First Order Logic

Outline

- First-Order Logic (FOL)
  - Syntax
  - Semantics
  - Representation

Objects, Relations and Functions

- First Order Logic resembles natural language in dealing with objects and relations between objects
- Objects: people, houses, colors etc.
  - E.g. John, Red, Ball
- Relations: Verbs or Phrases that relate objects to each other
  - Some relations are **unary** or **properties**: they state some fact about a single object: Round(ball).
  - **n-ary relations** state facts about two or more objects: Married(John, Mary), LargerThan(3,2).

Ontological Commitment

- What the language assumes about the nature of reality
- FOL vs Propositional logic
  - Propositional logic assumes that there are facts that either hold or do not hold
  - FOL assumes there are objects in the world, and there are relations among them that do or do not hold
- Temporal logic
  - Facts hold at particular times
- Higher-order logic
  - Relations and functions are also objects
  - More expressive than FOL

Epistemological Commitments

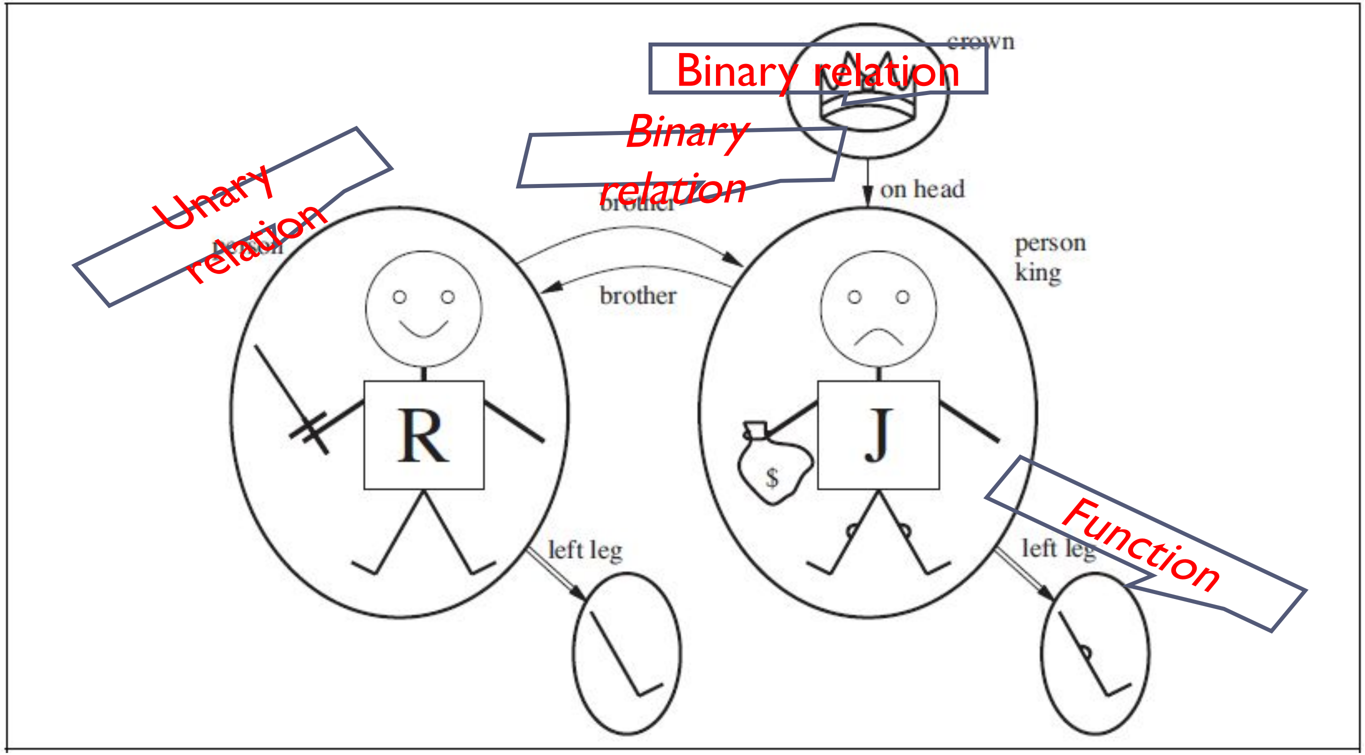
- Possible states of knowledge a logic allows wrt each fact

