Question 1

There are an infinite number of natural numbers. It follows from the following facts.

(i) There is at least one natural number.

(ii) For each natural number there is a distinct number which is its successor, i.e., for each number x there is a distinct number y such that y stands in the successor relation to x.

(iii) No two natural numbers have the same successor.

(iv) There is a natural number, namely 0, that is not the successor of any number.

Required: Find a formula that is satisfiable by a valuation only if the domain of the valuation is infinite.

We will symbolize the statements above. However, notice that statement (i) is actually redundant to symbolize since (iv) implies the existence of at least one natural number as well.

So we will symbolize (ii), (iii) and (iv). We will define a language with some nonlogical vocabulary.

Let $\mathcal{L}$ be a language with equality and one unary function symbol s and one constant symbol 0.

Symbolizing (ii) Consider the formula $\forall x \exists y (y \neq x \wedge y = sx)$.

Symbolizing (iii) Consider the formula $\forall x \forall y (x \neq y \rightarrow sx \neq sy)$.

Symbolizing (iv) Consider the formula $\forall x (sx \neq 0)$.

Now, we will take the conjunction of the three formulas above. i.e. our formula is

$$\forall x \exists y (y \neq x \wedge y = sx) \wedge \forall x \forall y (x \neq y \rightarrow sx \neq sy) \wedge \forall x (sx \neq 0)$$

Question 2

Assume σ is an arbitrary valuation based on a structure $\mathcal{U}$ with domain U.

(i)

Show: $\sigma \models \alpha \to \forall x \alpha$, where x does not occur free in α.

Assume for the sake of contradiction that $\sigma \not\models \alpha \to \forall x \alpha$, where x does not occur free in α.

Hence, $(\alpha \to \forall x \alpha)^\sigma = \bot$. Hence, we have $\alpha^\sigma = \top$ and $(\forall x \alpha)^\sigma = \bot$.

Since $(\forall x \alpha)^\sigma = \bot$, we know that for some $u \in U$, $\alpha^{\sigma(x/u)} = \bot$.

We know that σ and $\sigma^{(x/u)}$ agree on all variables except possibly x. Recall that x does not occur free in α. Hence, σ and $\sigma^{(x/u)}$ agree on all free variables of α. By Theorem 8.5.8 in the textbook, since σ and $\sigma^{(x/u)}$ agree on all free variables of α and all extralogical symbols, we have that $\alpha^\sigma = \alpha^{\sigma(x/u)}$.

Since $\alpha^{\sigma(x/u)} = \bot$ and $\alpha^\sigma = \alpha^{\sigma(x/u)}$, we have that $\alpha^\sigma = \bot$.

But $\alpha^\sigma = \bot$ contradicts our earlier result that $\alpha^\sigma = \top$.

Therefore, our initial assumption was wrong. Therefore, $\sigma \models \alpha \to \forall x \alpha$, as required.

(ii)

Show: $\sigma \models s_1 = t_1 \to ... \to s_n = t_n \to f s_1 ... s_n = f t_1 ... t_n$

Assume for the sake of contradiction $\sigma \not\models s_1 = t_1 \to ... \to s_n = t_n \to f s_1 ... s_n = f t_1 ... t_n$.

Hence, $(s_1 = t_1 \to ... \to s_n = t_n \to f s_1 ... s_n = f t_1 ... t_n)^\sigma = \bot$.

Hence, $(s_1 = t_1)^\sigma = \top$ and $(s_2 = t_2 \to ... \to s_n = t_n \to f s_1 ... s_n = f t_1 ... t_n)^\sigma = \bot$.

Repeating the above step $n-1$ more times we eventually get $(s_i = t_i)^\sigma = \top$ for all $i = 1, ..., n$ and $(f s_1 ... s_n = f t_1 ... t_n)^\sigma = \bot$.

This implies that $\langle s_i^\sigma, t_i^\sigma \rangle \in id_U$ for all $i = 1, ..., n$ and that $\langle f^\sigma(s_1^\sigma ... s_n^\sigma), f^\sigma(t_1^\sigma ... t_n^\sigma) \rangle \notin id_U$.

