Exercise 1

Fix a language L.

Required: Show that for every model M of L, there is a distinct model M' of L such that M' is isomorphic to M.

Proof. Let $M = \langle D, I \rangle$ be a model for L. We will construct a distinct model $M' = \langle D', I' \rangle$ such that $M \approx M'$.

Let $\mathcal{P}(D)$ be the powerset of D.

Let $D' = \{x \in \mathcal{P}(D) : \exists y \in D, x = \{y\}\}$.

In other words, D' is the set containing all the singleton sets of elements of D. i.e. $d \in D$ if and only if $\{d\} \in D'$.

Let $M' = \langle D', I' \rangle$ where I' is defined as follows.

If c is a constant symbol of L, then $I'(c) = \{I(c)\}$.

If f is an n-ary function symbol of L, then $I'(f)$ is defined as follows.

For $\{d_1\}, ..., \{d_n\} \in D'$ where $d_1, ..., d_n \in D$, we have $I'(f)(\{d_1\}, ..., \{d_n\}) = \{I(f)(d_1, ..., d_n)\}$.

If R is an n-ary predicate symbol of L, then $I'(R)$ is defined as follows.

For $\{d_1\}, ..., \{d_n\} \in D'$ where $d_1, ..., d_n \in D$, we have $\langle \{d_1\}, ..., \{d_n\} \rangle \in I'(R)$ iff $\langle d_1, ..., d_n \rangle \in I(R)$.

Clearly $M \neq M'$ since $D \neq D'$. Now we will define an isomorphism h from M onto M'.

Let $h : D \to D'$ be defined as follows. For $x \in D$, define $h(x) = \{x\}$.

Now we will check that h satisfies the three conditions for homomorphism in Definition 6.2.1.

Let c be a constant symbol for c.

Show: $h(I(c)) = I'(c)$.

$$\begin{aligned} h(I(c)) &= \{I(c)\} && \text{By definition of } h \\ &= I'(c) && \text{By definition of } I'(c) \end{aligned}$$

Let f be an n-ary function symbol of L.

Show: $h(I(f)(d_1, ..., d_n)) = I'(f)(h(d_1), ..., h(d_n))$

$$\begin{aligned} I'(f)(h(d_1), ..., h(d_n)) &= I'(f)(\{d_1\}, ..., \{d_n\}) && \text{By definition of } h \\ &= \{I(f)(d_1, ..., d_n)\} && \text{By definition of } I'(f) \\ &= h(I(f)(d_1, ..., d_n)) && \text{By definition of } h \end{aligned}$$

Let R be an n-ary predicate symbol of L.

Show: $\langle d_1, ..., d_n\rangle \in I(R)$ iff $\langle h(d_1), ..., h(d_n)\rangle \in I'(R)$.

$$\begin{aligned} \langle h(d_1), ..., h(d_n)\rangle \in I'(R) \ &\text{iff} \ \langle \{d_1\}, ..., \{d_n\}\rangle \in I'(R) && \text{By definition of } h \\ &\text{iff} \ \langle d_1, ..., d_n\rangle \in I(R) && \text{By definition of } I'(R) \end{aligned}$$

Therefore, h is a homomorphism of M into M'.

Now we will show h is one-to-one and onto.

One-To-One: If $h(x) = h(y)$, then $\{x\} = \{y\}$ which implies that $x = y$. This shows that h is one-to-one.

Onto: Let $y \in D'$. Then $y = \{x\}$ for some $x \in D$. Hence, $h(x) = \{x\} = y$. This shows that h is onto.

Therefore, h is an isomorphism of M onto M'.

Therefore, $M \approx M'$, as required. $\square$

Exercise 2

Fix a language L without the equals sign. Show that there is no sentence ϕ of L satisfying the following: For every model $M = \langle D, I \rangle$:

$$M \models \phi \text{ iff } D \text{ has only one member}$$

Proof. Assume for the sake of contradiction that there is a sentence ϕ satisfying the above statement.

Consider the following model $M = \langle D, I \rangle$ defined as follows.

$D = \{0\}$

If c is a constant symbol of L, then $I(c) = 0$.

If f is an n-ary function symbol of L, then $I(f)(d_1, ..., d_n) = 0$ for all $d_1, ..., d_n \in D$.

If R is an n-ary function symbol of L, then $I(R) = \emptyset$.

So clearly $M = \langle D, I \rangle$ is a model of L such that D has one element. Therefore, by assumption we have that $M \models \phi$.

Now, we will construct a new model $M' = \langle D', I' \rangle$.

$D' = \{0, 1\}$.

If c is a constant symbol of L, then $I'(c) = 0$.

If f is an n-ary function symbol of L, then $I'(f)(d_1, ..., d_n) = 0$ for all $d_1, ..., d_n \in D'$.

