2r - 1 = 0$ $r = 1/2$

$b_n^p = d$

$2d - d = 3$

$d = 3.$

$b_n = c\left(\frac{1}{2}\right)^n + 3$

$a_n = 2^{b_n}$

$a_n = 2^{\left[c\left(\frac{1}{2}\right)^n + 3\right]}$

$a_1 = 5 \Rightarrow 5 = 2^{[c/2+3]} \Rightarrow \underline{c = 2(\log 5 - 3)}$

$$\boxed{a_n = 2^n\left[c\left(\frac{1}{2}\right)^n + 3\right]}$$

Indeterminate - determinate

$a_n = 5a_{n/3} + 6 \qquad (a_1 = 5)$

Order $n - n/3 = 2n/3$ Indeterminate form.

$n = 3^k$

$\frac{n}{3} = 3^{k-1} \qquad k = \log_3 n$

Because we have ÷ by 3 in the recurrence relation.

$a_{3^k} = 5a_{3^{k-1}} + 6 \qquad b_k = a_{3^k}$.

$b_k = 5b_{k-1} + 6 \xrightarrow{\text{solve}} b_k = c_1 5^k - \frac{3}{2}$

$a_{3^k} = c_1 5^k - \frac{3}{2}$

$\boxed{a_n = c_1 5^{\log_3 n} - 3/2}$ we can solve for c_1 using a_1 $c_1 = 13/2$

10.14 GENERATING FUNCTIONS FOR COUNTING

(eg) Dice sum problem 3 dice and we want sum = 12.

$$x_1 + x_2 + x_3 = 12 \qquad 1 \le x_1 \le x_2 \le x_3 \le 6.$$

Much better approach than the counting sticks is by using generating function method.

$$A(x) = (x^1 + x^2 + x^3 + x^4 + x^5 + x^6) \times$$
$$(x^1 + x^2 + x^3 + x^4 + x^5 + x^6) \times$$
$$(x^1 + x^2 + x^3 + x^4 + x^5 + x^6).$$

coeff of $x^{12} = x^1 x^5 x^6$
$+ x^1 x^6 x^5$
$+ 1 \cdot x^2 x^6 x^5$
$+ 1 \cdot x^2 x^4 x^6$
$+ \ldots$

The coefficient of x^{12} is the no of ways we get sum = 12.

A problem of combinatorics

$\downarrow$ Reduced to

A problem in Algebra.

$$A(x) = (x + x^2 + \dots x^6)^3$$

$$= x^3 (1 + x + \dots + x^5)^3$$

Geometric series $a + ar + ar^2 + ar^3 + \dots + ar^{n-1}$

$$= a\left(\frac{1-r^n}{1-r}\right) \quad r \neq 1$$

$a = 1$

$$= 1\left(\frac{1-r^n}{1-r}\right)$$

$$= x^3 \left(\frac{1-x^6}{1-x}\right)^3.$$

$$= x^3 (1-x^6)^3 \frac{1}{(1-x)^3}$$

$$\boxed{\frac{1}{(1-x)^n} = (1-x)^{-n} = \sum_{r\geq 0} {}^{n+r-1}C_r\, x^r \quad n\geq 0.}$$

$$A(x) = x^3\left\{ {}^3C_0 + {}^3C_1(-x^6)^1 + {}^3C_2(-x^6)^2 + {}^3C_3(-x^6)^3 \right\} \times$$

$$= \sum_{r\geq 0} {}^{r+2}C_r\, x^r.$$

$$= \left(x^3 - 3x^9 + 3x^{15} - x^{21}\right) \times \sum_{r\geq 0} {}^{r+2}C_r\, x^r$$

Terms which can generate x^{12}

$$A'(x) = x^3\, {}^{r+2}C_r\, x^r + \left(-3x^9 \cdot {}^{r+2}C_r\, x^r\right)$$

$$r = 12 \qquad\qquad r = 3$$

$$= {}^{14}C_{12} - 3\, {}^5C_3$$

Some Important Formulae

(1) $1 + x + x^2 + \cdots\cdots = \frac{1}{(1-x)}$.

$$\sum_{r=0}^{\infty} x^r = \frac{1}{(1-x)}$$

(2) $$\sum_{r=0}^{\infty} (-1)^r x^r = \frac{1}{1+x}.$$

(3) $$\sum_{r=0}^{\infty} a^r x^r = \frac{1}{1-ax}$$ The coeff of the expansion are in a geometric series.