Since $\langle s_i^\sigma, t_i^\sigma \rangle \in id_U$ for all $i = 1, ..., n$, we have that $s_i^\sigma = t_i^\sigma$ for all $i = 1, ..., n$. Since f^σ is an n-ary operation, we have that $f^\sigma(s_1^\sigma ... s_n^\sigma) = f^\sigma(t_1^\sigma ... t_n^\sigma)$. Hence, $\langle f^\sigma(s_1^\sigma ... s_n^\sigma), f^\sigma(t_1^\sigma ... t_n^\sigma) \rangle \in id_U$.

But $\langle f^\sigma(s_1^\sigma ... s_n^\sigma), f^\sigma(t_1^\sigma ... t_n^\sigma) \rangle \in id_U$ contradicts our earlier result that $\langle f^\sigma(s_1^\sigma ... s_n^\sigma), f^\sigma(t_1^\sigma ... t_n^\sigma) \rangle \notin id_U$.

Therefore, our initial assumption was wrong.

Therefore, $\sigma \models s_1 = t_1 \to ... \to s_n = t_n \to fs_1...s_n = ft_1...t_n$, as required.

(iii)

Show: $\sigma \models \forall x\alpha \to \alpha(x/t)$.

Fact: $\alpha(x/t)^\sigma = \alpha^{\sigma(x/t^\sigma)}$

Assume for the sake of contradiction that $\sigma \not\models \forall x\alpha \to \alpha(x/t)$.

Hence, $(\forall x\alpha \to \alpha(x/t))^\sigma = \bot$. Hence, $(\forall x\alpha)^\sigma = \top$ and $\alpha(x/t)^\sigma = \bot$.

Since $(\forall x\alpha)^\sigma = \top$, we know that for all $u \in U$, $\alpha^{\sigma(x/u)} = \top$. Hence, in particular we know that for $u = t^\sigma$, we have that $\alpha^{\sigma(x/t^\sigma)} = \top$.

Since $\alpha^{\sigma(x/t^\sigma)} = \top$ and we know that $\alpha(x/t)^\sigma = \alpha^{\sigma(x/t^\sigma)}$, we have that $\alpha(x/t)^\sigma = \top$.

But $\alpha(x/t)^\sigma = \top$ contradicts our earlier result that $\alpha(x/t)^\sigma = \bot$.

Therefore, our initial assumption was wrong. Therefore, $\sigma \models \forall x\alpha \to \alpha(x/t)$, as required.

Question 3

Prove that $\sigma \models \exists x \forall y(\phi \leftrightarrow x = y)$ just in case $\sigma(y/u) \models \phi$ for exactly one $u \in U$.

Note: We will assume that x is not free in ϕ.

First consider the following. We will assume obvious properties of when existential and biconditional formulas are satisfied.

$$\begin{aligned}
&\sigma \models \exists x \forall y(\phi \leftrightarrow x = y) \\
\text{iff} \quad &(\exists x \forall y(\phi \leftrightarrow x = y))^{\sigma} = \top \\
\text{iff} \quad &\text{For some } u \in U, \ (\forall y(\phi \leftrightarrow x = y))^{\sigma(x/u)} \\
\text{iff} \quad &\text{For some } u \in U \text{ and for all } v \in U, \ (\phi \leftrightarrow x = y)^{\sigma(x/u)(y/v)} \\
\text{iff} \quad &\text{For some } u \in U \text{ and for all } v \in U, \ \phi^{\sigma(x/u)(y/v)} = \top \ \text{ iff } \ \langle x^{\sigma(x/u)(y/v)}, y^{\sigma(x/u)(y/v)} \rangle \in id_U \\
\text{iff} \quad &\text{For some } u \in U \text{ and for all } v \in U, \ \phi^{\sigma(x/u)(y/v)} = \top \ \text{ iff } \ \langle u, v \rangle \in id_U
\end{aligned}$$

Call the above Biconditional 1.

