Question 18, Page 158

Required: Prove that the following is equivalent to the axiom of choice:

Statement 1: For any set A whose members are nonempty sets, there is a function f with domain A such that $f(X) \in X$ for all $X \in A$

Proof. ($\Rightarrow$) Assume statement 1. We will show that the statement implies a version of the axiom of choice.

i.e. Let A be an arbitrary set.

Let $B = P(A) - \{\emptyset\}$. Thus, B is a set whose members are nonempty sets.

By Statement 1, there exists a function f with domain B such that $f(X) \in X$ for all $X \in B$.

This is equivalent to saying there exists a function f with domain being the set of all nonempty subsets of A such that $f(X) \in X$ for all nonempty $X \subseteq A$.

This is exactly the axiom of choice version (3) on page 151 of Enderton. i.e. that there exists a choice function.

($\Leftarrow$)
Assume the axiom of choice version (2) on page 151 of Enderton.

i.e. Let A be a set of nonempty sets. Then, by the axiom of choice version (2), the cartesian product of the elements in A is also non-empty.

i.e. $\Pi_{X \in A} X \neq \emptyset$

By definition of Cartesian product, there exists a function $f : A \to \bigcup A$ such that $f(X) \in X$ for all $X \in A$, where A is a set of nonempty sets.

This is exactly Statement 1 that we needed to prove.

Therefore Statement 1 is equivalent to the axiom of choice.

Question 19, Page 158

Assume that H is a function with finite domain I and that $H(i) \neq \emptyset$ for each $i \in I$.

Required: Without using axiom of choice, show that Statement 1 holds. Use induction on card(I).

Statement 1: There is a function f with domain I such that $f(i) \in H(i)$ for each $i \in I$.

Notation: We will write $|I|$ instead of card(I) for simplicity.

Let $T = \{n \in \omega | n = 0 \vee n = |I| \text{ and Statement 1 holds}\}$

So $0 \in T$.

If $|I| = 1$, then there exists exactly one $i \in I$. So $H(i) \neq \emptyset$.

Let $x \in H(i)$. Let $f_1(i) = x \in H(i)$.

So $1 \in T$.

Assume that $k \in T$ for $k \in \omega$.

Now consider $k^+ \in \omega$ such that $|I| = k^+$.

Let $j \in I$ be arbitrary.

Now consider $I - \{\mathrm{j}\}$. So $|I - \{\mathrm{j}\}| = k$.

Since $k \in T$, there exists a function f_k with domain $I - \{\mathrm{j}\}$ and $|I - \{\mathrm{j}\}| = k$ such that $f_k(i) \in H(i)$ for each $i \in I$.

Now consider any $z \in H(j)$.

Define $f_{k^+} = f_k \cup \{< j, z >\}$.

So, for all $i \in I$ we have $f_{k^+}(i) \in H(i)$.

Therefore, $k^+ \in T$.

Therefore, $T = \omega$ by induction.

This completes the proof, as required. $\square$

Question 20, Page 158

Assume that A is a nonempty set and R is a relation such that $(\forall x \in A)(\exists y \in A)yRx$.

Required: Show that there is a function $f : \omega \to A$ with $f(n^+)Rf(n)$ for all $n \in \omega$.

Case 1: If there is an $a \in A$ such that aRa, then let $f(n) = a$ for all $n \in \omega$.

Note, aRa and $f(n^+) = a$ and $f(n) = a$ for all $n \in \omega$.

It follows that $f(n^+)Rf(n)$ for all $n \in \omega$.

Case 2: If there is no $a \in A$ such that aRa, then by the axiom of choice let $g : A \to A$ be a function such that $g \subseteq R$ and $dom(g) = dom(R)$.

Similarly, by the axiom of choice, let $h : A \to A$ be a function such that that $h \subseteq R^{-1}$ and $dom(h) = dom(R^{-1})$

Now, define a function $i : A \to A$ such that $i(a) = g(h(a))$.

Clearly, $i(a)Ra$ for each $a \in A$

Now, let $x \in A$ be fixed.

