Exercise 8.2

a)

Show: $\phi \mathrel{\text{⥽}} \psi \models_{SC} \phi \mathrel{\Box\!\!\rightarrow} \psi$

Let $M = \langle W, \preceq, I \rangle$ be any SC-model.

We will show that for each $r \in W$, if $V_M(\phi \mathrel{\text{⥽}} \psi, r) = 1$, then $V_M(\phi \mathrel{\Box\!\!\rightarrow} \psi, r) = 1$.

i) Suppose for reductio that for some $r \in W$ we have that $V_M(\phi \mathrel{\text{⥽}} \psi, r) = 1$ and $V_M(\phi \mathrel{\Box\!\!\rightarrow} \psi, r) = 0$.

ii) From the latter in i) since $V_M(\phi \mathrel{\Box\!\!\rightarrow} \psi, r) = 0$, we know that there is some closest-to-r ϕ-world at which ψ is false. i.e. for some $a \in W$, we have the following.

a) $V_M(\phi, a) = 1$
b) For any $x \in W$, if $V_M(\phi, x) = 1$, then $a \preceq_r x$
c) $V_M(\psi, a) = 0$

iii) From the former in i) since $V_M(\phi \mathrel{\text{⥽}} \psi, r) = 1$, we have that $V_M(\Box(\phi \rightarrow \psi), r) = 1$ by definition of $\mathrel{\text{⥽}}$. Hence, $V_M(\phi \rightarrow \psi), a) = 1$. Hence, $V_M(\phi, a) = 0$ or $V_M(\psi, a) = 1$. Since from line ii) part a) we have $V_M(\phi, a) = 1$, we must have that $V_M(\psi, a) = 1$.

iv) From line ii) part c) we have $V_M(\psi, a) = 0$ and from iii) we have $V_M(\psi, a) = 1$ which is a contradiction.

Therefore, our initial assumption was wrong. Therefore, $\phi \mathrel{\text{⥽}} \psi \models_{SC} \phi \mathrel{\Box\!\!\rightarrow} \psi$, as required.

b)

Show: $\phi \mathrel{\Box\!\!\rightarrow} \psi \models_{SC} \phi \rightarrow \psi$

Let $M = \langle W, \preceq, I \rangle$ be any SC-model.

We will show that for each $r \in W$, if $V_M(\phi \mathrel{\Box\!\!\rightarrow} \psi, r) = 1$, then $V_M(\phi \rightarrow \psi, r) = 1$.

i) Suppose for reductio that for some $r \in W$ we have that $V_M(\phi \mathrel{\Box\!\!\rightarrow} \psi, r) = 1$ and $V_M(\phi \rightarrow \psi, r) = 0$.

ii) From the latter in i) since $V_M(\phi \rightarrow \psi, r) = 0$, we know that $V_M(\phi, r) = 1$ and $V_M(\psi, r) = 0$.

iii) From the former in ii) we have $V_M(\phi, r) = 1$. And we also know by the "Base" constraint that for any $a \in W$ we have $r \preceq_r a$. Hence, immediately we know r is a closest-to-r ϕ-world.

iv) From the former in i) we have that $V_M(\phi \mathrel{\Box\!\!\rightarrow} \psi, r) = 1$ and from iii) we have that r is a closest-to-r ϕ-world. Hence, $V_M(\psi, r) = 1$.

v) From the latter in ii) we have $V_M(\psi, r) = 0$ and from iv) we have $V_M(\psi, r) = 1$ which is a contradiction.

Therefore, our initial assumption was wrong. Therefore, $\phi \mathrel{\Box\!\!\rightarrow} \psi \models_{SC} \phi \rightarrow \psi$, as required.

Exercise 8.3

c)

CORRECTION: Exercise 8.3 c) was corrected in Sider's errata. We will solve the corrected version below.

Show: $(P \vee Q) \,\Box\!\!\rightarrow R \not\models_{SC} P \,\Box\!\!\rightarrow R$

Consider the following countermodel $M = \langle W, \preceq, I \rangle$.

As in Sider, we will only specify parts of the model that are relevant.

$W = \{r, a, b\}$

$\preceq_r = \{\langle b, a \rangle, ...\}$

$I(Q, b) = I(R, b) = I(P, a) = 1$. And all else is 0.

We have that $V_M((P \vee Q) \,\Box\!\!\rightarrow R, r) = 1$ and $V_M(P \,\Box\!\!\rightarrow R, r) = 0$.

