Question 1

Required: Show that if $\vdash_I \alpha$, then $\models_{G(n)} \alpha$ for every $n \geq 1$.

Proof. Proof by Induction on the length of proof of α. Assume $\vdash_I \alpha$. where our proof is a sequence of formulas $\alpha_1....\alpha_m$ where $\alpha_m = \alpha$.

Base Case: A proof of length 1 is just an axiom of $G(n)$.

We will show that all 10 axiom schemes of $G(n)$ are valid. Let $\mathcal{I}$ be an arbitrary valuation.

1. Show: $\models_{G(n)} (\alpha \to (\beta \to \alpha))$.

Case 1: Suppose $V_{\mathcal{I}}(\beta) \geq V_{\mathcal{I}}(\alpha)$. Then, $V_{\mathcal{I}}(\beta \to \alpha) = 0$. Hence, $V_{\mathcal{I}}(\alpha) \geq V_{\mathcal{I}}(\beta \to \alpha)$. Hence, $V_{\mathcal{I}}(\alpha \to (\beta \to \alpha)) = 0$.

Case 2: Suppose $V_{\mathcal{I}}(\beta) < V_{\mathcal{I}}(\alpha)$. Then, $V_{\mathcal{I}}(\beta \to \alpha) = V_{\mathcal{I}}(\alpha)$. Hence, $V_{\mathcal{I}}(\alpha) \geq V_{\mathcal{I}}(\beta \to \alpha)$. Hence, $V_{\mathcal{I}}(\alpha \to (\beta \to \alpha)) = 0$.

Hence, in both our cases we have $V_{\mathcal{I}}(\alpha \to (\beta \to \alpha)) = 0$.

Since $\mathcal{I}$ was an arbitrary valuation, we have $\models_{G(n)} (\alpha \to (\beta \to \chi))$.

2. Show: $\models_{G(n)} (\alpha \to (\beta \to \chi)) \to ((\alpha \to \beta) \to (\alpha \to \chi))$.

Case 1: Suppose $V_{\mathcal{I}}(\alpha) \geq V_{\mathcal{I}}(\chi)$. Hence, $V_{\mathcal{I}}(\alpha \to \chi) = 0$. Hence, $V_{\mathcal{I}}(\alpha \to \beta) \geq V_{\mathcal{I}}(\alpha \to \chi)$. Hence, $V_{\mathcal{I}}((\alpha \to \beta) \to (\alpha \to \chi)) = 0$. Hence, $V_{\mathcal{I}}(\alpha \to (\beta \to \chi)) \geq V_{\mathcal{I}}((\alpha \to \beta) \to (\alpha \to \chi))$. Hence, $V_{\mathcal{I}}((\alpha \to (\beta \to \chi)) \to ((\alpha \to \beta) \to (\alpha \to \chi))) = 0$.

Case 2: Suppose $V_{\mathcal{I}}(\alpha) < V_{\mathcal{I}}(\chi)$. Hence, $V_{\mathcal{I}}(\alpha \to \chi) = V_{\mathcal{I}}(\chi)$. We will consider two further subcases.

Sub-Case i) Suppose $V_{\mathcal{I}}(\beta) \geq V_{\mathcal{I}}(\chi)$. Then $V_{\mathcal{I}}(\alpha) < V_{\mathcal{I}}(\chi) \leq V_{\mathcal{I}}(\beta)$. Hence, $V_{\mathcal{I}}(\alpha \to \beta) = V_{\mathcal{I}}(\beta)$. And we have $V_{\mathcal{I}}(\beta) \geq V_{\mathcal{I}}(\chi) = V_{\mathcal{I}}(\alpha \to \chi)$. Hence, $V_{\mathcal{I}}(\alpha \to \beta) \geq V_{\mathcal{I}}(\alpha \to \chi)$. Hence, we have $V_{\mathcal{I}}((\alpha \to \beta) \to (\alpha \to \chi)) = 0$. Hence, $V_{\mathcal{I}}(\alpha \to (\beta \to \chi)) \geq V_{\mathcal{I}}(\alpha \to \beta) \to (\alpha \to \chi)$. Hence, $V_{\mathcal{I}}((\alpha \to (\beta \to \chi)) \to ((\alpha \to \beta) \to (\alpha \to \chi))) = 0$.

Sub-Case ii) Suppose $V_{\mathcal{I}}(\beta) < V_{\mathcal{I}}(\chi)$. Then, we know either $V_{\mathcal{I}}(\alpha \to \beta) = 0$ or $V_{\mathcal{I}}(\alpha \to \beta) = V_{\mathcal{I}}(\beta)$. Hence, $V_{\mathcal{I}}(\alpha \to \beta) \leq V_{\mathcal{I}}(\beta)$. Hence, $V_{\mathcal{I}}(\alpha \to \beta) \leq V_{\mathcal{I}}(\beta) < V_{\mathcal{I}}(\chi)$. And we know that $V_{\mathcal{I}}(\alpha \to \chi) = V_{\mathcal{I}}(\chi)$. Hence, we have $V_{\mathcal{I}}(\alpha \to \beta) < V_{\mathcal{I}}(\alpha \to \chi)$. Hence, $V_{\mathcal{I}}((\alpha \to \beta) \to (\alpha \to \chi)) = V_{\mathcal{I}}(\alpha \to \chi) = V_{\mathcal{I}}(\chi)$.

