Question 1

Part a)

Required: Show how to solve the maximum flow with losses problem using linear programming. Give a detailed description of your linear program and justify clearly and carefully that it solves the problem.

Let $c : E \to \mathbb{R}^{\geq 0}$ be the capacity function.

Let f_{uv} be a variable intended to represent the flow on edge $(u, v) \in E$. Consider the following linear program which we will call L.

$$\begin{array}{llcll} \text{Maximize} & \sum\limits_{(s,v)\in E} f_{sv} & & & \\ & & & & \\ \text{Subject to} & f_{uv} & \leq & c(u,v) & \text{for all } (u,v) \in E \\ & \sum\limits_{(u,v)\in E} f_{uv} & = & (1-\epsilon_u) \sum\limits_{(v,u)\in E} f_{vu} & \text{for all } u \in V \setminus \{s,t\} \\ & f_{uv} & \geq & 0 & \text{for all } (u,v) \in E \end{array}$$

Justification of correctness of L:

Let OPT be the maximum flow value corresponding to any max flow function. Let OBJ be the optimal objective value of L.

Want to Show: $OPT = OBJ$

We will show that $OPT \leq OBJ$ and $OPT \geq OBJ$.

Show: $OPT \leq OBJ$

Let f be any flow function such that $v(f) = OPT$. For each variable f_{uv} where $(u, v) \in E$, let $f_{uv} = f(u, v)$. We know that the following three facts hold since f is a valid flow with losses. The first fact is our capacity constraints, the second fact is flow conservation with losses, and the third fact is our non-negative flow requirements.

1. $f(u, v) \leq c(u, v)$ for all $(u, v) \in E$.

2. $\sum\limits_{(u,v)\in E} f(u, v) = (1 - \epsilon_u) \sum\limits_{(v,u)\in E} f(v, u)$ for all $u \in V \setminus \{s, t\}$

3. $f(u, v) \geq 0$ for all $(u, v) \in E$

But since we assigned $f_{uv} = f(u,v)$ for all $(u,v) \in E$, we know that the following holds.

1. $f_{uv} \leq c(u,v)$ for all $(u,v) \in E$.
2. $\sum\limits_{(u,v)\in E} f_{uv} = (1-\epsilon_u) \sum\limits_{(v,u)\in E} f_{vu}$ for all $u \in V \setminus \{s,t\}$
3. $f_{uv} \geq 0$ for all $(u,v) \in E$

Hence, all our constraints of L are satisfied. And $\sum\limits_{(s,v)\in E} f_{sv} = \sum\limits_{(s,v)\in E} f(s,v) = v(f) = OPT$. So we have a feasible solution with objective value equal to OPT.

Since our optimal objective value is the maximum objective value across all feasible solutions, we have that $OPT \leq OBJ$.

Show: $OPT \geq OBJ$

Consider an optimal feasible solution to L with objective value OBJ. Consider the values to each variable f_{uv} where $(u,v) \in E$. We know that the following holds from our constraints.

1. $f_{uv} \leq c(u,v)$ for all $(u,v) \in E$.
2. $\sum\limits_{(u,v)\in E} f_{uv} = (1-\epsilon_u) \sum\limits_{(v,u)\in E} f_{vu}$ for all $u \in V \setminus \{s,t\}$
3. $f_{uv} \geq 0$ for all $(u,v) \in E$

Now, consider the function $f : E \to \mathbb{R}^{\geq 0}$ defined by $f(u,v) = f_{uv}$. Hence, the following facts hold.

1. $f(u,v) \leq c(u,v)$ for all $(u,v) \in E$.
2. $\sum\limits_{(u,v)\in E} f(u,v) = (1-\epsilon_u) \sum\limits_{(v,u)\in E} f(v,u)$ for all $u \in V \setminus \{s,t\}$
3. $f(u,v) \geq 0$ for all $(u,v) \in E$

These facts show f is a valid flow with losses. And $v(f) = \sum\limits_{(s,v)\in E} f(s,v) = \sum\limits_{(s,v)\in E} f_{sv} = OBJ$.

Since OPT is the maximum flow value, we have that $OPT \geq v(f) = OBJ$. i.e. $OPT \geq OBJ$.