By the Recursion Theorem, there exists a unique function $f : \omega \to A$ such that $f(0) = x$ and $f(n^+) = i(f(n))$ for all $n \in \omega$.

Since $i(a)Ra$ for each $a \in A$, it follows that $f(n^+)Rf(n)$ for all $n \in \omega$.

Question 21, Page 158

Assume that A is a nonempty set such that for every set B,

$$B \in A \Leftrightarrow \text{every finite subset of B is a member of } A$$

Required: Show that A has a maximal element.

Let $B \subseteq A$ be a chain.

Let $D \subseteq \bigcup B$ be finite.

Enuermate the elements of D such that $D = \{d_1, ..., d_n\}$.

Since $D \subseteq \bigcup B$, there exists sets $B_1, ..., B_n \in B$, not necessarily all distinct, such that $d_i \in B_i$ for each $i \in \{1, .., n\}$.

But since B is a chain, we have a set B_j, where $j \in \{1, ..., n\}$ that contains all other B_i where $i \in \{1, ..., n\}$.

So, $d_1, .., d_n \in B_j$.

So $D \subseteq B_j$.

But $B_j \in B$ and $B \subseteq A$. So $B_j \in A$.

Since $B_j \in A$, by (1), every finite subset of B_j is a member of A.

Therefore, $D \in A$.

Since a finite $D \subseteq \bigcup B$ was arbitrary, we have that every finite subset of $\bigcup B$ is an element of A.

By (1), we have that $\bigcup B \in A$.

Thus, we have shown that for every chain $B \subseteq A$,
we have that $\bigcup B \in A$.

Therefore, by applying Zorn's Lemma, A contains a maximal element.

Question 32, Page 165

Let $F(A)$ be the collection of all finite subsets of A.

Required: Show that if A is infinite, then $A \approx F(A)$.

First consider the following function $f : A \to F(A)$ defined by $f(x) = \{x\}$.

i.e. f maps an element of A to the singleton containing that element.

Clearly f is injective. Let $x, y \in A$.

$$f(x) = f(y) \Rightarrow \{x\} = \{y\} \Rightarrow x = y$$

Therefore, we have that $A \preccurlyeq F(A)$

Now, we will show that $card(F(A)) \leq card(A)$ which would show that $F(A) \preccurlyeq A$.

Notation: For simplicity, we will write $card(X)$ as $|X|$.

First we will partition the set $F(A)$.

Let $F(A)_n$ be the set of finite subsets of A of cardinality $n \in \omega$.

Therefore, $F(A) = \bigcup_{n\in\omega} F(A)_n$

Notice that $|F(A)_n| \leq |A|^n$ since we can consider any subset of A of size n to be a selection of n elements of A. If we consider our selections to allow for repetitions, we can bound $|F(A)_n|$ by $|A|^n$. Call this Fact 1.

Now, consider the following.

$$\begin{aligned}
|F(A)| &= \Big|\bigcup_{n\in\omega} F(A)_n\Big| \\
&= \sum_{n\in\omega} |F(A)_n| && \text{Since each } F(A)_n \text{ is disjoint for } n \in \omega \\
&\leq \sum_{n\in\omega} |A|^n && \text{By Fact 1} \\
&= \sum_{n\in\omega} |A| && \text{By } n \text{ applications of Lemma 6R since } A \text{ is infinite} \\
&= \aleph_0 |A| && \text{Since we have a countable summation over } \omega \\
&= \max(\aleph_0, |A|) && \text{By absorption law} \\
&= |A| && \text{Since } A \text{ is infinite and } \aleph_0 \leq K \text{ for any infinite cardinal } K \text{ by Thm 6N}
\end{aligned}$$

Therefore, we have that $card(F(A)) \leq card(A)$.

Thus, we also know that $F(A) \preccurlyeq A$

Since $A \preccurlyeq F(A)$ and $F(A) \preccurlyeq A$, by Cantor-Schroder-Bernstein Theorem, we have that $A \approx F(A)$, as required.

Question 33, Page 165

Assume that A is an infinite set. Prove that $A \approx Sq(A)$.

Proof. First we will show that $A \preccurlyeq Sq(A)$.