The

(4) $$\sum_{r=0}^{\infty} (x^2)^r = \frac{1}{1-x^2} = 1+x^2+x^4+x^6+x^8 \cdots\cdots$$

(5) $$\sum_{r=0}^{\infty} (x^3)^r = \frac{1}{1-x^3} = 1+x^3+x^6+x^9+\cdots\cdots$$

(6) $$\frac{1}{(1-x)^n} = (1-x)^n = \sum_{r=0}^{\infty} {}^{n+r-1}C_r \, x^r$$

(7) $$(1+x)^n = \sum_{r=0}^{n} {}^{n}C_r \, x^r$$

(8) $\sum_{r=0}^{\infty} \frac{1}{r!} x^r = e^x$

10.15 Generating Functions

More Examples

(9) $1, 1/3, 1/9, 1/27, \ldots\ldots$

What is the mathematical expression which generates this?

$$\sum \left(\frac{1}{3}\right)^r (x)^r$$

$$= \sum_{r \geq 0} \left(\frac{x}{3}\right)^r$$

$$= \frac{1}{1-\frac{x}{3}} = \frac{3}{3-x}$$

What is the coeff of x^{21} in this expression above?

$$= \left(\frac{1}{3}\right)^{21}$$

(Q) $\sum_{r\geq 0}\left(2^r - \frac{1}{3^r}\right)x^r = \sum 2^r x^r - \sum \frac{1}{3^r}x^r.$

$$= \sum (2x)^r - \sum \left(\frac{x}{3}\right)^r$$

$$= \frac{1}{1-2x} - \frac{3}{3-x}$$

$$= \frac{5x}{(1-2x)(3-x)}.$$

— Coefficient of x^{10} in the above expansion.

Coeff of x^{10} is

$$= \left(2^{10} - \frac{1}{3^{10}}\right)$$

(Q) Coeff of x^7 in $(1+3x)^{10}$.

$$= \sum_{r=1}^{10} {}^{10}C_r \, 1^r (3x)^{10-r}$$

$$= {}^{10}C_3 \cdot 1 \cdot 3^7$$

Ph: +91 844-844-0102

(Q) $\sum_{r \geq 0} (-1)^r \, {}^{10}C_r \, 3^r x^r$

What expression generates the terms/series above.

$$\sum_{r \geq 0} {}^{10}C_r (-3x)^r \cdot 1^{10-r}$$

$$= (1-3x)^{-10}$$

(Q) $\sum_{r \geq 0} {}^{10+r}C_r \, x^r =$

$$= \sum_{r \geq 0} {}^{11-1+r}C_r \, x^r = \frac{1}{(1-x)^{11}}$$

(Q) $\frac{1}{(1+x)^{15}}$

$$= \frac{1}{(1-(-x))^{15}} = \sum_{r \geq 0} {}^{15+r-1}C_r \, (-1)^r x^r.$$

(Q) $\frac{1}{1-5x^3}$ $\qquad 5x^3 = y$.

$$\boxed{\frac{1}{1-x} = \sum_{r \geq 0} x^r} \qquad = \sum_{r \geq 0} (5x)^r.$$

gatecse@appliedcourse.com

(eg) $\frac{1}{(1+3x)^{11}}$ coeff of x^{12} = ?

