Exercise 6.5 g)

Note, Sider provides solutions for Exercise 6.5 g) at the back of the textbook. I have made some changes to Sider's proof to make it more clear to read by using a few different propositional logic tautologies in the deduction.

Show $\vdash_K \Diamond(P \to Q) \leftrightarrow (\Box P \to \Diamond Q)$

1.	$P \to ((P \to Q) \to Q)$	PL
2.	$\Box P \to \Box((P \to Q) \to Q)$	1, Nec, K, MP
3.	$\Box((P \to Q) \to Q) \to (\Diamond(P \to Q) \to \Diamond Q)$	K$\Diamond$
4.	$\Box P \to (\Diamond(P \to Q) \to \Diamond Q)$	2,3, PL
5.	$\Diamond(P \to Q) \to (\Box P \to \Diamond Q)$	4, PL
6.	$\sim P \to (P \to Q)$	PL
7.	$\Diamond \sim P \to \Diamond(P \to Q)$	6, Nec, K$\Diamond$, MP
8.	$\sim \Box P \to \Diamond \sim P$	MN
9.	$\sim \Box P \to \Diamond(P \to Q)$	7,8, PL
10.	$Q \to (P \to Q)$	PL
11.	$\Diamond Q \to \Diamond(P \to Q)$	10, Nec, K$\Diamond$, MP
12.	$\sim \Box P \vee \Diamond Q \to \Diamond(P \to Q)$	9,11, PL
13.	$(\Box P \to \Diamond Q) \to (\sim \Box P \vee \Diamond Q)$	PL
14.	$(\Box P \to \Diamond Q) \to \Diamond(P \to Q)$	12,13, PL
15.	$\Diamond(P \to Q) \leftrightarrow (\Box P \to \Diamond Q)$	5,14, PL

Exercise 6.5 h)

Show $\vdash_K \Diamond P \to (\Box Q \to \Diamond Q)$

1.	$Q \to (P \to Q)$	PL
2.	$\Box Q \to \Box(P \to Q)$	1, Nec, K, MP
3.	$\Box(P \to Q) \to (\Diamond P \to \Diamond Q)$	K$\Diamond$
4.	$\Box Q \to (\Diamond P \to \Diamond Q)$	2,3, PL
5.	$\Diamond P \to (\Box Q \to \Diamond Q)$	4, PL

Exercise 6.7 a)

Show $\vdash_T \Diamond\Box P \to \Diamond(P \vee Q)$

1.	$\Box P \to P$	T Axiom
2.	$P \to P \vee Q$	PL
3.	$\Box P \to (P \vee Q)$	1,2, PL
4.	$\Box(\Box P \to (P \vee Q))$	3, Nec
5.	$\Box(\Box P \to (P \vee Q)) \to (\Diamond\Box P \to \Diamond(P \vee Q))$	K$\Diamond$
6.	$\Diamond\Box P \to \Diamond(P \vee Q)$	4,5, MP

Exercise 6.8 b)

Show $\vdash_B \Box\Box(P \to \Box P) \to \Box(\sim P \to \Box \sim P)$

1.	$\Diamond\Box P \to P$	B Axiom
2.	$\Box(P \to \Box P) \to (\Diamond P \to \Diamond\Box P)$	K$\Diamond$
3.	$\Box(P \to \Box P) \to (\Diamond P \to P)$	1,2, PL
4.	$\Box(P \to \Box P) \to (\sim P \to \sim \Diamond P)$	3, PL
5.	$\sim \Diamond P \to \Box \sim P$	MN
6.	$\Box(P \to \Box P) \to (\sim P \to \Box \sim P)$	4,5, PL
7.	$\Box\Box(P \to \Box P) \to \Box(\sim P \to \Box \sim P)$	6, Nec, K, MP

Exercise 6.9 a)

Show $\vdash_{S4} \Box P \to \Box\Diamond\Box P$

1.	$\Box \to \Diamond\Box P$	T$\Diamond$
2.	$\Box\Box P \to \Box\Diamond\Box P$	1, Nec, K, MP
3.	$\Box P \to \Box\Box P$	$S4$ Axiom
4.	$\Box P \to \Box\Diamond\Box P$	2,3, PL

Exercise 6.9 b)