Furthermore, since $V_{\mathcal{I}}(\beta) < V_{\mathcal{I}}(\chi)$, we have that $V_{\mathcal{I}}(\beta \to \chi) = V_{\mathcal{I}}(\chi)$. Since $V_{\mathcal{I}}(\alpha) < V_{\mathcal{I}}(\chi)$, we have $V_{\mathcal{I}}(\alpha) < V_{\mathcal{I}}(\chi) = V_{\mathcal{I}}(\beta \to \chi)$. Since $V_{\mathcal{I}}(\alpha) < V_{\mathcal{I}}(\beta \to \chi)$, we have $V_{\mathcal{I}}(\alpha \to (\beta \to$

$\chi)) = V_{\mathcal{I}}(\beta \to \chi) = V_{\mathcal{I}}(\chi)$.

Since we have $V_{\mathcal{I}}(\alpha \to (\beta \to \chi)) = V_{\mathcal{I}}(\chi)$ and $V_{\mathcal{I}}((\alpha \to \beta) \to (\alpha \to \chi)) = V_{\mathcal{I}}(\chi)$, we have that $V_{\mathcal{I}}(\alpha \to (\beta \to \chi)) = V_{\mathcal{I}}((\alpha \to \beta) \to (\alpha \to \chi))$. Hence, $V_{\mathcal{I}}(\alpha \to (\beta \to \chi)) \geq V_{\mathcal{I}}((\alpha \to \beta) \to (\alpha \to \chi))$. Hence, $V_{\mathcal{I}}((\alpha \to (\beta \to \chi)) \to ((\alpha \to \beta) \to (\alpha \to \chi))) = 0$.

In both cases we have $V_{\mathcal{I}}((\alpha \to (\beta \to \chi)) \to ((\alpha \to \beta) \to (\alpha \to \chi))) = 0$.

Since $\mathcal{I}$ was an arbitrary valuation, we have $\models_{G(n)} (\alpha \to (\beta \to \chi)) \to ((\alpha \to \beta) \to (\alpha \to \chi))$.

3. Show: $\models_{G(n)} (\alpha \wedge \beta) \to \alpha$

We know $V_{\mathcal{I}}(\alpha \wedge \beta) = \max(V_{\mathcal{I}}(\alpha), V_{\mathcal{I}}(\beta)) \geq V_{\mathcal{I}}(\alpha)$.

Since $V_{\mathcal{I}}(\alpha \wedge \beta) \geq V_{\mathcal{I}}(\alpha)$, we have that $V_{\mathcal{I}}((\alpha \wedge \beta) \to \alpha) = 0$.

Since $\mathcal{I}$ is was arbitrary valuation, we have $\models_{G(n)} (\alpha \wedge \beta) \to \alpha$.

4. Show: $\models_{G(n)} (\alpha \wedge \beta) \to \beta$

We know $V_{\mathcal{I}}(\alpha \wedge \beta) = \max(V_{\mathcal{I}}(\alpha), V_{\mathcal{I}}(\beta)) \geq V_{\mathcal{I}}(\beta)$.

Since $V_{\mathcal{I}}(\alpha \wedge \beta) \geq V_{\mathcal{I}}(\beta)$, we have that $V_{\mathcal{I}}((\alpha \wedge \beta) \to \beta) = 0$.

Since $\mathcal{I}$ was an arbitrary valuation, we have $\models_{G(n)} (\alpha \wedge \beta) \to \beta$.

5. Show: $\models_{G(n)} \alpha \to (\beta \to (\alpha \wedge \beta))$

Case 1: Suppose $V_{\mathcal{I}}(\beta) \geq V_{\mathcal{I}}(\alpha)$. Hence, $V_{\mathcal{I}}(\alpha \wedge \beta) = \max(V_{\mathcal{I}}(\alpha), V_{\mathcal{I}}(\beta)) = V_{\mathcal{I}}(\beta)$. Hence, $V_{\mathcal{I}}(\beta) \geq V_{\mathcal{I}}(\alpha \wedge \beta)$. Hence, $V_{\mathcal{I}}(\beta \to (\alpha \wedge \beta)) = 0$. Hence, $V_{\mathcal{I}}(\alpha) \geq V_{\mathcal{I}}(\beta \to (\alpha \wedge \beta))$. Hence, $V_{\mathcal{I}}(\alpha \to (\beta \to (\alpha \wedge \beta))) = 0$.

Case 2: Suppose $V_{\mathcal{I}}(\beta) < V_{\mathcal{I}}(\alpha)$. Hence, $V_{\mathcal{I}}(\alpha \wedge \beta) = \max(V_{\mathcal{I}}(\alpha), V_{\mathcal{I}}(\beta)) = V_{\mathcal{I}}(\alpha)$. Hence, $V_{\mathcal{I}}(\beta) < V_{\mathcal{I}}(\alpha) = V_{\mathcal{I}}(\alpha \wedge \beta)$. Hence, $V_{\mathcal{I}}(\beta \to (\alpha \wedge \beta)) = V_{\mathcal{I}}(\alpha \wedge \beta) = V_{\mathcal{I}}(\alpha)$. Since $V_{\mathcal{I}}(\alpha) = V_{\mathcal{I}}(\beta \to (\alpha \wedge \beta))$, we have $V_{\mathcal{I}}(\alpha) \geq V_{\mathcal{I}}(\beta \to (\alpha \wedge \beta))$. Hence, $V_{\mathcal{I}}(\alpha \to (\beta \to (\alpha \wedge \beta))) = 0$.

In both our cases we have $V_{\mathcal{I}}(\alpha \to (\beta \to (\alpha \wedge \beta))) = 0$.

Since $\mathcal{I}$ was an arbitrary valuation, we have $\models_{G(n)} \alpha \to (\beta \to (\alpha \wedge \beta))$.