$$\frac{1}{(1-(-3x))^{11}} = \sum_{r\geq 0} {}^{11+r-1}C_r (-3x)^r.$$

$$= \sum {}^{10+r}C_r (-1)^r 3^r x^r.$$

coeff of x^{12} $\Rightarrow r = 12$

$$= \underline{\underline{{}^{22}C_{12} \times 3^{12}}}$$

(eg) $\sum_{r\geq 0} (-1)^r 3^r \cdot 5 \frac{x^r}{r!}$

$$= 5 \cdot \sum_{r\geq 0} \frac{(-3x)^r}{r!}$$

$$= \underline{\underline{5e^{-3x}}}$$

**
(eg) $\frac{x}{(1-x)^2} = \sum_{r\geq 0} x^1 \, {}^{2+r-1}C_r x^r = \sum_{r\geq 0} {}^{r+1}C_r x^{r+1}$

$$= \underline{\underline{\sum (r+1)x^{r+1}}}$$

$\left\{ \begin{array}{l} x^0 = 0 \\ x^1 = 1 \\ x^2 = 2 \\ x^3 = 3 \\ \vdots \end{array} \right.$

** If we want to generate $1^2, 2^2, \ldots$

$$\frac{x(1+x)}{(1-x)^3} = \sum_{r\geq 0} x(1+x)\, {}^{r+2}C_r\, x^r$$

$$= \sum_{r\geq 0} (x+x^2)\, x^r\, {}^{r+2}C_r .$$

Coeff of $x^0 : 0$

$x^1 : 1^2$

$x^2 = 2^2$

$x^3 = 3^2$

$$= \sum_{r\geq 0} r^2 x^r .$$

** $\boxed{\dfrac{x(1+x)}{(1-x)^3} = \sum r^2 x^r}$

eg $\sum_{r\geq 0} (r+1)x^r = \sum r x^r + \sum x^r$

$$= \frac{x}{(1-x)^2} + \frac{1}{1-x} = \frac{1}{1-x} .$$

<u>Shifting Property</u>.

If a_0, a_1, a, a_2, a_3 is generated by $A(x)$.

then $0, a_0, a_1, a_2, a_3$. are generated by $x A(x)$.

(eg) 0, 0, 1, 1, 1, 1, 1, 1 is generated by ?

$$A(x) = \frac{1}{1-x} = \sum_{r \geq 1} 1 x^r \qquad 1, 1, 1, 1, \ldots$$

$$x A(x) = 0, 1, 1, 1, \ldots$$

$$x^2 A(x) = 0, 0, 1, 1, 1, 1, \ldots$$

$\frac{x^2}{1-x}$ generates $0, 0, 1, 1, 1, \ldots$

eg $x^2 e^x = ?$

$$e^x = \sum_{r \geq 0} \frac{x^r}{r!} \qquad a_r = \frac{1}{r!} \qquad \frac{1}{0!}, \frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}, \ldots$$

$$x e^x = 0, \frac{1}{0!}, \frac{1}{1!}, \frac{1}{2!}, \ldots$$

$$x^2 e^x = 0, 0, \frac{1}{0!}, \frac{1}{1!}, \frac{1}{2!}, \ldots$$

$a_r = 0$ if $r < 2$

$a_r = \frac{1}{(r-2)!}$ if $r \geq 2$

(eg) $\dfrac{x^4(1+x)}{(1-x)^3} = x^3\left\{\dfrac{x(1+x)}{(1-x)^3}\right\}$.

$= x^3 \sum_{r\geq 0} r^2 x^r$.

$= \sum_{r\geq 1} a_r x^r$ $\begin{cases} a_r = 0 & \text{if } r < 3 \\ a_r = (r-3)^2 & \text{if } r \geq 3 \end{cases}$

(eg) $\dfrac{1}{3x+2}$.

