Exercise 1

Let L be a language with identity and two unary predicates F and G. For any model $M = \langle D, I \rangle$ for L, we say that F dominates G in M iff there is a one-to-one function f from $I(G)$ into $I(F)$, but not vice versa. Informally, this means that, in M, there are more F's than G's. Let K_{dom} be the following class of models for L: $K_{dom} = \{M : F \text{ dominates } G\}$.

Required: Show that K_{dom} is not EC_Δ.

Proof. Assume for the sake of contradiction that K_{dom} is EC_Δ.

Then there is a set $\Sigma \subseteq Sent_L$ such that for any model M of L, we have $M \models \Sigma$ iff F dominates G in M.

Let L' be a language just like L except that L' has a countably infinite set $C = \{c_1, ..., c_n, ...\}$ of constants.

Let $\Gamma_n = \{\sim = c_i c_j : i, j \in \{1, ..., n\} \text{ and } i \neq j\}$.

Let $\Delta_n = \{Gc_i : i \in \{1, ..., n\}\}$.

Finally, let $\Gamma = \bigcup_n \Gamma_n$ and let $\Delta = \bigcup_n \Delta_n$.

Step 1

Show: $\Sigma \cup \Gamma \cup \Delta$ is satisfiable in the language L'.

Let $\Omega \subseteq \Sigma \cup \Gamma \cup \Delta$ be finite.

Hence, for some $n \geq 1$ we have that $\Omega \subseteq \Sigma \cup \Gamma_n \cup \Delta_n$.

Let $M_0 = \langle D_0, I_0 \rangle$ be a model for L' defined as follows.

$D_0 = \{1, ..., n, n+1\}$.

$I_0(c_i) = i$ for each $i \in \{1, ..., n\}$.

$I_0(c_j) = 1$ for each $j > n$.

$I_0(G) = \{1, ..., n\}$.

$I_0(F) = \{1, ..., n, n+1\}$.

We know that $f : I_0(G) \to I_0(F)$ defined by $f(x) = x$ is one-to-one since if $f(x) = f(y)$, then $x = y$.

We will show there is no one-to-one function $g : I_0(F) \rightarrow I_0(G)$. If there were such a g, we know that $g(1), ..., g(n), g(n+1) \in I(G)$. But since $card(I(G)) = n$, we know there exists $i, j \in \{1, ..., n, n+1\}$ such that $g(i) = g(j)$ but $i \neq j$ which shows that g is not one-to-one. Hence, there cannot exist such a g.

Hence, F dominates G in M_0. Hence, $M_0 \models \Sigma$.

To show $M_0 \models \Gamma_n$, let $\sim = c_i c_j \in \Gamma_n$ be arbitrary for some $i, j \in \{1, ..., n\}$ such that $i \neq j$.

Then, $M_0 \models \sim = c_i c_j$ iff $I_0(c_i) \neq I_0(c_j)$.

And we know that $I_0(c_i) = i \neq j = I_0(c_j)$ since $i, j \in \{1, ..., n\}$ and $i \neq j$.

Hence, $M_0 \models \sim = c_i c_j$. Since $\sim = c_i c_j \in \Gamma_n$ was arbitrary, we have that $M_0 \models \Gamma_n$.

To show $M_0 \models \Delta_n$, let $Gc_i \in \Delta_n$ be arbitrary for some $i \in \{1, ..., n\}$.

Then, $M_0 \models Gc_i$ iff $I_0(c_i) \in I_0(G)$. And we know that $I_0(c_i) = i \in I_0(G)$ since $i \in \{1, ..., n\}$.

Hence, $M_0 \models Gc_i$. Since $Gc_i \in \Delta_n$ was arbitrary, we have that $M_0 \models \Delta_n$.

Therefore, we have that $M_0 \models \Sigma \cup \Gamma_n \cup \Delta_n$.

Since $\Omega \subseteq \Sigma \cup \Gamma_n \cup \Delta_n$, we have that $M_0 \models \Omega$.

Since $\Omega \subseteq \Sigma \cup \Gamma \cup \Delta$ was an arbitrary finite subset and is satisfiable, by the Compactness Theorem we have that $\Sigma \cup \Gamma \cup \Delta$ is satisfiable.

Step 2

In Step 1 we showed that $\Sigma \cup \Gamma \cup \Delta$ was satisfiable. Hence, there exists a model $M_1 = \langle D_1, I_1 \rangle$ for L' such that $M_1 \models \Sigma \cup \Gamma \cup \Delta$.

Note that $I_1(G)$ is infinite since $M_1 \models \Gamma \cup \Delta$. Hence, $card(D_1) \geq card(I_1(G)) \geq \aleph_0$ since $I_1(G)$ could possibly be uncountably infinite.

Hence, $card(D_1) \geq \aleph_0$. Now we have two cases to consider.