Since $OPT \leq OBJ$ and $OPT \geq OBJ$, we have that $OPT = OBJ$. Hence, L is correct, as required.

Please see the next page.

Part b)

First we will rewrite L so that the RHS are all constants, the variables are all on the LHS, and we will change equalities to inequalities.

$$\text{Maximize} \quad \sum_{(s,v)\in E} f_{sv}$$

$$\begin{array}{llll}
\text{Subject to} & f_{uv} & \leq c(u,v) & \text{for all } (u,v) \in E \\
& \sum_{(u,v)\in E} f_{uv} \text{ - } (1-\epsilon_u) \sum_{(v,u)\in E} f_{vu} & \leq \quad 0 & \text{for all } u \in V \setminus \{s,t\} \\
& (1-\epsilon_u) \sum_{(v,u)\in E} f_{vu} \text{ - } \sum_{(u,v)\in E} f_{uv} & \leq \quad 0 & \text{for all } u \in V \setminus \{s,t\} \\
& f_{uv} & \geq \quad 0 & \text{for all } (u,v) \in E
\end{array}$$

Fix an enumeration of the set E which is finite. i.e. Assume we can write $E = \{e_1, e_2, ..., e_n\}$, where each $e_i = (u,v)$ for some $u, v \in V$.

Also fix an enumeration of $V \setminus \{s,t\}$ so that $V \setminus \{s,t\} = \{v_1, v_2, ..., v_k\}$.

Let $x_i = f_{e_i}$ for $i \in \{1, ..., n\}$. i.e. We just renamed our original variables.

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$ be an $n \times 1$ matrix of variables. Note, $n = |E|$.

Let $c = \begin{bmatrix} c_1 \\ c_2 \\ \cdots \\ c_n \end{bmatrix}$ be an $n \times 1$ matrix where each entry c_i is such that $c_i = 1$ if $e_i = (s,v)$ for some $v \in V$, and $c_i = 0$ otherwise.

We will define our matrix A and our vector b. But first we will define m. Since m represents the number of constraints, we know that we must count the number of constraints of L.

Looking at $f_{uv} \leq c(u,v)$ for all $(u,v) \in E$, we know that there are exactly $|E|$ such constraints.

Similarly, for the second and third class of constraints of L, we have exactly $|V \setminus \{s,t\}|$ many constraints each.

Summing up, let $m = |E| + 2 \cdot |V \setminus \{s,t\}| = n + 2k$.

Let $b = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{bmatrix}$ be an $m \times 1$ vector defined as follows. For the first $|E|$ rows of b, let $b_i = c(e_i)$, where $i \in \{1, ..., |E|\}$ and c is the capacity function. Let every other entry of b equal 0. This exactly corresponds to the RHS of each constraint of L.

Finally, we define A to be an $m \times n$ matrix as follows. Note, since $m = n + 2k$, we know $m > n$.

The first $|E|$ many rows of A will be such that the i-th row, where $i \in \{1, ..., |E|\}$ will contain all 0s except with a 1 at the i-th entry of that row. These first $|E|$ rows correspond to the first class of constraints (the capacity constraints) of L.

The next $|V \setminus \{s, t\}|$ many rows of A will represent the second class of constraints. These rows will come immediately after the rows representing the first class of constraints described earlier. We will have a new row corresponding to each $v_i \in V \setminus \{s, t\} = \{v_1, v_2, ..., v_k\}$. The next i-th such row in this class of constraints corresponding to v_i will be defined as follows. The next i-th such row is a $1 \times n$ vector. Call this row r_i. For each $j \in \{1, ..., n\}$, let the j-th entry of r_i be equal to 1 if $e_j = (v_i, v)$ for some $v \in V$, or equal to $-(1 - \epsilon_{v_i})$ if $e_j = (v, v_i)$ for some $v \in V$, or equal to 0 otherwise.