Now we will prove the following claim.

Claim: $\sigma \models \exists x \forall y(\phi \leftrightarrow x = y)$ if and only if $\sigma(y/u) \models \phi$ for exactly one $u \in U$.

Proof. $(\Rightarrow)$: Assume $\sigma \models \exists x \forall y(\phi \leftrightarrow x = y)$.

By Biconditional 1 we get, For some $u \in U$ and for all $v \in U$, $\phi^{\sigma(x/u)(y/v)} = \top$ iff $\langle u, v \rangle \in id_U$.

So consider such a $u \in U$. Now, let $v = u$. Hence, we have that $\langle u, v \rangle \in id_U$. Hence, by our biconditional we have that $\phi^{\sigma(x/u)(y/v)} = \top$.

Now, we know that x is not free in ϕ. Hence, by Thm 8.5.8, since $\sigma(x/u)(y/v)$ and $\phi(y/v)$ agree on all free variables of ϕ and on all extralogical symbols, we have that $\phi^{\sigma(x/u)(y/v)} = \phi^{\sigma(y/v)}$.

Since $\phi^{\sigma(x/u)(y/v)} = \phi^{\sigma(y/v)}$ and $\phi^{\sigma(x/u)(y/v)} = \top$, we have that $\phi^{\sigma(y/v)} = \top$.

Since $\phi^{\sigma(y/v)} = \top$ and $u = v$, we have that $\phi^{\sigma(y/u)} = \top$. Hence, $\sigma(y/u) \models \phi$ for at least our $u \in U$.

Now, assume for the sake of contradiction that there exists some $w \in U$ such that $w \neq u$ and $\sigma(y/w) \models \phi$. Hence, $\phi^{\sigma(y/w)} = \top$.

Since $w \neq u$, we have that $\langle u, w \rangle \notin id_U$. Hence, by our earlier biconditional we have that $\phi^{\sigma(x/u)(y/w)} = \bot$. Recall x is not free in ϕ. Hence, by Thm 8.5.8, since $\sigma(x/u)(y/w)$

and $\sigma(y/w)$ agree on all free variables of ϕ and on all extralogical symbols, we have that $\phi^{\sigma(x/u)(y/w)} = \phi^{\sigma(y/w)}$.

Since $\phi^{\sigma(x/u)(y/w)} = \phi^{\sigma(y/w)}$ and $\phi^{\sigma(x/u)(y/w)} = \perp$, we have that $\phi^{\sigma(y/w)} = \perp$.

But $\phi^{\sigma(y/w)} = \perp$ contradicts the fact that $\phi^{\sigma(y/w)} = \top$. Hence, we cannot have such a $w \in U$ where $w \neq u$ and $\phi(y/w) \models \phi$.

Hence, we must have that $\sigma(y/u) \models \phi$ for only our $u \in U$.

Therefore, we have shown $\sigma(y/u) \models \phi$ for exactly one $u \in U$.

$(\Leftarrow)$: Assume $\sigma(y/u) \models \phi$ for exactly one $u \in U$.

Hence, $\phi^{\sigma(y/u)} = \top$ for exactly one $u \in U$. So consider such a $u \in U$.

Now, let $v \in U$ be arbitrary.

If $\phi^{\sigma(x/u)(y/v)} = \top$, then since x is not free in ϕ, we know that $\sigma(x/u)(y/v)$ and $\sigma(y/v)$ agree on all free variables of ϕ and all extralogical symbols. Hence, $\phi^{\sigma(x/u)(y/v)} = \phi^{\sigma(y/v)}$ by Thm 8.5.8. Since $\phi^{\sigma(x/u)(y/v)} = \top$, we have that $\phi^{\sigma(y/v)} = \top$. Since $\phi^{\sigma(y/v)} = \top$ and we know that $\phi^{\sigma(y/u)} = \top$ for exactly our $u \in U$, we must have that $u = v$. Hence, $\langle u, v\rangle \in id_U$.

