Exercise 2.1 Page 42

Required: Given the definitions of the defined symbols $\vee$ and $\leftrightarrow$, show that for any PL-interpretation, $\mathcal{I}$, and any wffs ϕ and χ,

1. $V_{\mathcal{I}}(\phi \vee \chi) = 1$ iff either $V_{\mathcal{I}}(\phi) = 1$ or $V_{\mathcal{I}}(\chi) = 1$.

2. $V_{\mathcal{I}}(\phi \leftrightarrow \chi) = 1$ iff $V_{\mathcal{I}}(\phi) = V_{\mathcal{I}}(\chi)$.

Proof of 1

($\Rightarrow$): Assume $V_{\mathcal{I}}(\phi \vee \chi) = 1$.

This is shorthand for saying $V_{\mathcal{I}}(\sim \phi \rightarrow \chi) = 1$.

By definition of valuation, this implies that $V_I(\sim \phi) = 0$ or $V_I(\chi) = 1$.

By definition of valuation, this implies $V_I(\phi) = 1$ or $V_I(\chi) = 1$.

($\Leftarrow$): Assume $V_I(\phi) = 1$ or $V_I(\chi) = 1$.

By definition of valuation, this implies that $V_I(\sim \phi) = 0$ or $V_I(\chi) = 1$.

By definition of valuation, this implies that $V_{\mathcal{I}}(\sim \phi \rightarrow \chi) = 1$.

This is shorthand for saying $V_{\mathcal{I}}(\phi \vee \chi) = 1$.

Therefore, combining our above results, we have proven that $V_{\mathcal{I}}(\phi \vee \chi) = 1$ iff either $V_{\mathcal{I}}(\phi) = 1$ or $V_{\mathcal{I}}(\chi) = 1$.

Proof of 2

($\Rightarrow$): Assume $V_{\mathcal{I}}(\phi \leftrightarrow \chi) = 1$.

This is shorthand for saying $V_{\mathcal{I}}(\sim ((\phi \rightarrow \chi) \rightarrow \sim (\chi \rightarrow \phi))) = 1$.

By definition of valuation, this implies $V_{\mathcal{I}}((\phi \rightarrow \chi) \rightarrow \sim (\chi \rightarrow \phi)) = 0$.

By definition of valuation, this implies $V_{\mathcal{I}}(\phi \rightarrow \chi) = 0$ or $V_{\mathcal{I}}(\sim (\chi \rightarrow \phi)) = 1$.

By definition of valuation, this implies $V_{\mathcal{I}}(\phi \rightarrow \chi) = 0$ or $V_{\mathcal{I}}(\chi \rightarrow \phi) = 0$.

By definition of valuation, this implies that $V_{\mathcal{I}}(\phi) = 0$ or $V_{\mathcal{I}}(\chi) = 1$, and that $V_{\mathcal{I}}(\chi) = 0$ or $V_{\mathcal{I}}(\phi) = 1$.

We have two cases to consider. Either $V_{\mathcal{I}}(\phi) = 1$ or $V_{\mathcal{I}}(\phi) \neq 1$.

Case 1: Consider $V_{\mathcal{I}}(\phi) = 1$. Since $V_{\mathcal{I}}(\phi) = 0$ or $V_{\mathcal{I}}(\chi) = 1$, it must be the case that $V_{\mathcal{I}}(\chi) = 1$. Hence, we have $V_{\mathcal{I}}(\phi) = V_{\mathcal{I}}(\chi) = 1$.

Case 2: Consider $V_{\mathcal{I}}(\phi) \neq 1$. Hence, $V_{\mathcal{I}}(\phi) = 0$. Since $V_{\mathcal{I}}(\chi) = 0$ or $V_{\mathcal{I}}(\phi) = 1$, it must be the case that $V_{\mathcal{I}}(\chi) = 0$. Hence, we have $V_{\mathcal{I}}(\phi) = V_{\mathcal{I}}(\chi) = 0$.

In either case we have $V_{\mathcal{I}}(\phi) = V_{\mathcal{I}}(\chi)$.

$(\Leftarrow)$: Now, assume $V_{\mathcal{I}}(\phi) = V_{\mathcal{I}}(\chi)$.