Language	Ontological Commitment (What exists in the world)	Epistemological Commitment (What an agent believes about facts)
Propositional logic	facts	true/false/unknown
First-order logic	facts, objects, relations	true/false/unknown
Temporal logic	facts, objects, relations, times	true/false/unknown
Probability theory	facts	degree of belief $\in [0, 1]$
Fuzzy logic	facts with degree of truth $\in [0, 1]$	known interval value

Models of FOL

- Domain of a model- set of objects it contains
- Objects are also called domain elements
- Relations are sets of tuples of objects that are related
- Functions should be total functions

Models for FOL



Symbols

- Constant symbols
  - For objects e.g. Richard, John
- Predicate symbols
  - For relations e.g. Brother, OnHead,Person
- Function symbols
  - For functions e.g. LeftLeg

Specify
arity

Interpretations

- An intended interpretation specifies
  - Which object is referred by a constant symbol
    - *Richard* refers to Richard the Lionheart
  - Which relation is referred by a predicate symbol
    - *Brother* refers to brotherhood relation
  - Which function is referred by a function symbol
    - *leftLeg* refers to the left leg function
- Truth should be defined in terms of all possible models and all possible interpretations

Syntax of FOL: basic elements

- Constants Dan, UoM, 5, …
- Predicates Brother, >, Before…
- Functions Sqrt, Length, …
- Variables x, y, a, b,…
- Connectives $\Rightarrow$, $\wedge$, $\vee$, $\Leftrightarrow$
- Equality =
- Quantifiers $\forall$, $\exists$

Terms

- Logical expressions that refer to objects
  - *John*
  - *LeftLeg(John)*
  - *LeftLeg(x)*

Syntax of FOL: Sentence

- Sentence →AtomicSentence

  | Sentence *Connective* Sentence

  | Quantifier Variable, … Sentence

  | ¬Sentence

- AtomicSentence →

  Predicate(Term, …) | Term = Term

- Term →

  Function(Term, …) | Constant | Variable

- Connective → ⇒ | ∧ | ∨ | ⇔
- Quantifier → ∀ | ∃

Sentences

- Atomic sentences
  - Formed by a predicate symbol followed by a parenthesized list of terms
    - *Brother(Richard, John)*
  - May have complex terms as arguments
    - *Married(Father(Richard), Mother(John))*
- Complex sentences
  - Constructed by using logical connectives
  - Similar to sentences in propositional logic
    - ¬Brother(LeftLeg(Richard), John)
    - Brother(Richard, John) ∧ Brother(John, Richard)
    - King(Richard) ∨ King(John)
    - ¬King(Richard) ⇒ King(John)

Quantifiers

- Allows us to express properties of collections of objects instead of enumerating objects by name

- Universal: "for all" $\forall$
- Existential: "there exists" $\exists$

Universal quantification

$\forall$ <*variables*> <*sentence*>

Everyone at UoM is smart:

$$\forall x \;\; At(x, UoM) \Rightarrow Smart(x)$$

$\forall x$ *P* is true in a model *m* iff *P* is true with *x* being ***each*** possible object in the model

Roughly speaking, equivalent to the conjunction of instantiations of *P*

$$\begin{aligned} & At(Dan, UoM) \Rightarrow Smart(Dan) \\ \wedge \;\; & At(Richard, UoM) \Rightarrow Smart(Richard) \\ \wedge \;\; & At(Ben, UoM) \Rightarrow Smart(Ben) \\ \wedge \;\; & \ldots \end{aligned}$$

A common mistake to avoid

- Typically, $\Rightarrow$ is the main connective with $\forall$
  - A universal quantifier is also equivalent to a set of implications over all objects

- Common mistake: using $\wedge$ as the main connective with $\forall$:

  $\forall x\, At(x, UoM) \wedge Smart(x)$

  means "Everyone is at UoM and everyone is smart"

  Leads to overly strong statements

Existential quantification

$\exists$ <*variables*> <*sentence*>

Someone at UoM is smart:

$\exists x\, At(x, UoM) \land Smart(x)$

$\exists x\, P$ is true in a model m iff P is true with x being ***some*** possible object in the model

- Roughly speaking, equivalent to the disjunction of instantiations of P

$$
\begin{aligned}
& At(Dan, UoM) \land Smart(Dan) \\
\lor\ & At(Richard, UoM) \land Smart(Richard) \\
\lor\ & At(Ben, UoM) \land Smart(Ben) \\
\lor\ & \ldots
\end{aligned}
$$

Another common mistake to avoid

- Typically, $\wedge$ is the main connective with $\exists$
- Common mistake: using $\Rightarrow$ as the main connective with $\exists$:

$$\exists\, x\ At(x, UoM) \Rightarrow Smart(x)$$

is true even for someone who is not at UoM!