Conversely, if $\langle u, v\rangle \in id_U$, then $u = v$. Hence, trivially $\phi^{\sigma(y/v)} = \phi^{\sigma(y/u)} = \top$. Since x is not free in ϕ, we know that $\sigma(x/u)(y/v)$ and $\sigma(y/v)$ agree on all free variables of ϕ and all extralogical symbols. Hence, by Thm 8.5.8 we have $\phi^{\sigma(x/u)(y/v)} = \phi^{\sigma(y/v)}$. Since $\phi^{\sigma(y/v)} = \top$, we have that $\phi^{\sigma(x/u)(y/v)} = \top$.

Combining the above two observations we get that for all $v \in U$, $\phi^{\sigma(x/u)(y/v)} = \top$ iff $\langle u, v\rangle \in id_U$.

Hence, we have shown that, For some $u \in U$ and for all $v \in U$, $\phi^{\sigma(x/u)(y/v)} = \top$ iff $\langle u, v\rangle \in id_U$.

By Biconditional 1 we have that $\sigma \models \exists x \forall y(\phi \leftrightarrow x = y)$.

Therefore, we have proven the **Claim**.

i.e. We proved that, $\sigma \models \exists x \forall y(\phi \leftrightarrow x = y)$ if and only if $\sigma(y/u) \models \phi$ for exactly one $u \in U$.

$\square$

Question 4

For a first-order language $\mathcal{L}$, we define by induction on the degree of α, the parity of a formula α, $pr(\alpha)$, which is either 1 or 0 as follows:

- If α is atomic, then $pr(\alpha) = 0$
- If $\alpha = \neg\beta$, then $pr(\alpha) = 1 - pr(\beta)$
- If $\alpha = \beta \to \gamma$, then $pr(\alpha) = (1 - pr(\beta)) \times pr(\gamma)$
- $\alpha = \forall x\beta$, then $pr(\alpha) = pr(\beta)$

(i)

Required: Show that the set of all formulas whose parity is 0 is a Hintikka set.

Before we prove our main result, we will first prove a quick lemma that we will use later.

Lemma: For any formula α and any term t, we have $pr(\alpha) = pr(\alpha(x/t))$.

Proof by induction on complexity of α.

Base Case: If α is atomic, then clearly $\alpha(x/t)$ is also atomic. Hence, $pr(\alpha) = 0$ and $pr(\alpha(x/t)) = 0$. Hence, $pr(\alpha) = pr(\alpha(x/t))$.

Inductive Hypothesis: Assume $pr(\alpha) = pr(\alpha(x/t))$ for every formula α with $deg(\alpha) < n$.

We will show that $pr(\alpha) = pr(\alpha(x/t))$ for every formula α when $deg(\alpha) = n$.

Case 1: $\alpha = \neg\beta$ where $deg(\beta) < n$.

$$\begin{aligned}
pr(\alpha) &= pr(\neg\beta) \\
&= 1 - pr(\beta) \\
&= 1 - pr(\beta(x/t)) && \text{By Inductive Hypothesis} \\
&= pr(\neg\beta(x/t)) \\
&= pr((\neg\beta)(x/t)) \\
&= pr(\alpha(x/t))
\end{aligned}$$

Case 2: $\alpha = \beta \to \gamma$ where $deg(\beta) < n$ and $deg(\gamma) < n$.

$$\begin{aligned}
pr(\alpha) &= pr(\beta \to \gamma) \\
&= (1 - pr(\beta)) \times pr(\gamma) \\
&= (1 - pr(\beta(x/t))) \times pr(\gamma(x/t)) && \text{By Inductive Hypothesis} \\
&= pr(\beta(x/t) \to \gamma(x/t)) \\
&= pr((\beta \to \gamma)(x/t)) \\
&= pr(\alpha(x/t))
\end{aligned}$$