Show $\vdash_{S4} \Box\Diamond\Box\Diamond P \to \Box\Diamond P$

1.	$\Box\Diamond P \to \Diamond P$	T Axiom
2.	$\Box(\Box\Diamond P \to \Diamond P)$	1, Nec
3.	$\Box(\Box\Diamond P \to \Diamond P) \to (\Diamond\Box\Diamond P \to \Diamond\Diamond P)$	K$\Diamond$
4.	$\Diamond\Box\Diamond P \to \Diamond\Diamond P$	2,3, MP
5.	$\Diamond\Diamond P \to \Diamond P$	S4$\Diamond$
6.	$\Diamond\Box\Diamond P \to \Diamond P$	4,5, PL
7.	$\Box\Diamond\Box\Diamond P \to \Box\Diamond P$	6, Nec, K, MP

Exercise 6.17 Sider

Required: Where S is any of our modal systems, show that if Δ is an S-consistent set of wffs containing the formula $\Diamond\phi$, then $\Box^{-}(\Delta) \cup \{\phi\}$ is also S-consistent. You may appeal to lemmas and theorems proved so far.

Proof. Assume Δ is an S-consistent set of wffs containing the formula $\Diamond\phi$.

Recall that $\Diamond\phi$ is just an abbreviation for $\sim \Box \sim \phi$.

Since $\Diamond\phi \in \Delta$, we have that $\sim \Box \sim \phi \in \Delta$.

Since Δ is S-consistent, by Theorem 6.4 we know that there exists some maximal S-consistent set Δ' such that $\Delta \subseteq \Delta'$.

Since $\sim \Box \sim \phi \in \Delta$ and $\Delta \subseteq \Delta'$, we have that $\sim \Box \sim \phi \in \Delta'$.

Now, recall Lemma 6.6 which we will restate below verbatim.

Lemma 6.6: If Δ is a maximal S-consistent set of wffs containing $\sim \Box\phi$, then there exists a maximal S-consistent set of wffs Γ such that $\Box^{-}(\Delta) \subseteq \Gamma$ and $\sim \phi \in \Gamma$.

Since Δ' is maximal S-consistent and $\sim \Box \sim \phi \in \Delta'$, by Lemma 6.6 there exists a maximal S-consistent set of wffs Γ such that $\Box^{-}(\Delta') \subseteq \Gamma$ and $\sim\sim \phi \in \Gamma$.

Now, since Γ is maximal S-consistent, by Lemma 6.5a either $\sim \phi \in \Gamma$ or $\sim\sim \phi \in \Gamma$ but not both. Since we showed earlier that $\sim\sim \phi \in \Gamma$, we have that $\sim \phi \notin \Gamma$. We also know by Lemma 6.5a that either $\phi \in \Gamma$ or $\sim \phi \in \Gamma$ but not both. Since $\sim \phi \notin \Gamma$, we must have that $\phi \in \Gamma$.

Now, since $\Delta \subseteq \Delta'$, we trivially have that $\Box^{-}(\Delta) \subseteq \Box^{-}(\Delta')$. Since $\Box^{-}(\Delta) \subseteq \Box^{-}(\Delta')$ and $\Box^{-}(\Delta') \subseteq \Gamma$, we have that $\Box^{-}(\Delta) \subseteq \Gamma$.

Since $\phi \in \Gamma$ and $\Box^{-}(\Delta) \subseteq \Gamma$, we have that $\Box^{-}(\Delta) \cup \{\phi\} \subseteq \Gamma$.

Now, assume for reductio that $\Box^{-}(\Delta) \cup \{\phi\}$ was inconsistent. Hence, we have a proof of a contradiction with premises from $\Box^{-}(\Delta) \cup \{\phi\}$. But since $\Box^{-}(\Delta) \cup \{\phi\} \subseteq \Gamma$, we know that this proof can be viewed as a proof with premises from Γ. Hence, Γ proves a contradiction. Hence, Γ is inconsistent which is impossible since Γ is maximal S-consistent.

Therefore, $\Box^{-}(\Delta) \cup \{\phi\}$ is S-consistent, completing the proof, as required.

$\Box$

Exercise 6.20 Sider

Consider the system that results from adding to K every axiom of the form $\Diamond\phi \to \Box\phi$. Let the models for this system be defined as those whose accessibility relation meets the following condition: every world can see at most one world. Prove completeness for this (strange) system.

Proof. Call this (strange) system S. Consider the following canonical model for S which we take from Sider (page 227).