Consider the following injective function $f : A \to Sq(A)$

For any $x \in A$, define $f(x) = \{< 0, x >\}$.

i.e. f maps any element x in A to the function (sequence) which maps 0 to x.

Clearly, for any $x, y \in A$,

$$\begin{aligned} f(x) = f(y) &\Rightarrow \{< 0, x >\} = \{< 0, y >\} \\ &\Rightarrow < 0, x > = < 0, y > \\ &\Rightarrow x = y \end{aligned}$$

So f is injective. Therefore, $A \preccurlyeq Sq(A)$

Now we will show that $card(Sq(A)) \leq card(A)$ which would show that $Sq(A) \preccurlyeq A$

Notation: For simplicity, we will write $card(X)$ as $|X|$.

We know that $Sq(A) = {}^0A \cup {}^1A \cup {}^2A \cup ... = \bigcup_{n\in\omega} {}^nA$. It follows that,

$$\begin{aligned} |Sq(A)| &= \left| \bigcup_{n\in\omega} {}^nA \right| \\ &= \sum_{n\in\omega} |{}^nA| && \text{Since each } {}^nA \text{ is disjoint} \\ &= \sum_{n\in\omega} |A|^n && \text{Since } |{}^nA| = |A|^n \\ &= \sum_{n\in\omega} |A| && \text{By } n \text{ applications of Lemma 6R since } A \text{ is infinite} \\ &= \aleph_0 |A| && \text{Since we have a countable summation over } \omega \\ &= \max(\aleph_0, |A|) && \text{By absorption law} \\ &= |A| && \text{Since } A \text{ is infinite and } \aleph_0 \leq K \text{ for any infinite cardinal } K \text{ by Thm 6N} \end{aligned}$$

Therefore, $card(Sq(A)) \leq card(A)$.

Thus, $Sq(A) \preccurlyeq A$.

Since $A \preccurlyeq Sq(A)$ and $Sq(A) \preccurlyeq A$, by Cantor-Schroder-Bernstein Theorem, we have that $A \approx Sq(A)$, as required. $\square$

Question 34, Page 165

Assume that $2 \leq \kappa \leq \lambda$. Prove that $\kappa^\lambda = 2^\lambda$

Proof. Since $2 \leq \kappa$, by Theorem 6L, we have that $2^\lambda \leq \kappa^\lambda$

Now, we must show that $\kappa^\lambda \leq 2^\lambda$. Consider the following.

$$
\begin{aligned}
\kappa &\leq 2^\kappa && \text{Obvious fact of cardinals} \\
\kappa^\lambda &\leq (2^\kappa)^\lambda && \text{By Theorem 6L, since } \kappa \leq 2^\kappa \\
&= 2^{\kappa\cdot\lambda} && \text{By Theorem 6I} \\
&= 2^\lambda && \text{Since } \kappa \leq \lambda \text{, so by absorption law, } \kappa \cdot \lambda = \lambda
\end{aligned}
$$

Therefore, $k^\lambda \leq 2^\lambda$.

Since $2^\lambda \leq \kappa^\lambda$ and $k^\lambda \leq 2^\lambda$, we have that $\kappa^\lambda = 2^\lambda$.

This completes the proof, as required. $\square$

Question 35, Page 165

Required: Find a collection A of $2^{\aleph_0}$ sets of natural numbers such that any two distinct members of A have finite intersection. Suggestion: Start with the collection of infinite set of primes.

Let $\mathbb{P}$ be the infinite set of prime numbers. We know that this set is well-ordered.

Consider any subset $Q \subseteq \mathbb{P}$.

Q could either be finite or infinite.

Regardless, we can enumerate the elements by $<$ such that $Q = \{p_1, p_2, p_3, ...\}$.

Now consider the function $f : P(\mathbb{P}) \to P(\mathbb{P})$ defined by,

$f(Q) = \{p_1, p_1p_2, p_1p_2p_3, ...\}$.

So, f takes an ordered subset of $\mathbb{P}$ and multiplies the ith element of A by each of the elements with indices less than i.

Now, consider the following set A.

$A = \{f(Q) \subseteq \mathbb{N} | Q \subseteq \mathbb{P}\}$.