6. Show: $\models_{G(n)} \alpha \to (\alpha \vee \beta)$

We know $V_{\mathcal{I}}(\alpha) \geq \min(V_{\mathcal{I}}(\alpha), V_{\mathcal{I}}(\beta)) = V_{\mathcal{I}}(\alpha \vee \beta)$. Since $V_{\mathcal{I}}(\alpha) \geq V_{\mathcal{I}}(\alpha \vee \beta)$, we have $V_{\mathcal{I}}(\alpha \to (\alpha \vee \beta)) = 0$.

Since $\mathcal{I}$ was an arbitrary valuation, we have $\models_{G(n)} \alpha \rightarrow (\alpha \vee \beta)$.

7. Show: $\models_{G(n)} \beta \rightarrow (\alpha \vee \beta)$

We know $V_{\mathcal{I}}(\beta) \geq \min(V_{\mathcal{I}}(\alpha), V_{\mathcal{I}}(\beta)) = V_{\mathcal{I}}(\alpha \vee \beta)$. Since $V_{\mathcal{I}}(\beta) \geq V_{\mathcal{I}}(\alpha \vee \beta)$, we have $V_{\mathcal{I}}(\beta \rightarrow (\alpha \vee \beta)) = 0$.

Since $\mathcal{I}$ was an arbitrary valuation, we have $\models_{G(n)} \beta \rightarrow (\alpha \vee \beta)$.

8. Show: $\models_{G(n)} (\alpha \rightarrow \chi) \rightarrow ((\beta \rightarrow \chi) \rightarrow ((\alpha \vee \beta) \rightarrow \chi))$

We know either $V_{\mathcal{I}}(\alpha \vee \beta) = V_{\mathcal{I}}(\beta)$ or $V_{\mathcal{I}}(\alpha \vee \beta) = V_{\mathcal{I}}(\alpha)$.

Case 1: Suppose $V_{\mathcal{I}}(\alpha \vee \beta) = V_{\mathcal{I}}(\beta)$. Hence, we have $V_{\mathcal{I}}(\beta \rightarrow \chi) = V_{\mathcal{I}}((\alpha \vee \beta) \rightarrow \chi)$. Hence, $V_{\mathcal{I}}(\beta \rightarrow \chi) \geq V_{\mathcal{I}}((\alpha \vee \beta) \rightarrow \chi)$. Hence, $V_{\mathcal{I}}((\beta \rightarrow \chi) \rightarrow ((\alpha \vee \beta) \rightarrow \chi)) = 0$. Hence, $V_{\mathcal{I}}(\alpha \rightarrow \chi) \geq V_{\mathcal{I}}((\beta \rightarrow \chi) \rightarrow ((\alpha \vee \beta) \rightarrow \chi))$. Hence, $V_{\mathcal{I}}((\alpha \rightarrow \chi) \rightarrow ((\beta \rightarrow \chi) \rightarrow ((\alpha \vee \beta) \rightarrow \chi))) = 0$.

Case 2: Suppose $V_{\mathcal{I}}(\alpha \vee \beta) = V_{\mathcal{I}}(\alpha)$. Hence, we have $V_{\mathcal{I}}((\alpha \rightarrow \chi) \rightarrow ((\beta \rightarrow \chi) \rightarrow ((\alpha \vee \beta) \rightarrow \chi))) = V_{\mathcal{I}}((\alpha \rightarrow \chi) \rightarrow ((\beta \rightarrow \chi) \rightarrow (\alpha \rightarrow \chi)))$. And we know that $(\alpha \rightarrow \chi) \rightarrow ((\beta \rightarrow \chi) \rightarrow (\alpha \rightarrow \chi))$ is an instance of Axiom 1 which we showed earlier was valid. Hence, $V_{\mathcal{I}}((\alpha \rightarrow \chi) \rightarrow ((\beta \rightarrow \chi) \rightarrow (\alpha \rightarrow \chi))) = 0$. Since $V_{\mathcal{I}}((\alpha \rightarrow \chi) \rightarrow ((\beta \rightarrow \chi) \rightarrow ((\alpha \vee \beta) \rightarrow \chi))) = V_{\mathcal{I}}((\alpha \rightarrow \chi) \rightarrow ((\beta \rightarrow \chi) \rightarrow (\alpha \rightarrow \chi)))$, we have $V_{\mathcal{I}}((\alpha \rightarrow \chi) \rightarrow ((\beta \rightarrow \chi) \rightarrow ((\alpha \vee \beta) \rightarrow \chi))) = 0$.

In either case we have $V_{\mathcal{I}}((\alpha \rightarrow \chi) \rightarrow ((\beta \rightarrow \chi) \rightarrow ((\alpha \vee \beta) \rightarrow \chi))) = 0$.

Since $\mathcal{I}$ was an arbitrary valuation, we have $\models_{G(n)} (\alpha \rightarrow \chi) \rightarrow ((\beta \rightarrow \chi) \rightarrow ((\alpha \vee \beta) \rightarrow \chi))$.

9. Show: $\models_{G(n)} (\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \sim \beta) \rightarrow \sim \alpha)$

We know that either $V_{\mathcal{I}}(\sim \alpha) = 0$ or $V_{\mathcal{I}}(\sim \alpha) = n$.

Case 1: Suppose $V_{\mathcal{I}}(\sim \alpha) = 0$. Hence, $V_{\mathcal{I}}(\alpha \rightarrow \sim \beta) \geq V_{\mathcal{I}}(\sim \alpha)$. Hence, $V_{\mathcal{I}}((\alpha \rightarrow \sim \beta) \rightarrow \sim \alpha) = 0$. Hence, $V_{\mathcal{I}}(\alpha \rightarrow \beta) \geq V_{\mathcal{I}}((\alpha \rightarrow \sim \beta) \rightarrow \sim \alpha)$. Hence, $V_{\mathcal{I}}((\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \sim \beta) \rightarrow \sim \alpha)) = 0$.

