Exercise 4.3

a)

Show: $\nvDash \forall x(Fx \rightarrow Gx) \rightarrow \forall x(Gx \rightarrow Fx)$

Consider the following countermodel $M = \langle D, I \rangle$.

$D = \{u, v\}$

$I(F) = \{u\}$

$I(G) = \{u, v\}$

Let g be the variable assignment $g(x) = u$ for all variables x.

Clearly, $V_{M,g}(\forall x(Fx \rightarrow Gx) \rightarrow \forall x(Gx \rightarrow Fx)) = 0$.

Therefore, $\nvDash \forall x(Fx \rightarrow Gx) \rightarrow \forall x(Gx \rightarrow Fx)$.

b)

Show: $\nvDash \forall x(Fx \vee \sim Gx) \rightarrow (\forall x Fx \vee \sim \exists x Gx)$

Consider the following countermodel $M = \langle D, I \rangle$.

$D = \{u, v\}$

$I(F) = \{u\}$

$I(G) = \{u\}$

Let g be the variable assignment $g(x) = u$ for all variables x.

Clearly, $V_{M,g}(\forall x(Fx \vee \sim Gx) \rightarrow (\forall x Fx \vee \sim \exists x Gx)) = 0$.

Therefore, $\nvDash \forall x(Fx \vee \sim Gx) \rightarrow (\forall x Fx \vee \sim \exists x Gx)$.

c)

Show: $Rab \nvDash \exists x Rxx$.

Consider the following countermodel $M = \langle D, I \rangle$.

$D = \{u, v\}$

$I(a) = u$

$I(b) = v$

$I(R) = \{\langle u, v\rangle\}$

Let g be the variable assignment $g(x) = u$ for all variables x.

Clearly, $V_{M,g}(Rab) = 1$, but $V_{M,g}(\exists x Rxx) = 0$.

Therefore, $Rab \not\models \exists x Rxx$.

d)

Show: $Fx \not\models \forall x Fx$.

Consider the following countermodel $M = \langle D, I\rangle$.

$D = \{u, v\}$

$I(F) = \{u\}$

Let g be the variable assignment $g(x) = u$ for all variables x.

Clearly, $V_{M,g}(Fx) = 1$, but $V_{M,g}(\forall x Fx) = 0$.

Therefore, $Fx \not\models \forall x Fx$.

e)

Show: $\forall x \forall y \forall z[(Rxy \wedge Ryz) \rightarrow Rxz], \forall x \exists y Rxy \not\models \exists x Rxx$.

Consider the following countermodel $M = \langle D, I\rangle$.

$D = \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, 3, ...\}$ is the set of natural numbers.

$I(R) = \{\langle u, v\rangle : u < v\}$ where $<$ is the standard ordering on the natural numbers.

Let g be the variable assignment $g(x) = u$ for all variables x. Clearly, $V_{M,g}(\forall x \forall y \forall z[(Rxy \wedge Ryz) \rightarrow Rxz]) = 1$ and $V_{M,g}(\forall x \exists y Rxy) = 1$, but $V_{M,g}(\exists x Rxx) = 0$.

Therefore, $\forall x \forall y \forall z[(Rxy \wedge Ryz) \rightarrow Rxz], \forall x \exists y Rxy \not\models \exists x Rxx$.

Exercise 4.4

We will use the derived rules given in the Lecture 10 notes.

a)

Show: $\forall x(Fx \to Gx), \forall x(Gx \to Hx) \vdash_{PC} \forall x(Fx \to Hx)$

1.	$\forall x(Fx \to Gx)$	premise
2.	$\forall x(Fx \to Gx) \to (Fx \to Gx)$	PC1
3.	$Fx \to Gx$	1,2, MP
4.	$\forall x(Gx \to Hx)$	premise
5.	$\forall x(Gx \to Hx) \to (Gx \to Hx)$	PC1
6.	$Gx \to Hx$	4,5, MP
7.	$Fx \to Hx$	3,6, PL
8.	$\forall x(Fx \to Hx)$	7, UG

b)

Show: $\vdash_{PC} Fa \to \exists x Fx$

1.	$Fa \to \exists x Fx$	EG (Existential Generalization)

c)

Show: $\vdash_{PC} \forall x Rax \to \forall x \exists y Ryx$

First we will show that $\forall x Rax \vdash_{PC} \forall x \exists y Ryx$ and then use the Deduction Theorem.

