Question 1

Part (a)

Show that the set $\{1, 2, 4, 8, 16, ...\}$ of powers of 2 is Δ-definable.

The set can be defined by $\phi(x) :\equiv (\exists y < x)(SS0Ey = x)$ which is a Δ-formula.

Part (b)

Suppose $A \subseteq \mathbb{N}$ is a Σ-definable set.

Required: Show that the complement $\mathbb{N} \setminus A$ is Π-definable.

Let $\phi(x)$ be the Σ-formula such that,

1. $\forall a \in A, \mathfrak{N} \models \phi(\overline{a})$ and

2. $\forall b \notin A, \mathfrak{N} \models \neg\phi(\overline{b})$.

We know that the negation of any Σ-formula is logically equivalent to a Π-formula, since the set of Σ-formulas and the set of Π-formulas are the same, except that Σ-formulas can begin with unbounded $\exists$ quantifiers and Π-formulas can begin with unbounded $\forall$ quantifiers.

And the negation of Σ-formulas starting with an unbounded $\exists$ quantifier is logically equivalent to a Π-formula beginning with an unbounded $\forall$, when we distribute the negation down throughout the formula.

Consider $\neg\phi(x)$ which is logically equivalent to a Π-formula from the above discussion. We will show that the Π-formula logically equivalent to $\neg\phi(x)$ defines $\mathbb{N} \setminus A$.

Let $a \in \mathbb{N} \setminus A$. This implies that $a \notin A$.

By 2. this implies that $\mathfrak{N} \models \neg\phi(\overline{a})$.

Since a was arbitrary, we have that $\forall a \in \mathbb{N} \setminus A, \mathfrak{N} \models \neg\phi(\overline{a})$.

Let $b \notin \mathbb{N} \setminus A$. This implies that $b \in A$.

By 1. this implies that $\mathfrak{N} \models \phi(\overline{b})$ which implies that $\mathfrak{N} \not\models \neg\phi(\overline{b})$.

Since b was arbitrary, we have that $\forall b \notin \mathbb{N} \setminus A, \mathfrak{N} \not\models \neg\phi(\overline{b})$.

Therefore, we have shown that the Π-formula logically equivalent to $\neg\phi(x)$ defines $\mathbb{N} \setminus A$.

Part (c)

Let $B \subseteq \mathbb{N} \times \mathbb{N}$ be a Σ-definable set.

Let $B_1 \subseteq \mathbb{N}$ be the set $B_1 = \{b_1 \in \mathbb{N} : (b_1, b_2) \in B \text{ for some } b_2 \in \mathbb{N}\}$

Required: Show that B_1 is Σ-definable.

Let $\phi(x, y)$ be the Σ-formula that defines B.

Now consider the following Σ-formula that defines the set B_1.

$\sigma(x) :\equiv (\exists y)\phi(x, y)$

Question 2

Part (a)

Compute the following natural numbers.

Note: $16910355000 = 2^3 \cdot 3 \cdot 5^4 \cdot 7 \cdot 11^5$

(i) $< 3, 0, 4, 2, 1 >= 2^4 \cdot 3^1 \cdot 5^5 \cdot 7^3 \cdot 11^2 = 6225450000$

(ii) $(16910355000)_3 = 3$

(iii) $|16910355000| = 5$

(iv) $(16910355000)_{42} = 0$

(v) $17 ^\frown 42 = 0$

Part (b)

(i) $(5, 13) \in$ ITHPRIME is FALSE because 13 is the 6th prime, not the 5th prime.

(ii) $(1200, 3) \in$ LENGTH is TRUE because $1200 =< 3, 0, 1 >$ and thus, $|1200| = 3$.

(iii) IthElement$(\overline{1}, \overline{2}, \overline{3630})$ is FALSE because $3630 = 2 \cdot 3 \cdot 5 \cdot 11^2$ which implies that 3630 is not even a code number since it does not have 7 as a prime factor.