The canonical model for S is the MPL-model $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{I} \rangle$ where:

- $\mathcal{W}$ is the set of all maximal S-consistent sets of wffs.
- $\mathcal{R}ww'$ iff $\Box^{-}(w) \subseteq w'$
- $\mathcal{I}(\alpha, w) = 1$ iff $\alpha \in w$ for each sentence letter α and each $w \in \mathcal{W}$.
- $\Box^{-}(\Delta)$ is defined as the set of wffs ϕ such that $\Box\phi \in \Delta$.

Want to Show: The accessibility relation for the canonical model for our strange system S satisfies the condition: every world can see at most one world.

Assume for the sake of contradiction that the relation $\mathcal{R}$ in our canonical model is such that it is not the case that every world can see at most one world.

Hence, there exists a world $w_1 \in \mathcal{W}$ that can see more than one world. i.e. There exists $w_2 \in \mathcal{W}$ and $w_3 \in \mathcal{W}$ such that $\mathcal{R}w_1w_2$ and $\mathcal{R}w_1w_3$ with $w_2 \neq w_3$.

By Lemma 6.5c, we know that for any maximal consistent set Γ, if $\vdash_S \phi$, then $\phi \in \Gamma$.

Since $\vdash_S \Diamond\phi \to \Box\phi$ for any formula ϕ, we know that $\Diamond\phi \to \Box\phi \in \Gamma$ for any maximal consistent set Γ and any formula ϕ. Hence, any maximal consistent set in $\mathcal{W}$ is nonempty.

In particular, w_1, w_2, w_3 are nonempty.

Consider the following argument. Let $\phi \in w_2$ be arbitrary.

By Theorem 6.7 we know that $V_{\mathcal{M}}(\phi, w_2) = 1$ iff $\phi \in w_2$. Since $\phi \in w_2$, we have that $V_{\mathcal{M}}(\phi, w_2) = 1$.

Since $\mathcal{R}w_1w_2$ and $V_{\mathcal{M}}(\phi, w_2) = 1$, we have that $V_{\mathcal{M}}(\Diamond\phi, w_1) = 1$.

By Theorem 6.7 we know that $V_{\mathcal{M}}(\Diamond\phi, w_1) = 1$ iff $\Diamond\phi \in w_1$. Since $V_{\mathcal{M}}(\Diamond\phi, w_1) = 1$, we have that $\Diamond\phi \in w_1$.

By Lemma 6.5d we know that if $\vdash_S \Diamond\phi \to \Box\phi$ and $\Diamond\phi \in w_1$, then $\Box\phi \in w_1$. Since $\vdash_S \Diamond\phi \to \Box\phi$ and we've shown that $\Diamond\phi \in w_1$, we have that $\Box\phi \in w_1$.

By Theorem 6.7 we know that $V_{\mathcal{M}}(\Box\phi, w_1) = 1$ iff $\Box\phi \in w_1$. Since $\Box\phi \in w_1$, we have that $V_{\mathcal{M}}(\Box\phi, w_1) = 1$.

Now, we know that Rw_1w_3. Since $V_{\mathcal{M}}(\Box\phi, w_1) = 1$, we have that $V_{\mathcal{M}}(\phi, w_3) = 1$.

By Theorem 6.7 we know $V_{\mathcal{M}}(\phi, w_3) = 1$ iff $\phi \in w_3$. Since $V_{\mathcal{M}}(\phi, w_3) = 1$, we have $\phi \in w_3$.

And recall that $\phi \in w_2$ was arbitrary. We just showed that $\phi \in w_3$. Hence, we've shown that $w_2 \subseteq w_3$.

An entirely symmetric argument with the roles of w_2 and w_3 reversed also demonstrates that $w_3 \subseteq w_2$.

Since $w_2 \subseteq w_3$ and $w_3 \subseteq w_2$, we have that $w_2 = w_3$.

But we assumed that $w_2 \neq w_3$. This is a contradiction.

Hence, our assumption that there exists a world w_1 that sees more than one world was wrong. Therefore, every world sees at most one world.

Therefore, the accessibility relation of the canonical model for our system S satisfies the required condition: every world sees at most one world.

Want to Show: Completeness for our (strange) system S.

Let ϕ be any S-valid formula. Hence, ϕ is valid in all S-models.