1.	$\forall x Rax$	premise
2.	$\forall x Rax \to Rax$	PC1
3.	Rax	1,2, MP
4.	$Rax \to \exists y Ryx$	EG (Existential Generalization)
5.	$\exists y Ryx$	3,4, MP
6.	$\forall x \exists y Ryx$	5, UG

Since we have shown that $\forall x Rax \vdash_{PC} \forall x \exists y Ryx$, then by the Deduction Theorem we have that $\vdash_{PC} \forall x Rax \to \forall x \exists y Ryx$.

d)

Show: $\exists x Rax, \forall y(Ray \to \forall z Rzy) \vdash_{PC} \exists x \forall z Rzx$

1.	$\exists x Rax$	premise
2.	Rad	1, EI (Existential Instantiation/Rule C)
3.	$\forall y(Ray \to \forall z Rzy)$	premise
4.	$\forall y(Ray \to \forall z Rzy) \to (Rad \to \forall z Rzd)$	PC1
5.	$Rad \to \forall z Rzd$	3,4, MP
6.	$\forall z Rzd$	2,5, MP
7.	$\forall z Rzd \to \exists x \forall z Rzx$	EG (Existential Generalization)
8.	$\exists x \forall z Rzx$	6,7, MP

Exercise 9.1

For each formula, give a validity proof if the wff is SQML-valid, and a countermodel if it is invalid.

a)

Sider gave solutions for a) in the back of the book.

b)

Show: $\models_{SQML} \Diamond\forall xFx \to \exists x\Diamond Fx$

Let $M = \langle W, D, I\rangle$ be any SQML-model.

We will show that for any variable assignment g and any $w_1 \in W$, we have $V_{M,g}(\Diamond\forall xFx \to \exists x\Diamond Fx, w_1) = 1$.

i) Suppose for reductio that for some variable assignment g and some $w_1 \in W$, we have $V_{M,g}(\Diamond\forall xFx \to \exists x\Diamond Fx, w_1) = 0$. Hence, $V_{M,g}(\Diamond\forall xFx, w_1) = 1$ and $V_{M,g}(\exists x\Diamond Fx, w_1) = 0$.

ii) From the former in i) since $V_{M,g}(\Diamond\forall xFx, w_1) = 1$, we know that for some $w_2 \in W$ we have $V_{M,g}(\forall xFx, w_2) = 1$.

iii) From the latter in i) since $V_{M,g}(\exists x\Diamond Fx, w_1) = 0$ we know that for every $u \in U$, $V_{M,g^x_u}(\Diamond Fx, w_1) = 0$. Since $D \neq \emptyset$, for some particular $v \in D$ we know $V_{M,g^x_v}(\Diamond Fx, w_1) = 0$. Hence, $V_{M,g^x_v}(Fx, w_2) = 0$.

iv) From ii) since $V_{M,g}(\forall xFx, w_2) = 1$, we know that $V_{M,g^x_v}(Fx, w_2) = 1$.

v) From iii) we have $V_{M,g^x_v}(Fx, w_2) = 0$ and from iv) we have $V_{M,g^x_v}(Fx, w_2) = 1$ which is a contradiction.

Therefore, our initial assumption was wrong. Therefore, $\models_{SQML} \Diamond\forall xFx \to \exists x\Diamond Fx$, as required.

c)

Sider gave solutions for c) in the back of the book.

d)

Show: $\models_{SQML} \Box\forall x(Fx \to Gx) \to (\forall x\Box Fx \to \Box\forall xGx)$

Let $M = \langle W, D, I \rangle$ be any SQML-model.

We will show that for any variable assignment g and any $w_1 \in W$, we have $V_{M,g}(\Box\forall x(Fx \to Gx) \to (\forall x\Box Fx \to \Box\forall xGx), w_1) = 1$.

i) Suppose for reductio that for some variable assignment g and some $w_1 \in W$, we have $V_{M,g}(\Box\forall x(Fx \to Gx) \to (\forall x\Box Fx \to \Box\forall xGx), w_1) = 0$.

ii) From i) we get $V_{M,g}(\Box\forall x(Fx \to Gx), w_1) = 1$ and $V_{M,g}(\forall x\Box Fx \to \Box\forall xGx, w_1) = 0$.