Either $V_{\mathcal{I}}(\phi) = V_{\mathcal{I}}(\chi) = 1$ or $V_{\mathcal{I}}(\phi) = V_{\mathcal{I}}(\chi) = 0$. We'll consider both cases separately.

Case 1: Consider $V_{\mathcal{I}}(\phi) = V_{\mathcal{I}}(\chi) = 1$. Then, $V_{\mathcal{I}}(\phi \rightarrow \chi) = 1$ and $V_{\mathcal{I}}(\chi \rightarrow \phi) = 1$. This implies that $V_{\mathcal{I}}(\phi \rightarrow \chi) = 1$ and $V_{\mathcal{I}}(\sim (\chi \rightarrow \phi)) = 0$. This implies that $V_{\mathcal{I}}((\phi \rightarrow \chi) \rightarrow \sim (\chi \rightarrow \phi)) = 0$. This implies that $V_{\mathcal{I}}(\sim ((\phi \rightarrow \chi) \rightarrow \sim (\chi \rightarrow \phi))) = 1$. Finally, this is shorthand for saying $V_{\mathcal{I}}(\phi \leftrightarrow \chi) = 1$.

Case 2: Consider $V_{\mathcal{I}}(\phi) = V_{\mathcal{I}}(\chi) = 0$. Then, $V_{\mathcal{I}}(\phi \rightarrow \chi) = 1$ and $V_{\mathcal{I}}(\chi \rightarrow \phi) = 1$. This implies that $V_{\mathcal{I}}(\phi \rightarrow \chi) = 1$ and $V_{\mathcal{I}}(\sim (\chi \rightarrow \phi)) = 0$. This implies that $V_{\mathcal{I}}((\phi \rightarrow \chi) \rightarrow \sim (\chi \rightarrow \phi)) = 0$. This implies that $V_{\mathcal{I}}(\sim ((\phi \rightarrow \chi) \rightarrow \sim (\chi \rightarrow \phi))) = 1$. Finally, this is shorthand for saying $V_{\mathcal{I}}(\phi \leftrightarrow \chi) = 1$.

In either case we have $V_{\mathcal{I}}(\phi \leftrightarrow \chi) = 1$.

Therefore, combining our above results, we've proven that $V_{\mathcal{I}}(\phi \leftrightarrow \chi) = 1$ iff $V_{\mathcal{I}}(\phi) = V_{\mathcal{I}}(\chi)$.

Section 2.3 Page 57

(a) Prove $P \to (Q \to R) \Rightarrow (Q \wedge \sim R) \to \sim P$

1.	$P \Rightarrow P$	RA
2.	$P \to (Q \to R) \Rightarrow P \to (Q \to R)$	RA
3.	$Q \wedge \sim R \Rightarrow Q \wedge \sim R$	RA
4.	$Q \wedge \sim R \Rightarrow Q$	3, $\wedge$E
5.	$P, P \to (Q \to R) \Rightarrow Q \to R$	1,2, $\to$E
6.	$P, P \to (Q \to R), Q \wedge \sim R \Rightarrow R$	4,5, $\to$E
7.	$Q \wedge \sim R \Rightarrow \sim R$	3, $\wedge$E
8.	$P, P \to (Q \to R), Q \wedge \sim R \Rightarrow R \wedge \sim R$	6,7, $\wedge$I
9.	$P \to (Q \to R), Q \wedge \sim R \Rightarrow \sim P$	8, RAA
10.	$P \to (Q \to R) \Rightarrow (Q \wedge \sim R) \to \sim P$	9, $\to$I

(b) Prove $P, Q, R \Rightarrow P$

1.	$P \Rightarrow P$	RA
2.	$Q \Rightarrow Q$	RA
3.	$R \Rightarrow R$	RA
4.	$P, Q \Rightarrow P \wedge Q$	1,2, $\wedge$I
5.	$P, Q \Rightarrow P$	4, $\wedge$E
6.	$P, Q, R \Rightarrow P \wedge R$	3,5, $\wedge$I
7.	$P, Q, R \Rightarrow P$	6, $\wedge$E