Case 2: Suppose $V_{\mathcal{I}}(\sim \alpha) = n$. Hence, we must have $V_{\mathcal{I}}(\alpha) < n$. Now we will consider two further subcases. We know either $V_{\mathcal{I}}(\sim \beta) = 0$ or $V_{\mathcal{I}}(\sim \beta) = n$.

Sub-Case i) Suppose $V_{\mathcal{I}}(\sim \beta) = 0$. Hence, $V_{\mathcal{I}}(\alpha) \geq V_{\mathcal{I}}(\sim \beta)$. Hence, $V_{\mathcal{I}}(\alpha \rightarrow \sim \beta) = 0$ Hence, $V_{\mathcal{I}}(\alpha \rightarrow \sim \beta) < V_{\mathcal{I}}(\sim \alpha) = n$. Hence, $V_{\mathcal{I}}((\alpha \rightarrow \sim \beta) \rightarrow (\sim \alpha)) = V_{\mathcal{I}}(\sim \alpha) = n$. And we know that since $V_{\mathcal{I}}(\sim \beta) = 0$, we must have that $V_{\mathcal{I}}(\beta) = n$. Hence, $V_{\mathcal{I}}(\alpha) < V_{\mathcal{I}}(\beta) = n$. Hence, $V_{\mathcal{I}}(\alpha \rightarrow \beta) = V_{\mathcal{I}}(\beta) = n$.

So we have $V_{\mathcal{I}}(\alpha \rightarrow \beta) = n$ and $V_{\mathcal{I}}((\alpha \rightarrow \sim \beta) \rightarrow (\sim \alpha)) = n$. Hence, $V_{\mathcal{I}}(\alpha \rightarrow \beta) =$

$V_{\mathcal{I}}((\alpha \rightarrow \sim \beta) \rightarrow (\sim \alpha))$. Hence, $V_{\mathcal{I}}(\alpha \rightarrow \beta) \geq V_{\mathcal{I}}((\alpha \rightarrow \sim \beta) \rightarrow (\sim \alpha))$. Hence, $V_{\mathcal{I}}((\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \sim \beta) \rightarrow \sim \alpha)) = 0$.

Sub-Case ii) Suppose $V_{\mathcal{I}}(\sim \beta) = n$. Since we also know $V_{\mathcal{I}}(\alpha) < n$, we have that $V_{\mathcal{I}}(\alpha \rightarrow \sim \beta) = V_{\mathcal{I}}(\sim \beta) = n$. Since $V_{\mathcal{I}}(\sim \alpha) = n$, we have that $V_{\mathcal{I}}(\alpha \rightarrow \sim \beta) = V_{\mathcal{I}}(\sim \alpha) = n$. Hence, $V_{\mathcal{I}}(\alpha \rightarrow \sim \beta) \geq V_{\mathcal{I}}(\sim \alpha)$. Hence, $V_{\mathcal{I}}((\alpha \rightarrow \sim \beta) \rightarrow \sim \alpha) = 0$. Hence, $V_{\mathcal{I}}(\alpha \rightarrow \beta) \geq V_{\mathcal{I}}((\alpha \rightarrow \sim \beta) \rightarrow \sim \alpha)$. Hence, $V_{\mathcal{I}}((\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \sim \beta) \rightarrow \sim \alpha)) = 0$.

In both our cases we have $V_{\mathcal{I}}((\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \sim \beta) \rightarrow \sim \alpha)) = 0$.

Since $\mathcal{I}$ was an arbitrary valuation, we have $\models_{G(n)} (\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \sim \beta) \rightarrow \sim \alpha)$

10. Show: $\models_{G(n)} \sim \alpha \rightarrow (\alpha \rightarrow \beta)$.

We know either $V_{\mathcal{I}}(\alpha) = n$ or $V_{\mathcal{I}}(\alpha) \neq n$.

Case 1: Suppose $V_{\mathcal{I}}(\alpha) = n$. Hence, $n = V_{\mathcal{I}}(\alpha) \geq V_{\mathcal{I}}(\beta)$. Hence, $V_{\mathcal{I}}(\alpha \rightarrow \beta) = 0$. Hence, $V_{\mathcal{I}}(\sim \alpha) \geq V_{\mathcal{I}}(\alpha \rightarrow \beta)$. Hence, $V_{\mathcal{I}}(\sim \alpha \rightarrow (\alpha \rightarrow \beta)) = 0$.

Case 2: Suppose $V_{\mathcal{I}}(\alpha) \neq n$. Hence, we have $V_{\mathcal{I}}(\sim \alpha) = n$. Hence, we have $n = V_{\mathcal{I}}(\sim \alpha) \geq V_{\mathcal{I}}(\alpha \rightarrow \beta)$. Hence, $V_{\mathcal{I}}(\sim \alpha \rightarrow (\alpha \rightarrow \beta)) = 0$.

In both cases we have $V_{\mathcal{I}}(\sim \alpha \rightarrow (\alpha \rightarrow \beta)) = 0$.

Since $\mathcal{I}$ was an arbitrary valuation, we have $\models_{G(n)} \sim \alpha \rightarrow (\alpha \rightarrow \beta)$.

Hence, we have shown that all 10 Axiom Schemes of $G(n)$ are valid.

Inductive Step

Recall that $\vdash_I \alpha$ where our proof is a sequence of formulas $\alpha_1 .... \alpha_m$ where $\alpha_m = \alpha$.

