Question 1

Suppose that Σ is a set of sentences such that at least one sentence from Σ is true in each $\mathcal{L}$-structure.

Required: Show that there exists finitely many $\sigma_1, ..., \sigma_n \in \Sigma$ such that the sentence $\sigma_1 \vee ... \vee \sigma_n$ is valid.

Proof. Assume for sake of contradiction that every finitely many $\sigma_1, ..., \sigma_n \in \Sigma$ is such that the corresponding sentence $\sigma_1 \vee ... \vee \sigma_n$ is not valid.

i.e. For every finite set of sentences $\sigma_1, ..., \sigma_n \in \Sigma$, there exists an $\mathcal{L}$-structure $\mathfrak{A}$ such that $\mathfrak{A} \not\models (\sigma_1 \vee ... \vee \sigma_n)$.

Let $\sigma_1, ..., \sigma_n \in \Sigma$ be arbitrary, and let $\mathfrak{A}$ be the corresponding structure such that $\mathfrak{A} \not\models (\sigma_1 \vee ... \vee \sigma_n)$.

It follows that,

$$\begin{aligned}
\mathfrak{A} \not\models (\sigma_1 \vee ... \vee \sigma_n) &\Rightarrow \mathfrak{A} \models \neg(\sigma_1 \vee ... \vee \sigma_n) \\
&\Rightarrow \mathfrak{A} \models (\neg\sigma_1 \wedge ... \wedge \neg\sigma_n) \\
&\Rightarrow \mathfrak{A} \models \neg\sigma_1 \quad \text{and...and} \quad \mathfrak{A} \models \neg\sigma_n
\end{aligned}$$

Let $\Gamma_n = \{\neg\sigma_1, ...\neg\sigma_n\}$. Clearly we have that $\mathfrak{A} \models \Gamma_n$ from the lines above.

Now, let $\Gamma = \{\neg\sigma : \sigma \in \Sigma\}$. We have that $\forall n \in \mathbb{N}$, $\Gamma_n \subseteq \Gamma$ where Γ_n is a set containing any n sentences from Γ.

From the work above, we have shown that every Γ_n has some corresponding model $\mathfrak{A}$. Thus, every finite subset $\Gamma_n \subseteq \Gamma$ has a model. By the Compactness Theorem, this implies that Γ has a model.

i.e. there exists an $\mathcal{L}$-structure $\mathfrak{B}$ such that $\mathfrak{B} \models \Gamma$.

But, this implies that $\mathfrak{B} \models \neg\sigma$ for each $\neg\sigma \in \Gamma$.

And this implies that $\mathfrak{B} \not\models \sigma$ for each $\sigma \in \Sigma$.

But, we supposed that there was at least one sentence in Σ that is true in each $\mathcal{L}$-structure.

But, there is no sentence in Σ that is true in $\mathfrak{B}$ which is a contradiction.

This completes the proof, as required. $\square$

Question 2

a)

Required: Show that if α or β is a sentence, then the implication $\mathfrak{A} \models \alpha \vee \beta \Rightarrow (\mathfrak{A} \models \alpha$ or $\mathfrak{A} \models \beta)$ holds for all structures $\mathfrak{A}$.

Proof. Without loss of generality, assume that α is a sentence. The case when β is a sentence, and the case when both α, β are sentences are similar.

Fix a structure $\mathfrak{A}$. Let A be the universe of $\mathfrak{A}$.

If $\mathfrak{A} \not\models \alpha \vee \beta$, then we're done. So, assume that $\mathfrak{A} \models \alpha \vee \beta$.

Required to Prove: $(\mathfrak{A} \models \alpha$ or $\mathfrak{A} \models \beta)$

From assumption, we have that, $\mathfrak{A} \models \alpha \vee \beta[s]$, for all $s : Vars \to A$.

By definition, this implies that $\mathfrak{A} \models \alpha[s]$ or $\mathfrak{A} \models \beta[s]$, for all $s : Vars \to A$.