(c) Prove $P \to Q, R \to Q \Rightarrow (P \vee R) \to Q$

1.	$P \to Q \Rightarrow P \to Q$	RA
2.	$R \to Q \Rightarrow R \to Q$	RA
3.	$P \Rightarrow P$	RA
4.	$R \Rightarrow R$	RA
5.	$P \to Q, P \Rightarrow Q$	1,3, $\to$E
6.	$R \to Q, R \Rightarrow Q$	2,4, $\to$E
7.	$P \vee R \Rightarrow P \vee R$	RA
8.	$P \vee R, P \to Q, R \to Q \Rightarrow Q$	5,6,7, $\vee$E
9.	$P \to Q, R \to Q \Rightarrow (P \vee R) \to Q$	8, $\to$I

Exercise 2.4 Page 62

(a)

Prove $\vdash P \to P$

1.	$P \to ((P \to P) \to P)$	PL1
2.	$(P \to ((P \to P) \to P)) \to ((P \to (P \to P)) \to (P \to P))$	PL2
3.	$P \to (P \to P)$	PL1
4.	$(P \to (P \to P)) \to (P \to P)$	1,2 MP
5.	$P \to P$	3,4 MP

(b)

Prove $\vdash (\sim P \to P) \to P$

1.	$\sim P \to ((\sim P \to\sim P) \to\sim P)$	PL1
2.	$(\sim P \to ((\sim P \to\sim P) \to\sim P)) \to ((\sim P \to (\sim P \to\sim P)) \to (\sim P \to\sim P))$	PL2
3.	$\sim P \to (\sim P \to\sim P)$	PL1
4.	$(\sim P \to (\sim P \to\sim P)) \to (\sim P \to\sim P)$	1,2 MP
5.	$\sim P \to\sim P$	3,4 MP
6.	$(\sim P \to\sim P) \to (\sim P \to P) \to P$	PL3
7.	$(\sim P \to P) \to P$	5,6 MP

(c)

Prove $\sim\sim P \vdash P$

1.	$\sim\sim P \to (\sim P \to\sim\sim P)$	PL1
2.	$\sim\sim P$	premise
3.	$\sim P \to\sim\sim P$	1,2 MP
4.	$(\sim P \to\sim\sim P) \to ((\sim P \to\sim P) \to P)$	PL3
5.	$(\sim P \to\sim P) \to P$	3,4 MP
6.	$\sim P \to ((\sim P \to\sim P) \to\sim P)$	PL1
7.	$(\sim P \to ((\sim P \to\sim P) \to\sim P)) \to ((\sim P \to (\sim P \to\sim P)) \to (\sim P \to\sim P))$	PL2
8.	$\sim P \to (\sim P \to\sim P)$	PL1
9.	$(\sim P \to (\sim P \to\sim P)) \to (\sim P \to\sim P)$	6,7 MP
10.	$\sim P \to\sim P$	8,9 MP
11.	P	5,10 MP

Exercise 2.7 Page 70

Required: Show by induction that the truth value of a wff depends only on the truth values of its sentence letters. That is, show that for any wff ϕ and any PL-interpretations $\mathcal{I}$ and $\mathcal{I}'$, if $\mathcal{I}(\alpha) = \mathcal{I}'(\alpha)$ for each sentence letter α in ϕ, then $V_{\mathcal{I}}(\phi) = V_{\mathcal{I}'}(\phi)$.

Assume $\mathcal{I}(\alpha) = \mathcal{I}'(\alpha)$ for each sentence letter α in any wff ϕ.

Claim: $V_{\mathcal{I}}(\phi) = V_{\mathcal{I}'}(\phi)$.

Proof. Proof by induction on the complexity of wffs.

Base Case: Consider the wff P for some sentence letter P. We know that $\mathcal{I}(P) = \mathcal{I}'(P)$ by assumption. Hence,

$$V_{\mathcal{I}}(P) = \mathcal{I}(P) = \mathcal{I}'(P) = V_{\mathcal{I}'}(P)$$

Inductive Hypothesis: For wffs ϕ and ψ, assume $V_{\mathcal{I}}(\phi) = V_{\mathcal{I}'}(\phi)$ and $V_{\mathcal{I}}(\psi) = V_{\mathcal{I}'}(\psi)$.

Show: $V_{\mathcal{I}}(\sim \phi) = V_{\mathcal{I}'}(\sim \phi)$.