Inductive Hypothesis: Assume all lines of the proof of α up to a certain point are valid. Say, the first $(k-1)$ many lines of the proof are valid. i.e. Assume that for each $i \in \{1, ..., k-1\}$ such that $\vdash_I \alpha_i$, we have $\models_{G(n)} \alpha_i$.

Now consider $\vdash_{G(n)} \alpha_k$.

If α_k is an axiom, then by our base case we know $\models_{G(n)} \alpha_k$.

If α_k is not an axiom, then α_k is a result of modus ponens from two earlier lines.

Hence, in some earlier lines in the proof there are formulas α_i and $\alpha_j = \alpha_i \rightarrow \alpha_k$ where $i, j < k$ such that $\vdash_I \alpha_i$ and $\vdash_I \alpha_i \rightarrow \alpha_k$.

By Inductive Hypothesis we have $\models_{G(n)} \alpha_i$ and $\models_{G(n)} \alpha_i \to \alpha_k$.

We want to show that $\models_{G(n)} \alpha_k$.

Let $\mathcal{I}$ be an arbitrary valuation.

Since $\models_{G(n)} \alpha_i \to \alpha_k$, we have $V_{\mathcal{I}}(\alpha_i \to \alpha_k) = 0$. Hence, we have $V_{\mathcal{I}}(\alpha_i) \geq V_{\mathcal{I}}(\alpha_k)$.

Since $\models_{G(n)} \alpha_i$, we have $V_{\mathcal{I}}(\alpha_i) = 0$.

So we have $V_{\mathcal{I}}(\alpha_i) \geq V_{\mathcal{I}}(\alpha_k)$ and we have $V_{\mathcal{I}}(\alpha_i) = 0$.

Hence, $0 \geq V_{\mathcal{I}}(\alpha_k)$. And, by definition of our semantics we must have that $V_{\mathcal{I}}(\alpha_k) \geq 0$.

Since $0 \geq V_{\mathcal{I}}(\alpha_k)$ and $V_{\mathcal{I}}(\alpha_k) \geq 0$, we must have that $V_{\mathcal{I}}(\alpha_k) = 0$.

Since $\mathcal{I}$ was an arbitrary valuation, we have that $\models_{G(n)} \alpha_k$.

Therefore, by induction on the complexity of proofs we have proven our desired claim.

$\square$

Question 2

1) Prove $\Rightarrow (\alpha \to \beta) \to (\sim \beta \to \sim \alpha)$

1.	$\alpha \to \beta \Rightarrow \alpha \to \beta$	RA
2.	$\alpha \Rightarrow \alpha$	RA
3.	$\sim \beta \Rightarrow \sim \beta$	RA
4.	$\alpha \to \beta, \alpha \Rightarrow \beta$	1,2, $\to$E
5.	$\alpha \to \beta, \alpha, \sim \beta \Rightarrow \beta \wedge \sim \beta$	3,4, $\wedge$I
6.	$\alpha \to \beta, \sim \beta \Rightarrow \sim \alpha$	5, RAA
7.	$\alpha \to \beta \Rightarrow \sim \beta \to \sim \alpha$	6, $\to$I
8.	$\Rightarrow (\alpha \to \beta) \to (\sim \beta \to \sim \alpha)$	7, $\to$I

2) Prove $\Rightarrow \alpha \to (\sim \alpha \to \beta)$

1.	$\alpha \Rightarrow \alpha$	RA
2.	$\sim \alpha \Rightarrow \sim \alpha$	RA
3.	$\alpha, \sim \alpha \Rightarrow \alpha \wedge \sim \alpha$	1,2, $\wedge$I
4.	$\alpha, \sim \alpha \Rightarrow \beta$	3, EF
5.	$\alpha \Rightarrow \sim \alpha \to \beta$	4, $\to$I
6.	$\Rightarrow \alpha \to (\sim \alpha \to \beta)$	5, $\to$I

3) Prove $\Rightarrow \sim\sim (\sim\sim \alpha \to \alpha)$. We will use instances of Part 1) and Part 2) from above with suitable substituions for α and β.

1.	$\sim (\sim\sim \alpha \to \alpha) \Rightarrow \sim (\sim\sim \alpha \to \alpha)$	RA
2.	$\Rightarrow \sim \alpha \to (\sim\sim \alpha \to \alpha)$	Part 2)
3.	$\Rightarrow (\sim \alpha \to (\sim\sim \alpha \to \alpha)) \to (\sim (\sim\sim \alpha \to \alpha) \to \sim\sim \alpha)$	Part 1)
4.	$\Rightarrow \sim (\sim\sim \alpha \to \alpha) \to \sim\sim \alpha$	2,3, $\to$E
5.	$\sim (\sim\sim \alpha \to \alpha) \Rightarrow \sim\sim \alpha$	1,4, $\to$E
6.	$\alpha \Rightarrow \alpha$	RA
7.	$\sim\sim \alpha \Rightarrow \sim\sim \alpha$	RA
8.	$\alpha, \sim\sim \alpha \Rightarrow \alpha \wedge \sim\sim \alpha$	6,7, $\wedge$I
9.	$\alpha, \sim\sim \alpha \Rightarrow \alpha$	8, $\wedge$E
10.	$\alpha \Rightarrow \sim\sim \alpha \to \alpha$	9, $\to$I
11.	$\sim (\sim\sim \alpha \to \alpha), \alpha \Rightarrow (\sim\sim \alpha \to \alpha) \wedge \sim (\sim\sim \alpha \to \alpha)$	1,10 $\wedge$I
12.	$\sim (\sim\sim \alpha \to \alpha) \Rightarrow \sim \alpha$	11, RAA
13.	$\sim (\sim\sim \alpha \to \alpha) \Rightarrow \sim \alpha \wedge \sim\sim \alpha$	5,12, $\wedge$I
14.	$\Rightarrow \sim\sim (\sim\sim \alpha \to \alpha)$	13, RAA