Therefore, $(P \vee Q) \,\Box\!\!\rightarrow R \not\models_{SC} P \,\Box\!\!\rightarrow R$.

d)

Show: $(P \wedge Q) \,\Box\!\!\rightarrow R \not\models_{SC} P \,\Box\!\!\rightarrow (Q \,\Box\!\!\rightarrow R)$

Consider the following countermodel $M = \langle W, \preceq, I \rangle$.

As in Sider, we will only specify parts of the model that are relevant.

$W = \{r, a, b\}$

$\preceq_r = \{\langle a, b \rangle, ...\}$

$\preceq_a = \{\langle b, r \rangle, ...\}$

$I(P, a) = I(Q, b) = 1$. And all else is 0.

We have that $V_M((P \wedge Q) \,\Box\!\!\rightarrow R, r) = 1$ and $V_M(P \,\Box\!\!\rightarrow (Q \,\Box\!\!\rightarrow R), r) = 0$.

Therefore, $(P \wedge Q) \,\Box\!\!\rightarrow R \not\models_{SC} P \,\Box\!\!\rightarrow (Q \,\Box\!\!\rightarrow R)$.

Exercise 8.4

a)

CORRECTION: Exercise 8.4 a) was corrected in Sider's errata. We will solve the corrected version. Sider's correction removes the antecedent $\Diamond P$ in the wff below.

Show: $\vDash_{SC} \sim (P \,\Box\!\!\rightarrow \sim Q) \rightarrow (P \,\Box\!\!\rightarrow Q)$

Let $M = \langle W, \preceq, I \rangle$ be any SC-model.

We will show that for each $r \in W$ we have $V_M(\sim (P \,\Box\!\!\rightarrow \sim Q) \rightarrow (P \,\Box\!\!\rightarrow Q), r) = 1$.

i) Suppose for reductio that for some $r \in W$, $V_M(\sim (P \,\Box\!\!\rightarrow \sim Q) \rightarrow (P \,\Box\!\!\rightarrow Q), r) = 0$.

ii) From i) we get $V_M(\sim (P \,\Box\!\!\rightarrow \sim Q), r) = 1$ and $V_M(P \,\Box\!\!\rightarrow Q, r) = 0$.

iii) From the former in ii) since $V_M(\sim (P \,\Box\!\!\rightarrow \sim Q), r) = 1$, we get $V_M(P \,\Box\!\!\rightarrow \sim Q, r) = 0$. Hence, there is some closest-to-r P-world at which $\sim Q$ is false. i.e. for some $a \in W$, we have the following.

a) $V_M(P, a) = 1$
b) For any $x \in W$, if $V_M(P, x) = 1$, then $a \preceq_r x$
c) $V_M(\sim Q, a) = 0$

iv) From the latter in ii) since $V_M(P \,\Box\!\!\rightarrow Q, r) = 0$ we know there is some closest-to-r P-world at which Q is false. i.e. for some $b \in W$, we have the following.

a) $V_M(P, b) = 1$
b) For any $x \in W$, if $V_M(P, x) = 1$, then $b \preceq_r x$
c) $V_M(Q, b) = 0$

v) From line iii) part a) we have $V_M(P, a) = 1$. Hence, by line iv) part b) since $V_M(P, a) = 1$, we get that $b \preceq_r a$. From line iv) part a) we have $V_M(P, b) = 1$. Hence, by line iii) part b) since $V_M(P, b) = 1$, we get that $a \preceq_r b$. Since $a \preceq_r b$ and $b \preceq_r a$, by anti-symmetry we get that $a = b$.

vi) Since $a = b$ from v) and since $V_M(\sim Q, a) = 0$ from line iii) part c), we get that $V_M(\sim Q, b) = 0$. Hence, $V_M(Q, b) = 1$.

vii) From vi) we have $V_M(Q, b) = 1$ and from line iv) part c) we have $V_M(Q, b) = 0$ which is a contradiction.

So our initial assumption was wrong. Therefore, $\vDash_{SC} \sim (P \,\Box\!\!\rightarrow \sim Q) \rightarrow (P \,\Box\!\!\rightarrow Q)$.

Note that since we have shown $\models_{SC} \sim (P \mathrel{\Box\!\!\rightarrow} \sim Q) \to (P \mathrel{\Box\!\!\rightarrow} Q)$, then trivially we also have that $\models_{SC} \Diamond P \to [\sim (P \mathrel{\Box\!\!\rightarrow} \sim Q) \to (P \mathrel{\Box\!\!\rightarrow} Q)]$. Hence, Sider's original formula for Exercise 8.4 a) and the corrected formula in the errata are actually both valid. But it is clear that the antecedent $\Diamond P$ is redundant.

b)

Show: $\not\models_{SC} [P \mathrel{\Box\!\!\rightarrow} (Q \to R)] \to [(P \wedge Q) \mathrel{\Box\!\!\rightarrow} R]$

Consider the following countermodel $M = \langle W, \preceq, I \rangle$.