Since α is a sentence, there are no free variables in α, which means that $\alpha[s] \equiv \alpha[t]$, for all $s, t : Vars \to A$.

Thus, we have that $\mathfrak{A} \models \alpha[t]$ for all t or $\mathfrak{A} \models \beta[s]$ for all s.

Finally, by definition, we have that $(\mathfrak{A} \models \alpha$ or $\mathfrak{A} \models \beta)$, completing the proof, as required. $\square$

Note, the case where β is a sentence has the exact same proof as above, with the role of α and β reversed.

The case where both α and β are sentences is similar to the first two cases. However, since there are no free variables in α, β, we have that $\alpha[s] \equiv \alpha[t]$ and $\beta[s] \equiv \beta[u]$, for all $s, t, u : Vars \to A$.

b)

Required: Write down an example of formulas α and β such that $\mathfrak{A} \models \alpha \vee \beta \not\Rightarrow (\mathfrak{A} \models \alpha$ or $\mathfrak{A} \models \beta)$.

For simplicity, consider the language $\mathcal{L}$ with a single 1-ary relation symbol P.

Consider the structure $\mathfrak{A} = (A, P)$ where $A = \{0, 1\}$ and $P^{\mathfrak{A}} = \{0\}$. i.e. only the element 0 stands in relation to P.

Define $\alpha :\equiv P(x)$ and $\beta :\equiv \neg P(x)$.

We will show that for our α, β, the hypothesis of the implication holds, but the conclusion does not.

We must show that for any variable assignment function $s : Vars \to A$, that $\mathfrak{A} \models \alpha \vee \beta[s]$ holds. i.e. that $(\mathfrak{A} \models \alpha[s]$ or $\mathfrak{A} \models \beta[s])$ holds.

There are only two possible variable assignment functions s. Either $s(x) = 0$ or $s(x) = 1$.

If $s(x) = 0$, then we have that $\mathfrak{A} \models \alpha[s]$ holds, which implies that $(\mathfrak{A} \models \alpha[s]$ or $\mathfrak{A} \models \beta[s])$ holds, which implies that $\mathfrak{A} \models \alpha \vee \beta[s]$ holds.

If $s(x) = 1$, then we have that $\mathfrak{A} \models \beta[s]$ holds, which implies that $(\mathfrak{A} \models \alpha[s]$ or $\mathfrak{A} \models \beta[s])$ holds, which implies that $\mathfrak{A} \models \alpha \vee \beta[s]$ holds.

However, we will show that the conclusion $(\mathfrak{A} \models \alpha$ or $\mathfrak{A} \models \beta)$ does not hold.

i.e. We will show that $(\mathfrak{A} \models \alpha[s_1]$ or $\mathfrak{A} \models \beta[s_2])$ does not hold for all $s_1, s_2 : Vars \to A$.

i.e. We will show that there exists $s_1, s_2 : Vars \to A$ such that $(\mathfrak{A} \not\models \alpha[s_1]$ and $\mathfrak{A} \not\models \beta[s_2])$.

If $s_1(x) = 1$, then $\mathfrak{A} \not\models \alpha[s_1]$.

If $s_2(x) = 0$, then $\mathfrak{A} \not\models \beta[s_2]$.

So there exists s_1, s_2 such that $(\mathfrak{A} \not\models \alpha[s_1]$ and $\mathfrak{A} \not\models \beta[s_2])$.

So we have shown that $(\mathfrak{A} \models \alpha$ or $\mathfrak{A} \models \beta)$ does not hold.

Therefore, we have proven that $\mathfrak{A} \models \alpha \vee \beta \not\Rightarrow (\mathfrak{A} \models \alpha$ or $\mathfrak{A} \models \beta)$ for our α and β.

Question 3

a)

Required: Define an isomorphism i other than the identity map $i(q) = q$ to show that $(\mathbb{Q}, 0, 1, <) \cong (\mathbb{Q}, 0, 1, <)$.