Hence, ϕ is valid in all models whose accessibility relation is such that every world sees at most one world. And we showed that the accessibility relation of the canonical model for S is such that every world sees at most one world.

Hence, ϕ is valid in the canonical model.

And by Corollary 6.8 we know that ϕ is valid in the canonical model for S iff $\vdash_S \phi$. Since ϕ is valid in the canonical model for our system S, we have that $\vdash_S \phi$.

This proves completeness for our (strange) system S, as required. $\Box$

Question 4 (Hughes and Cresswell)

We will format our validity proofs below similar to how Sider formats his validity proofs.

a)

Consider a frame in which R satisfies the condition that if an arbitrary world u_1 sees worlds u_2 and u_3 then there must be a world u_4 which both u_2 and u_3 see.

Show: $\Diamond\Box P \to \Box\Diamond P$ is valid on such a frame.

Let $M = \langle W, R, I \rangle$ be any model where R satisfies the above property.

We will show that for any world $u_1 \in W$, we have $V_M(\Diamond\Box P \to \Box\Diamond P, u_1) = 1$.

i) Suppose for reductio that $V_M(\Diamond\Box P \to \Box\Diamond P, u_1) = 0$. Hence, $V_M(\Diamond\Box P, u_1) = 1$ and $V_M(\Box\Diamond P, u_1) = 0$.

ii) Given the former in i), for some $u_2 \in W$ such that $u_1 R u_2$ we have $V_M(\Box P, u_2) = 1$.

iii) Given the latter in i), for some $u_3 \in W$ such that $u_1 R u_3$ we have $V_M(\Diamond P, u_3) = 0$.

iv) Since $u_1 R u_2$ and $u_1 R u_3$ we know that we must have some $u_4 \in W$ such that $u_2 R u_4$ and $u_3 R u_4$.

v) From ii) we have that $V_M(\Box P, u_2) = 1$. Since from iv) we know that $u_2 R u_4$, we have that $V_M(P, u_4) = 1$.

vi) From iii) we have that $V_M(\Diamond P, u_3) = 0$. Since from iv) we know that $u_3 R u_4$, we have that $V_M(P, u_4) = 0$.

vii) From v) we have $V_M(P, u_4) = 1$ and from vi) we have $V_M(P, u_4) = 0$ which is a contradiction.

Therefore, our initial assumption was wrong. Therefore, $\Diamond\Box P \to \Box\Diamond P$ is valid on our frame.

b)

Consider a frame in which R satisfies the condition that if an arbitrary world u_1 sees worlds u_2 and u_3, then $u_2 R u_3$ or $u_3 R u_2$.

Show: $\Box(\Box P \to Q) \vee \Box(\Box Q \to P)$ is valid on such a frame.

Let $M = \langle W, R, I \rangle$ be any model where R satisfies the above property.

We will show that for any world $u_1 \in W$, we have $V_M(\Box(\Box P \to Q) \vee \Box(\Box Q \to P), u_1) = 1$.

i) Suppose for reductio that $V_M(\Box(\Box P \to Q) \vee \Box(\Box Q \to P), u_1) = 0$. Then, $V_M(\Box(\Box P \to Q), u_1) = 0$ and $V_M(\Box(\Box Q \to P), u_1) = 0$.

ii) Given the former in i), for some $u_2 \in W$ such that $u_1 R u_2$ we have $V_M(\Box P \to Q, u_2) = 0$.

iii) Given the latter in ii), for some $u_3 \in W$ such that $u_1 R u_3$ we have $V_M(\Box Q \to P, u_3) = 0$

iv) From ii) we get that $V_M(\Box P, u_2) = 1$ and $V_M(Q, u_2) = 0$.

v) From iii) we get that $V_M(\Box Q, u_3) = 1$ and $V_M(P, u_3) = 0$.

vi) Since $u_1 R u_2$ and $u_1 R u_3$, we know that either $u_2 R u_3$ or $u_3 R u_2$.

vii) If $u_2 R u_3$, then from iv) we get that $V_M(P, u_3) = 1$ which would contradict the fact that $V_M(P, u_3) = 0$ in v). If $u_3 R u_2$, then from v) we get that $V_M(Q, u_2) = 1$ which would contradict the fact that $V_M(Q, u_2) = 0$ in iv). In either case we have a contradiction.

Therefore, our initial assumption was wrong. Therefore, $\Box(\Box P \to Q) \vee \Box(\Box Q \to P)$ is valid on our frame.

