Exercise 1

Claim: For every $n \in \mathbb{N}$, $h(S^n(p)) = n$.

Proof. We will prove the claim by induction on n.

Base Case: $n = 1$

We want to show $h(S(p)) = 1$. This was done in the booklet which we will restate. Consider the following.

$$\begin{aligned} h(p) &= h(I^*(\mathbf{c})) && \text{Since } p = I^*(\mathbf{c}) \\ &= I(\mathbf{c}) && \text{Since } h \text{ is a homomorphism} \\ &= 0 \end{aligned}$$

Hence,

$$\begin{aligned} h(S(p)) &= h(I^*(\mathbf{f})(p)) && \text{Since } I^*(\mathbf{f}) = S \\ &= I(\mathbf{f})(h(p)) && \text{Since } h \text{ is a homomorphism} \\ &= I(\mathbf{f})(0) && \text{Since } h(p) = 0 \\ &= \mathit{successor}(0) && \text{Since } I(\mathbf{f}) = \mathit{successor} \\ &= 1 \end{aligned}$$

This completes the base case.

IH: $h(S^n(p)) = n$
Show: $h(S^{n+1}(p)) = n + 1$

Consider the following.

$$\begin{aligned} h(S^{n+1}(p)) &= h(S(S^n(p))) \\ &= h(I^*(\mathbf{f})(S^n(p))) && \text{Since } I^*(\mathbf{f}) = S \\ &= I(\mathbf{f})(h(S^n(p))) && \text{Since } h \text{ is a homomorphism} \\ &= I(\mathbf{f})(n) && \text{By } \textbf{IH} \\ &= \mathit{successor}(n) && \text{Since } I(\mathbf{f}) = \mathit{successor} \\ &= n + 1 \end{aligned}$$

This completes the proof, as required. □

Exercise 3

Suppose that the language L has no equals sign, one unary predicate symbol F, one unary function symbol f, and no other predicate symbols, function symbols or constant symbols.

Conjecture: For every model M for L, there is a finite model M' such that M' is elementarily equivalent to M.

We will show that the **Conjecture** is false.

Proof. Let $M = \langle D, I \rangle$ be the following model.

$D = \mathbb{N}$.

$I(F) = \{x \in \mathbb{N} : \exists y \in \mathbb{N}, x = 2^y\}$.

$I(f)(n) = n + 1$ for every $n \in D$.

We want to show that there is no finite model M' such that $M \equiv M'$.

Assume for the sake of contradiction that there is a finite model $M' = \langle D', I' \rangle$ where $card(D') = k$ for some $k \in \mathbb{N}$ such that $M \equiv M'$.

Note that $2 \in I(F)$. Hence, $I(F) \neq \emptyset$. Hence, $M \models \exists x Fx$. Since $M \equiv M'$, we have that $M' \models \exists x Fx$. Hence, $I'(F) \neq \emptyset$. So we will not have issues involving empty sets.

We know that the gaps between the powers of two within the natural numbers become arbitrarily large.

Hence, for some $j > k$, there exists a sequence of j-many consecutive natural numbers that are not powers of two.

Hence, for some $s \in \mathbb{N}$, we have that $s \in I(F)$, but $s + 1, s + 2, ..., s + j \notin I(F)$, and $s + j + 1 \in I(F)$. In other words, we have a power of two, followed by j many consecutive natural numbers that are not powers of two, followed by another power of two.

Notation 1: For any term t, let $\mathbf{f^0}t$ be t itself. And define $\mathbf{f^{n+1}}t$ to be $\mathbf{ff^n}t$.

Hence,

$$M \models \exists \mathbf{x}\big(\mathbf{Fx}\left(\bigwedge_{\mathbf{n=1}}^{\mathbf{j}} \sim \mathbf{Ff^n x}\right) \wedge \mathbf{F^{j+1}x}\big)$$

Note, the above sentence is written informally. Since there are only finitely many conjunctions and finite indices, the sentence is well-formed (but written informally).