Let $i : \mathbb{Q} \to \mathbb{Q}$ be defined as

$$i(q) = \begin{cases} q & \text{if } q \geq 0 \\ \frac{1}{2}q & \text{if } q < 0 \end{cases}$$

We will first show that i is injective and surjective, and thus bijective.

Injectivity: Assume $i(q) = i(r)$, where $q, r \in \mathbb{Q}$. We have 2 cases.

Case 1: $q, r \geq 0$

$$\begin{aligned} i(q) &= i(r) \\ q &= q \end{aligned}$$

Therefore, i is injective in case 1.

Case 2: $q, r < 0$

$$\begin{aligned} i(q) &= i(r) \\ \frac{1}{2}q &= \frac{1}{2}r \\ q &= r \end{aligned}$$

Therefore, i is injective in case 2.

Surjectivity: Let $r \in \mathbb{Q}$. We will show that always exists a $q \in \mathbb{Q}$ such that $i(q) = r$.

Case 1: For $r \geq 0$, let $q = r$. Clearly $i(q) = i(r) = r$.

Case 2: For $r < 0$, let $q = 2r$. Clearly $i(q) = i(2r) = \frac{1}{2}(2r) = r$.

Therefore, i is surjective.

Since i is injective and surjective, i is bijective.

Now, we will show that $(\mathbb{Q}, 0, 1, <) \cong (\mathbb{Q}, 0, 1, <)$ with isomorphism i.

For notation, call $\mathfrak{A} = (\mathbb{Q}, 0, 1, <)$. So we will show that $\mathfrak{A} \cong \mathfrak{A}$ with isomorphism i.

Constants 0 and 1

Clearly $i(0^{\mathfrak{A}}) = 0^{\mathfrak{A}}$ and $i(1^{\mathfrak{A}}) = 1^{\mathfrak{A}}$.

Relation $<$

We must now show that for all $x, y \in \mathbb{Q}$, we have $x <^{\mathfrak{A}} y$ if and only if $i(x) <^{\mathfrak{A}} i(y)$.

We will consider 3 cases, and for brevity, we will write $<$ instead of $<^{\mathfrak{A}}$.

Case 1: $x, y \geq 0$

We have that $x < y$ if and only if $i(x) = x < y = i(y)$ is trivial as i is identity on this interval.

Case 2: $x, y < 0$

Clearly, $x < y$ if and only if $i(x) = \frac{1}{2}x < \frac{1}{2}y = i(y)$ is trivial, as multiplying or dividing by 2, which is positive, does not affect linear ordering. So both directions hold.

Case 3: $x < 0$ and $y \geq 0$

Clearly, $x < y$ always holds, and $i(x) = x < 2y = i(y)$ trivially holds, since the value of a positive y is simply multiplied by a positive 2 when applying i, and thus linear order is still preserved.

We have checked all conditions of isomorphism.

Therefore, we have that $(\mathbb{Q}, 0, 1, <) \cong (\mathbb{Q}, 0, 1, <)$ with isomorphism i.

b)

There DOES NOT exist an isomorphism from $(\mathbb{Z}, 0, 1, <)$ to $(\mathbb{Q}, 0, 1, <)$ because there is no bijective function $i : \mathbb{Z} \to \mathbb{Q}$ that preserves order.

Assume for contradiction that there was an isomorphism i from $(\mathbb{Z}, 0, 1, <)$ to $(\mathbb{Q}, 0, 1, <)$.

For notation, call $\mathfrak{A} = (\mathbb{Z}, 0, 1, <)$ and $\mathfrak{B} = (\mathbb{Q}, 0, 1, <)$. Consider the $<$ relation.

We must have that $0 <^{\mathfrak{A}} 1$ if and only if $i(0) <^{\mathfrak{B}} i(1)$.

However, if $i(0) \in \mathbb{Q}$ and $i(1) \in \mathbb{Q}$, then clearly $\frac{i(0)+i(1)}{2} \in \mathbb{Q}$ since the rationals are dense.