Part (c)

Required: An $\mathcal{L}_{NT}$-term $t(x)$ such that $\mathfrak{N} \models t(\overline{\ulcorner \phi \urcorner}) = \overline{\ulcorner \neg \phi \urcorner}$.

Let $t(x) :\equiv \cdot SS0ESS0SSS0ESx$

In more informal notation, $t(x) :\equiv \overline{2}^{\overline{2}} \cdot \overline{3}^{Sx}$.

Now let $s : Vars \to A$ be a variable assignment function where A is the universe of $\mathfrak{N}$.

We will check that, $\mathfrak{N} \models t(\overline{\ulcorner \phi \urcorner}) = \overline{\ulcorner \neg \phi \urcorner}[s]$. We will use Lemma 2.8.4 below.

Clearly $s(t(\overline{\ulcorner \phi \urcorner})) = s(\overline{2}^{\overline{2}} \cdot \overline{3}^{S\overline{\ulcorner \phi \urcorner}}) = \overline{2}^{\overline{2}} \cdot \overline{3}^{S\overline{\ulcorner \phi \urcorner}}$

Similarly, $s(\overline{\ulcorner \neg \phi \urcorner}) = \overline{\ulcorner \neg \phi \urcorner} = \overline{< 1, \ulcorner \phi \urcorner >} = \overline{2^2 \cdot 3^{\ulcorner \phi \urcorner + 1}} = \overline{2^2} \cdot \overline{3^{\ulcorner \phi \urcorner + 1}} = \overline{2}^{\overline{2}} \cdot \overline{3}^{\overline{\ulcorner \phi \urcorner + 1}} = \overline{2}^{\overline{2}} \cdot \overline{3}^{S\overline{\ulcorner \phi \urcorner}}$

Therefore, we have found a satisfactory $t(x)$.

Question 4

Part (a)

Required: Express Goldbach's conjecture as a Π-sentence. You may use $Even(x)$ and $Prime(x)$ as subformulas.

The following sentence ϕ is a Π-sentence that expresses Goldbach's conjecture.

$$\phi :\equiv (\forall x)\Big((Even(x) \land x > SS0) \to (\exists y < x)(\exists z < x)(Prime(y) \land Prime(z) \land x = y + z)\Big)$$

Part (b)

Let N be the axioms of Robinson Arithmetic.

Required: Show that Goldbach's Conjecture is true if, and only if, $N \nvdash \neg Goldbach$.

($\Rightarrow$)

Consider the following proof by contradiction.

Assume Goldbach's conjecture is true. i.e. $\mathfrak{N} \models Goldbach$.

Assume for sake of contradiction that $N \vdash \neg Goldbach$.

Since $N \vdash \neg Goldbach$, by Soundness Theorem, we have that $N \models \neg Goldbach$.

But, we know that $\mathfrak{N} \models N$, so we have that $\mathfrak{N} \models \neg Goldbach$

This is equivalent to saying $\mathfrak{N} \not\models Goldbach$.

So we have that $\mathfrak{N} \models Goldbach$ and $\mathfrak{N} \not\models Goldbach$ which is a contradiction.

Therefore, our assumption that $N \vdash \neg Goldbach$ was wrong, and so $N \nvdash \neg Goldbach$.

($\Leftarrow$)

Consider the following proof by contraposition.

Assume $N \vdash \neg Goldbach$

Required to Prove: Goldbach's conjecture is false.

By Soundness Theorem, we have that $N \models \neg Goldbach$.

But, we know that $\mathfrak{N} \models N$, so we have that $\mathfrak{N} \models \neg Goldbach$

This is equivalent to saying $\mathfrak{N} \not\models Goldbach$.

Therefore, we have shown that Goldbach's conjecture is false.

Since we have proven both directions, we have proven that Goldbach's Conjecture is true if, and only if, $N \nvdash \neg Goldbach$, as required.

Question 5

Suppose θ is an $\mathcal{L}_{NT}$-sentence such that,

$N \vdash [\theta \leftrightarrow Thm_N(\overline{\ulcorner \neg\theta \urcorner})]$.