4) Prove $\sim\sim(\alpha \to \beta), \sim\sim\alpha \Rightarrow \sim\sim\beta$

1.	$\sim\sim(\alpha \to \beta) \Rightarrow \sim\sim(\alpha \to \beta)$	RA
2.	$\alpha \to \beta \Rightarrow \alpha \to \beta$	RA
3.	$\alpha \Rightarrow \alpha$	RA
4.	$\alpha \to \beta, \alpha \Rightarrow \beta$	2,3 $\to$E
5.	$\sim\beta \Rightarrow \sim\beta$	RA
6.	$\alpha \to \beta, \alpha, \sim\beta \Rightarrow \beta \wedge \sim\beta$	4,5, $\wedge$I
7.	$\alpha \to \beta, \sim\beta \Rightarrow \sim\alpha$	6, RAA
8.	$\sim\sim\alpha \Rightarrow \sim\sim\alpha$	RA
9.	$\alpha \to \beta, \sim\beta, \sim\sim\alpha \Rightarrow \sim\alpha \wedge \sim\sim\alpha$	7,8 $\wedge$I
10.	$\sim\beta, \sim\sim\alpha \Rightarrow \sim(\alpha \to \beta)$	9, RAA
11.	$\sim\sim(\alpha \to \beta), \sim\beta, \sim\sim\alpha \Rightarrow \sim(\alpha \to \beta) \wedge \sim\sim(\alpha \to \beta)$	1,10 $\wedge$I
12.	$\sim\sim(\alpha \to \beta), \sim\sim\alpha \Rightarrow \sim\sim\beta$	11, RAA

5) Prove $\Rightarrow \sim\sim\sim\alpha \to \sim\alpha$

1.	$\sim\sim\sim\alpha \Rightarrow \sim\sim\sim\alpha$	RA
2.	$\alpha \Rightarrow \alpha$	RA
3.	$\alpha \Rightarrow \sim\sim\alpha$	2, DNI
4.	$\sim\sim\sim\alpha, \alpha \Rightarrow \sim\sim\alpha \wedge \sim\sim\sim\alpha$	1,3, $\wedge$I
5.	$\sim\sim\sim\alpha \Rightarrow \sim\alpha$	4, RAA
6.	$\Rightarrow \sim\sim\sim\alpha \to \sim\alpha$	5, $\to$I

The final proof is on the next page.

6) Prove $\sim\sim(\alpha \wedge \beta) \Rightarrow \sim\sim \alpha \wedge \sim\sim \beta$

1.	$\sim\sim(\alpha \wedge \beta) \Rightarrow \sim\sim(\alpha \wedge \beta)$	RA
2.	$\alpha \wedge \beta \Rightarrow \alpha \wedge \beta$	RA
3.	$\alpha \wedge \beta \Rightarrow \alpha$	2, $\wedge$E
4.	$\sim \alpha \Rightarrow \sim \alpha$	RA
5.	$\alpha \wedge \beta, \sim \alpha \Rightarrow \alpha \wedge \sim \alpha$	3,4, $\wedge$I
6.	$\sim \alpha \Rightarrow \sim(\alpha \wedge \beta)$	5, RAA
7.	$\sim \alpha, \sim\sim(\alpha \wedge \beta) \Rightarrow \sim(\alpha \wedge \beta) \wedge \sim\sim(\alpha \wedge \beta)$	1,6, $\wedge$I
8.	$\sim\sim(\alpha \wedge \beta) \Rightarrow \sim\sim \alpha$	7, RAA
9.	$\alpha \wedge \beta \Rightarrow \beta$	2, $\wedge$E
10.	$\sim \beta \Rightarrow \sim \beta$	RA
11.	$\alpha \wedge \beta, \sim \beta \Rightarrow \beta \wedge \sim \beta$	9,10, $\wedge$I
12.	$\sim \beta \Rightarrow \sim(\alpha \wedge \beta)$	11, RAA
13.	$\sim \beta, \sim\sim(\alpha \wedge \beta) \Rightarrow \sim(\alpha \wedge \beta) \wedge \sim\sim(\alpha \wedge \beta)$	1,12, $\wedge$I
14.	$\sim\sim(\alpha \wedge \beta) \Rightarrow \sim\sim \beta$	13, RAA
15.	$\sim\sim(\alpha \wedge \beta) \Rightarrow \sim\sim \alpha \wedge \sim\sim \beta$	8,14 $\wedge$I

Exercise 3.5 Sider Page 98

Required: We noted that it seems in-principle possible for a formula to be "never-false", given the Lukasiewicz tables, without being "always-true". Give an example of such a formula.

Show: The formula $P \vee \neg P$ is never-false.

P	$\neg P$	P $\vee\neg P$
1	0	1
0	1	1
#	#	#

Since $P\vee\neg P$ only attains truth values 1 or #, we have that $P\vee\neg P$ is never-false, as required.

Exercise 3.6 Page 98

Required: Show that no wff ϕ whose sentence letters are just P and Q and which has no connectives other than $\wedge$, $\vee$, and $\sim$ has the same Lukasiewicz truth table as $P \rightarrow Q$. i.e, that for no such ϕ is $LV_{\mathcal{I}}(\phi) = LV_{\mathcal{I}}(P \rightarrow Q)$ for each trivalent interpretation $\mathcal{I}$.

We will prove the following claim.