And clearly $i(0) <^{\mathfrak{B}} \frac{i(0)+i(1)}{2} <^{\mathfrak{B}} i(1)$.

So there must be a $z \in \mathbb{Z}$ such that, $0 <^{\mathfrak{A}} z <^{\mathfrak{A}} 1$ and $i(z) = \frac{i(0)+i(1)}{2}$.

But clearly there is no $z \in \mathbb{Z}$ such that $0 <^{\mathfrak{A}} z <^{\mathfrak{A}} 1$ since 0 and 1 are successive integers and the integers are not dense.

This is a contradiction.

Therefore, there is no isomorphism i from $(\mathbb{Z}, 0, 1, <)$ to $(\mathbb{Q}, 0, 1, <)$.

c)

There DOES NOT exist an isomorphism from $(\mathbb{Q}, 0, 1, <)$ to $(\mathbb{R}, 0, 1, <)$ because there is no bijective function $i : \mathbb{Q} \to \mathbb{R}$.

We know that for sets A and B, we have that $|A| = |B|$ if and only if there exists a bijection $f : A \to B$.

But we know that $\aleph_0 = |\mathbb{Q}| < |\mathbb{R}| = c$ from basic results about cardinality of infinite sets.

So there is no bijective function $i : \mathbb{Q} \to \mathbb{R}$.

Therefore, there is no isomorphism from $(\mathbb{Q}, 0, 1, <)$ to $(\mathbb{R}, 0, 1, <)$.

Question 4

Let $\mathfrak{B}$ be any model of N.

Required: Show that $\mathfrak{B}$ contains a substructure isomorphic to the natural numbers $\mathfrak{N} = (\mathbb{N}, 0, S, +, \cdot, E, <)$.

Let B be the universe of the model $\mathfrak{B}$.

Notation: $\bar{a} = \underbrace{SSS...S}_{aS's}0$

By Lemma 2.8.4 part 2, we know that if $a \neq b$, then $N \vdash \bar{a} \neq \bar{b}$.

By Soundness Theorem, we have for $a \neq b, N \models \bar{a} \neq \bar{b}$. Since $\mathfrak{B} \models N$, we have for $a \neq b, \mathfrak{B} \models \bar{a} \neq \bar{b}$.

This implies that the universe B must contain infinitely many distinct elements such that,

$\{0^{\mathfrak{B}}, S^{\mathfrak{B}}0^{\mathfrak{B}}, S^{\mathfrak{B}}S^{\mathfrak{B}}0^{\mathfrak{B}}, ...\} \subseteq B$ since $\neq$ is transitive.

Define $A = \{0^{\mathfrak{B}}, S^{\mathfrak{B}}0^{\mathfrak{B}}, S^{\mathfrak{B}}S^{\mathfrak{B}}0^{\mathfrak{B}}, ...\} \subseteq B$.

Define $\mathfrak{A} = (A, 0, S, +, \cdot, E, <)$ such that,

$0^{\mathfrak{A}} = 0^{\mathfrak{B}}$

$S^{\mathfrak{A}} = S^{\mathfrak{B}}|_A$

$+^{\mathfrak{A}} = +^{\mathfrak{B}}|_{A^2}$

$\cdot^{\mathfrak{A}} = \cdot^{\mathfrak{B}}|_{A^2}$

$E^{\mathfrak{A}} = E^{\mathfrak{B}}|_{A^2}$

$<^{\mathfrak{A}} = <^{\mathfrak{B}} \cap A^2$

$\mathfrak{A}$ is a substructure of $\mathfrak{B}$ by definition. i.e. $\mathfrak{A} \subseteq \mathfrak{B}$

We will now show that $\mathfrak{A}$ is isomorphic to the natural numbers $\mathfrak{N} = (\mathbb{N}, 0, S, +, \cdot, E, <)$.

Define a bijective function $i : A \rightarrow \mathbb{N}$ by $i(a) = n$, where n is the number of $S^{\mathfrak{B}\prime}s$ in a.