Since $M \equiv M'$, we have that,

$$M' \models \exists \mathbf{x}\big(\mathbf{Fx}\Big(\bigwedge_{\mathbf{n=1}}^{\mathbf{j}} \sim \mathbf{Ff^n x}\Big) \wedge \mathbf{F^{j+1}x}\big) \tag{1}$$

Notation 2: Let $S = I'(f)$. For any $d \in D'$, let $S^0(d) = d$. And define $S^{n+1}(d) = S(S^n(d))$. Notice that $S^1(d) = S(d)$.

Hence, by (1), there exists $d \in D'$ such that $d \in I'(F)$, but $S^1(d), S^2(d), ..., S^j(d) \notin I'(F)$ and $S^{j+1}(d) \in I'(F)$.

We want to show that $S^{j+1}(d) \notin I'(F)$ which will contradict $S^{j+1}(d) \in I'(F)$.

Let $A = \{S^1(d), S^2(d), ..., S^j(d)\}$. Notice that for all $x \in A$, we have that $x \notin I'(F)$.

Recall that $j > k$ where $k = card(D')$. Hence, $j > card(D')$.

Hence, we know that at least two of $S^1(d), S^2(d), ..., S^j(d) \in A$ are equal.

Hence, for some $p, q \in \mathbb{N}$ such that $1 \leq p < q \leq j$, we have that $S^p(d) = S^q(d)$.

Consider the set $B = \{S^p(d), S^{p+1}(d), ..., S^{q-1}(d)\} \subseteq A$. We will prove the following Lemma.

Lemma 1: B is a closed set under applications of S. i.e. For every $x \in B$, we have $S(x) \in B$.

Let $x \in B$. Hence, $x = S^r(d)$ for some $r \in \{p, p+1, ..., q-1\}$.

Case 1: If $r \in \{p, p+1, ..., q-2\}$, then clearly $S(x) = S(S^r(d)) = S^{r+1}(d) \in B$ since $r+1 \in \{p+1, p+2, ..., q-1\}$.

Case 2: If $r = q-1$, then $S(x) = S(S^r(d)) = S^{r+1}(d) = S^{(q-1)+1} = S^q(d) = S^p(d) \in B$.

In either case, $S(x) \in B$, proving **Lemma 1**.

Lemma 2: For every $x \in B$ and every $n \geq 1$, we have $S^n(x) \in B$.

Let $x \in B$. We will use induction on n.

Base Case: $n = 1$. Clearly $S^1(x) = S(x) \in B$ by **Lemma 1**.
IH: $S^n(x) \in B$
Show: $S^{n+1}(x) \in B$

We know $S^n(x) \in B$ by **IH**. Hence, by **Lemma 1**, we have that $S(S^n(x)) = S^{n+1}(x) \in B$, proving **Lemma 2**.

Lemma 3: For every $m, n \geq 1$, we have that $S^n(S^m(d)) = S^{m+n}(d)$.

Let $m \in \mathbb{N}$ such that $m \geq 1$ be arbitrary. We will use induction on n.

Base Case: $n = 1$. Clearly $S(S^m(d)) = S^{m+1}(d)$ by **Notation 2**.
IH: $S^n(S^m(d)) = S^{m+n}(d)$.
Show: $S^{n+1}(S^m(d)) = S^{m+(n+1)}(d)$

Consider the following.

$$\begin{aligned} S^{n+1}(S^m(d)) &= S(S^n(S^m(d)) && \text{By } \textbf{Notation 2} \\ &= S(S^{m+n}(d)) && \text{By } \textbf{IH} \\ &= S^{(m+n)+1}(d) && \text{By } \textbf{Notation 2} \\ &= S^{m+(n+1)}(d) \end{aligned}$$

This completes the proof of **Lemma 3**.

Since $S^{q-1}(d) \in B$, we have that $S(S^{q-1}(d)) = S^q \in B$ by **Lemma 1**.

We know that $q \leq j$. Hence, $q < j + 1$. Hence, $j + 1 - q > 0$. Hence, $j + 1 - q \geq 1$.

Since $S^q(d) \in B$, by **Lemma 2** we have that $S^{(j+1-q)}(S^q(d)) \in B$.

By **Lemma 3**, $S^{(j+1-q)}(S^q(d)) = S^{q+(j+1-q)}(d)$. Hence, $S^{q+(j+1-q)}(d) \in B$,

And clearly $q + (j + 1 - q) = j + 1$. Hence, $S^{j+1}(d) \in B$.