By inductive hypothesis, either $V_{\mathcal{I}}(\phi) = V_{\mathcal{I}'}(\phi) = 1$ or $V_{\mathcal{I}}(\phi) = V_{\mathcal{I}'}(\phi) = 0$.

Case 1: If $V_{\mathcal{I}}(\phi) = V_{\mathcal{I}'}(\phi) = 1$, then clearly $V_{\mathcal{I}}(\sim \phi) = V_{\mathcal{I}'}(\sim \phi) = 0$.

Case 2: If $V_{\mathcal{I}}(\phi) = V_{\mathcal{I}'}(\phi) = 0$, then clearly $V_{\mathcal{I}}(\sim \phi) = V_{\mathcal{I}'}(\sim \phi) = 1$.

In either case, we have $V_{\mathcal{I}}(\sim \phi) = V_{\mathcal{I}'}(\sim \phi)$.

Show: $V_{\mathcal{I}}(\phi \to \psi) = V_{\mathcal{I}'}(\phi \to \psi)$.

By inductive hypothesis, we know $V_{\mathcal{I}}(\phi) = V_{\mathcal{I}'}(\phi)$ and $V_{\mathcal{I}}(\psi) = V_{\mathcal{I}'}(\psi)$.

Either $V_{\mathcal{I}}(\phi) = V_{\mathcal{I}'}(\phi) = 0$ or $V_{\mathcal{I}}(\phi) = V_{\mathcal{I}'}(\phi) = 1$.

Case 1: If $V_{\mathcal{I}}(\phi) = V_{\mathcal{I}'}(\phi) = 0$, then we have $V_{\mathcal{I}}(\phi \to \psi) = V_{\mathcal{I}'}(\phi \to \psi) = 1$.

Case 2: If $V_{\mathcal{I}}(\phi) = V_{\mathcal{I}'}(\phi) = 1$, then we must consider two subcases. If $V_{\mathcal{I}}(\psi) = V_{\mathcal{I}'}(\psi) = 1$, then since $V_{\mathcal{I}}(\phi) = V_{\mathcal{I}'}(\phi) = 1$, clearly we have $V_{\mathcal{I}}(\phi \to \psi) = V_{\mathcal{I}'}(\phi \to \psi) = 1$. If $V_{\mathcal{I}}(\psi) = V_{\mathcal{I}'}(\psi) = 0$, then since $V_{\mathcal{I}}(\phi) = V_{\mathcal{I}'}(\phi) = 1$, clearly we have $V_{\mathcal{I}}(\phi \to \psi) = V_{\mathcal{I}'}(\phi \to \psi) = 0$. Regardless, we have $V_{\mathcal{I}}(\phi \to \psi) = V_{\mathcal{I}'}(\phi \to \psi)$.

In either case, we have $V_{\mathcal{I}}(\phi \to \psi) = V_{\mathcal{I}'}(\phi \to \psi)$.

And Sider only officially includes $\{\sim, \to\}$ as logical connectives. The other connectives are interdefined using these two logical connectives. Therefore, by induction on the complexity of wffs, we have proven the **Claim**. $\square$

Exercise 2.8 Page 70

Required: Suppose that a wff ϕ has no repetitions of sentence letters (i.e., each sentence letter occurs at most once in ϕ.). Show that ϕ is not PL-valid.

We will prove a stronger claim first. Consider the following definition.

Definition: A wff ϕ is considered **contingent** if there is an interpretation $\mathcal{I}$ such that $V_{\mathcal{I}}(\phi) = 1$ and there is an interpretation $\mathcal{I}'$ such that $V_{\mathcal{I}'}(\phi) = 0$.

Now we will prove the following claim.

Claim: For each wff ϕ, if ϕ has no repetitions of sentence letters, then ϕ is contingent.

Proof. Proof by induction on the complexity of wffs.

Base Case: For a wff P where P is a sentence letter, then let $\mathcal{I}$ be an interpretation such that $\mathcal{I}(P) = 1$, and let $\mathcal{I}'$ be an interpretation such that $\mathcal{I}'(P) = 0$.

Hence, $V_{\mathcal{I}}(P) = \mathcal{I}(P) = 1$.