Case 3: $\alpha = \forall y\beta$ where $deg(\beta) < n$. Without loss of generality, assume that variable(s) in α are already appropriately relabelled if variable(s) in t are captured by the quantifier of y in the substitution $\alpha(x/t)$.

$$\begin{aligned}
pr(\alpha) &= pr(\forall y\beta) \\
&= pr(\beta) \\
&= pr(\beta(x/t)) && \text{By Inductive Hypothesis} \\
&= pr(\forall y(\beta(x/t))) \\
&= pr((\forall y\beta)(x/t)) \\
&= pr(\alpha(x/t))
\end{aligned}$$

Therefore, by induction on the complexity of formulas we have proven the **Lemma**.

Let Φ be the set of all formulas whose parity is 0. We will show that Φ is a Hintikka set.

We will verify all **9 conditions** given in Def 8.7.1 for our set ϕ.

1. Show: If α is any atomic formula such that $\alpha \in \Phi$, then $\neg\alpha \notin \Phi$.

Assume $\alpha \in \Phi$. Hence, $pr(\alpha) = 0$. Hence, $pr(\neg\alpha) = 1 - pr(\alpha) = 1 - 0 = 1$. Since $pr(\neg\alpha) = 1$, we have $\neg\alpha \notin \Phi$.

2. Show: If α is any formula such that $\neg\neg\alpha \in \Phi$, then also $\alpha \in \Phi$.

Assume $\neg\neg\alpha \in \Phi$. Hence, $pr(\neg\neg\alpha) = 0$. Assume for reductio that $\alpha \notin \Phi$. Hence, $pr(\alpha) = 1$. Hence, $pr(\neg\alpha) = 1 - pr(\alpha) = 1 - 1 = 0$. Hence, $pr(\neg\neg\alpha) = 1 - pr(\neg\alpha) = 1 - 0 = 1$. But $pr(\neg\neg\alpha) = 1$ contradicts $pr(\neg\neg\alpha) = 0$. Hence, we must have $\alpha \in \Phi$.

3. Show: If α and β are any formulas such that $\alpha \to \beta \in \Phi$, then also $\neg\alpha \in \Phi$ or $\beta \in \Phi$.

Assume $\alpha \to \beta \in \Phi$. Hence, $pr(\alpha \to \beta) = 0$. Assume for reductio that $\neg\alpha \notin \Phi$ and $\beta \notin \Phi$. Hence, $pr(\neg\alpha) = 1$ and $pr(\beta) = 1$. Notice, $1 = pr(\neg\alpha) = 1 - pr(\alpha)$. Hence, $pr(\alpha \to \beta) = (1 - pr(\alpha)) \times pr(\beta) = 1 \times 1 = 1$. But $pr(\alpha \to \beta) = 1$ contradicts $pr(\alpha \to \beta) = 0$. Hence, we must have $\neg\alpha \in \Phi$ or $\beta \in \Phi$.

4. Show: If α and β are any formulas such that $\neg(\alpha \to \beta) \in \Phi$, then $\alpha \in \Phi$ and $\neg\beta \in \Phi$.

Assume $\neg(\alpha \to \beta) \in \Phi$. Hence, $pr(\neg(\alpha \to \beta)) = 0$. Notice, $0 = pr(\neg(\alpha \to \beta)) = 1 - pr(\alpha \to \beta)$. Rearranging, we get $pr(\alpha \to \beta) = 1$. Notice, $1 = pr(\alpha \to \beta) = (1 - pr(\alpha)) \times pr(\beta)$. Since $1 = (1 - pr(\alpha)) \times pr(\beta)$, we must have that $1 - pr(\alpha) = 1$ and $pr(\beta) = 1$. Since $1 - pr(\alpha) = 1$, we have $pr(\alpha) = 0$. Since $pr(\beta) = 1$, we have $pr(\neg\beta) = 1 - pr(\beta) = 1 - 1 = 0$. Since $pr(\alpha) = 0$ and $pr(\neg\beta) = 0$, we have $\alpha \in \Phi$ and $\neg\beta \in \Phi$.

