Question 1, Page 172

Assume that $<_A$ and $<_B$ are partial orderings on A and B, respectively, and that f is a function from A into B satisfying,

$$x <_A y \Rightarrow f(x) <_B f(y)$$

for all $x, y \in A$.

(a)

Can we conclude that f is one-to-one?

No, we cannot conclude f is one-to-one? Consider the following counterexample.

Let $A = \{1, 2, 3, 4, 5\}$ and let $B = \{1, 2, 3, 4\}$.

Let $<_A = \{\langle 1, 2\rangle\}$

Notice, 3, 4 and 5 are not compared in $<_A$.

Let $<_B = \{\langle 1, 2\rangle, \langle 3, 4\rangle\}$.

Notice, 4 and 5 are not compared in $<_B$.

Now consider the following function $f : A \to B$.

$$f(x) = \begin{cases} x & \text{if } x \in \{1, 2, 3, 4\} \\ 4 & \text{if } x = 5 \end{cases}$$

Clearly f satisfies the condition since we have that $1 <_A 2$ and $1 = f(1) <_B f(2) = 2$.

However, f is not one-to-one since $f(4) = f(5) = 4$ but $4 \neq 5$.

(b)

Can we conclude that $x <_A y \Leftrightarrow f(x) <_B f(y)$?

No, our counterexample in part (a) also applies here.

We have that $3 = f(3) <_B f(4) = 4$ but $3 \not<_A 4$.

Question 2, Page 172

Assume that R is a partial ordering on a set A. Prove that R^{-1} is also a partial ordering on A.

Proof. We know that R is irreflexive. We will show that R^{-1} is irreflexive.

Assume for the sake of contradiction that R^{-1} is not irreflexive.

Then $\exists a \in A$ such that $\langle a, a \rangle \in R^{-1}$.

But this means that $\langle a, a \rangle \in R$ which contradicts the fact that R is irreflexive.

Therefore, R^{-1} must also be irreflexive.

We know that R is transitive. We will show that R^{-1} is transitive.

Assume for the sake of contradiction that R^{-1} is not transitive.

Then $\exists x, y, z \in A$ such that $\langle x, y \rangle \in R^{-1}$ and $\langle y, z \rangle \in R^{-1}$ but $\langle x, z \rangle \notin R^{-1}$.

But this implies that $\langle z, y \rangle \in R$ and $\langle y, x \rangle \in R$ but $\langle z, x \rangle \notin R$ which is impossible since it contradicts the fact that R is transitive.

Therefore, R^{-1} must also be transitive.

Since we have shown that R^{-1} is irreflexive and transitive, we have shown that R^{-1} is also a partial ordering, completing the proof, as required. $\square$

Question 3, Page 172

Assume that S is a finite set having n elements. Prove that a linear ordering on S contains $\frac{n(n-1)}{2}$ pairs.

Proof. Let R be a linear ordering on S. Then the elements of S can be labelled and written in a chain as follows.

$$s_1 < s_2 < ... < s_{n-1} < s_n$$

From left to right we see that s_1 is compared with $(n-1)$ other elements of S.

We also see that s_2 is compared with $(n-1)$ elements of S. But since we already counted $s_1 < s_2$ above, we should not double count this comparison. So, s_2 has $(n-2)$ new comparisons not mentioned earlier.

Similarly, s_3 has $(n-3)$ comparisons not mentioned earlier. And s_4 has $(n-4)$ comparisons not mentioned earlier.

Adding up all of the comparisons we have $(n-1)+(n-2)+(n-3)+...+(n-n)$ total comparisons in R.

Now, let $A = (n-1)+(n-2)+(n-3)+...+(n-n)$ be the number of total pairs.

We will manipulate A so that we get the desired expression.

We can rewrite A as $A = 0+1+2+3+...+(n-2)+(n-1)$. Call this version 1.

We can also rewrite A as $A = (n-1)+(n-2)+...+3+2+1+0$. Call this version 2.

Adding up version 1 and version 2 we get that,

$$\begin{aligned} 2A &= \underbrace{(n-1)+(n-1)+...+(n-1)}_{n \text{ times}} \\ &= n(n-1) \end{aligned}$$

Dividing by 2 we get that,

$$A = \frac{n(n-1)}{2}$$

Therefore the number of pairs is indeed $\frac{n(n-1)}{2}$, completing the proof, as required. $\square$

Question 8, Page 182

Required: Prove that the subset axioms are provable from the other axioms.

Subset Axioms: For each formula $\phi(x)$ not containing the letter B, the following is an axiom.

$$\forall t_1, ..., \forall t_k \forall A \exists B \forall x (x \in B \Leftrightarrow x \in A \wedge \phi(x))$$

We will show that the subset axioms follow from the replacement axioms.

