Question 1

Let $\mathcal{L}$ be our first-order language supplemented with primitive vocabulary for talking about arithmetic. Let f be an embedding of R in ${}^*R$. For any valuation σ based on R, let $f\sigma$ be the valuation based on ${}^*R$ such that, for each variable y, we have $y^{f\sigma} = f(y^\sigma)$.

(i)

Show: $t^{f\sigma} = f(t^\sigma)$, for every term t.

Proof by induction on the complexity of terms.

Convention: When dealing with the structure ${}^*R$, we will use the convention in 10.3.2 (page 218). i.e. ^{*0} is $0^{{}^*R}$ and ${}^*s$ is $s^{{}^*R}$ and ${}^*+$ is $+^{{}^*R}$ and ${}^*\times$ is $\times^{{}^*R}$. This convention is also given in lecture notes 10.

Base Case: $deg(t) = 0$. Hence, $t = y$, where y is a variable. By definition we know that $y^{f\sigma} = f(y^\sigma)$.

Inductive Hypothesis: Assume for every term t such that $deg(t) < n$, we have that $t^{f\sigma} = f(t^\sigma)$.

Now we will show $t^{f\sigma} = f(t^\sigma)$ when $deg(t) = n$. We will consider several cases.

Case 1: $t = 0$.

$$\begin{aligned} f(0^\sigma) &= f(0^R) && \\ &= {}^*0 && \text{By def of embedding} \\ &= 0^{f\sigma} && \end{aligned}$$

Case 2: $t = sr$ where $deg(r) < n$.

By inductive hypothesis we know that $r^{f\sigma} = f(r^\sigma)$. Consider the following.

$$\begin{aligned} f((sr)^\sigma) &= f(s^\sigma(r^\sigma)) && \\ &= f(r^\sigma + 1) && + \text{ is standard addition on } N \\ &= {}^*s(f(r^\sigma)) && \text{By def of embedding} \\ &= {}^*s(r^{f\sigma}) && \text{By Inductive Hypothesis} \\ &= s^{f\sigma}(r^{f\sigma}) && \text{Since } s^{f\sigma} = {}^*s \\ &= (sr)^{f\sigma} && \end{aligned}$$

Case 3: $t = r_1 + r_2$ where $deg(r_1) < n$ and $deg(r_2) < n$.

Note, we are using informal infix notation for simplicity and readability.

By inductive hypothesis we know that $r_1^{f\sigma} = f(r_1^\sigma)$ and $r_2^{f\sigma} = f(r_2^\sigma)$. Consider the following.

$$
\begin{aligned}
f((r_1 + r_2)^\sigma) &= f(r_1^\sigma +^\sigma r_2^\sigma) \\
&= f(r_1^\sigma + r_2^\sigma) && + \text{ is standard addition on } N \\
&= f(r_1^\sigma) \, {}^*\!+ (r_2^\sigma) && \text{By def of embedding} \\
&= r_1^{f\sigma} \, {}^*\!+ r_2^{f\sigma} && \text{By Inductive Hypothesis} \\
&= r_1^{f\sigma} +^{f\sigma} r_2^{f\sigma} && \text{Since } +^{f\sigma} = {}^*\!+ \\
&= (r_1 + r_2)^{f\sigma}
\end{aligned}
$$

Case 4: $t = r_1 \times r_2$ where $deg(r_1) < n$ and $deg(r_2) < n$.

Note, we are using informal infix notation for simplicity and readability.

By inductive hypothesis we know that $r_1^{f\sigma} = f(r_1^\sigma)$ and $r_2^{f\sigma} = f(r_2^\sigma)$. Consider the following.

$$
\begin{aligned}
f((r_1 \times r_2)^\sigma) &= f(r_1^\sigma \times^\sigma r_2^\sigma) \\
&= f(r_1^\sigma r_2^\sigma) && \text{standard multiplication on } N \\
&= f(r_1^\sigma) \, {}^*\!\times f(r_2^\sigma) && \text{By def of embedding} \\
&= r_1^{f\sigma} \, {}^*\!\times r_2^{f\sigma} && \text{By Inductive Hypothesis} \\
&= r_1^{f\sigma} \times^{f\sigma} r_2^{f\sigma} && \text{Since } \times^{f\sigma} = {}^*\!\times \\
&= (r_1 \times r_2)^{f\sigma}
\end{aligned}
$$

Therefore, by induction on the complexity of terms we have proven that $t^{f\sigma} = f(t^\sigma)$ for every term t.

(ii)

Show: $f[\sigma(x/n)] = (f\sigma)(x/fn)$, for any variable x and number n.

Let x be an arbitrary variable. Let n by any number.