Since $S^{j+1}(d) \in B$ and $B \subseteq A$, we have that $S^{j+1}(d) \in A$.

But we know that for all $x \in A$, we have that $x \notin I'(F)$.

Hence, $S^{j+1}(d) \notin I'(F)$. But this contradicts our earlier result that $S^{j+1}(d) \in I'(F)$.

Therefore, our assumption was wrong and there is no finite model M' for L such that $M \equiv M'$.

Therefore, the conjecture is false, completing the proof, as required.

$\square$

Exercise 4

Suppose that the language L has the equals sign, one binary predicate P, and no other predicate symbols, function symbols, or constant symbols. Let $M = \langle \mathbb{Q}, I \rangle$, where $I(P) =<_{\mathbb{Q}}$ and $M' = \langle \mathbb{R}, I' \rangle$, where $I'(P) =<_{\mathbb{R}}$. Sometimes we write these as $M = (\mathbb{Q}, <)$ and $M' = (\mathbb{R}, <)$. Note that M and M' are not isomorphic.

Required: Show that M and M' are elementarily equivalent.

Let Γ be the the set of sentences of DLOWE on page 70 of the booklet. Note that the sentences in the booklet have a binary predicate R instead of P. So let Γ be the set of sentences of DLOWE that replaces every instance of R with P.

And we know that $M \models \Gamma$ and $M' \models \Gamma$.

First we will prove the following Lemma.

Lemma: Γ is complete. i.e. For every $\phi \in Sent_L$, either $\Gamma \models \phi$ or $\Gamma \models \sim \phi$.

Proof. Assume for the sake of contradiction that there exists a $\phi \in Sent_L$ such that $\Gamma \not\models \phi$ and $\Gamma \not\models \sim \phi$.

Hence, there exists a model M_0 for L such that $M_0 \models \Gamma$ and $M_0 \not\models \phi$.

And, there exists a model M_1 for L such that $M_1 \models \Gamma$ and $M_1 \not\models \sim \phi$.

Equivalently, we have that $M_0 \models \Gamma$ and $M_0 \models \sim \phi$ and we have that $M_1 \models \Gamma$ and $M_1 \models \phi$.

Now, by the Downward Lowhenheim Skolem Theorem, there exists a countable model M_0' such that $M_0' \models \Gamma$ and $M_0' \models \sim \phi$.

And, by the Downward Lowhenheim Skolem Theorem, there exists a countable model M_1' such that $M_1' \models \Gamma$ and $M_1' \models \phi$.

Notice that we have $M_0' \models \sim \phi$ and $M_1' \models \phi$.

Equivalently, $M_0' \not\models \phi$ and $M_1' \models \phi$.

By Theorem 7.4.2 we know that any two countable models of DLOWE are isomorphic. Since M_0' and M_1' are countable models, and $M_0' \models \Gamma$ and $M_1' \models \Gamma$, we have that $M_0' \approx M_1'$.

By the Isomorphism Theorem (6.2.6), since $M_0' \approx M_1'$, we have that $M_0' \equiv M_1'$.

This means that for any $\sigma \in Sent_L$, we have $M_0' \models \sigma$ iff $M_1' \models \sigma$.

But we have that $M_0' \not\models \phi$ and $M_1' \models \phi$ which is a contradiction. Therefore, our initial assumption was wrong.

This completes the proof of the Lemma. $\square$

Proof that $M \equiv M'$

Proof. We know that $M \models \Gamma$ and $M' \models \Gamma$.

We want to show that for all $\phi \in Sent_L$, we have $M \models \phi \Leftrightarrow M' \models \phi$.

Note: We're just using the symbol $\Leftrightarrow$ as a shorthand for 'iff' in the metalanguage. And $\Rightarrow$ is just 'if' etc.

Let $\phi \in Sent_L$.

Show: $M \models \phi \Leftrightarrow M' \models \phi$.

By our **Lemma**, we know that either $\Gamma \models \phi$ or $\Gamma \models \sim \phi$. We will consider both cases.