Required: Determine whether θ is true or false in $\mathfrak{N}$ and justify your answer.

We will show that θ is FALSE in $\mathfrak{N}$.

Proof. Assume for sake of contradiction that θ is true in $\mathfrak{N}$. i.e. $\mathfrak{N} \models \theta$.

Since $N \vdash [\theta \leftrightarrow Thm_N(\overline{\ulcorner \neg\theta \urcorner})$, by the Soundness Theorem, we have that,

$N \models [\theta \leftrightarrow Thm_N(\overline{\ulcorner \neg\theta \urcorner})]$.

Since $\mathfrak{N} \models N$, we have that $\mathfrak{N} \models [\theta \leftrightarrow Thm_N(\overline{\ulcorner \neg\theta \urcorner})]$.

This implies that $\mathfrak{N} \models \theta$ if and only if $\mathfrak{N} \models Thm_N(\overline{\ulcorner \neg\theta \urcorner})$.

Since we assumed that $\mathfrak{N} \models \theta$, we have that $\mathfrak{N} \models Thm_N(\overline{\ulcorner \neg\theta \urcorner})$.

This says that it is true that $\ulcorner \neg\theta \urcorner$ is the Godel number of the formula $\neg\theta$ that is a theorem of N.

i.e. $N \vdash \neg\theta$.

By Soundness Theorem, we have that $N \models \neg\theta$.

Since $\mathfrak{N} \models N$, we have that $\mathfrak{N} \models \neg\theta$.

This implies that $\mathfrak{N} \not\models \theta$.

So, we have that $\mathfrak{N} \models \theta$ and $\mathfrak{N} \not\models \theta$.

This is a contradiction. Therefore our assumption that θ was true in $\mathfrak{N}$ was wrong.

Therefore θ is FALSE in $\mathfrak{N}$, as required. $\square$

Alternative Proof

Suppose θ is an $\mathcal{L}_{NT}$-sentence such that,

$N \vdash [\theta \leftrightarrow Thm_N(\overline{\ulcorner \neg\theta \urcorner})]$.

Required: Determine whether θ is true or false in $\mathfrak{N}$ and justify your answer.

We will show that θ is FALSE in $\mathfrak{N}$.

First, as a sub-proof, we will show that $N \nvdash \theta$. Assume for sake of contradiction that $N \vdash \theta$.

Call $N \vdash \theta$ "result". Now consider the following deduction.

1.	$N \vdash [\theta \leftrightarrow Thm_N(\overline{\ulcorner \neg\theta \urcorner})]$	
2.	$N \vdash \theta$	result
3.	$N \vdash Thm_N(\overline{\ulcorner \neg\theta \urcorner})$	1,2, (PC)

From line 3 and the Soundness Theorem, we that $N \models Thm_N(\overline{\ulcorner \neg\theta \urcorner})$.

Since $\mathfrak{N} \models N$, we have that $\mathfrak{N} \models Thm_N(\overline{\ulcorner \neg\theta \urcorner})$

This says that it is true that $\ulcorner \neg\theta \urcorner$ is the Godel number of the formula $\neg\theta$ that is a theorem of N.

i.e. we have that $N \vdash \neg\theta$.

Since $N \vdash \theta$ and $N \vdash \neg\theta$, we have that $N \vdash \perp$ by (PC).

By Soundness Theorem, this implies that $N \models \perp$. Since, $\mathfrak{N} \models N$, we have that $\mathfrak{N} \models \perp$.

This is a contradiction, as $\perp$ cannot have a model. Therefore, our assumption that $N \vdash \theta$ was wrong.

Therefore $N \nvdash \theta$. Now we will prove our main result.

Proof. Since $N \nvdash \theta$, by the contrapositive of the Completeness Theorem, we have that $N \not\models \theta$.

This implies that $N \models \neg\theta$.

Since, $\mathfrak{N} \models N$, we have that $\mathfrak{N} \models \neg\theta$ which implies that $\mathfrak{N} \not\models \theta$.