As in Sider, we will only specify parts of the model that are relevant.

$W = \{r, a, b\}$

$\preceq_r = \{\langle b, a \rangle, ...\}$

$I(P, a) = I(Q, a) = I(P, b) = 1$. And all else is 0.

We have that $V_M([P \mathrel{\Box\!\!\rightarrow} (Q \to R)] \to [(P \wedge Q) \mathrel{\Box\!\!\rightarrow} R], r) = 0$.

Therefore, $\not\models_{SC} [P \mathrel{\Box\!\!\rightarrow} (Q \to R)] \to [(P \wedge Q) \mathrel{\Box\!\!\rightarrow} R]$.

Exercise 8.5

Show: $\not\models_{LC} [\phi \mathrel{\Box\!\!\rightarrow} (\psi \vee \chi)] \rightarrow [(\phi \mathrel{\Box\!\!\rightarrow} \psi) \vee (\phi \mathrel{\Box\!\!\rightarrow} \chi)]$

Assume for the sake of contradiction that $\models_{LC} [\phi \mathrel{\Box\!\!\rightarrow} (\psi \vee \chi)] \rightarrow [(\phi \mathrel{\Box\!\!\rightarrow} \psi) \vee (\phi \mathrel{\Box\!\!\rightarrow} \chi)]$ for all wffs ϕ, ψ and χ.

Hence, when ϕ is P and ψ is Q and χ is R, where P, Q, and R are sentence letters, we must have $\models_{LC} [P \mathrel{\Box\!\!\rightarrow} (Q \vee R)] \rightarrow [(P \mathrel{\Box\!\!\rightarrow} Q) \vee (P \mathrel{\Box\!\!\rightarrow} R)]$.

However, consider the following countermodel $M = \langle W, \preceq, I \rangle$ for the above instance.

As in Sider, we will only specify parts of the model that are relevant.

$W = \{r, a, b\}$

$\preceq_r = \{\langle a, b \rangle, \langle b, a \rangle, ...\}$

$I(P, a) = I(R, a) = I(P, b) = I(Q, b) = 1$. And all else is 0.

We have that $LV_M([P \mathrel{\Box\!\!\rightarrow} (Q \vee R)] \rightarrow [(P \mathrel{\Box\!\!\rightarrow} Q) \vee (P \mathrel{\Box\!\!\rightarrow} R)], r) = 0$.

Hence, $\not\models_{LC} [P \mathrel{\Box\!\!\rightarrow} (Q \vee R)] \rightarrow [(P \mathrel{\Box\!\!\rightarrow} Q) \vee (P \mathrel{\Box\!\!\rightarrow} R)]$ which is a contradiction.

Therefore, our initial assmuption was wrong.

Therefore, $\not\models_{LC} [\phi \mathrel{\Box\!\!\rightarrow} (\psi \vee \chi)] \rightarrow [(\phi \mathrel{\Box\!\!\rightarrow} \psi) \vee (\phi \mathrel{\Box\!\!\rightarrow} \chi)]$.

Exercise 8.6

Required: Show that every LC-valid wff is SC-valid.

We will use the Hint provided by Sider at the back of the book.

Definition: An LC model is "Stalnaker-acceptable" iff it obeys the limit and anti-symmetry assumptions.

First we will prove the following claim.

Claim: For every LC model $M = \langle W, \preceq, I \rangle$ that is Stalnaker-acceptable, for every $w \in W$, and every wff ϕ, we have that $LV_M(\phi, w) = 1$ iff $SV_M(\phi, w) = 1$.

Note, our notation is such that LV is the valuation under Lewis' semantics, and SV is the valuation under Stalnaker's semantics.

Proof. Let $M = \langle W, \preceq, I \rangle$ be an LC model that is Stalnaker-acceptable. Let $w \in W$.

Show: $LV_M(\phi, w) = 1$ iff $SV_M(\phi, w) = 1$ for every wff ϕ.

We will show this by induction on the complexity of ϕ.

Base Case: Consider the case of a sentence letter P.

We know that $LV_M(P, w) = I(P) = SV_M(P, w)$.

Hence, trivially $LV_M(P, w) = 1$ iff $SV_M(P, w) = 1$.