5. Show: If α is any formula and x is any variable such that $\forall x\alpha \in \Phi$, then $\alpha(x/t) \in \Phi$ for every term t.

Assume $\forall x\alpha \in \Phi$. Hence, $pr(\forall x\alpha) = 0$. Hence, $0 = pr(\forall x\alpha) = pr(\alpha)$. Let t be a term. We know by our **Lemma** that $pr(\alpha) = pr(\alpha(x/t))$. Hence, $pr(\alpha(x/t)) = pr(\alpha) = 0$. Hence, $\alpha(x/t) \in \Phi$. Since t was arbitrary, we have that $\alpha(x/t) \in \Phi$ for every term t.

6. Show: If α is any formula and x is any variable such that $\neg\forall x\alpha \in \Phi$, then $\neg\alpha(x/t) \in \Phi$ for some term t.

Assume $\neg\forall x\alpha \in \Phi$. Hence, $pr(\neg\forall x\alpha) = 0$. Hence, $0 = pr(\neg\forall x\alpha) = 1 - pr(\forall x\alpha)$. Rearranging, we get $pr(\forall x\alpha) = 1$. Since $pr(\forall x\alpha) = pr(\alpha)$, we have that $pr(\alpha) = 1$. Now, let t simply be the term x. We could have chosen any term, but t being x will suffice. By our **Lemma** we know that $pr(\alpha) = pr(\alpha(x/t))$. Since $pr(\alpha) = 1$, we have that $pr(\alpha(x/t)) = 1$. Hence, $pr(\neg\alpha(x/t)) = 1 - pr(\alpha(x/t)) = 1 - 1 = 0$. Hence, $\neg\alpha(x/t) \in \Phi$ for our t.

7. Show: If our language has equality, then $t = t \in \Phi$ for every term t.

Assume our language has equality and let t be a term. We know that $t = t$ is atomic. Hence, $pr(t = t) = 0$. Hence, $t = t \in \Phi$.

8. Show: If $n \geq 1$ and $s_1, ..., s_n$ and $t_1...t_n$ are any $2n$ terms such that for each $i = 1, ..., n$ we have $s_i = t_i \in \Phi$, then for every n-ary function symbol f we have $fs_1...s_n = ft_1...t_n \in \Phi$.

Assume for $n \geq 1$, $s_1, ..., s_n$ and $t_1...t_n$ are any $2n$ terms such that for each $i = 1, ..., n$ we have $s_i = t_i \in \Phi$. Let f be an n-ary function symbol.

Notice, $fs_1...s_n = ft_1...t_n$ is atomic. Hence, $pr(fs_1...s_n = ft_1...t_n) = 0$. Hence, we have $fs_1...s_n = ft_1...t_n \in \Phi$.

9. Show: If $n \geq 1$ and $s_1, ..., s_n$ and $t_1...t_n$ are any $2n$ terms such that for each $i = 1, ..., n$ we have $s_i = t_i \in \Phi$ and if P is an n-ary predicate symbol such that $Ps_1...s_n \in \Phi$, then $Pt_1...t_n \in \Phi$.

Assume for $n \geq 1$, $s_1, ..., s_n$ and $t_1...t_n$ are any $2n$ terms such that for each $i = 1, ..., n$, we have $s_i = t_i \in \Phi$. Let P be an n-ary predicate symbol and assume $Ps_1...s_n \in \Phi$.

Notice, $Pt_1...t_n$ is atomic. Hence, $pr(Pt_1...t_n) = 0$. Hence, $Pt_1...t_n \in \Phi$.

Therefore, we have verified all **9 conditions** of Def 8.7.1 of a Hintikka set for Φ.

Therefore, our set Φ which is the set of all formulas whose parity is 0 is indeed a Hintikka set, as required.

(ii)

Required: Define a valuation σ and prove that $\sigma \models \alpha$ iff α has parity 0.