Therefore, θ is FALSE in $\mathfrak{N}$, as required. $\square$

Question 6

Part (a)

Suppose that $\phi(x)$ is a formula that weakly represents A, that is,

if $a \in A$, then $N \vdash \phi(\overline{a})$

if $b \notin A$, then $N \nvdash \phi(\bar{b})$

Required: Show that A is semi-calculable by describing a suitable computer program P_0.

Consider the following computer program P_0.

For each $n \in \mathbb{N}$ as an input, the program P_0 will search for a deduction of $\phi(\overline{n})$ from N, and if a deduction is found, will return "yes".

Note, a deduction of $\phi(\overline{n})$ from N is a finite sequence of logical axioms, non-logical axioms N, and rules of inference. The program P_0 will search for a finite sequence that is a deduction of $\phi(\overline{n})$ from N. Since $\mathcal{L}_{NT}$ is a countable language, all possible deductions in $\mathcal{L}_{NT}$ can be enumerated. Thus, P_0 will check every possible deduction in an enumeration of all possible finite deductions in $\mathcal{L}_{NT}$.

If $n \in A$, then since $N \vdash \phi(\overline{n})$, and deductions are finite, P_0 will eventually halt once it finds a finite deduction as we can enumerate all possible deductions. Once P_0 finds a deduction, it will return "yes".

If $n \notin A$, we have that since since $N \nvdash \phi(\overline{n})$, there is no finite deduction of $\phi(\overline{n})$ from N. Since there are countably infinitely many possible finite deductions, P_0 will keep running forever to try and find a deduction, but will never halt, as no deduction can ever be found.

Therefore, we have a program P_0 such that $\forall a \in A$, P_0 will eventually return "yes" on input a, and $\forall b \notin A$, P_0 will never halt (i.e. run forever) on input b.

Thus, A is semi-decidable, as required.

Part (b)

Required: Show that a set $A \subseteq \mathbb{N}$ is calculable if and only if both A and $\mathbb{N} \setminus A$ are semi-calculable.

$(\Rightarrow)$

Assume $A \subseteq \mathbb{N}$ is calculable. i.e. There exists a computer program P such that,

If $a \in A$, program P returns "yes" on input a.

If $b \notin A$, program P returns "no" on input b.

Now consider the following program P_0 that will show that A is semi-calculable.

For any $n \in \mathbb{N}$ as an input, the program P_0 will then input n into P.

If $n \in A$, then P will return "yes". The program P_0 will then also return "yes".

If $n \notin A$, then P will return "no". However, the program P_0 will then run forever (i.e. not halt). We can do this by simply making P_0 go on an endless loop.

Thus, our program P_0 shows that A is semi-calculable.

Now consider the following program P_1 that will show that $\mathbb{N} \setminus A$ is semi-calculable.

For any $n \in \mathbb{N}$ as an input, the program P_1 will then input n into P.

If $n \in \mathbb{N} \setminus A$, then $n \notin A$. So P will return "no". The program P_1 will then return "yes".

If $n \notin \mathbb{N} \setminus A$, then $n \in A$. So P will return "yes". However, the program P_1 will then run forever (i.e. not halt). We can do this by simply making P_1 go on an endless loop.

Thus, our program P_1 shows that $\mathbb{N} \setminus A$ is semi-calculable.

($\Leftarrow$)

Assume that A and $\mathbb{N} \setminus A$ are semi-calculable. i.e. There exists two computer programs P_0 and P_1 such that,

If $a \in A$, program P_0 returns "yes" on input a.

If $b \notin A$, program P_0 does not halt (i.e. runs forever) when given input b.

And,

If $a \in \mathbb{N} \setminus A$, program P_1 returns "yes" on input a.

If $b \notin \mathbb{N} \setminus A$, program P_1 does not halt (i.e. runs forever) when given input b.

Now consider the following program P that will show that A is calculable.