Let y be an arbitrary variable. Either $y = x$ or $y \neq x$. We will consider each case separately.

Case 1: $y = x$

Consider the following.

$$
\begin{aligned}
y^{f[\sigma(x/n)]} &= x^{f[\sigma(x/n)]} && \text{Since } y = x \\
&= f(x^{\sigma(x/n)}) \\
&= fn \\
&= x^{(f\sigma)(x/fn)} \\
&= y^{(f\sigma)(x/fn)} && \text{Since } y = x
\end{aligned}
$$

Case 2: $y \neq x$

Consider the following.

$$
\begin{aligned}
y^{f[\sigma(x/n)]} &= f(y^{\sigma(x/n)}) \\
&= f(y^{\sigma}) && \text{Since } y \neq x \\
&= y^{f\sigma} \\
&= y^{(f\sigma)(x/fn)} && \text{Since } y \neq x
\end{aligned}
$$

In either case we have that $y^{f[\sigma(x/n)]} = y^{(f\sigma)(x/fn)}$.

Since y was an arbitrary variable, we have that $f[\sigma(x/n)] = (f\sigma)(x/fn)$, as required.

(iii)

Required to Prove: If f is an isomorphism between R and ${}^*R$, then $\alpha^{f\sigma} = \alpha^\sigma$ for every formula α.

Assume f is an isomorphism between R and ${}^*R$.

To show our required result, we will first prove the following **Claim**.

Claim: $\alpha^{f\sigma} = \top$ iff $\alpha^\sigma = \top$ for every formula α and every valuation σ based on R.

Proof by induction on the complexity of formulas.

Base Case: $deg(\alpha) = 0$. Hence, α is atomic. The only predicate symbol in our language is equality. Hence, α is of the form $t_1 = t_2$ where t_1 and t_2 are terms.

Consider the following.

$$\begin{aligned}
\alpha^{f\sigma} = \top \text{ iff } & (t_1 = t_2)^{f\sigma} = \top \\
\text{iff } & \langle t_1^{f\sigma}, t_2^{f\sigma} \rangle \in id_{^*N} \\
\text{iff } & \langle f(t_1^\sigma), f(t_2^\sigma) \rangle \in id_{^*N} && \text{By part (i)} \\
\text{iff } & \langle t_1^\sigma, t_2^\sigma \rangle \in id_N && \text{By injectivity of } f \\
\text{iff } & (t_1 = t_2)^\sigma = \top \\
\text{iff } & \alpha^\sigma = \top
\end{aligned}$$

Therefore, our claim holds in the **Base Case**.

Inductive Hypothesis: Assume $\alpha^{f\sigma} = \top$ iff $\alpha^\sigma = \top$ for every formula α such that $deg(\alpha) < n$ and every valuation σ based on R.

Now we will show $\alpha^{f\sigma} = \top$ iff $\alpha^\sigma = \top$ when $deg(\alpha) = n$ for any valuation σ based on R.

We will consider several cases.

Case 1: $\alpha = \neg\beta$, where $deg(\beta) = n - 1$.

Consider the following.

$$\begin{aligned}
\alpha^{f\sigma} = \top \text{ iff } & (\neg\beta)^{f\sigma} = \top \\
\text{iff } & \beta^{f\sigma} = \bot \\
\text{iff } & \beta^\sigma = \bot && \text{By Inductive Hypothesis on } \beta \\
\text{iff } & (\neg\beta)^\sigma = \top \\
\text{iff } & \alpha^\sigma = \top
\end{aligned}$$

Therefore, our claim holds in **Case 1**.

Case 2: $\alpha = \beta \to \gamma$, where $deg(\beta) < n$ and $deg(\gamma) < n$.

Consider the following.

$$\begin{aligned}
\alpha^{f\sigma} = \top \ &\text{iff} \ (\beta \to \gamma)^{f\sigma} = \top \\
&\text{iff} \ \beta^{f\sigma} = \bot \ \text{ or } \ \gamma^{f\sigma} = \top \\
&\text{iff} \ \beta^{\sigma} = \bot \ \text{ or } \ \gamma^{\sigma} = \top \qquad\qquad \text{By Inductive Hypothesis on } \beta \text{ and } \gamma \\
&\text{iff} \ (\beta \to \gamma)^{\sigma} = \top \\
&\text{iff} \ \alpha^{\sigma} = \top
\end{aligned}$$

Therefore, our claim holds in **Case 2**.

Case 3: $\alpha = \forall x \beta$, where $deg(\beta) = n - 1$.

Show: $\alpha^{f\sigma} = \top$ iff $\alpha^{\sigma} = \top$ where $\alpha = \forall x \beta$ for any valuation σ based on R.

We will prove both directions of the biconditional separately.