Let σ be a valuation based on a structure $\mathcal{U}$ with domain $U = \{u\}$ and the interpretation function is such that:

$f^{\mathcal{U}}(u, ..., u) = u$ for each n-ary function symbol f. Note, since $U = \{u\}$, we have defined $f^{\mathcal{U}}$ entirely.

$P^{\mathcal{U}} = \{\langle u, ..., u\rangle\}$ for each n-ary predicate symbol P.

And let $\sigma(z) = u$ for each variable z.

First we will show the following.

Show: For every term t, we have $t^\sigma = u$.

Let t be an arbitrary term. We know that we must have $t^\sigma \in U$, where U is our domain. And $U = \{u\}$. Hence, $t^\sigma \in \{u\}$. Hence, $t^\sigma = u$. Since t was arbitrary, we have shown that for every term t, we have $t^\sigma = u$.

Now we will prove the following claim which is an equivalent restatement of what we are required to prove.

Claim: $\alpha^\sigma = \top$ iff $pr(\alpha) = 0$.

Proof by induction on the complexity of formulas.

Base Case: Let α be an atomic formula.

Since α is atomic we have $pr(\alpha) = 0$.

Since α is atomic, we know $\alpha = Pt_1...t_n$ for some n-ary predicate symbol P and terms $t_1...t_n$. Hence,

$$\begin{aligned}
\alpha^\sigma = \top \ &\text{iff} \ (Pt_1...t_n)^\sigma = \top \\
&\text{iff} \ \langle t_1^\sigma, ..., t_n^\sigma\rangle \in P^\sigma \\
&\text{iff} \ \langle u, ..., u\rangle \in P^\sigma && \text{Since } t^\sigma = u \text{ for every term } t
\end{aligned}$$

And we know that $\langle u, ..., u\rangle \in P^\sigma$ is true. Hence, we have $\alpha^\sigma = \top$.

Since $\alpha^\sigma = \top$ and $pr(\alpha) = 0$, we trivially have that $\alpha^\sigma = \top$ iff $pr(\alpha) = 0$.

This completes the Base Case.

Inductive Hypothesis: For every formula α such that $deg(\alpha) < n$, assume $\alpha^\sigma = \top$ iff $pr(\alpha) = 0$.

We will show that for α such that $deg(\alpha) = n$, we have $\alpha^\sigma = \top$ iff $pr(\alpha) = 0$.

Case 1: $\alpha = \neg\beta$, where $\deg(\beta) < n$.

Show: $(\neg\beta)^\sigma = \top$ iff $pr(\neg\beta) = 0$.

We know by inductive hypothesis that $\beta^\sigma = \top$ iff $pr(\beta) = 0$.

$(\Rightarrow)$: Assume $(\neg\beta)^\sigma = \top$. Hence, $\beta^\sigma = \bot$. Hence, by the inductive hypothesis, we have $pr(\beta) = 1$. Hence, $pr(\neg\beta) = 1 - pr(\beta) = 1 - 1 = 0$.

$(\Leftarrow)$: Assume $pr(\neg\beta) = 0$. Hence, $0 = pr(\neg\beta) = 1 - pr(\beta)$. Hence, $pr(\beta) = 1$. Hence, by the inductive hypothesis we get that $\beta^\sigma = \bot$. Hence, $(\neg\beta)^\sigma = \top$.

Combining the above two results we get $(\neg\beta)^\sigma = \top$ iff $pr(\neg\beta) = 0$.

Case 2: $\alpha = \beta \to \gamma$ where $deg(\beta) < n$ and $deg(\gamma) < n$.

Show: $(\beta \to \gamma)^\sigma = \top$ iff $pr(\beta \to \gamma) = 0$.

By inductive hypothesis we know that $\beta^\sigma = \top$ iff $pr(\beta) = 0$.

By inductive hypothesis we know that $\gamma^\sigma = \top$ iff $pr(\gamma) = 0$.