1. For any $n \in \mathbb{N}$ as an input, the program P will first input n into P_0 and have P_0 run for one minute.

2. If P_0 halts and returns "yes" within the first minute, we know that $n \in A$. So then P will also return "yes".

3. If P_0 does not halt within the first minute, we save the state of the program P_0. We will then input n into P_1 and have P_1 run for one minute.

4. If P_1 halts and returns "yes" within a minute, we know that $n \in \mathbb{N} \setminus A$. This implies that $n \notin A$. So then P will return "no".

5. If P_1 does not halt within a minute, we will save the state of the program P_1. We will then input n back into P_0 and have P_0 run for another 1 minute.

P will repeat steps 1-5 until P finally halts and outputs either "yes" or "no".

We know that for any input $n \in \mathbb{N}$, either $n \in A$ or $n \in \mathbb{N} \setminus A$.

If $n \in A$, then P_0 will eventually halt. If $n \in \mathbb{N} \setminus A$, then P_1 will eventually halt.

So we know that exactly one of P_0 and P_1 will always halt for each input $n \in \mathbb{N}$.

And therefore, from the above reasoning, P will always output either "yes" or "no".

Therefore, we have shown that A is calculable.

Question 7

The sequence of Fibonacci numbers is defined by $Fib(0) = Fib(1) = 1$ and $Fib(n) = Fib(n-2) + Fib(n-1)$ for all $n \geq 2$.

Required: Write down a Δ-formula defining the function $Fib : \mathbb{N} \to \mathbb{N}$.

We will write a Δ-formula $Fibon(a, b)$ defining the set $FIBON = \{(a, b) \in \mathbb{N}^2 : b = Fib(a)\}$.

We will first write a Δ-formula $FibonConSeq(a, b, c)$ defining the set

$FIBONCONSEQ = \{(a, b, c) \in \mathbb{N}^3 : b = Fib(a) \text{ and } c =< Fib(0), ..., Fib(a) >\}$

We will define the construction sequence, $< Fib(0), ..., Fib(a) >$ such that $Fib(0) = Fib(1) = 1$ and $Fib(n) = Fib(n-2) + Fib(n-1)$ for $2 \leq n \leq a$.

We need c to be a codenumber and the length of c to be $a + 1$ since $Fib(n)$ starts at $n = 0$. The first element of c needs to be 1 since $Fib(0) = 1$. If $a > 0$, we need the second element of c to be 1 since $Fib(1) = 1$. We need the last element of c to be b since $F(a) = b$. Finally, we need every element in c following the first two elements, if they exist, to be equal to the sum of the previous two elements.

$FibonConSeq(a, b, c) :\equiv$

$Codenumber(c) \wedge Length(c, Sa) \wedge IthElement(\overline{1}, \overline{1}, c) \wedge$

$(a > 0 \to IthElement(\overline{1}, \overline{2}, c)) \wedge IthElement(b, Sa, c)$

$\wedge (\forall i < a)\Big(i > \overline{1} \to (\exists x < c)(\exists y < c)(\exists z < c)[Ithelement(x, i, c) \wedge Ithelement(y, Si, c)$

$\wedge Ithelement(z, SSi, c) \wedge (z = x + y)]\Big)$

Note, in our Δ-formula for $FibonConSeq$, we added the extra condition that $i > \overline{1}$ because if $i = 0$, or $i = \overline{1}$, then there does not exist two prior elements to i in c. So we must add this extra condition.

Now we can define our Δ-formula $Fibon(a, b)$. We will ignore the bound for now.

$Fibon(a, b) :\equiv (\exists c < Bound) FibonConSeq(a, b, c).$

Now we will determine $Bound$. We will use the following two facts.

Fib Fact: For every $a \geq 0$, $Fib(a) \leq 2^a$. This is a trivial fact that can be proven by induction.

Lemma 5.8.7 (textbook): If $a \in \mathbb{N}$, such that $a \geq 1$ then $p_a \leq 2^{a^a}$, where p_a is the ath prime.

