Exercise 1

Theorem 1.5.3: Suppose that the language L and L' is an expansion of the language L. Then,

a) $Term_L \subseteq Term_{L'}$
b) $AtForm_L \subseteq AtForm_{L'}$
c) $Form_L \subseteq Form_{L'}$
d) $Sent_L \subseteq Sent_{L'}$

NOTE: We will format our induction proofs similar to Thm 2.1.11 in the booklet and the PHLC51 notes on induction.

Proof of a) $Term_L \subseteq Term_{L'}$

Proof. Proof by induction on $t \in Term_L$.

1. Base Case: $t \in Vble \cup Const_L$
Show: $t \in Term_{L'}$.

Case 1: If $t \in Vble$, then since the set of variable symbols are common to all first-order langauges, we know that $t \in Term_{L'}$.

Case 2: If $t \in Const_L$, then since L' is an expansion of L and every constant symbol of L is a constant symbol of L', we know that $t \in Const_{L'}$. Hence, $t \in Term_{L'}$.

2. Inductive Step
IH: $t_i \in Term_{L'}$ for $i = 1, ..., n$.
Show: $ft_1...t_n \in Term_{L'}$.

Let f be an n-ary function symbol of L. Since L' is an expansion of L and every function symbol of L is a function symbol of L', we know that f is a function symbol of L'. And by **IH**, since each $t_i \in Term_{L'}$, we conclude that $ft_1...t_n \in Term'_L$.

This completes the proof. □

Proof of b) $AtForm_L \subseteq AtForm_{L'}$

Proof. Let $\phi \in AtForm_L$. We want to show that $\phi \in AtForm_{L'}$.

We know that ϕ could have two possible forms.

Case 1: ϕ is of the form $Pt_1...t_n$ where P is some n-ary predicate symbol of L and each $t_i \in Term_L$. Since L' is an expansion of L and every predicate symbol of L is a predicate symbol of L', we know that P is a predicate symbol of L'. And in part a) we proved that $Term_L \subseteq Term_{L'}$. Since each $t_i \in Term_L$ we have that each $t_i \in Term_{L'}$. Hence,

$Pt_1...t_n \in AtForm_{L'}$. i.e. $\phi \in AtForm_{L'}$.

Case 2: If L has the equals sign, then ϕ could be of the form $= t_1t_2$ where $t_1, t_2 \in Term_L$. Since L' is an expansion of L and L contains the equals sign, we must have that L' contains the equals sign. And in part a) we proved that $Term_L \subseteq Term_{L'}$. Since each $t_i \in Term_L$ we have that each $t_i \in Term_{L'}$. Hence, $= t_1t_2 \in AtForm_{L'}$. i.e. $\phi \in AtForm_{L'}$.

In either case $\phi \in AtForm_{L'}$. This completes the proof. $\square$

Proof of c) $Form_L \subseteq Form_{L'}$

Proof. Proof by induction on $\phi \in Form_L$.

1. Base Case: $\phi \in AtForm_L$.
Show: $\phi \in Form_{L'}$.

In part b) we showed that $AtForm_L \subseteq AtForm_{L'}$. Since $\phi \in AtForm_L$, we have that $\phi \in AtForm_{L'}$. Also, $AtForm_{L'} \subset Form_{L'}$. Since $\phi \in AtForm_{L'}$ we have that $\phi \in Form_{L'}$.

2. Inductive Step $\sim$
IH: $\phi \in Form_{L'}$.
Show: $\sim \phi \in Form_{L'}$.

By **IH**, since $\phi \in Form_{L'}$, we have that $\sim \phi \in Form_{L'}$ by Def 1.2.4.

3. Inductive Step $\rightarrow$
IH1: $\phi \in Form_{L'}$.
IH2: $\psi \in Form_{L'}$.
Show: $(\phi \rightarrow \psi) \in Form_{L'}$.

By **IH1** and **IH2**, we have that $\phi, \psi \in Form_{L'}$. Hence, $(\phi \rightarrow \psi) \in Form_{L'}$ by Def 1.2.4.

4. Inductive Step $\leftrightarrow$
IH1: $\phi \in Form_{L'}$.
IH2: $\psi \in Form_{L'}$.
Show: $(\phi \leftrightarrow \psi) \in Form_{L'}$.

By **IH1** and **IH2**, we have that $\phi, \psi \in Form_{L'}$. Hence, $(\phi \leftrightarrow \psi) \in Form_{L'}$ by Def 1.2.4.

5. Inductive Step $\vee$
IH1: $\phi \in Form_{L'}$.
IH2: $\psi \in Form_{L'}$.
Show: $(\phi \vee \psi) \in Form_{L'}$.

By **IH1** and **IH2**, we have that $\phi, \psi \in Form_{L'}$. Hence, $(\phi \vee \psi) \in Form_{L'}$ by Def 1.2.4.

6. Inductive Step $\wedge$
IH1: $\phi \in Form_{L'}$.
IH2: $\psi \in Form_{L'}$.
Show: $(\phi \wedge \psi) \in Form_{L'}$.