$= \dfrac{1/2}{1+3/2\,x} = \dfrac{1}{2}\left[\dfrac{1}{1+3/2\,x}\right] = \dfrac{1}{2}\left[\dfrac{1}{1-\left(-\frac{3}{2}x\right)}\right]$

$= \dfrac{1}{2}\sum_{r\geq 0} x^r\left(-\dfrac{3}{2}\right)^r$

(eg) $\dfrac{1}{3x-2} = \dfrac{-1/2}{1-\frac{3x}{2}} = -\dfrac{1}{2}\left\{\dfrac{1}{1-\frac{3x}{2}}\right\}$

$= -\dfrac{1}{2}\sum_{r\geq 0} x^r\left(\dfrac{3}{2}\right)^r$

(eg) $\dfrac{1}{x^2-5x+6} = \dfrac{1}{(x-2)(x-3)} = \dfrac{A}{(x-2)} + \dfrac{B}{(x-3)}$ Partial Fractions.

$= \dfrac{A(x-3)+B(x-3)}{(x-2)(x-2)}$. Equating the coefficients and solving for A and B.

$A+B=0$

$-3A-2B=1$

$A=-1$

$B=1$

$= \dfrac{-1}{x-2} + \dfrac{1}{x-3}$.

$\Rightarrow \dfrac{1/2}{1-\frac{x}{2}} + \dfrac{(-1/3)}{1-\frac{x}{3}}$.

$= 1/2 \sum_{r\geq 0} \left(\frac{1}{2}\right)^r x^r + \left(-1/3\right) \sum_{r\geq 0} \left(\frac{1}{3}\right)^r x^r$.

(eg) $\dfrac{x^3}{x^2-5x+6} = x^3\left[\dfrac{1}{(x-3)(x-2)}\right]$

$\Rightarrow x^3\left[\dfrac{A}{(x-2)} + \dfrac{B}{(x-3)}\right]$ Same as previous problem.

series.

Shifting by 3 places.

Counting Examples.

8) If we have 3 digits.

# of ways we can choose the digits so that they sum to 12.

$0 \le x_1, x_2, x_3 \le 9$.

$x_1 + x_2 + x_3 = 12$.

$A(x) = (1 + x + x^2 + x^3 + x^4 + x^6 + \dots x^9) \times$

$(1 + x^2 + x^3 + \dots + x^9) \times$

$(1 + x^2 + x^3 + \dots + x^9)$.

$= (1 + x + x^2 + \dots + x^9)^3$

$= \left(\frac{1 - x^{10}}{1 - x}\right)^3$

$= (1 - x^{10})^3 \sum_{r \ge 1} {}^{2+r}C_r x^r$.

$= (1 - x^{30} - 3x^{10} + 3x^{20}) \sum {}^{2+r}C_r x^r$.

Coeff of x^{12}

$= \sum_{r=12} {}^{2+r}C_r x^r - 3 \sum_{r=2} {}^{2+r}C_r x^{10+r} = \underline{{}^{14}C_{12} - 3 \times {}^{4}C_2}$

(eg) # of ways we can place 3n identical balls into 2 boxes such that each box contains atmost 2n balls.

$x_1 + x_2 = 3n.$

$0 \le x_1, x_2 \le 2n.$

$A(x) = (1 + x^1 + x^2 + \dots x^{2n})^2.$ We need to find coeff of x^{3n}

$$= \left[\frac{1 - x^{2n+1}}{1 - x} \right]^2$$

$$= (1 - x^{2n+1})^2 \sum_{r \ge 0} {}^{r+1}C_r x^r.$$

$$= \left(\cancel{1} + x^{4n+2} - 2x^{2n+1} \right) \sum (r+1) x^r.$$

Cannot produce sum=3n

$$= \sum (r+1) x^r - 2(r+1) x^{2n+r+1}$$

$r = 3n.$ $\qquad$ $r = n-1$

$(3n+1)$ $\qquad$ $-2(n)$

$\therefore$ coeff of $x^{3n} = 3n+1-2n = \underline{n+1}$