We know that our codenumber c will look like $c =< Fib(0), Fib(1), ..., F(a) >$.

It follows that,

$$\begin{aligned}
c &=< Fib(0), ..., F(a) > \\
&= p_1^{Fib(0)+1} ... p_{a+1}^{Fib(a)+1} \\
&< \underbrace{p_{a+1}^{Fib(0)+1} ... p_{a+1}^{Fib(a)+1}}_{a+1 \text{ terms}} && \text{Since } p_{a+1} \text{ is the largest prime in this product} \\
&< \underbrace{p_{a+1}^{Fib(a)+1} ... p_{a+1}^{Fib(a)+1}}_{a+1 \text{ terms}} && \text{Since } Fib \text{ is an increasing function} \\
&= (p_{a+1})^{(Fib(a)+1)^{a+1}} \\
&\leq (p_{a+1})^{(2^a+1)^{(a+1)}} && \text{By Fib Fact} \\
&\leq \left(2^{(a+1)^{(a+1)}}\right)^{(2^a+1)^{(a+1)}} && \text{By Lemma 5.8.7}
\end{aligned}$$

Therefore, we have $Bound = \left(2^{(a+1)^{(a+1)}}\right)^{(2^a+1)^{(a+1)}}$.

Therefore, we have that,

$Fibon(a, b) :\equiv (\exists c < \left(\overline{2}^{Sa^{Sa}}\right)^{(\overline{2}^a+\overline{1})^{Sa}})FibonConSeq(a, b, c)$.

Thus, we have we found a Δ-formula defining the function $Fib : \mathbb{N} \to \mathbb{N}$.

Alternate Proof (Don't Submit)

Suppose θ is an $\mathcal{L}_{NT}$-sentence such that,

$N \vdash [\theta \leftrightarrow Thm_N(\overline{\ulcorner \neg\theta \urcorner})]$.

Required: Determine whether θ is true or false in $\mathfrak{N}$ and justify your answer.

We will show that θ is false in $\mathfrak{N}$.

Assume for sake of contradiction that θ is true in $\mathfrak{N}$. i.e. $\mathfrak{N} \models \theta$.

We must have that $N \models \theta$ because if $N \not\models \theta$, then $N \models \neg\theta$. Since $\mathfrak{N} \models N$ we would have that $\mathfrak{N} \models \neg\theta$. This implies that $\mathfrak{N} \not\models \theta$ which would contradict our assumption that $\mathfrak{N} \models \theta$.

By Completeness Theorem, since $N \models \theta$, we have that $N \vdash \theta$.

Call $N \vdash \theta$ "result". Now consider the following deduction.

1.	$N \vdash [\theta \leftrightarrow Thm_N(\overline{\ulcorner \neg\theta \urcorner})]$	
2.	$N \vdash \theta$	result
3.	$N \vdash Thm_N(\overline{\ulcorner \neg\theta \urcorner})$	1,2, (PC)

From line 4 and the Completeness Theorem, we that $N \models Thm_N(\overline{\ulcorner \neg\theta \urcorner})$.

Since $\mathfrak{N} \models N$, we have by transitivity of $\models$ that $\mathfrak{N} \models Thm_N(\overline{\ulcorner \neg\theta \urcorner})$

This implies that it is true that $\overline{\ulcorner \neg\theta \urcorner}$ is the Godel number of the formula $\neg\theta$ that is a theorem of N.

i.e. we have that $N \vdash \neg\theta$.

Since $N \vdash \theta$ and $N \vdash \neg\theta$, we have that $N \vdash \bot$ by (PC).

By Soundness Theorem, this implies that $N \models \bot$. Since, $\mathfrak{N} \models N$, by transitivity of $\models$, we have that $\mathfrak{N} \models \bot$. This is a contradiction, as $\bot$ cannot have a model.

Therefore, our assumption that θ was true in $\mathfrak{N}$ was false.

Therefore, θ is FALSE in $\mathfrak{N}$, as required.