$(\Rightarrow)$: Assume $(\beta \to \gamma)^\sigma = \top$. Hence, $\beta^\sigma = \bot$ or $\gamma^\sigma = \top$. If $\beta^\sigma = \bot$, then by inductive hypothesis we know that $pr(\beta) = 1$. Hence, $pr(\beta \to \gamma) = (1 - pr(\beta)) \times pr(\gamma) = (1 - 1) \times pr(\gamma) = 0 \times pr(\gamma) = 0$. If $\gamma^\sigma = \top$, then by inductive hypothesis we have that $pr(\gamma) = 0$. Hence, $pr(\beta \to \gamma) = (1 - pr(\beta)) \times pr(\gamma) = (1 - pr(\beta)) \times 0 = 0$. In either case we have $pr(\beta \to \gamma) = 0$.

$(\Leftarrow)$: Assume $pr(\beta \to \gamma) = 0$. Hence, $0 = pr(\beta \to \gamma) = (1 - pr(\beta)) \times pr(\gamma)$. Hence, $1 - pr(\beta) = 0$ or $pr(\gamma) = 0$. If $1 - pr(\beta) = 0$, then $pr(\beta) = 1$. Hence, by inductive hypothesis, $\beta^\sigma = \bot$. Hence, $(\beta \to \gamma)^\sigma = \top$. If $pr(\gamma) = 0$, then by inductive hypothesis $\gamma^\sigma = \top$. Hence, $(\beta \to \gamma)^\sigma = \top$. In either case we have $(\beta \to \gamma)^\sigma = \top$.

Combining the above two results we get $(\beta \to \gamma)^\sigma = \top$ iff $pr(\beta \to \gamma) = 0$.

Case 3: $\alpha = \forall x\beta$ where $deg(\beta) < n$.

Show: $(\forall x\beta)^\sigma = \top$ iff $pr(\forall x\beta) = 0$.

By inductive hypothesis we know that $\beta^\sigma = \top$ iff $pr(\beta) = 0$.

$(\Rightarrow)$: Assume $(\forall x\beta)^\sigma = \top$. Hence, for each $v \in U$, we have that $\beta^{\sigma(x/v)} = \top$. Since $U = \{u\}$, we have that $v = u$ for all $v \in U$. Since $\sigma(z) = u$ for every variable z, we know that $\sigma(x/v)$ and σ agree on all variables (and in particular all free variables of β) and on all extralogical symbols. By Theorem 8.5.8 we have that $\beta^{\sigma(x/v)} = \beta^\sigma$. Since $\beta^{\sigma(x/v)} = \top$, we have that $\beta^\sigma = \top$. Since $\beta^\sigma = \top$, by inductive hypothesis we have that $pr(\beta) = 0$. Hence, we have $pr(\forall x\beta) = pr(\beta) = 0$.

$(\Leftarrow)$: Assume $pr(\forall x\beta) = 0$. Hence, $0 = pr(\forall x\beta) = pr(\beta)$. Since $pr(\beta) = 0$, by inductive hypothesis we have that $\beta^\sigma = \top$. Now, let $v \in U$ be arbitrary. Since $U = \{u\}$, we have that $v = u$. Since $\sigma(z) = u$ for every variable z, we know that $\sigma(x/v)$ and σ agree on all variables (and in particular all free variables of β) and on all extralogical symbols. By Theorem 8.5.8 we have that $\beta^{\sigma(x/v)} = \beta^\sigma$. Since $\beta^\sigma = \top$, we have that $\beta^{\sigma(x/v)} = \top$. Since for every $v \in U$ we have $\beta^{\sigma(x/v)} = \top$, we conclude that $(\forall x\beta)^\sigma = \top$.

Combining the above two results we get $(\forall x\beta)^\sigma = \top$ iff $pr(\forall x\beta) = 0$.

By induction on the complexity of formulas we have proven our **Claim**.

i.e. we have shown that for every formula α, we have $\alpha^\sigma = \top$ iff $pr(\alpha) = 0$.

This is equivalent to saying for every formula α, we have $\sigma \models \alpha$ iff α has parity 0.

This completes the proof, as required.