The next $|V \setminus \{s, t\}|$ many rows of A will represent the third class of constraints. These rows will come immediately after the rows representing the second class of constraints described earlier. We will have a new row corresponding to each $v_i \in V \setminus \{s, t\} = \{v_1, v_2, ..., v_k\}$. The next i-th such row in this class of constraints corresponding to v_i will be defined as follows. The next i-th such row is a $1 \times n$ vector. Call this row r_i. For each $j \in \{1, ..., n\}$, let the j-th entry of r_i be equal to $(1 - \epsilon_{v_i})$ if $e_j = (v, v_i)$ for some $v \in V$, or equal to -1 if $e_j = (v_i, v)$ for some $v \in V$, or equal to 0 otherwise.

This completely describes our $m \times n$ matrix A.

Now we have completely defined our vectors and matrices x, c, A, b. We can now rewrite L using these matrices and vectors. i.e.

$$\begin{array}{ll} \text{Maximize} & c^T x \\ \text{Subject to} & Ax \leq b \\ & x \geq 0 \end{array}$$

Since all we have done was change equalities to inequalities in L, and rewrote L using matrix-vector notation, we know that this new representation of L is still fundamentally the same linear program. Hence, a solution to this matrix-vector representation of L is still a solution to our original L. And from part a), we know that our original L solves the max flow problem with losses, as required.

Question 2

Consider the following BILP (Binary Integer Linear Program). For $i \in \{1, ..., n\}$, let x_i be a binary variable representing a purchase of tool i. i.e. $x_i = 1$ if we purchase tool i, and $x_i = 0$ otherwise.

$$\text{Minimize} \quad \sum_{i=1}^{n} x_i c_i + \sum_{i=1}^{m} \sum_{j=1}^{n} (1 - x_j) d_{i,j}$$

$$\text{Subject to} \quad x_i + \sum_{j:\text{ tool } j \text{ incompatible with } i} x_j \quad \leq 1 \text{ for all } i \in \{1, ..., n\}$$

$$x_i \in \{0, 1\} \text{ for all } i \in \{1, ..., n\}$$

Let's denote our BILP by L.

Justification of correctness of L:

Let OPT be the minimum cost from any valid purchasing of tools. Let OBJ be the optimal objective value of L.

Want to Show: $OPT = OBJ$.

We will show that $OBJ \leq OPT$ and $OPT \leq OBJ$.

Show: $OBJ \leq OPT$

Consider any valid purchasing of tools with total cost equal to OPT. Let $x_i = 1$ if we purchase tool i, and let $x_i = 0$ otherwise. We will show that this is a feasible solution to L.

We know that tools that are incompatible are not purchased together. Hence, we must have purchased at most 1 tool amongst tool i and all the other tools that are incompatible with tool i, for each $i \in \{1, ..., n\}$. Hence, we must have that $x_i + \sum_{j:\text{ tool } j \text{ incompatible with } i} x_j \leq 1$ for each $i \in \{1, ..., n\}$. So the first constraint is satisfied.

And trivially we have that $x_i \in \{0, 1\}$ since either $x_i = 0$ or $x_i = 1$ for all $i \in \{1, ..., n\}$.

Our cost of tools is $\sum_{i=1}^{n} x_i c_i$ (where each x_i is an indicator variable for tool i). And our cost of our dependencies is $\sum_{i=1}^{m} \sum_{j=1}^{n} (1 - x_j) d_{i,j}$, where we don't have any dependency cost for $d_{i,j}$ if $x_j = 1$. Hence, the total cost is $OPT = \sum_{i=1}^{n} x_i c_i + \sum_{i=1}^{m} \sum_{j=1}^{n} (1 - x_j) d_{i,j}$.

Hence, we have a feasible solution with objective value equal to OPT.

Since L minimizes objective values across all feasible solutions, we have that $OBJ \leq OPT$.

Show: $OPT \leq OBJ$

Consider an optimal feasible solution to L with objective value equal to OBJ.

We will show that this solution corresponds to a valid purchasing of tools.

Purchase tool i if $x_i = 1$, and do not purchase tool i if $x_i = 0$.

From constraint 1, we know that for each tool i, we have $x_i + \sum_{j\text{: tool } j \text{ incompatible with } i} x_j \leq 1$. Hence, for each tool i, we know we have purchased at most 1 tool amongst tool i and all the other tools that are incompatible with tool i.