Case 1: If $card(D_1) = \aleph_0$, then simply let $M_2 = \langle D_2, I_2 \rangle = \langle D_1, I_1 \rangle = M_1$. Hence, we have that $M_2 \models \Sigma \cup \Gamma \cup \Delta$ and $card(D_2) = card(D_1) = \aleph_0$.

Case 2: If $card(D_1) > \aleph_0$, then since L' is of cardinality $\aleph_0$ and $M_1 \models \Sigma \cup \Gamma \cup \Delta$, by the Generalized Downward Lowhenheim-Skolem Theorem there exists a model $M_2 = \langle D_2, I_2 \rangle$ for L' such that $M_2 \models \Sigma \cup \Gamma \cup \Delta$ and $card(D_2) = \aleph_0$.

Note: We considered two cases since the version of the Generalized Downward Lowhenheim-Skolem Theorem (5.3.11) stated on page 51 **only applies to strictly greater cardinalities**, as opposed to "greater than or equal to". So we considered two cases for accuracy sake.

In either Case 1 and 2 above, we have a model $M_2 = \langle D_2, I_2 \rangle$ for L' such that $M_2 \models \Sigma \cup \Gamma \cup \Delta$ and $card(D_2) = \aleph_0$. And since $M_2 \models \Gamma \cup \Delta$, we have $I_2(G)$ is infinite. Hence, $\aleph_0 \leq card(I_2(G)) \leq card(D_2) = \aleph_0$. Hence, $card(I_2(G)) = \aleph_0$.

Now, consider the model $M_3 = \langle D_3, I_3 \rangle$ of the original language L such that $D_3 = D_2$ and $I_3(G) = I_2(G)$ and $I_3(F) = I_2(F)$.

Notice that M_2 is an expansion of M_3. By Theorem 2.4.5, we know that for each $\phi \in Sent_L$, we have $Val_{M_2}(\phi) = Val_{M_3}(\phi)$.

Recall that $\Sigma \subseteq Sent_L$. Since $M_2 \models \Sigma$, we have that $M_2 \models \sigma$ for each $\sigma \in \Sigma$. Hence, $Val_{M_2}(\sigma) = 1$ for each $\sigma \in \Sigma$.

Hence, by Theorem 2.4.5 we have that $Val_{M_3}(\sigma) = 1$ for each $\sigma \in \Sigma$. Hence, $M_3 \models \sigma$ for each $\sigma \in \Sigma$. Hence, $M_3 \models \Sigma$.

Now, notice that $M_3 \models \Sigma$ and $card(I_3(G)) = card(I_2(G)) = \aleph_0$.

Since F dominates G in M_3, we know that

$$\aleph_0 = card(I_3(G)) \leq card(I_3(F)) \tag{1}$$

And we know we must have

$$card(I_3(F)) \leq card(D_3) = \aleph_0 \tag{2}$$

Therefore, combining (1) and (2) we get,

$$\aleph_0 \leq card(I_3(F)) \leq \aleph_0$$

Therefore, we have that,

$$card(I_3(F)) = \aleph_0$$

But notice we have that $card(I_3(G)) = card(I_3(F)) = \aleph_0$. We know we can enumuerate both $I_3(G)$ and $I_3(F)$ to get a bijection $k : I_3(F) \to I_3(G)$. In particular, this means that $k : I_3(F) \to I_3(G)$ is one-to-one which implies that F does not dominate G in M_3.

Therefore, $M_3 \not\models \Sigma$ which contradicts our earlier result that $M_3 \models \Sigma$. Therefore, our initial assumption was wrong and K_{dom} is not EC_Δ, completing the proof, as required. □

Exercise 2

Let L be a first-order language with identity and one binary relation symbol R. Suppose that $M = \langle D, I \rangle$ is a finite model for L. Suppose $M' = \langle D', I' \rangle$ is another model for L such that $M \equiv M'$.

Required: Show that $M \approx M'$

Proof. We will restate some of the relevant parts of the given information in the problem.

Summary of Given Information

Suppose $D = \{d_1, ..., d_n\}$ where each $d_1, ..., d_n$ are distinct.

Let β_n be the sentence in Section 7.1 such that $M \models \beta_n$ iff M has exactly n members.

For any two $i, j \in \{1, ..., n\}$, let $\psi_{i,j}$ be defined as follows.

If $\langle d_i, d_j \rangle \in I(\mathbf{R})$, then $\psi_{i,j}$ is the formula $\mathbf{R v_1 v_j}$

If $\langle d_i, d_j \rangle \notin I(\mathbf{R})$, then $\psi_{i,j}$ is the formula $\sim \mathbf{R v_1 v_j}$

Let ϕ be the formula:

$$\exists \mathbf{v_1}, ..., \exists \mathbf{v_n} (\bigwedge_{\mathbf{i \neq j}} \mathbf{v_i} \neq \mathbf{v_j} \wedge \bigwedge_{\mathbf{i,j}} \psi_{\mathbf{i,j}})$$

And we have $M \models \beta_n \wedge \phi$. Since $M \equiv M$, we have $M' \models \beta_n \wedge \phi$. Hence, $M' \models \phi$.