For example, $S^{\mathfrak{B}}S^{\mathfrak{B}}0^{\mathfrak{B}} \mapsto 2$

We will show that i is surjective and injective, and thus, bijective.

Surjectivity: For each $n \in \mathbb{N}$, we can always choose $a = \underbrace{S^{\mathfrak{B}}...S^{\mathfrak{B}}}_{nS^{\mathfrak{B}}\prime s}0^{\mathfrak{B}}$ such that $i(a) = n$. Therefore, i is surjective.

Injectivity: Assume that $i(a) = i(b)$. This implies that $i(a) = i(b) = n$, for some $n \in \mathbb{N}$. This implies that a and b both have n many $S^{\mathfrak{B}}\prime s$. Therefore, $a = b = \underbrace{S^{\mathfrak{B}}...S^{\mathfrak{B}}}_{nS^{\mathfrak{B}}\prime s}0^{\mathfrak{B}}$.

Therefore i is injective.

Since i is surjective and injective, we have that i is bijective.

Now we have to check the remaining conditions for isomorphism between structures.

Constants 0

$i(0^{\mathfrak{A}}) = i(0^{\mathfrak{B}}) = 0^{\mathbb{N}}$. This completes the constant condition for isomorphism.

Let $\bar{x}, \bar{y} \in A$ be arbitrary, where $\bar{x} = \underbrace{S^{\mathfrak{B}}...S^{\mathfrak{B}}}_{xS^{\mathfrak{B}}\prime s}0^{\mathfrak{B}}$ and $\bar{y} = \underbrace{S^{\mathfrak{B}}...S^{\mathfrak{B}}}_{yS^{\mathfrak{B}}\prime s}0^{\mathfrak{B}}$ be arbitrary.

Note, for each of the below functions we will occasionally write $f^{\mathfrak{B}}|_A$ and $f^{\mathfrak{B}}|_{A^2}$ as simply $f^{\mathfrak{B}}$ for brevity. We will also implicitly be using axioms $N_1 - N_{11}$.

Function S

$$i(S^{\mathfrak{A}}(\bar{x})) = i(S^{\mathfrak{B}}(\underbrace{S^{\mathfrak{B}}...S^{\mathfrak{B}}}_{xS^{\mathfrak{B}}\prime s}0^{\mathfrak{B}})) = i(\underbrace{S^{\mathfrak{B}}...S^{\mathfrak{B}}}_{(x+1)S^{\mathfrak{B}}\prime s}0^{\mathfrak{B}}) = (x+1)^{\mathbb{N}}$$

Also,

$$S^{\mathbb{N}}(i(\bar{x})) = S^{\mathbb{N}}(i(\underbrace{S^{\mathfrak{B}}...S^{\mathfrak{B}}}_{xS^{\mathfrak{B}}\prime s}0^{\mathfrak{B}})) = S^{\mathbb{N}}(x^{\mathbb{N}}) = (x+1)^{\mathbb{N}}$$

Function +

We will explicitly use Lemma 2.8.4 part 5, $N \vdash \bar{a} + \bar{b} = \overline{a+b}$ for $a, b \in \mathbb{N}$ in our structures with soundness.

$$\begin{aligned}
i(\bar{x} +^{\mathfrak{A}} \bar{y}) &= i(\underbrace{S^{\mathfrak{B}}...S^{\mathfrak{B}}}_{xS^{\mathfrak{B}}{}'s} 0^{\mathfrak{B}} +^{\mathfrak{A}} \underbrace{S^{\mathfrak{B}}...S^{\mathfrak{B}}}_{yS^{\mathfrak{B}}{}'s} 0^{\mathfrak{B}}) \\
&= i(\underbrace{S^{\mathfrak{B}}...S^{\mathfrak{B}}}_{xS^{\mathfrak{B}}{}'s} 0^{\mathfrak{B}} +^{\mathfrak{B}} \underbrace{S^{\mathfrak{B}}...S^{\mathfrak{B}}}_{yS^{\mathfrak{B}}{}'s} 0^{\mathfrak{B}}) \\
&= i(\underbrace{S^{\mathfrak{B}}...S^{\mathfrak{B}}}_{(x+y)S^{\mathfrak{B}}{}'s} 0^{\mathfrak{B}}) \\
&= (x+y)^{\mathbb{N}}
\end{aligned}$$