Now consider any two elements $X, Y \in A$ such that $X \neq Y$.

By the fundamental theorem of arithmetic, we know that prime factorizations are unique.

Therefore, if we decompose each of the elements of X and Y into prime factors and compare them, we know that the first prime p_i that does not appear in the factorizations of an element in both X and Y changes the sequence of (possibly) infinite primes in the rest of the set X and Y.

Thus, X and Y can only have finitely many elements in common.

But A simply applies the function f to each subset of $\mathbb{P}$. But we know that $card(\mathbb{P}) = \aleph_0$. And $card(P(\mathbb{P})) = 2^{\aleph_0}$. So A must also have cardinality $2^{\aleph_0}$.

Therefore, A is a collection of $2^{\aleph_0}$ sets of natural numbers such that any two distinct members of A have finite intersection, as required.

Question 5, Page 178

Assume that $<$ is a well ordering on A and that $f : A \to A$ satisfies the condition,

$x < y \Rightarrow f(x) < f(y)$ for all $x, y \in A$. Call this condition 1.

Required: Prove that $x \leq f(x)$ for all $x \in A$.

Suggestion: Consider $f(f(x))$

Proof. Assume for the sake of contradiction that there exists a $z \in A$ such that $f(z) < z$.

Now let $B = \{y \in A | f(y) < y\}$

We know B is nonempty since $z \in B$.

Since $B \subseteq A$, there exists a least element $x \in B$.

So $f(x) < x$.

Now, as the suggestion says, let us apply f again using condition 1 .

i.e. $f(f(x)) < f(x)$

Therefore $f(x) \in B$.

Since x is the least element of B, we have that $x \leq f(x)$.

So we have that $f(x) < x$ and $x \leq f(x)$.

Therefore, we have a contradiction. Therefore, our assumption that there exists a $z \in A$ such that $f(z) < z$ was wrong.

Therefore, $x \leq f(x)$ for all $x \in A$, completing the proof, as required. $\square$

Question 7, Page 178

Let C be some fixed set. Apply transfinite recursion to ω (with its usual well ordering), using for $\gamma(x, y)$ the formula,

$$y = C \cup \bigcup\bigcup ran(x)$$

Let F be the γ-constructed function on ω.

(a)

Calculate $F(0), F(1)$, and $F(2)$. Make a good guess as to what $F(n)$ is.

$$\begin{aligned} F(0) &= C \cup \bigcup\bigcup ran(F \upharpoonright seg(0)) \\ &= C \cup \bigcup\bigcup ran(\emptyset) \\ &= C \cup \bigcup\bigcup \emptyset \\ &= C \end{aligned}$$

$$\begin{aligned} F(1) &= C \cup \bigcup\bigcup ran(F \upharpoonright seg(1)) \\ &= C \cup \bigcup\bigcup ran(\{< 0, F(0) >\}) \\ &= C \cup \bigcup\bigcup \{F(0)\} \end{aligned}$$

$$\begin{aligned} F(2) &= C \cup \bigcup\bigcup ran(F \upharpoonright seg(2)) \\ &= C \cup \bigcup\bigcup ran(\{< 0, F(0) >, < 1, F(1) >\}) \\ &= C \cup \bigcup\bigcup \{F(0), F(1)\} \end{aligned}$$

So, we can guess that,

$$\begin{aligned} F(n) &= C \cup \bigcup\bigcup ran(F \upharpoonright seg(n)) \\ &= C \cup \bigcup\bigcup ran(\{< 0, F(0) >, < 1, F(1) >, ..., < n-1, F(n-1) >\}) \\ &= C \cup \bigcup\bigcup \{F(0), F(1), ..., F(n-1)\} \end{aligned}$$

(b)

Show that if $a \in F(n)$, then $a \subseteq F(n^+)$.

Assume $a \in F(n) = C \cup \bigcup \bigcup \{F(0), F(1), ..., F(n-1)\}$.

Now, consider $F(n^+) = C \cup \bigcup \bigcup \{F(0), F(1), ..., F(n-1), F(n)\}$

Clearly, $a \subseteq F(n^+)$.