iii) From the latter in ii) since $V_{M,g}(\forall x\Box Fx \to \Box\forall xGx, w_1) = 0$ we get $V_{M,g}(\forall x\Box Fx, w_1) = 1$ and $V_{M,g}(\Box\forall xGx, w_1) = 0$.

iv) From the latter in iii) since $V_{M,g}(\Box\forall xGx, w_1) = 0$, we know for some $w_2 \in W$ we have $V_{M,g}(\forall xGx, w_2) = 0$. Hence, for some $u \in D$ we have $V_{M,g^x_u}(Gx, w_2) = 0$.

v) From the former in iii) since $V_{M,g}(\forall x\Box Fx, w_1) = 1$, we have $V_{M,g^x_u}(\Box Fx, w_1) = 1$. Hence, $V_{M,g^x_u}(Fx, w_2) = 1$.

vi) From the former in ii) since $V_{M,g}(\Box\forall x(Fx \to Gx), w_1) = 1$ we get $V_{M,g}(\forall x(Fx \to Gx), w_2) = 1$. Hence, $V_{M,g^x_u}(Fx \to Gx, w_2) = 1$. Hence, $V_{M,g^x_u}(Fx, w_2) = 0$ or $V_{M,g^x_u}(Gx, w_2) = 1$. Since from v) we have $V_{M,g^x_u}(Fx, w_2) = 1$, we must have that $V_{M,g^x_u}(Gx, w_2) = 1$.

vii) From iv) we have $V_{M,g^x_u}(Gx, w_2) = 0$ and from vi) we have $V_{M,g^x_u}(Gx, w_2) = 1$ which is a contradiction.

Therefore, our initial assumption was wrong. Therefore, $\models_{SQML} \Box\forall x(Fx \to Gx) \to (\forall x\Box Fx \to \Box\forall xGx)$, as required.

e)

Show: $\not\models_{SQML} \exists x(Nx \wedge \forall y(Ny \to y = x) \wedge \Box Ox) \to \Box\exists x(Nx \wedge \forall y(Ny \to y = x) \wedge Ox)$

Consider the following countermodel $M = \langle W, D, I \rangle$.

$W = \{a, b\}$

$D = \{u\}$

$I(N) = \{\langle u, a \rangle\}$

$I(O) = \{\langle u, a \rangle, \langle u, b \rangle\}$

Let g be the variable assignment $g(x) = u$ for all variables x.

Clearly, $V_{M,g}(\exists x(Nx \wedge \forall y(Ny \rightarrow y = x) \wedge \Box Ox) \rightarrow \Box \exists x(Nx \wedge \forall y(Ny \rightarrow y = x) \wedge Ox), a) = 0$.

Therefore, $\not\models_{SQML} \exists x(Nx \wedge \forall y(Ny \rightarrow y = x) \wedge \Box Ox) \rightarrow \Box \exists x(Nx \wedge \forall y(Ny \rightarrow y = x) \wedge Ox)$.

Exercise 9.4

Determine the VDQML-invalidity of the following formulas.

a)

Show: $\nvDash_{VDQML} \Box\forall x Fx \rightarrow \forall x\Box Fx$

Consider the following countermodel $M = \langle W, R, D, Q, I\rangle$ where

$W = \{a, b\}$

$R = \{\langle a, b\rangle\}$

$D = \{u, v\}$

$D_a = \{v\}$

$D_b = \{u\}$

$I(F) = \{\langle u, b\rangle\}$

Let g be the variable assignment $g(x) = u$ for all variables x.

Clearly, $V_{M,g}(\Box\forall x Fx \rightarrow \forall x\Box Fx, a) = 0$.

Therefore, $\nvDash_{VDQML} \Box\forall x Fx \rightarrow \forall x\Box Fx$.

b)

Show: $\nvDash_{VDQML} \exists x\Box Fx \rightarrow \Box\exists x Fx$

Consider the following countermodel $M = \langle W, R, D, Q, I\rangle$ where

$W = \{a, b\}$

$R = \{\langle a, b\rangle\}$

$D = \{u, v\}$

$D_a = \{u\}$

$D_b = \{v\}$

$I(F) = \{\langle u, b\rangle\}$

Let g be the variable assignment $g(x) = u$ for all variables x.

Clearly, $V_{M,g}(\exists x \Box Fx \to \Box \exists x Fx, a) = 0$.