Replacement Axioms: For each formula $\psi(x, y)$ not containing the letter B, the following is an axiom.

$$\forall t_1, ..., \forall t_k \forall A [(\forall x \in A) \forall y_1 \forall y_2 (\psi(x, y_1) \wedge \psi(x, y_2) \Rightarrow y_1 = y_2) \Rightarrow \exists B \forall y (y \in B \Leftrightarrow (\exists x \in A)(\psi(x, y))]$$

Proof. Let A be an arbitrary set. Let $\phi(x)$ be a formula that does not contain the letter B.

Note, $\phi(x)$ can also depend on A as well as arbitrary sets $t_1, ..., t_k$ that are not B.

Now, define $\psi(x, y) := x = y \wedge \phi(x)$.

Clearly, $\psi(x, y)$ is defines a function class with a unique y for each x.

Now, $\psi(x, y)$ is suitable and fits the criteria of the axiom of Replacement. Thus, $\psi(x, y)$ defines a function class, and there exists a set B such that,

$$\begin{aligned} B &= \{y | (\exists x \in A) \psi(x, y)\} \\ &= \{y | (\exists x \in A)[x = y \wedge \phi(x)]\} \\ &= \{x \in A | \phi(x)\} \end{aligned}$$

Therefore, we have a set B that contains all the elements of A that satisfies the formula $\phi(x)$.

This is exactly what the subset axioms achieves. Therefore, we have shown that the subset axioms follow from the replacement axioms. $\square$

Question 9, Page 182

Required: Prove that the pairing axiom is provable from the other axioms.

Pairing Axiom: $\forall u \forall v \exists B \forall x (x \in B \Leftrightarrow x = u \vee x = v)$

We will show that the subset axioms follows from the replacement axioms, the power set axiom and the empty set axiom.

Replacement Axioms: For each formula $\psi(x, y)$ not containing the letter B, the following is an axiom.

$\forall t_1, ..., \forall t_k \forall A[(\forall x \in A)\forall y_1 \forall y_2(\psi(x, y_1) \wedge \psi(x, y_2) \Rightarrow y_1 = y_2) \Rightarrow \exists B \forall y(y \in B \Leftrightarrow (\exists x \in A)(\psi(x, y))]$

Power Set Axiom: $\forall a \exists B \forall x (x \in B \Leftrightarrow x \subseteq a)$

Empty Set Axiom: $\exists B \forall x (x \notin B)$

Proof. Let u and v be arbitrary sets.

We know that $\emptyset$ is a set by the empty set axiom.

By the power set axiom, we know that we can create $P(\emptyset) = \{\emptyset\}$. By another application of the power set axiom, we know we can create $P(P(\emptyset)) = \{\emptyset, \{\emptyset\}\}$.

Now, define $\psi(x, y) := (x = \emptyset \wedge y = u) \vee (x = \{\emptyset\} \wedge y = v)$.

Clearly, $\psi(x, y)$ is defines a function class with a unique y for each x.

Now, we will use the replacement axiom on the set $P(P(\emptyset))$ and $\psi(x, y)$.

Note that $\psi(\emptyset, u)$ and $\psi(\{\emptyset\}, v)$ are both satisfied.

Therefore, there exists a set B such that $B = \{u, v\}$.

This is exactly what the pairing axiom achieves. Therefore, we have shown that the pairing axiom follows from the other axioms. $\square$

Question 10, Page 184

For any set S, we can define the relation $\in_S$ by the equation:

$$\in_S = \{\langle x, y\rangle \in S \times S | x \in y\}$$

(a)

Show that for any natural number n, the ϵ-image of $\langle n, \epsilon_n\rangle$ is n.

First consider the function E as described by Enderton on page 182 but on the structure $\langle n, \in_n\rangle$.

We know that $E(t) = \{E(x)|x \in_n t\}$.

So, we have that

$$E(0) = \{E(x)|x \in_n 0\} = \emptyset = 0$$

$$E(1) = \{E(x)|x \in_n 1\} = \{\emptyset\} = 1$$

$$E(2) = \{E(x)|x \in_n 2\} = \{\emptyset, \{\emptyset\}\} = 2$$

If we continue on we see that,

$$E(n-1) = \{E(x)|x \in_n n-1\} = n-1$$

Therefore, we have that $ran(E) = \{1, 2, ..., n-1\} = n$.

Therefore, the ϵ-image of $\langle n, \epsilon_n\rangle$ is n, as required.

b)

Find the ϵ-image of $\langle \omega, \in_\omega\rangle$.