If R is an n-ary function symbol of L, then $I'(R) = \emptyset$.

Now, we will define a homomorphism from M' onto M. Let $h : D' \to D$ be defined by $h(d') = 0$ for all $d' \in D'$.

Now we will check that h satisfies the three conditions for homomorphism in Definition 6.2.1.

Let c be a constant symbol for c.

Show: $h(I'(c)) = I(c)$.

$$\begin{aligned} h(I'(c)) &= h(0) && \text{By definition of } I'(c) \\ &= 0 && \text{By definition of } h \\ &= I(c) && \text{By definition of } I(c) \end{aligned}$$

Let f be an n-ary function symbol of L.

Show: $h(I'(f)(d_1, ..., d_n)) = I(f)(h(d_1), ..., h(d_n))$

$$\begin{aligned} h(I'(f)(d_1, ..., d_n)) &= h(0) && \text{By definition of } I'(f) \\ &= 0 && \text{By definition of } h \end{aligned}$$

And we have that,

$$\begin{aligned} I(f)(h(d_1), ..., h(d_n)) &= I(f)(0, ..., 0) && \text{By definition of } h \\ &= 0 && \text{By definition of } I(f) \end{aligned}$$

Hence, $h(I'(f)(d_1, ..., d_n)) = I(f)(h(d_1), ..., h(d_n))$.

Let R be an n-ary predicate symbol of L.

Show: $\langle d_1, ..., d_n \rangle \in I'(R)$ iff $\langle h(d_1), ..., h(d_n) \rangle \in I(R)$.

This holds trivially true since $I(R) = I'(R) = \emptyset$.

Therefore, h is a homomorphism from M' into M.

Now we will show that h is onto.

Onto: Let $d \in D$. Since $D = \{0\}$ we must have $d = 0$. Then let $d' = 0 \in D'$. Hence, we have that $h(d') = h(0) = 0 = d$. This shows that h is onto.

Therefore, h is a homomorphism from M' onto M.

By Theorem 6.6.3 part (3) we have that since h is a homomorphism from M' onto M and ϕ contains no occurences of the equals sign, then we must have that

$$Val_{M',s}(\phi) = Val_{M,(h \circ s)}(\phi), \text{ for every } s : Vble \to D' \tag{1}$$

Let $s : Vble \to D'$. Recall that ϕ does not contain the equals sign.

And notice that $h \circ s : Vble \to D$.

Since $M \models \phi$ and $\phi \in Sent_L$, by Theorem 2.2.15 we have that $M \models \phi[h \circ s]$. Hence, $Val_{M,(h \circ s)}(\phi) = 1$.

Hence, by (1) we have that $Val_{M',s}(\phi) = 1$. Hence, $M' \models \phi[s]$. Since $M' \models \phi[s]$ and $\phi \in Sent_L$, by Theorem 2.2.15 we have that $M' \models \phi$.

By our T-biconditional since $M' \models \phi$, we have that D' has one element. But we know D' has 2 elements since $D' = \{0, 1\}$. This is a contradiction.

Therefore, our assumption was wrong and there is no such sentence ϕ. This completes the proof, as required. $\square$

Exercise 3

Required: Complete the Inductive Step $\exists$ in Lemma 6.6.1.

Inductive Step 8. $\exists$
IH: $Val_{M,s}(\phi) = Val_{M',(h\circ s)}(\phi)$, for every $s : Vble \to D$.
Show: $Val_{M,s}(\exists x\phi) = Val_{M',(h\circ s)}(\exists x\phi)$, for every $s : Vble \to D$.

Proof. Choose $s : Vble \to D$. We want to show that $Val_{M,s}(\exists x\phi) = Val_{M',(h\circ s)}(\exists x\phi)$. It suffices to show that

$$Val_{M,s}(\exists x\phi) = 1 \text{ iff } Val_{M',(h\circ s)}(\exists x\phi) = 1$$

($\Rightarrow$): Assume $Val_{M,s}(\exists x\phi) = 1$. So

$$\max\{Val_{M,s^d_x}(\phi) : d \in D\} = 1$$

Hence, there exists a $d^* \in D$ such that

$$Val_{M,s^{d^*}_x}(\phi) = 1 \tag{2}$$

We want to show that $Val_{M',(h\circ s)}(\exists x\phi) = 1$. It suffices to show that,

$$\max\{Val_{M',(h\circ s)^d_x}(\phi) : d \in D'\} = 1$$

In other words, it suffices to show that

$$\text{there exists } d' \in D', Val_{M',(h\circ s)^{d'}_x}(\phi) = 1$$

Since $h : D \to D'$ is one-to-one and onto we know that $h^{-1} : D' \to D$ exists and is one-to-one and onto. Hence, there exists $d' \in D'$ such that $h^{-1}(d') = d^*$.