Hence, our choice of tools is indeed a valid purchasing. Our cost of tools is $\sum_{i=1}^{n} x_i c_i$ (where each x_i is an indicator variable for tool i). And the cost of our dependencies is $\sum_{i=1}^{m} \sum_{j=1}^{n} (1 - x_j) d_{i,j}$, where we don't have any dependency cost for $d_{i,j}$ if $x_j = 1$. Hence, the total cost is $OBJ = \sum_{i=1}^{n} x_i c_i + \sum_{i=1}^{m} \sum_{j=1}^{n} (1 - x_j) d_{i,j}$.

Hence, we have a valid purchasing of tools whose total cost equals OBJ.

Since OPT is the minimum cost from any valid purchasing of tools, we have that $OPT \leq OBJ$.

Since $OBJ \leq OPT$ and $OPT \leq OBJ$, we conclude that $OPT = OBJ$. Therefore, our BILP L is correct, as required.

Question 3

Part a)

The vertices of the feasible region of L in the form (x_1, x_2) are $(0, 0)$, $(0, 1.6)$, $(\frac{9}{11}, \frac{23}{11})$, $(1.7142857, 0)$. Note, we have that 1.7142857 is rounded as it was determined from a graphing program (as the hint suggested).

Part b)

The optimal solution of L is $(x_1, x_2) = (\frac{9}{11}, \frac{23}{11}) \approx (0.8182, 2.0909)$.

The optimal solutions of I are $(x_1, x_2) = (1, 1)$ and $(x_1, x_2) = (0, 1)$.

The optimal objective value corresponding to L is $x_2 = \frac{23}{11} \approx 2.0909$.

The optimal objective value corresponding to I is $x_2 = 1$.

Part c)

We will restate L below.

$$\begin{array}{ll} \text{Maximize} & x_2 \\ \text{Subject to} & -3x_1 + 5x_2 \leq 8 \\ & 7x_1 + 3x_2 \leq 12 \\ & x_1, x_2 \geq 0 \end{array}$$

Let $c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $b = \begin{bmatrix} 8 \\ 12 \end{bmatrix}$ and $A = \begin{bmatrix} -3 & 5 \\ 7 & 3 \end{bmatrix}$. Hence, in matrix notation, we can write L as follows.

$$\begin{array}{ll} \text{Maximize} & c^T x \\ \text{Subject to} & Ax \leq b \\ & x \geq 0 \end{array}$$

We have 2 constraints and 2 variables. Hence, our dual will also have 2 constraints and 2 variables.

Let $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, where y_1 is the dual variable corresponding to the constraint $-3x_1 + 5x_2 \leq 8$ and y_2 is the dual variable corresponding to the constraint $7x_1 + 3x_2 \leq 12$.

As from lecture, we know that the dual L' is of the following form.

$$\begin{array}{ll} \text{Minimize} & y^T b \\ \text{Subject to} & y^T A \geq c^T \\ & y \geq 0 \end{array}$$

We will now rewrite L' without using matrix notation. Please see the next page.

Rewriting L' without matrix notation we get,

$$\begin{array}{lll} \text{Minimize} & 8y_1 + 12y_2 & \\ \text{Subject to} & -3y_1 + 7y_2 & \geq 0 \\ & 5y_1 + 3y_2 & \geq 1 \\ & y_1, y_2 \geq 0 & \end{array}$$

And as we mentioned before, we have that $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, where y_1 is the dual variable corresponding to the constraint $-3x_1 + 5x_2 \leq 8$ and y_2 is the dual variable corresponding to the constraint $7x_1 + 3x_2 \leq 12$.

Part d)

The optimal solution of L' is $(y_1, y_2) = (\frac{7}{44}, \frac{3}{44}) \approx (0.1591, 0.0682)$.

The optimal objective value corresponding to L' is $8y_1 + 12y_2 = \frac{23}{11} \approx 2.0909$.

The optimal solution of I' is $(y_1, y_2) = (0, 1)$.

The optimal objective value corresponding to I' is $8y_1 + 12y_2 = 12$.

Question: Does strong duality hold for I and I'?

No, we have that the optimal objective value corresponding to I is 1, whereas the optimal objective value corresponding to I' is 12. Since $1 \neq 12$, we have that strong duality does not hold for I and I'.