Case 1: $\Gamma \models \phi$

($\Rightarrow$): Assume $M \models \phi$. Since $\Gamma \models \phi$ and $M' \models \Gamma$, we have that $M' \models \phi$.

($\Leftarrow$): Assume $M' \models \phi$. Since $\Gamma \models \phi$ and $M \models \Gamma$, we have that $M \models \phi$.

Case 2: $\Gamma \models \sim \phi$

($\Rightarrow$): We will show this by the contrapositive. Assume $M' \not\models \phi$. Since $\Gamma \models \sim \phi$ and $M \models \Gamma$, we have that $M \models \sim \phi$. Hence, $M \not\models \phi$.

($\Leftarrow$): We will show this by the contrapositive. Assume $M \not\models \phi$. Since $\Gamma \models \sim \phi$ and $M' \models \Gamma$, we have that $M' \models \sim \phi$. Hence, $M' \not\models \phi$.

Hence, in either case we have demonstrated the **Show** line.

Therefore, we have proven that for all $\phi \in Sent_L$, we have $M \models \phi \Leftrightarrow M' \models \phi$.

Hence, $M \equiv M'$. i.e. M and M' are elementarily equivalent, as required. $\square$

Exercise 5

Suppose that the language L has the equals sign and one binary predicate $\mathbf{P}$, and no other predicate symbols, function symbols, or constants. Let $M = \langle \mathbb{N}, I \rangle$, where $I(P) = <_{\mathbb{N}}$. Sometimes we write this as $M = \langle \mathbb{N}, < \rangle$.

ϕ_0 representing $\{0\}$

Required: Write down a formula ϕ_0 that represents $\{0\}$ with one free variable $\mathbf{v_1}$.

Note: We will be using infix notation.

Let ϕ_0 be the following formula with free variable $\mathbf{v_1}$.

$$\forall \mathbf{v_3}(\mathbf{v_1} \neq \mathbf{v_3} \rightarrow \mathbf{v_1 P v_3})$$

ϕ_1 representing $\{1\}$

Required: Write down a formula ϕ_1 that represents $\{1\}$ with one free variable $\mathbf{v_1}$.

Note: We will use free variable $\mathbf{v_2}$ as opposed to free variable $\mathbf{v_1}$ so that we could use our formula ϕ_0 and not have issues with scope. Our formula will still represent $\{1\}$. We will also be using infix notation.

Let ϕ_1 be the following formula with free variable $\mathbf{v_2}$.

$$\exists \mathbf{v_1}(\phi_0 \wedge \mathbf{v_1} \neq \mathbf{v_2} \wedge \forall \mathbf{v_4}(\mathbf{v_1} \neq \mathbf{v_4} \wedge \mathbf{v_2} \neq \mathbf{v_4} \rightarrow \mathbf{v_2 P v_4}))$$

Note: In our formula ϕ_1 above we used free variable $\mathbf{v_2}$ instead of $\mathbf{v_1}$ so that we could use our formula ϕ_0 which also contains free variable $\mathbf{v_1}$. This seemed like the intention of the exercise where we use previous formulas to define new formulas. This also seems necessary for Assignment 6 that was mentioned at the end of the handout for Assignment 5.

If we insist on using free variable $\mathbf{v_1}$ in ϕ_1, then we cannot use our original ϕ_0 and instead have to relabel the variables in ϕ_0.

i.e. If we **insist** on using free variable $\mathbf{v_1}$ in ϕ_1, then we define ϕ_1 and ϕ_0 as follows.

Let ϕ_0 be the formula $\forall \mathbf{v_3}(\mathbf{v_2} \neq \mathbf{v_3} \rightarrow \mathbf{v_2 P v_3})$ with free variable $\mathbf{v_2}$.

Then, let ϕ_1 be the following formula with free variable $\mathbf{v_1}$.

$$\exists \mathbf{v_2}(\phi_0 \wedge \mathbf{v_2} \neq \mathbf{v_1} \wedge \forall \mathbf{v_4}(\mathbf{v_2} \neq \mathbf{v_4} \wedge \mathbf{v_1} \neq \mathbf{v_4} \rightarrow \mathbf{v_1 P v_4}))$$