Also,

$$\begin{aligned}
i(\bar{x}) +^{\mathbb{N}} i(\bar{y}) &= i(\underbrace{S^{\mathfrak{B}}...S^{\mathfrak{B}}}_{xS^{\mathfrak{B}}{}'s} 0^{\mathfrak{B}}) +^{\mathbb{N}} i(\underbrace{S^{\mathfrak{B}}...S^{\mathfrak{B}}}_{yS^{\mathfrak{B}}{}'s} 0^{\mathfrak{B}}) \\
&= x^{\mathbb{N}} +^{\mathbb{N}} y^{\mathbb{N}} \\
&= (x+y)^{\mathbb{N}}
\end{aligned}$$

Function $\cdot$

We will explicitly use Lemma 2.8.4 part 6, $N \vdash \bar{a} \cdot \bar{b} = \overline{a \cdot b}$ for $a, b \in \mathbb{N}$ in our structures with soundness.

$$\begin{aligned}
i(\bar{x} \cdot^{\mathfrak{A}} \bar{y}) &= i(\overline{x \cdot y}) \\
&= i(\underbrace{S^{\mathfrak{B}}...S^{\mathfrak{B}}}_{x \cdot yS^{\mathfrak{B}}{}'s} 0^{\mathfrak{B}}) \\
&= (x \cdot y)^{\mathbb{N}}
\end{aligned}$$

Also,

$$\begin{aligned}
i(\bar{x}) \cdot^{\mathbb{N}} i(\bar{y}) &= i(\underbrace{S^{\mathfrak{B}}...S^{\mathfrak{B}}}_{xS^{\mathfrak{B}}{}'s} 0^{\mathfrak{B}}) \cdot^{\mathbb{N}} i(\underbrace{S^{\mathfrak{B}}...S^{\mathfrak{B}}}_{yS^{\mathfrak{B}}{}'s} 0^{\mathfrak{B}}) \\
&= x^{\mathbb{N}} \cdot^{\mathbb{N}} y^{\mathbb{N}} \\
&= (x \cdot y)^{\mathbb{N}}
\end{aligned}$$

Function E

We will explicitly use Lemma 2.8.4 part 4, $N \vdash \bar{a} E \bar{b} = \overline{a^b}$ for $a, b \in \mathbb{N}$ in our structures with soundness.

$$\begin{aligned}
i(\bar{x} E^{\mathfrak{A}} \bar{y}) &= i(\overline{x^y}) \\
&= i(\underbrace{S^{\mathfrak{B}}...S^{\mathfrak{B}}}_{x^y S^{\mathfrak{B}}{}'s} 0^{\mathfrak{B}}) \\
&= (x^y)^{\mathbb{N}}
\end{aligned}$$

Also,

$$\begin{aligned} i(\bar{x})E^{\mathbb{N}}i(\bar{y}) &= i(\underbrace{S^{\mathfrak{B}}...S^{\mathfrak{B}}}_{xS^{\mathfrak{B}}'s}0^{\mathfrak{B}})E^{\mathbb{N}}(\underbrace{S^{\mathfrak{B}}...S^{\mathfrak{B}}}_{yS^{\mathfrak{B}}'s}0^{\mathfrak{B}}) \\ &= x^{\mathbb{N}}E^{\mathbb{N}}y^{\mathbb{N}} \\ &= (x^y)^{\mathbb{N}} \end{aligned}$$

Relation $<$

We must have that $\bar{x} <^{\mathfrak{A}} \bar{y}$ if and only if $i(\bar{x}) <^{\mathbb{N}} i(\bar{y})$.