($\Rightarrow$): Assume $\alpha^{f\sigma} = \top$. Hence, $(\forall x \beta)^{f\sigma} = \top$. Hence, for each $u \in {}^*N$ we have that $\beta^{(f\sigma)(x/u)} = \top$.

Since f is an isomorphism, we know that f is a bijection from N to ${}^*N$. Hence, we know that ${}^*N = \{fn : n \in N\}$.

Hence, for each $n \in N$ we have that $\beta^{(f\sigma)(x/fn)} = \top$.

By part (ii) we know that $f[\sigma(x/n)] = (f\sigma)(x/fn)$, for any variable x and number n.

Hence, for each $n \in N$ we have that $\beta^{f[\sigma(x/n)]} = \top$.

By Inductive Hypothesis, we know that $\beta^{f[\sigma(x/n)]} = \top$ iff $\beta^{\sigma(x/n)} = \top$ for each $n \in N$.

Hence, for each $n \in N$ we have that $\beta^{\sigma(x/n)} = \top$.

Hence, $(\forall x \beta)^{\sigma} = \top$.

Hence, $\alpha^{\sigma} = \top$.

($\Leftarrow$): Assume $\alpha^{\sigma} = \top$. Hence, $(\forall x \beta)^{\sigma} = \top$. Hence, for each $n \in N$ we have that $\beta^{\sigma(x/n)} = \top$.

By Inductive Hypothesis, we know that $\beta^{f[\sigma(x/n)]} = \top$ iff $\beta^{\sigma(x/n)} = \top$ for each $n \in N$.

Hence, for each $n \in N$ we have that $\beta^{f[\sigma(x/n)]} = \top$.

By part (ii) we know that $f[\sigma(x/n)] = (f\sigma)(x/fn)$, for any variable x and number n.

Hence, for each $n \in N$ we have that $\beta^{(f\sigma)(x/fn)} = \top$.

Since f is an isomorphism, we know that f is a bijection from N to ${}^*N$. Hence, we know that ${}^*N = \{fn : n \in N\}$.

Hence, for each $u \in {}^*N$ we have that $\beta^{(f\sigma)(x/u)} = \top$.

Hence, $(\forall x \beta)^{f\sigma} = \top$.

Hence, $\alpha^{f\sigma} = \top$.

Combining our two results we get that $\alpha^{f\sigma} = \top$ iff $\alpha^{\sigma} = \top$.

Therefore, our claim holds in **Case 3**.

Therefore, by induction on the complexity of formulas we have proven our **Claim**.

Our **Claim** states that $\alpha^{f\sigma} = \top$ iff $\alpha^{\sigma} = \top$ for every formula α and every valuation σ based on R.

This implies that for every formula α and every valuation σ based on R, we have that $\alpha^{f\sigma} = \alpha^{\sigma}$.

This is exactly what we wanted to show, as required.

Question 2

Let ${}^*R$ be any model for Ω. Let f be a mapping from N to ${}^*N$, defined by: $f(n) = s_n^{{}^*R}$, for all $n \in N$.

(i)

Show: f is injective. i.e. for each $m, n \in N$, if $m \neq n$, then $f(m) \neq f(n)$.

Let $m, n \in N$. Assume $m \neq n$. We will show that $f(m) \neq f(n)$.

Consider the formula $s_m \neq s_n$. Let σ be an arbitrary valuation based on the standard model R.

We know that $(s_m \neq s_n)^\sigma = \top$ iff $\langle m, n \rangle \notin id_N$. Since $m \neq n$, we know that $\langle m, n \rangle \notin id_N$. Hence, by our biconditional, we have that $(s_m \neq s_n)^\sigma = \top$.

Since σ was an arbitrary valuation based on R, we have that $R \models s_m \neq s_n$. Hence, $s_m \neq s_n \in \Omega$.

By assumption, we know that ${}^*R$ is a model for Ω. i.e. ${}^*R \models \Omega$. Since $s_m \neq s_n \in \Omega$, we have ${}^*R \models s_m \neq s_n$.

Let τ be an arbitrary valuation based on ${}^*R$. We know $(s_m \neq s_n)^\tau = \top$ iff $\langle s_m^\tau, s_n^\tau \rangle \notin id_{{}^*N}$.

Since ${}^*R \models s_m \neq s_n$, we have that $(s_m \neq s_n)^\tau = \top$. Hence, by our biconditional we have that $\langle s_m^\tau, s_n^\tau \rangle \notin id_{{}^*N}$. Hence, $s_m^\tau \neq s_n^\tau$.

And $s_m^\tau = s_m^{{}^*R} = f(m)$.

And $s_n^\tau = s_n^{{}^*R} = f(n)$.