Hence, $V_{\mathcal{I}'}(P) = \mathcal{I}'(P) = 0$.

Therefore, P is contingent.

Inductive Hypothesis: Assume ϕ is a wff with no repetitions of sentence letters and ϕ is contingent. Assume ψ is a wff with no repetitions of sentence letters and ψ is contingent.

Show: $\sim \phi$ is contingent where $\sim \phi$ has no repeated sentence letters.

Since $\sim \phi$ has no repeated sentence letters, we know that ϕ has no repeated sentence letters.

By inductive hypothesis, we know that there exists interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ such that $V_{\mathcal{I}_1}(\phi) = 1$ and $V_{\mathcal{I}_2}(\phi) = 0$.

Hence, $V_{\mathcal{I}_1}(\sim \phi) = 0$ and $V_{\mathcal{I}_2}(\sim \phi) = 1$.

Hence, $\sim \phi$ is contingent.

Show: $\phi \to \psi$ is contingent where $\phi \to \psi$ has no repeated sentence letters.

Since $\phi \to \psi$ has no repeated sentence letters, we have that ϕ and ψ each have no repeated sentence letters within themselves.

By inductive hypothesis, there exists interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ such that $V_{\mathcal{I}_1}(\phi) = 1$ and $V_{\mathcal{I}_2}(\phi) = 0$. And, there exists interpretations $\mathcal{I}_3$ and $\mathcal{I}_4$ such that $V_{\mathcal{I}_3}(\psi) = 1$ and $V_{\mathcal{I}_4}(\psi) = 0$.

Let SL_ϕ be the set of sentence letters in ϕ. Let SL_ψ be the set of sentence letters in ψ. Since we know that $\phi \to \psi$ has no repetitions of sentence letters, we know that $SL_\phi \cap SL_\psi = \emptyset$.

Consider the following interpretation $\mathcal{J}$.

$$\mathcal{J}(P) = \begin{cases} \mathcal{I}_1(P) & \text{if } P \in SL_\phi \\ \mathcal{I}_3(P) & \text{if } P \in SL_\psi \end{cases}$$

Hence, $V_\mathcal{J}(\phi) = V_{\mathcal{I}_1}(\phi) = 1$.

Hence, $V_\mathcal{J}(\psi) = V_{\mathcal{I}_3}(\psi) = 1$.

Therefore, $V_\mathcal{J}(\phi \to \psi) = 1$.

Consider the following interpretation $\mathcal{J}'$.

$$\mathcal{J}'(P) = \begin{cases} \mathcal{I}_1(P) & \text{if } P \in SL_\phi \\ \mathcal{I}_4(P) & \text{if } P \in SL_\psi \end{cases}$$

Hence, $V_{\mathcal{J}'}(\phi) = V_{\mathcal{I}_1}(\phi) = 1$.

Hence, $V_{\mathcal{J}'}(\psi) = V_{\mathcal{I}_4}(\psi) = 0$.

Therefore, $V_{\mathcal{J}'}(\phi \to \psi) = 0$.

Notice, the interpretation J is such that $V_\mathcal{J}(\phi \to \psi) = 1$ and the interpretation J' is such that $V_{\mathcal{J}'}(\phi \to \psi) = 0$. Hence, $\phi \to \psi$ is contingent.

Therefore, by induction on the complexity of formulas, we have shown that every wff without repetitions of sentence letters is contingent.

The **Claim** we just proved says that every wff ϕ without repetitions of sentence letters is contingent. So let ϕ be a wff without repetitions of sentence letters. Hence, there is an interpretation $\mathcal{I}$ such that $V_\mathcal{I}(\phi) = 1$ and an interpretation $\mathcal{I}'$ such that $V_{\mathcal{I}'}(\phi) = 0$. In particular, since $V_{\mathcal{I}'}(\phi) = 0$, we have that ϕ is not PL-valid.

Therefore, we've proven that every wff ϕ without repetitions of sentence letters is not PL-valid. This completes the proof, as required. $\square$

Extra Problem

Required: Prove that $\{\rightarrow, \vee\}$ is not an adequate set of sentential connectives, that is to say prove that there exists a truth-function of two variables $f : \{0,1\}^2 \rightarrow \{0,1\}$ that cannot be expressed as the truth-function of any wff constructed with just the conditional and the disjunction.