Now assume that R satisfies the condition that if an arbitrary world u_1 sees worlds u_2 and u_3, then $u_2 R u_3$ or $u_3 R u_2$. And assume that R is also transitive.

Show: $\Box(\Box P \to \Box Q) \vee \Box(\Box Q \to \Box P)$ is valid on such a frame.

Let $M = \langle W, R, I \rangle$ be any model where R satisfies the above property.

We will show that for any world $u_1 \in W$, we have $V_M(\Box(\Box P \to \Box Q) \vee \Box(\Box Q \to \Box P), u_1) = 1$.

i) Suppose for reductio that $V_M(\Box(\Box P \to \Box Q) \vee \Box(\Box Q \to \Box P), u_1) = 0$. Then, $V_M(\Box(\Box P \to \Box Q), u_1) = 0$ and $V_M(\Box(\Box Q \to \Box P), u_1) = 0$.

ii) Given the former in i), for some $u_2 \in W$ such that $u_1 R u_2$ we have $V_M(\Box P \to \Box Q, u_2) = 0$.

iii) Given the latter in ii), for some $u_3 \in W$ such that $u_1 R u_3$ we have $V_M(\Box Q \to \Box P, u_3) = 0$

iv) From ii) we get that $V_M(\Box P, u_2) = 1$ and $V_M(\Box Q, u_2) = 0$.

v) From iii) we get that $V_M(\Box Q, u_3) = 1$ and $V_M(\Box P, u_3) = 0$.

vi) From the latter in iv) we know for some $u_4 \in W$ such that $u_2 R u_4$, we have $V_M(Q, u_4) = 0$.

vii) From the latter in v) we know for some $u_5 \in W$ such that u_3Ru_5, we have $V_M(P, u_5) = 0$.

viii) We know that since u_1Ru_2 and u_1Ru_3, we have that u_2Ru_3 or u_3Ru_2.

ix) If u_2Ru_3, then since u_3Ru_5 in vii), by transitivity we have u_2Ru_5. From the former in iv) we know that $V_M(\Box P, u_2) = 1$. Since u_2Ru_5, we have that $V_M(P, u_5) = 1$ which would contradict the fact that $V_M(P, u_5) = 0$ in vii). If u_3Ru_2, then since u_2Ru_4 in vi), by transitivity we have u_3Ru_4. From the former in v) we know that $V_M(\Box Q, u_3) = 1$. Since u_3Ru_4, we have that $V_M(Q, u_4) = 1$ which would contradict the fact that $V_M(Q, u_4) = 0$ in vi). In either case we have a contradiction.

Therefore, our initial assumption was wrong. Therefore, $\Box(\Box P \to \Box Q) \vee \Box(\Box Q \to \Box P)$ is valid on our frame.

c)

Consider a frame in which R satisfies the condition that if an arbitrary world u_1 sees two worlds u_2 and u_3, then u_2Ru_3 and u_3Ru_2.

Show: $\Diamond P \to \Box\Diamond P$ is valid on such a frame.

Let $M = \langle W, R, I\rangle$ be any model where R satisfies the above property.

We will show that for any world $u_1 \in W$, we have $V_M(\Diamond P \to \Box\Diamond P, u_1) = 1$.

i) Suppose for reductio that $V_M(\Diamond P \to \Box\Diamond P, u_1) = 0$. Then, $V_M(\Diamond P, u_1) = 1$ and $V_M(\Box\Diamond P, u_1) = 0$.

ii) Given the former in i), for some $u_2 \in W$ such that u_1Ru_2 we have $V_M(P, u_2) = 1$.

iii) Given the latter in i), for some $u_3 \in W$ such that u_1Ru_3 we have $V_M(\Diamond P, u_3) = 0$.

iv) Since u_1Ru_2 and u_1Ru_3, we know that u_2Ru_3 and u_3Ru_2.

v) From iii) we have $V_M(\Diamond P, u_3) = 0$ and from iv) we have u_3Ru_2. Hence, we have that $V_M(P, u_2) = 0$.

vi) From ii) we have $V_M(P, u_2) = 1$ and from v) we have $V_M(P, u_2) = 0$ which is a contradiction.

Therefore, our initial assumption was wrong. Therefore, $\Diamond P \to \Box\Diamond P$ is valid on our frame.