Claim: If $\mathcal{I}$ is a trivalent interpretation such that $\mathcal{I}(P) = \mathcal{I}(Q) = \#$, then for every wff ϕ whose sentence letters are just P and Q and which has no connectives other than $\wedge$, $\vee$, and $\sim$ is such that $LV_{\mathcal{I}}(\phi) = \#$.

Proof by induction on the complexity of wffs whose sentence letters are just P and Q and which has no connectives other than $\wedge$, $\vee$, and $\sim$.

Let $\mathcal{I}$ be a trivalent interpretation such that $\mathcal{I}(\alpha) = \#$ for any sentence letter α. In particular, $\mathcal{I}(P) = \mathcal{I}(Q) = \#$.

Base Case: Consider the case of atomic formulas.

If our formula is just P, then since $\mathcal{I}(P) = \#$, we have $LV_{\mathcal{I}}(P) = \mathcal{I}(P) = \#$.

If our formula is just Q, then since $\mathcal{I}(Q) = \#$, we have $LV_{\mathcal{I}}(Q) = \mathcal{I}(Q) = \#$.

Inductive Hypothesis: Assume ϕ and ψ are wffs whose sentence letters are just P and Q and which has no connectives other than $\wedge$, $\vee$, and $\sim$ with $LV_{\mathcal{I}}(\phi) = \#$ and $LV_{\mathcal{I}}(\psi) = \#$.

Show: $LV_{\mathcal{I}}(\sim \phi) = \#$.

By Inductive Hypothesis, we know $LV_{\mathcal{I}}(\phi) = \#$. Then by definition of valuation we have that $LV_{\mathcal{I}}(\sim \phi) = \#$.

Show: $LV_{\mathcal{I}}(\phi \wedge \psi) = \#$.

By Inductive Hypothesis we know $LV_{\mathcal{I}}(\phi) = \#$ and $LV_{\mathcal{I}}(\psi) = \#$. Then by definition of valuation we have that $LV_{\mathcal{I}}(\phi \wedge \psi) = \#$.

Show: $LV_{\mathcal{I}}(\phi \vee \psi) = \#$.

By Inductive Hypothesis we know $LV_{\mathcal{I}}(\phi) = \#$ and $LV_{\mathcal{I}}(\psi) = \#$. Then by definition of valuation we have that $LV_{\mathcal{I}}(\phi \vee \psi) = \#$.

Therefore, by induction on the complexity of wffs whose sentence letters are just P and Q and which has no connectives other than $\wedge$, $\vee$, and $\sim$, we have proven the claim. Now we will apply our Claim to Exercise 3.6.

Consider the evaluation $\mathcal{I}$ such that $\mathcal{I}(\alpha) = \#$ for any sentence letter α. In particular, $\mathcal{I}(P) = \mathcal{I}(Q) = \#$.

By our claim, we know that any wff ϕ whose sentence letters are just P and Q and which has no connectives other than $\wedge$, $\vee$, and $\sim$ is such that $LV_{\mathcal{I}}(\phi) = \#$.

But since $\mathcal{I}(P) = \mathcal{I}(Q) = \#$, by definition of valuation we know that $LV_{\mathcal{I}}(P \to Q) = 1$.

Hence, there is at least one truth valuation, namely $\mathcal{I}$, where the valuation of $P \to Q$ differs from the valuation of any formula ϕ whose sentence letters are just P and Q and which has no connectives other than $\wedge$, $\vee$, and $\sim$.

Therefore, there is no wff ϕ whose sentence letters are just P and Q and which has no connectives other than $\wedge$, $\vee$, and $\sim$ that has the same Lukasiewicz truth table as $P \to Q$, completing the proof, as required.

Exercise 3.8 Page 100

Say that one trivalent interpretation $\mathcal{I}$ refines another, $\mathcal{J}$, iff for any sentence letter α, if $\mathcal{I}(\alpha) = 1$ then $\mathcal{J}(\alpha) = 1$ and if $\mathcal{I}(\alpha) = 0$ then $\mathcal{J}(\alpha) = 0$.

Required: Show that refining a trivalent interpretation preserves classical values for all wffs, given the Kleene tables. That is, if $\mathcal{J}$ refines $\mathcal{I}$, then for every wff, ϕ, if $KV_{\mathcal{I}}(\phi) = 1$ then $KV_{\mathcal{J}}(\phi) = 1$, and if $KV_{\mathcal{I}}(\phi) = 0$ then $KV_{\mathcal{J}}(\phi) = 0$.

Proof. Proof by induction on the complexity of wffs. Assume $\mathcal{J}$ refines $\mathcal{I}$.

Base Case: Consider the formula α, where α is some sentence letter.

If $KV_{\mathcal{I}}(\alpha) = 1$, then $KV_{\mathcal{I}}(\alpha) = \mathcal{I}(\alpha) = 1$. Since $\mathcal{J}$ is a refinement of $\mathcal{I}$ with $\mathcal{I}(\alpha) = 1$, we have $\mathcal{J}(\alpha) = 1$. Hence, $KV_{\mathcal{J}}(\alpha) = \mathcal{J}(\alpha) = 1$.

If $KV_{\mathcal{I}}(\alpha) = 0$, then $KV_{\mathcal{I}}(\alpha) = \mathcal{I}(\alpha) = 0$. Since $\mathcal{J}$ is a refinement of $\mathcal{I}$ with $\mathcal{I}(\alpha) = 0$, we have $\mathcal{J}(\alpha) = 0$. Hence, $KV_{\mathcal{J}}(\alpha) = \mathcal{J}(\alpha) = 0$.