Inductive Hypothesis: Assume ϕ is such that $LV_M(\phi, w) = 1$ iff $SV_M(\phi, w) = 1$. And assume ψ is such that $LV_M(\psi, w) = 1$ iff $SV_M(\psi, w) = 1$.

Show: $LV_M(\sim \phi, w) = 1$ iff $SV_M(\sim \phi, w) = 1$

$$\begin{aligned} LV_M(\sim \phi, w) = 1 \ \text{ iff } \ & LV_M(\phi, w) = 0 \\ \text{iff } \ & SV_M(\phi, w) = 0 \qquad\qquad \text{By Inductive Hypothesis} \\ \text{iff } \ & SV_M(\sim \phi, w) = 1 \end{aligned}$$

Show: $LV_M(\phi \to \psi, w) = 1$ iff $SV_M(\phi \to \psi, w) = 1$

$$\begin{aligned} LV_M(\phi \to \psi, w) = 1 \ \text{ iff } \ & LV_M(\phi, w) = 0 \ \text{ or } \ LV_M(\psi, w) = 1 \\ \text{iff } \ & SV_M(\phi, w) = 0 \ \text{ or } \ SV_M(\psi, w) = 1 \quad \text{By Inductive Hypothesis} \\ \text{iff } \ & SV_M(\phi \to \psi, w) = 1 \end{aligned}$$

Show: $LV_M(\Box\phi, w) = 1$ iff $SV_M(\Box\phi, w) = 1$

$$\begin{aligned} LV_M(\Box\phi, w) = 1 \ & \text{iff for every } \ u \in W, \ LV_M(\phi, u) = 1 \\ & \text{iff for every } \ u \in W, \ SV_M(\phi, u) = 1 \qquad \text{By Inductive Hypothesis} \\ & \text{iff } \ SV_M(\Box\phi, w) = 1 \end{aligned}$$

Show: $LV_M(\phi \mathbin{\Box\!\!\rightarrow} \psi, w) = 1$ iff $SV_M(\phi \mathbin{\Box\!\!\rightarrow} \psi, w) = 1$

Assume for the sake of contradiction that that the above were false.

Without loss of generality, assume $LV_M(\phi \mathbin{\Box\!\!\rightarrow} \psi, w) = 1$, and $SV_M(\phi \mathbin{\Box\!\!\rightarrow} \psi, w) = 0$. The other case is similar.

Since $SV_M(\phi \mathbin{\Box\!\!\rightarrow} \psi, w) = 0$, we know there is some closest-to-w ϕ world where ψ is false. Hence, there is some $a \in W$ such that,

i) $SV_M(\phi, a) = 1$
ii) For any $x \in W$, if $SV_M(\phi, x) = 1$ then $a \preceq_w x$
iii) $SV(\psi, a) = 0$

From i), since $LV_M(\phi \mathbin{\Box\!\!\rightarrow} \psi, w) = 1$, then we know that EITHER,

a) ϕ is true at no worlds. OR
b) There is some $x \in W$ such that $LV_M(\phi, x) = 1$ and for all $y \in W$, if $y \preceq_w x$ then $LV_M(\phi \rightarrow \psi, y) = 1$.

Since $SV_M(\phi, a) = 1$, by Inductive Hypothesis we have that $LV_M(\phi, a) = 1$. Hence, a) cannot hold.

Hence, b) must hold. So there exists some $x \in W$ such that $LV_M(\phi, x) = 1$. By Inductive Hypothesis, $SV_M(\phi, x) = 1$. By ii) we get that $a \preceq_w x$.

Since $a \preceq_w x$, then from b) we get that $LV_M(\phi \rightarrow \psi, a) = 1$.

Hence, $LV_M(\phi, a) = 0$ or $LV_M(\psi, a) = 1$. By Inductive Hypothesis we get that $SV_M(\phi, a) = 0$ or $SV_M(\psi, a) = 1$. Since $SV_M(\phi, a) = 1$ from i), we must have that $SV_M(\psi, a) = 1$.

But $SV_M(\psi, a) = 1$ contradicts the fact that $SV(\psi, a) = 0$ from iii).

By induction on the complexity of wffs, we have proven the **Claim**. $\Box$

Now we will show that every LC valid wff is SC valid. Let ϕ be an LC valid wff. Hence, ϕ holds in all LC models. In particular, ϕ holds in all Stalnaker-acceptable LC-models. By our **Claim** the truth conditions of both Stalnaker and Lewis semantics in Stalnaker-acceptable LC-models are the same. And every SC model is a Stalnaker-acceptable LC model. Hence, ϕ is valid in all SC models.