Let $s : Vble \to D'$. So $M' \models \phi[s]$. So there are $e_1, ..., e_n \in D'$ such that

$$M' \models \bigwedge_{\mathbf{i \neq j}} \mathbf{v_i} \neq \mathbf{v_j} \wedge \bigwedge_{\mathbf{i,j}} \psi_{\mathbf{i,j}} \, [\mathbf{s^{e_1 ... e_n}_{v_1 ... v_n}}]$$

If $i \neq j$, then $M' \models \mathbf{v_i} \neq \mathbf{v_j} \, [s^{e_1 ... e_n}_{v_1 ... v_n}]$ in which case $e_i \neq e_j$. So $e_1, ..., e_n$ are all distinct.

Since $M \models \beta_n$, the domain D' has exactly n members. So $D' = \{e_1, ..., e_n\}$.

Defining an Isomorphism

The problem asks for an isomorphism from M' onto M.

Instead, we will construct an isomorphism from M onto M' for convenience. This will still show that $M \approx M'$.

Let $h : D \to D'$ be defined as $h(d_i) = e_i$ for each $i \in \{1, ..., n\}$.

One-to-one: Assume $d_i \neq d_j$. Then $f(d_i) = e_i \neq e_j = f(d_j)$ since each $e_1, ..., e_n$ are distinct. This shows that h is one-to-one.

Note, we used the contrapositive of the usual definition of a function being one-to-one above.

Onto: Assume $e_i \in D'$. Then choose $d_i \in D$. Hence, $h(d_i) = e_i$ which shows h is onto.

Now we will check that h is a homomorphism. Since L only has one relation symbol, we only have to show the following.

Show:$\langle d_i, d_j \rangle \in I(\mathbf{R})$ iff $\langle h(d_i), h(d_j) \rangle \in I'(\mathbf{R})$

($\Rightarrow$): Assume $\langle d_i, d_j \rangle \in I(\mathbf{R})$. We want to show $\langle h(d_i), h(d_j) \rangle \in I'(\mathbf{R})$.

Since $\langle d_i, d_j \rangle \in I(\mathbf{R})$, we know that the formula $\psi_{i,j}$ is the formula $\mathbf{Rv_iv_j}$.

We know from our earlier work that $M' \models \phi\, [s^{e_1 \ldots e_n}_{v_1 \ldots v_n}]$. Hence, $M' \models \psi_{i,j}\, [s^{e_1 \ldots e_n}_{v_1 \ldots v_n}]$.

Hence, $M' \models \mathbf{Rv_iv_j}\, [s^{e_1 \ldots e_n}_{v_1 \ldots v_n}]$. And we know $M' \models \mathbf{Rv_iv_j}\, [s^{e_1 \ldots e_n}_{v_1 \ldots v_n}]$ iff $\langle e_i, e_j \rangle \in I'(\mathbf{R})$.

Hence, $\langle e_i, e_j \rangle \in I'(\mathbf{R})$. But we know that $h(d_i) = e_i$ and $h(d_j) = e_j$.

Hence, $\langle h(d_i), h(d_j) \rangle \in I'(\mathbf{R})$.

($\Leftarrow$): We will prove this direction by the contrapositive.

Assume $\langle d_i, d_j \rangle \notin I(\mathbf{R})$. We want to show $\langle h(d_i), h(d_j) \rangle \notin I'(\mathbf{R})$.

Since $\langle d_i, d_j \rangle \notin I(\mathbf{R})$, we know that the formula $\psi_{i,j}$ is the formula $\sim \mathbf{Rv_iv_j}$.

We know from our earlier work that $M' \models \phi\, [s^{e_1 \ldots e_n}_{v_1 \ldots v_n}]$. Hence, $M' \models \psi_{i,j}\, [s^{e_1 \ldots e_n}_{v_1 \ldots v_n}]$.

Hence, $M' \models \sim \mathbf{Rv_iv_j}\, [s^{e_1 \ldots e_n}_{v_1 \ldots v_n}]$. And we know $M' \models \sim \mathbf{Rv_iv_j}\, [s^{e_1 \ldots e_n}_{v_1 \ldots v_n}]$ iff $\langle e_i, e_j \rangle \notin I'(\mathbf{R})$.

Hence, $\langle e_i, e_j \rangle \notin I'(\mathbf{R})$. But we know that $h(d_i) = e_i$ and $h(d_j) = e_j$.

Hence, $\langle h(d_i), h(d_j) \rangle \notin I'(\mathbf{R})$.

Therefore, h is a homomorphism. Since we showed earlier that h is one-to-one and onto, we have that h is an isomorphism from M onto M'.

Therefore, $M \approx M'$, completing the proof, as required. $\square$