Inductive Hypothesis: Assume ϕ is a wff such that if $KV_{\mathcal{I}}(\phi) = 1$ then $KV_{\mathcal{J}}(\phi) = 1$, and if $KV_{\mathcal{I}}(\phi) = 0$ then $KV_{\mathcal{J}}(\phi) = 0$. Assume ψ is a wff such that if $KV_{\mathcal{I}}(\psi) = 1$ then $KV_{\mathcal{J}}(\psi) = 1$, and if $KV_{\mathcal{I}}(\psi) = 0$ then $KV_{\mathcal{J}}(\psi) = 0$.

Show: If $KV_{\mathcal{I}}(\sim \phi) = 1$ then $KV_{\mathcal{J}}(\sim \phi) = 1$ and if $KV_{\mathcal{I}}(\sim \phi) = 0$ then $KV_{\mathcal{J}}(\sim \phi) = 0$.

First, assume $KV_{\mathcal{I}}(\sim \phi) = 1$. Hence, by Kleene tables we have $KV_{\mathcal{I}}(\phi) = 0$. By Inductive Hypothesis, we have $KV_{\mathcal{J}}(\phi) = 0$. Hence, by Kleene tables we have $KV_{\mathcal{J}}(\sim \phi) = 1$.

Next, assume $KV_{\mathcal{I}}(\sim \phi) = 0$. Hence, by Kleene tables we have $KV_{\mathcal{I}}(\phi) = 1$. By Inductive Hypothesis, we have $KV_{\mathcal{J}}(\phi) = 1$. Hence, by Kleene tables we have $KV_{\mathcal{J}}(\sim \phi) = 0$.

Show: If $KV_{\mathcal{I}}(\phi \wedge \psi) = 1$ then $KV_{\mathcal{J}}(\phi \wedge \psi) = 1$, and if $KV_{\mathcal{I}}(\phi \wedge \psi) = 0$ then $KV_{\mathcal{J}}(\phi \wedge \psi) = 0$.

First, assume $KV_{\mathcal{I}}(\phi \wedge \psi) = 1$. Hence, by Kleene tables we have $KV_{\mathcal{I}}(\phi) = 1$ and $KV_{\mathcal{I}}(\psi) = 1$. By Inductive Hypothesis, we have $KV_{\mathcal{J}}(\phi) = 1$ and $KV_{\mathcal{J}}(\psi) = 1$. Hence, by Kleene tables we have $KV_{\mathcal{J}}(\phi \wedge \psi) = 1$.

Next, assume $KV_{\mathcal{I}}(\phi \wedge \psi) = 0$. Hence, by Kleene tables we have $KV_{\mathcal{I}}(\phi) = 0$ or $KV_{\mathcal{I}}(\psi) = 0$. By Inductive Hypothesis, we have $KV_{\mathcal{J}}(\phi) = 0$ or $KV_{\mathcal{J}}(\psi) = 0$. Hence, by Kleene tables we have $KV_{\mathcal{J}}(\phi \wedge \psi) = 0$.

Show: If $KV_{\mathcal{I}}(\phi \vee \psi) = 1$ then $KV_{\mathcal{J}}(\phi \vee \psi) = 1$, and if $KV_{\mathcal{I}}(\phi \vee \psi) = 0$ then $KV_{\mathcal{J}}(\phi \vee \psi) = 0$.

First, assume $KV_{\mathcal{I}}(\phi \vee \psi) = 1$. Hence, by Kleene tables we have $KV_{\mathcal{I}}(\phi) = 1$ or $KV_{\mathcal{I}}(\psi) = 1$. By Inductive Hypothesis, we have $KV_{\mathcal{J}}(\phi) = 1$ or $KV_{\mathcal{J}}(\psi) = 1$. Hence, by Kleene tables we have $KV_{\mathcal{J}}(\phi \vee \psi) = 1$.

Next, assume $KV_{\mathcal{I}}(\phi \vee \psi) = 0$. Hence, by Kleene tables we have $KV_{\mathcal{I}}(\phi) = 0$ and $KV_{\mathcal{I}}(\psi) = 0$. By Inductive Hypothesis, we have $KV_{\mathcal{J}}(\phi) = 0$ and $KV_{\mathcal{J}}(\psi) = 0$. Hence, by Kleene tables we have $KV_{\mathcal{J}}(\phi \vee \psi) = 0$.

Show: If $KV_{\mathcal{I}}(\phi \to \psi) = 1$ then $KV_{\mathcal{J}}(\phi \to \psi) = 1$, and if $KV_{\mathcal{I}}(\phi \to \psi) = 0$ then $KV_{\mathcal{J}}(\phi \to \psi) = 0$.

First, assume $KV_{\mathcal{I}}(\phi \to \psi) = 1$. Hence, by Kleene tables we have $KV_{\mathcal{I}}(\phi) = 0$ or $KV_{\mathcal{I}}(\psi) = 1$. Hence, by Inductive Hypothesis, we have $KV_{\mathcal{J}}(\phi) = 0$ or $KV_{\mathcal{J}}(\psi) = 1$. Hence, by Kleene Tables we have $KV_{\mathcal{J}}(\phi \to \psi) = 1$.

Next, assume $KV_{\mathcal{I}}(\phi \to \psi) = 0$. Hence, by Kleene Tables we have $KV_{\mathcal{I}}(\phi) = 1$ and $KV_{\mathcal{I}}(\psi) = 0$. Hence, by Inductive Hypothesis, we have $KV_{\mathcal{J}}(\phi) = 1$ and $KV_{\mathcal{J}}(\psi) = 0$. Hence, by Kleene Tables we have $KV_{\mathcal{J}}(\phi \to \psi) = 0$.

Therefore, by induction on the complexity of wffs, we have proven our Claim.

$\square$