Therefore, $\not\models_{VDQML} \exists x \Box Fx \to \Box \exists x Fx$.

c)

Show: $\not\models_{VDQML} \forall x \Box \exists y(y = x)$

Consider the following countermodel $M = \langle W, R, D, Q, I\rangle$ where

$W = \{a, b\}$

$R = \{\langle a, b\rangle\}$

$D = \{u, v\}$

$D_a = \{u\}$

$D_b = \{v\}$

Let g be the variable assignment $g(x) = u$ for all variables x.

Clearly, $V_{M,g}(\forall x \Box \exists y(y = x), a) = 0$.

Therefore, $\not\models_{VDQML} \forall x \Box \exists y(y = x)$.

Exercise 9.6

a)

Sider provides solutions for a) in the back of the book.

b)

Show: $(\Box\forall xFx \wedge \Diamond\forall xGx) \to \Diamond\forall x(Fx \wedge Gx)$

1.	$Fx \wedge Gx \to Fx \wedge Gx$	PL
2.	$Fx \to (Gx \to (Fx \wedge Gx))$	1,PL (Import/Export)
3.	$\forall x(Fx \to (Gx \to (Fx \wedge Gx)))$	2, UG
4.	$\forall xFx \to \forall x(Gx \to (Fx \wedge Gx))$	Distribution, 3, MP
5.	$\forall x(Gx \to (Fx \wedge Gx)) \to (\forall xGx \to \forall x(Fx \wedge Gx))$	Distribution
6.	$\forall xFx \to (\forall xGx \to \forall x(Fx \wedge Gx))$	4,5, PL
7.	$\Box\forall xFx \to \Box(\forall xGx \to \forall x(Fx \wedge Gx))$	6, Nec, K, MP
8.	$\Box(\forall xGx \to \forall x(Fx \wedge Gx)) \to (\Diamond\forall xGx \to \Diamond\forall x(Fx \wedge Gx))$	K$\Diamond$
9.	$\Box\forall xFx \to (\Diamond\forall xGx \to \Diamond\forall x(Fx \wedge Gx))$	7,8, PL
10.	$(\Box\forall xFx \wedge \Diamond\forall xGx) \to \Diamond\forall x(Fx \wedge Gx)$	9, PL (Import/Export)

c)

Show: $(\forall x\Box(Fx \to Gx) \wedge \exists x\Diamond Fx) \to \exists x\Diamond Gx$

1.	$\forall x\Box(Fx \to Gx) \to \Box(Fx \to Gx)$	PC1
2.	$\Box(Fx \to Gx) \to (\Diamond Fx \to \Diamond Gx)$	K$\Diamond$
3.	$\forall x\Box(Fx \to Gx) \to (\Diamond Fx \to \Diamond Gx)$	1,2,PL
4.	$\forall x\Box(Fx \to Gx) \to (\sim\Diamond Gx \to \sim\Diamond Fx)$	3, PL
5.	$\forall x(\forall x\Box(Fx \to Gx) \to (\sim\Diamond Gx \to \sim\Diamond Fx))$	4, UG
6.	$\forall x\Box(Fx \to Gx) \to \forall x(\sim\Diamond Gx \to \sim\Diamond Fx)$	PC2, 5, MP
7.	$\forall x(\sim\Diamond Gx \to \sim\Diamond Fx) \to (\forall x\sim\Diamond Gx \to \forall x\sim\Diamond Fx)$	Distribution
8.	$\forall x\Box(Fx \to Gx) \to (\forall x\sim\Diamond Gx \to \forall x\sim\Diamond Fx)$	6,7, PL
9.	$\forall x\Box(Fx \to Gx) \to (\sim\forall x\sim\Diamond Fx \to \sim\forall x\sim\Diamond Gx)$	8, PL
10.	$\forall x\Box(Fx \to Gx) \to (\exists x\Diamond Fx \to \exists x\Diamond Gx)$	9. def of $\exists$
11.	$(\forall x\Box(Fx \to Gx) \wedge \exists x\Diamond Fx) \to \exists x\Diamond Gx$	10, PL (Import/Export)

d)

Show: $\forall y \Box \exists x(x = y)$

1.	$y = y$	RX
2.	$\forall x \sim (x = y) \rightarrow \sim (y = y)$	PC1
3.	$\sim \forall x \sim (x = y)$	1,2, PL
4.	$\exists x(x = y)$	3, def of $\exists$
5.	$\Box \exists x(x = y)$	4, Nec
6.	$\forall y \Box \exists x(x = y)$	5, UG