We will explicitly use Lemma 2.8.4 part 3, if $a < b$, then $N \vdash \bar{a} < \bar{b}$ in our structures with soundness.

($\Rightarrow$) This trivially holds since by definition, i outputs the number of $S^{\mathfrak{B}}$ in $\bar{x}$ and $\bar{y}$, so linear order in $<^{\mathfrak{A}}$ is preserved in $<^{\mathbb{N}}$ when applying i.

($\Leftarrow$) Similarly, the reverse holds since if numbers a, b are such that $a^{\mathbb{N}} <^{\mathbb{N}} b^{\mathbb{N}}$, then clearly, the corresponding $\bar{a}, \bar{b}$ such that $i(\bar{a}) = a$ and $i(\bar{b}) = b$ has $\bar{a} <^{\mathfrak{A}} \bar{b}$ by Lemma 2.8.4 since the number of $S^{\mathfrak{B}}$'s in $\bar{a}$ is less than the number of $S^{\mathfrak{B}}$'s in $\bar{b}$.

Therefore, we have shown that i is indeed an isomorphism from the substructure $\mathfrak{A} \subseteq \mathfrak{B}$ to the natural numbers $\mathfrak{N}$, as required.

Therefore, we have shown that for any structure $\mathfrak{B}$ of N, there exists a substructure $\mathfrak{A} \subseteq \mathfrak{B}$ that is isomorphic to $\mathfrak{N}$.

Question 5

Required: Infinite Ramsey Theorem $\Rightarrow$ Finite Ramsey Theorem

Proof. We will prove the contrapositive. Assume the negation of the Finite Ramsey Theorem.

Negation of Finite Ramsey Theorem: There exists $k \in \mathbb{N}$, such that for all $n \in \mathbb{N}$, every graph with at least n vertices, does not have a clique and does not have a co-clique of size k.

We will prove that this implies the negation of the Infinite Ramsey Theorem.

Negation of Infinite Ramsey Theorem: There exists an infinite graph with no infinite clique and no infinite co-clique.

Consider the language $\mathcal{L}$ with a symmetric, anti-reflexive binary relation A that describes adjacency between vertices.

Symmetric: $(\forall x)(\forall y)(A(x,y) \to A(y,x))$ Anti-Reflexive: $(\forall x)(\neg A(x,x))$

Consider the following set of sentences Σ which contains 3 sentence schemas.

$$\Sigma = \begin{cases} (\exists x_1)...(\exists x_n)(\bigwedge_{1 \leq i < j \leq n}(x_i \neq x_j) & \text{for all } \ n \in \mathbb{N} \\ (\forall x_1)...(\forall x_n)(\bigvee_{1 \leq i < j \leq n} A(x_i, x_j)) & \text{for all } \ n \in \mathbb{N} \\ (\forall x_1)...(\forall x_n)(\bigvee_{1 \leq i < j \leq n} \neg A(x_i, x_j)) & \text{for all } \ n \in \mathbb{N} \end{cases}$$

The first sentence schema is to assert that there are infinitely many distinct vertices. Note, that $\neq$ is symmetric. The second and third sentence schemas are to assert that there are no infinite co-cliques and no infinite cliques.

The set Σ is describes an infinite graph with no infinite cliques and no infinite co-cliques.

Consider any finite subset $\Sigma_0 \subseteq \Sigma$.

If Σ_0 contains sentences from the first sentence schema, then we can clearly construct a model with a sufficient number of finitely many distinct elements.

Since we assumed the negation of the Finite Ramsey Theorem, if Σ_0 contains sentences from the second or third sentence schemas, we know there is a finite model with sufficient number of elements, i.e. a graph with a sufficient number of finite vertices with no cliques or co-cliques that models the sentences in Σ_0.

By the Compactness Theorem, since every finite $\Sigma_0 \subseteq \Sigma$ has a model, this implies that Σ also has a model, i.e. a graph that characterizes the sentences in Σ. Therefore there exists an infinite graph with no infinite clique and no infinite co-clique.

$\square$