Since $s_m^\tau \neq s_n^\tau$, we have that $f(m) \neq f(n)$.

Therefore, f is injective, as required.

(ii)

Show: f is an embedding.

Convention: When dealing with the structure ${}^*R$, we will use the convention in 10.3.2 (page 218). i.e. ^{*0} is $0^{{}^*R}$ and ${}^*s$ is $s^{{}^*R}$ and ${}^*+$ is $+^{{}^*R}$ and ${}^*\times$ is $\times^{{}^*R}$. This convention is also given in lecture notes 10.

In particular, ${}^*s_n$ is $s_n^{{}^*R}$.

We already know f is injective from part (i). Now we will show that f satisfies the additional 4 conditions for an embedding given in Definition 10.3.4 (page 219). i.e. for every $m, n \in N$,

Condition 1: $f0 = {}^*0$
Condition 2: $f(m+1) = {}^*s(fm)$
Condition 3: $f(m+n) = fm \; {}^*\!+ fn$
Condition 4: $f(mn) = fm \; {}^*\!\times fn$

We will prove each of these facts in turn. Note the use of the convention mentioned earlier.

Proof of Condition 1: $f0 = {}^*0$

$$f0 = {}^*s_0 = {}^*0$$

Proof of Condition 2: $f(m+1) = {}^*s(fm)$

Let $m \in N$ be arbitrary.

$$\begin{aligned} f(m+1) &= {}^*s_{m+1} \\ &= {}^*s\,{}^*s_m \\ &= {}^*s(fm) \end{aligned}$$

Proof of Condition 3: $f(m+n) = fm \; {}^*\!+ fn$

Fix an arbitrary $m \in N$. We will use induction on n.

Base Case: $n = 0$

$$\begin{aligned} f(m+0) &= f(m) && \text{Since } \forall v_1(v_1 + 0 = v_1) \in \Omega \\ &= f(m) \; {}^*\!+ {}^*0 && \text{Since } \forall v_1(v_1 + 0 = v_1) \in \Omega \\ &= fm + f0 && \text{By Condition 1} \end{aligned}$$

Inductive Hypothesis: Assume $f(m+n) = fm \; {}^*\!+ fn$ for any $n \in N$

Show: $f(m+(n+1)) = fm \,^*\!+ f(n+1)$

$$\begin{aligned} f(m+(n+1)) &= f((m+n)+1) && \text{Since } \forall v_1 \forall v_2 (v_1 + sv_2 = s(v_1+v_2)) \in \Omega \\ &= {}^*s(f(m+n)) && \text{By Condition 2} \\ &= {}^*s(fm + fn) && \text{By Inductive Hypothesis} \\ &= fm \,^*\!+ {}^*s(fn) && \text{Since } \forall v_1 \forall v_2 (s(v_1+v_2) = v_1 + sv_2) \in \Omega \\ &= fm \,^*\!+ f(n+1) && \text{By Condition 2} \end{aligned}$$

Therefore, by induction on n we have proven **Condition 3**.

Proof of Condition 4: $f(mn) = fm \,^*\!\times fn$

Fix an arbitrary $m \in N$. We will use induction on n.

Base Case: $n = 0$

$$\begin{aligned} f(m0) &= f0 && \text{Since } \forall v_1 (v_1 \times 0 = 0) \in \Omega \\ &= {}^*0 && \text{By Condition 1} \\ &= fm \,^*\!\times {}^*0 && \text{Since } \forall v_1 (v_1 \times 0 = 0) \in \Omega \\ &= fm \,^*\!\times f0 && \text{By Condition 1} \end{aligned}$$

Inductive Hypothesis: Assume $f(mn) = fm \,^*\!\times fn$ for any $n \in N$

Show: $f(m(n+1)) = fm \,^*\!\times f(n+1)$

$$\begin{aligned} f(m(n+1)) &= f(mn+m) && \text{Since } \forall v_1 \forall v_2 (v_1 \times sv_2 = (v_1 \times v_2) + v_1) \in \Omega \\ &= f(mn) \,^*\!+ fm && \text{By Condition 3} \\ &= (fm \,^*\!\times fn) \,^*\!+ fm && \text{By Inductive Hypothesis} \\ &= fm \,^*\!\times {}^*s(fn) && \text{Since } \forall v_1 \forall v_2 ((v_1 \times v_2) + v_1 = v_1 \times sv_2) \in \Omega \\ &= fm \,^*\!\times f(n+1) && \text{By Condition 2} \end{aligned}$$

Therefore, by induction on n we have proven **Condition 4**.

We have verified all 4 conditions given in Definition 10.3.4 (page 219). And we showed in part (i) that f is injective. Therefore, f is an embedding, as required.