We know that $s^{d^*}_x : Vble \to D$. Thus, by (2) and the **IH** we have that

$$Val_{M',(h\circ s^{d^*}_x)}(\phi) = 1$$

By Lemma 6.5.1 we know that $(h \circ s^{d^*}_x)$ and $(h \circ s)^{h(d^*)}_x$ are identical. Hence,

$$Val_{M',(h\circ s)^{h(d^*)}_x}(\phi) = 1$$

And we know that $h(d^*) = h(h^{-1}(d')) = d'$. Hence,

$$Val_{M',(h\circ s)^{d'}_x}(\phi) = 1$$

This completes this direction of the proof.

($\Leftarrow$): Assume $Val_{M',(h \circ s)}(\exists x \phi) = 1$. So

$$\max\{Val_{M',(h \circ s)^d_x}(\phi) : d \in D\} = 1$$

Hence, there exists a $d' \in D'$ such that

$$Val_{M',(h \circ s)^{d'}_x}(\phi) = 1 \tag{3}$$

We want to show that $Val_{M,s}(\exists x \phi) = 1$. It suffices to show that,

$$\max\{Val_{M,s^d_x}(\phi) : d \in D\} = 1$$

In other words, it suffices to show that

$$\text{there exists } d^* \in D, Val_{M,s^{d^*}_x}(\phi) = 1$$

Since $h : D \to D'$ is one-to-one and onto we know that for $d' \in D'$, there exists $d^* \in D$ such that $h(d^*) = d'$. Looking at (3) we get that,

$$Val_{M',(h \circ s)^{h(d^*)}_x}(\phi) = 1$$

By Lemma 6.5.1 we know that $(h \circ s^{d^*}_x)$ and $(h \circ s)^{h(d^*)}_x$ are identical. Hence,

$$Val_{M',(h \circ s^{d^*}_x)}(\phi) = 1$$

Since $s^{d^*}_x : Vble \to D$, by **IH** we have that,

$$Val_{M,s^{d^*}_x}(\phi) = 1$$

This completes the other direction of the proof.

This completes the proof, as required. $\square$

Exercise 4

Suppose that L is a language without the equals and with one unary predicate symbol F. And suppose that $M = \langle D, I \rangle$ is a model for L. Show that there is a model $M' = \langle D', I' \rangle$ for L whose domain has at most four members and which is elementarily equivalent to M.

Proof. We have 3 cases of $M = \langle D, I \rangle$ to consider.

Case 1: $I(F) = D$
Case 2: $I(F) = \emptyset$
Case 3: $I(F) \subset D$ such that $I(F) \neq \emptyset$ and $I(F) \neq D$

Case 1: $I(F) = D$

$M = \langle D, I \rangle$ where D is arbitrary and I defined as,

$I(F) = D$.

Define $M' = \langle D', I' \rangle$ such that,

$D' = \{0\}$

$I'(F) = D' = \{0\}$

Now consider $h : D \to D'$ defined by $h(d) = 0$ for each $d \in D$. We will check that h satisfies the conditions for homomorphism in Definition 6.2.1.

Since L only contains one unary predicate symbol, we only have to check the predicate condition in Definition 6.2.1.

Show: $d \in I(F)$ iff $h(d) \in I'(F)$

$$\begin{aligned} d \in I(F) &\text{ iff } d \in D && \text{Since } I(F) = D \\ &\text{ iff } h(d) \in D' && \text{Since } h : D \to D' \\ &\text{ iff } h(d) \in I'(F) && \text{Since } I'(F) = D' \end{aligned}$$

Hence, h is a homomorphism from M into M. Now we will show that h is onto.

Onto: Let $d' \in D'$. Since $D' = \{0\}$, we have $d' = 0$. So consider any $d \in D$. Hence, by definition of h, we have that $h(d) = 0 = d'$ which shows h is onto.

Hence, h is a homomorphism from M onto M'.

Case 2: $I(F) = \emptyset$

$M = \langle D, I \rangle$ where D arbitrary and I defined as,

$I(F) = \emptyset$.

Define $M' = \langle D', I' \rangle$ such that,

$D' = \{0\}$

$I'(F) = \emptyset$

Now consider $h : D \to D'$ defined by $h(d) = 0$ for each $d \in D$. We will check that h satisfies the conditions for homomorphism in Definition 6.2.1.

Since L only contains one unary predicate symbol, we only have to check the predicate condition in Definition 6.2.1.

Show: $d \in I(F)$ iff $h(d) \in I'(F)$

This holds trivially since $I(F) = I'(F) = \emptyset$.

Hence, h is a homomorphism from M into M'. Now we will show that h is onto.