By **IH1** and **IH2**, we have that $\phi, \psi \in Form_{L'}$. Hence, $(\phi \wedge \psi) \in Form_{L'}$ by Def 1.2.4.

7. Inductive Step $\forall$
IH: $\phi \in Form_{L'}$.
Show: $\forall x\phi \in Form_{L'}$.

By **IH**, since $\phi \in Form_{L'}$, we have that $\forall x\phi \in Form_{L'}$ by Def 1.2.4.

8. Inductive Step $\exists$
IH: $\phi \in Form_{L'}$.
Show: $\exists x\phi \in Form_{L'}$.

By **IH**, since $\phi \in Form_{L'}$, we have that $\exists x\phi \in Form_{L'}$ by Def 1.2.4.

This completes the proof. $\square$

Proof of d) $Sent_L \subseteq Sent_{L'}$

Proof. Assume $\phi \in Sent_L$. We want to show that $\phi \in Sent_{L'}$.

We know that $Sent_L \subset Form_L$ since every L-sentence is an L-formula with no free variables. Hence, $\phi \in Form_L$.

In part c) we proved that $Form_L \subseteq Form_{L'}$. Since $\phi \in Form_L$, we have that $\phi \in Form_{L'}$.

But we know that every variable in $\phi \in Form_{L'}$ is under the scope of a quantifier since we assumed that ϕ was an L-sentence. i.e. there are no free variables in ϕ.

Since $\phi \in Form_{L'}$ has no free variables, we have that $\phi \in Sent_{L'}$.

This completes the proof, as required. $\square$

Exercise 2

First Sentence

Given Infix Notation: $\forall \mathbf{v_3}(((\mathbf{v_3}+\sharp\mathbf{o})=\sharp(\mathbf{v_3}+\mathbf{o}))\rightarrow\exists\mathbf{v_2}((\mathbf{v_2}\star\sharp\mathbf{v_2})\triangleright((\mathbf{v_3}\star\mathbf{o})\star\mathbf{v_4})))$

Prefix Notation: $\forall\mathbf{v_3}(=+\mathbf{v_3}\sharp\mathbf{o}\sharp+\mathbf{v_3}\mathbf{o}\rightarrow\exists\mathbf{v_2}\triangleright\star\mathbf{v_2}\sharp\mathbf{v_2}\star\star\mathbf{v_3}\mathbf{o}\mathbf{v_4})$

Second Sentence

Given Prefix Notation: $\forall\mathbf{v_4}\exists\mathbf{v_1}\triangleright\sharp+\sharp\mathbf{v_4}+\mathbf{o}\mathbf{v_1}++\sharp\mathbf{v_3}\mathbf{v_3}+\sharp\mathbf{o}+\mathbf{v_5}\mathbf{v_5}$

Infix Notation: $\forall\mathbf{v_4}\exists\mathbf{v_1}(\sharp(\sharp\mathbf{v_4}+(\mathbf{o}+\mathbf{v_1}))\triangleright((\sharp\mathbf{v_3}+\mathbf{v_3})+(\sharp\mathbf{o}+(\mathbf{v_5}+\mathbf{v_5}))))$

Exercise 3

NOTE: As in the assignment outline, we will be using $\forall$ and $\exists$ ambiguously in the metalanguage for metalinguistic universal and existential quantification.

NOTE: We will also be formatting the argument below similar to the format in the booklet on T-conditionals.

Consider the following.