Note: For a truth function $f : \{0,1\}^2 \rightarrow \{0,1\}$, we let the first entry represent the truth value of P, and the second entry represent the truth value of Q.

First we will prove the following claim.

Claim: Any wff of two sentence letters P and Q using connectives in the set $\{\rightarrow, \vee\}$ expresses a truth function $f : \{0,1\}^2 \rightarrow \{0,1\}$ such that $f(\langle 1,1\rangle) = 1$.

Proof. Proof by induction on the complexity of wffs of sentences containing sentence letters P and Q and using connectives in the set $\{\rightarrow, \vee\}$.

Base Case: Consider the case of atomic sentences. Either our wff is P or it is Q.

If our wff is P, then its truth function f satisfies $f(\langle 1,1\rangle) = 1$ since P has truth value 1.

If our wff is Q, then its truth function f satisfies $f(\langle 1,1\rangle) = 1$ since Q has truth value 1.

Inductive Hypothesis: Assume ϕ and ψ are wffs of the sentence letters P and Q such that the truth function f_ϕ that represents ϕ is such that $f_\phi(\langle 1,1\rangle) = 1$ and the truth function f_ψ that represents ψ is such that $f_\psi(\langle 1,1\rangle) = 1$.

Show: The truth function g_1 that expresses $\phi \rightarrow \psi$ is such that $g_1(\langle 1,1\rangle) = 1$.

Consider $g_1(\langle 1,1\rangle)$ which expresses $\phi \rightarrow \psi$. This implies that P is assigned the truth value 1 and Q is assigned the truth value 1.

Hence, by inductive hypothesis, we have that $f_\phi(\langle 1,1\rangle) = 1$ and $f_\psi(\langle 1,1\rangle) = 1$.

Hence, when P and Q each have truth value 1, we have that ϕ has truth value 1 and ψ has truth value 1. Hence, when P and Q each have truth value 1, we have that $\phi \rightarrow \psi$ has truth value 1.

Therefore, $g_1(\langle 1,1\rangle) = 1$.

Show: The truth function g_2 that expresses $\phi \vee \psi$ is such that $g_2(\langle 1,1\rangle) = 1$.

Consider $g_2(\langle 1,1\rangle)$ which expresses $\phi \vee \psi$. This implies that P is assigned the truth value 1 and Q is assigned the truth value 1.

Hence, by inductive hypothesis, we have that $f_\phi(\langle 1,1\rangle) = 1$ and $f_\psi(\langle 1,1\rangle) = 1$.

Hence, when P and Q each have truth value 1, we have that ϕ has truth value 1 and ψ has truth value 1. Hence, when P and Q each have truth value 1, we have that $\phi \vee \psi$ has truth value 1.

Therefore, $g_2(\langle 1,1\rangle) = 1$.

By induction on the complexity of wff using sentence letters P and Q and connectives in $\{\rightarrow, \vee\}$, we have proven our **Claim**.

$\square$

Show $\{\rightarrow, \vee\}$ not adequate

Now, consider the following truth function $f' : \{0,1\}^2 \rightarrow \{0,1\}$ defined by $f'(\langle x,y\rangle) = 0$ for all $\langle x,y\rangle \in \{0,1\}^2$. i.e. f' is the truth function that always maps elements of its domain to 0.

In particular, $f'(\langle 1,1\rangle) = 0$.

And by our **Claim** we know that any wff ϕ made up of sentence letters P and Q and the connectives $\{\rightarrow, \vee\}$ is such that the truth function f that represents ϕ satisfies $f(\langle 1,1\rangle) = 1$.

Therefore, there is no wff ϕ using P, Q and the connectives $\{\rightarrow, \vee\}$ that could possibly represent the truth function $f' : \{0,1\}^2 \rightarrow \{0,1\}$ defined by $f'(\langle x,y\rangle) = 0$ for all $\langle x,y\rangle \in \{0,1\}^2$ since we have $f'(\langle 1,1\rangle) = 0 \neq 1$.

This shows that $\{\rightarrow, \vee\}$ is not an adequate set of connectives, as required.