Onto: Let $d' \in D'$. Since $D' = \{0\}$, we have $d' = 0$. So consider any $d \in D$. Hence, by definition of h, we have that $h(d) = 0 = d'$ which shows h is onto.

Hence, h is a homomorphism from M onto M'.

Case 3: $I(F) \subset D$ such that $I(F) \neq \emptyset$ and $I(F) \neq D$

$M = \langle D, I \rangle$ where D is arbitrary and I defined as,

$I(F) \subset D$ such that $I(F) \neq \emptyset$ and $I(F) \neq D$.

Define $M' = \langle D', I' \rangle$ such that,

$D' = \{0, 1\}$

$I'(F) = \{0\}$

Now consider $h : D \to D'$ defined by $h(d) = 0$ if $d \in I(F)$ and $h(d) = 1$ if $d \in D \setminus I(F)$.

We will check that h satisfies the conditions for homomorphism in Definition 6.2.1.

Since L only contains one unary predicate symbol, we only have to check the predicate condition in Definition 6.2.1.

Show: $d \in I(F)$ iff $h(d) \in I'(F)$

($\Rightarrow$): Assume $d \in I(F)$. Then by definition of h we have that $h(d) = 0 \in I'(F)$. So $h(d) \in I'(F)$.

($\Leftarrow$): Assume $h(d) \in I'(F)$. Since $I'(F) = \{0\}$, we have that $h(d) = 0$. Then by definition of h, we must have $d \in I(F)$.

Hence, h is a homomorphism from M into M'. Now we will show that h is onto.

Onto: Let $d' \in D'$. Since $D' = \{0, 1\}$, either $d' = 0$ or $d' = 1$. If $d' = 0$, let $d \in I(F)$ so that $h(d) = 0 = d'$. If $d' = 1$, let $d \in D \setminus I(F)$ so that $h(d) = 1 = d'$.

Hence, h is a homomorphism from M onto M'.

Proof that $M \equiv M'$

The rest of the following proof applies equally to each of the three cases above. So the proof below completes each of the three cases above. We could have restated the work below 3 times for each of the 3 cases but that would be tedious to read.

So, consider $M = \langle D, I\rangle$, $M' = \langle D', I'\rangle$ and $h : D \to D'$ from ANY OF THE ABOVE THREE CASES.

By Theorem 6.6.3 part (3) we have that for every $\phi \in Form_L$, if ϕ contains no occurrences of the equals sign then we have

$$Val_{M,s}(\phi) = Val_{M',(h\circ s)}(\phi), \text{ for every } s : Vble \to D \tag{4}$$

Show: For every $\phi \in Sent_L$, we have $M \models \phi$ iff $M' \models \phi$.

Let $\phi \in Sent_L$. Notice that ϕ contains no occcurences of the equals sign since L does not contain the equals sign.

($\Rightarrow$): Assume $M \models \phi$. Let $s : Vble \to D$. By Theorem 2.2.15, since $\phi \in Sent_L$ we have that $M \models \phi[s]$.

Hence, $Val_{M,s}(\phi) = 1$. By (4) since $\phi \in Sent_L \subseteq Form_L$ does not contain the equals sign and $Val_{M,s}(\phi) = 1$, we have that $Val_{M',(h\circ s)}(\phi) = 1$. Hence, $M' \models \phi[h \circ s]$.

By Theorem 2.2.15, since $M' \models \phi[h \circ s]$ and $\phi \in Sent_L$ we have that $M' \models \phi$.

($\Leftarrow$): Assume $M' \models \phi$. Now let $s : Vble \to D$. Hence, $(h \circ s) : Vble \to D'$.

By Theorem 2.2.15, since $M' \models \phi$ and $\phi \in Sent_L$ we have that $M' \models \phi[h \circ s]$. Hence, $Val_{M',(h \circ s)}(\phi) = 1$.

By (4) since $\phi \in Sent_L \subseteq Form_L$ does not contain the equals sign and $Val_{M',(h \circ s)}(\phi) = 1$, we have that $Val_{M,s}(\phi) = 1$. Hence, $M \models \phi[s]$.

By Theorem 2.2.15, since $M \models \phi[s]$ and $\phi \in Sent_L$ we have that $M \models \phi$.

Therefore, we have shown that for every $\phi \in Sent_L$, we have $M \models \phi$ iff $M' \models \phi$.

Hence, $M \equiv M'$.

Note that this section of the proof applies to each $M = \langle D, I \rangle$, $M' = \langle D', I' \rangle$ and $h : D \to D'$ from **Case 1, 2 and 3**.

Hence, we have shown that $M \equiv M'$ in all three cases.

And M' in all three cases had at most 2 elements in its domain (and hence at most 4 elements in its domain).

This completes the proof, as required.

$\square$