Consider the function E as described by Enderton on page 182 but on the structure $\langle \omega, \in_\omega\rangle$.

We know that $E(t) = \{E(x)|x \in_\omega t\}$.

So, we have that

$$E(0) = \{E(x)|x \in_\omega 0\} = \emptyset = 0$$

$$E(1) = \{E(x)|x \in_\omega 1\} = \{\emptyset\} = 1$$

$$E(2) = \{E(x)|x \in_\omega 2\} = \{\emptyset, \{\emptyset\}\} = 2$$

If we continue on we see that for $n \in \omega$,

$$E(n) = \{E(x)|x \in_\omega n\} = n$$

Therefore, we have that $ran(E) = \{0, 1, 2, ...\} = \omega$.

Therefore, the ϵ-image of $\langle \omega, \epsilon_\omega \rangle$ is ω, as required.

Question 11, Page 184

(a)

Although the set $\mathbb{Z}$ of integers is not well ordered by its normal ordering, show that the ordering

$$0, 1, 2, ..., -1, -2, -3, ...$$

is a well ordering on $\mathbb{Z}$.

First we will show that the above ordering is a linear ordering.

Notation: I will refer to the set of positive integers by $\mathbb{N}$ and the set of negative integers by $\mathbb{Z} \setminus \omega$. However, strictly speaking this is incorrect since the integers are defined as pairs of natural numbers. But this is simply for notation.

Call the above ordering $<$. To show transitivity, we will consider several cases.

Case 1: If $x, y, z \in \omega$, then if $x < y$ and $y < z$, then $x < z$ by the usual ordering of the positive integers (the naturals) which coincides with $<$.

Case 2: If $x, y, z \in \mathbb{Z} \setminus \omega$, then if $x < y$ and $y < z$, then clearly $x < z$ since the ordering $<$ restricted to the negative integers is isomorphic to the usual ordering of $\omega \setminus \{0\}$. Thus, transitivity holds.

Case 3: If $x, y \in \omega$ and $z \in \omega \setminus \{0\}$, then if $x < y$ and $y < z$, then clearly $x < z$ since the positive integers always precede the negative integers in this ordering.

Case 4: If $x \in \omega$ and $y, z \in \omega \setminus \{0\}$ and if $x < y$ and $y < z$, then clearly $x < z$ since the positive integers always precede the negative integers in this ordering.

Therefore, transitivity holds.

Trichotomy trivially holds since we have written all the integers on a chain. Thus, every element in $\mathbb{Z}$ is being compared.

Now, consider any $B \subseteq \mathbb{Z}$ such that $B \neq \emptyset$.

Case 1: If B contains positive integers, then we know that the positive integers are less than all the negative integers in this ordering. But, the positive integers are just the natural numbers which we know is well-ordered by the usual ordering of the naturals, which coincides with $<$. Therefore, a least element exists.

Case 2: If B contains no positive integers, then B only contains negative integers. But we know that $<$ restricted to the negative integers is isomorphic to the usual ordering of $\omega \setminus \{0\}$.

Therefore, a least element exists for every nonempty subset.

Thus, we have shown that $<$ is transitive, satisfies trichotomy and has a least element for every nonempty subset.

Therefore, the ordering $<$ is a well-ordering, as required.

Question 21, Page 195

Required: Prove the following version of Zorn's lemma. Assume that $<$ is a partial ordering on A. Assume that whenever $C \subseteq A$ for which $<^\circ$ is a linear ordering on C, then C has an upper bound in A. Then there exists a maximal element of A.

Proof. Assume for the sake of contradiction that there is no maximal element of A.

By the axiom of choice, for every chain C, we can define a function f such that $f(C)$ produces an element of A that is larger than all the elements in the chain C.

We know that $\emptyset \subseteq A$ must have some upper bound $x_0 \in A$.

So let $C = \{x_0\} \subseteq A$ such that $<^\circ$ is a linear ordering on C and C has an upper bound $f(C) = x_1 \in A$.

So we can write the following chain.

$$x_0 <^\circ x_1$$

We can find a larger element for every element in $C \cup \{x_1\}$ by our function f. i.e. $f(C \cup \{x_1\})$

If we continue on like this we will have a chain that never ends.

$$x_0 <^\circ x_1 <^\circ x_2 <^\circ ... <^\circ x_\omega <^\circ ....$$

.

This will create a chain with an element in A indexed by each ordinal.

Note that $x_\alpha = f(X)$ where $X = \{a_\beta | \beta \in \alpha\}$.

But we know that the collection of ordinals is a proper class. But A is a set. So we have that the elements of the set A are indexed by the proper class of ordinals.

This is a contradiction. Therefore, there must be a maximal element of A. This completes the proof, as required.

□