Properties of quantifiers

$\forall x \ \forall y$ is the same as $\forall y \ \forall x$

$\exists x \ \exists y$ is the same as $\exists y \ \exists x$

$\exists x \ \forall y$ is *not* the same as $\forall y \ \exists x$

$\exists x \ \forall y$ Loves(x,y)

- "There is a person who loves everyone in the world"

$\forall y \ \exists x$ Loves(x,y)

- "Everyone in the world is loved by at least one person"

- Quantifier duality: each can be expressed using the other

  $\forall x$ Likes(x,IceCream)

  $\neg \exists x \ \neg$Likes(x,IceCream)

  $\exists x$ Likes(x,Broccoli)

  $\neg \forall x \ \neg$Likes(x,Broccoli)

De Morgan Rules

- Universal quantifier is a conjunction
- Existential quantifier is a disjunction

$$
\begin{aligned}
\forall x\ \neg P &\equiv \neg\exists x\ P & \neg(P \vee Q) &\equiv \neg P \wedge \neg Q \\
\neg\forall x\ P &\equiv \exists x\ \neg P & \neg(P \wedge Q) &\equiv \neg P \vee \neg Q \\
\forall x\ P &\equiv \neg\exists x\ \neg P & P \wedge Q &\equiv \neg(\neg P \vee \neg Q) \\
\exists x\ P &\equiv \neg\forall x\ \neg P & P \vee Q &\equiv \neg(\neg P \wedge \neg Q)
\end{aligned}
$$

Equality

- $term_1 = term_2$ is true under a given interpretation if and only if $term_1$ and $term_2$ refer to the same object

- E.g., definition of *Sibling* in terms of *Parent*:
  $\exists\ x,y\ Brother(x, Richard) \land Brother(y, Richard) \land \neg(x=y)$

Using First Order Logic

- Assertions
  - Sentences that are added to the KB using TELL
    - *TELL(KB, King(John))*
    - *TELL(KB,* $\forall x$ *King(x)* $\Rightarrow$ *Person(x))*
- Queries (goals)
  - Ask questions of the KB using ASK
    - *ASK(KB, King(John))*
  - Quantified queries (Answer is a substitution or a binding list)
    - *ASK(KB,* $\exists x$ *Person(x))*

Example: The Royal Kinship Domain

- Includes facts s.a.
  - Elizabeth is the mother of Charles
  - Charles is the father of William
  - William is the husband of Kate
  - One's grandmother is the mother of one's parent
- Unary predicates
  - Male, Female
- Binary predicates
  - Parent, Sibling, Brother, Sister, Child, Daughter, Son, Spouse, Wife, Husband, Grandparent, GrandChild, Cousin, Aunt, Uncle

Axioms

- Provide the basic factual information from which useful conclusions can be derived
- One's mother is one's female parent
  - $\forall m,c \;\; Mother(c) = m \Leftrightarrow (Female(m) \land Parent(m,c))$
- One's husband is one's male spouse
  - $\forall w, h \; Husband(h, w) \Leftrightarrow Male(h) \land Spouse(h, w)$
- Male and female are disjoint categories
  - $\forall x \; Male(x) \Leftrightarrow \neg Female(x)$
- Parent and child are inverse relations
  - $\forall p, c \; Parent(p, c) \Leftrightarrow Child(c, p)$

Axioms

- The above axioms are also definitions
- Some axioms are just plain facts
  - Male(Harry)
- Some sentences are theorems, which are entailed by axioms
  - $\forall x, y\ Sibling(x, y) \Leftrightarrow Sibling(y, x)$
    - *Logically follows from the axiom 'a sibling is another child of one's parent'*
    - $\forall x, y\ Sibling(x,y) \Leftrightarrow \neg(x=y) \land \exists p\ Parent(p,x) \land Parent(p,y)$

Summary

- First-order logic:
  - Much more expressive than propositional logic
  - Allows objects and relations as semantic primitives
  - Universal and existential quantifiers
  - Syntax