$$
\begin{array}{ll}
& M \models \forall \mathbf{v_1}(\triangleright \mathbf{v_1 o} \to \exists \mathbf{v_2}(\triangleright \mathbf{o v_2} \wedge = + \mathbf{v_2 v_1 o})) \\
\text{iff} & M \models \forall \mathbf{v_1}(\triangleright \mathbf{v_1 o} \to \exists \mathbf{v_2}(\triangleright \mathbf{o v_2} \wedge = + \mathbf{v_2 v_1 o}))[s] \\
\text{iff} & \forall q \in \mathbb{Q}^+,\ M \models \triangleright \mathbf{v_1 o} \to \exists \mathbf{v_2}(\triangleright \mathbf{o v_2} \wedge = + \mathbf{v_2 v_1 o})[s^q_{v_1}] \\
\text{iff} & \forall q \in \mathbb{Q}^+,\ M \not\models \triangleright \mathbf{v_1 o}[s^q_{v_1}] \ \text{ or } \ M \models \exists \mathbf{v_2}(\triangleright \mathbf{o v_2} \wedge = + \mathbf{v_2 v_1 o})[s^q_{v_1}] \\
\text{iff} & \forall q \in \mathbb{Q}^+, \text{ if } \ M \models \triangleright \mathbf{v_1 o}[s^q_{v_1}], \ \text{ then } \ M \models \exists \mathbf{v_2}(\triangleright \mathbf{o v_2} \wedge = + \mathbf{v_2 v_1 o})[s^q_{v_1}] \\
\text{iff} & \forall q \in \mathbb{Q}^+, \text{ if } \ M \models \triangleright \mathbf{v_1 o}[s^q_{v_1}], \ \text{ then } \exists p \in \mathbb{Q}^+,\ M \models \triangleright \mathbf{o v_2} \wedge = + \mathbf{v_2 v_1 o}[(s^q_{v_1})^p_{v_2}] \\
\text{iff} & \forall q \in \mathbb{Q}^+, \text{ if } \ M \models \triangleright \mathbf{v_1 o}[s^q_{v_1}], \ \text{ then } \exists p \in \mathbb{Q}^+,\ M \models \triangleright \mathbf{o v_2}[(s^q_{v_1})^p_{v_2}] \ \text{ and } \ M \models = + \mathbf{v_2 v_1 o}[(s^q_{v_1})^p_{v_2}] \\
\text{iff} & \forall q \in \mathbb{Q}^+, \text{ if } \ \langle Val_{M,s^q_{v_1}}(\mathbf{v_1}), Val_{M,s^q_{v_1}}(\mathbf{o})\rangle \in I(\triangleright), \ \text{ then } \exists p \in \mathbb{Q}^+, \text{ both} \\
& \langle Val_{M,(s^q_{v_1})^p_{v_2}}(\mathbf{o}), Val_{M,(s^q_{v_1})^p_{v_2}}(\mathbf{v_2})\rangle \in I(\triangleright) \ \text{ and } \ Val_{M,(s^q_{v_1})^p_{v_2}}(+\mathbf{v_2 v_1}) = Val_{M,(s^q_{v_1})^p_{v_2}}(\mathbf{o}) \\
\text{iff} & \forall q \in \mathbb{Q}^+, \text{ if } \ \langle s^q_{v_1}(\mathbf{v_1}), I(\mathbf{o})\rangle \in I(\triangleright), \ \text{ then } \exists p \in \mathbb{Q}^+, \text{ both } \ \langle (I(\mathbf{o}), (s^q_{v_1})^p_{v_2}(\mathbf{v_2})\rangle \in I(\triangleright) \\
& \quad \text{and } \ I(+)\Big(Val_{M,(s^q_{v_1})^p_{v_2}}(\mathbf{v_2}), Val_{M,(s^q_{v_1})^p_{v_2}}(\mathbf{v_1})\Big) = I(\mathbf{o}) \\
\text{iff} & \forall q \in \mathbb{Q}^+, \text{ if } \ \langle q, 1\rangle \in I(\triangleright), \ \text{ then } \exists p \in \mathbb{Q}^+, \text{ both } \ \langle 1, p\rangle \in I(\triangleright) \\
& \quad \text{and } \ I(+)\Big((s^q_{v_1})^p_{v_2}(\mathbf{v_2}), (s^q_{v_1})^p_{v_2}(\mathbf{v_1})\Big) = I(\mathbf{o}) \\
\text{iff} & \forall q \in \mathbb{Q}^+, \text{ if } \ \langle q, 1\rangle \in I(\triangleright), \ \text{ then } \exists p \in \mathbb{Q}^+, \text{ both } \ \langle 1, p\rangle \in I(\triangleright) \ \text{ and } \ I(+)(p, q) = 1 \\
\text{iff} & \forall q \in \mathbb{Q}^+, \text{ if } \ q \leq 1, \ \text{ then } \exists p \in \mathbb{Q}^+, \text{ both } \ 1 \leq p \ \text{ and } \ (p \times q) = 1
\end{array}
$$

Therefore, we have proven the following T-biconditional.

$$\forall \mathbf{v_1}(\triangleright \mathbf{v_1 o} \to \exists \mathbf{v_2}(\triangleright \mathbf{o v_2} \wedge = + \mathbf{v_2 v_1 o})) \ \text{ is true in } M$$

iff

$$\forall q \in \mathbb{Q}^+, \text{ if } \ q \leq 1, \ \text{ then } \exists p \in \mathbb{Q}^+, \text{ both } \ 1 \leq p \ \text{ and } \ (p \times q) = 1$$

Exercise 4

First Sentence

$$\forall \mathbf{v_1} \forall \mathbf{v_2} (\triangleright \mathbf{v_1} \mathbf{v_2} \rightarrow \triangleright \sharp \mathbf{v_2} \sharp \mathbf{v_1})$$

iff

$$\text{For every } p \in \mathbb{Q}^+, \text{for every } q \in \mathbb{Q}^+, \text{if } p \leq q, \text{ then } \frac{1}{q} \leq \frac{1}{p}$$

Second Sentence

$$\forall \mathbf{v_2} (\triangleright + \mathbf{v_2} \mathbf{v_2} \mathbf{v_2} \rightarrow \triangleright \mathbf{o} \sharp + \mathbf{v_2} \mathbf{o})$$

iff

$$\text{For every } p \in \mathbb{Q}^+, \text{if } (p \times p) \leq p, \text{ then } 1 \leq \frac{1}{p \times 1}$$