Question 1

Show that the following statements are equivalent.

a) A is a finite set.

b) Every total ordering on A is a well-ordering on A.

c) If R is a well-ordering on A, then R^{-1} is also a well-ordering on A.

We will make use of Theorem 7B given by Enderton on page 173.

Theorem 7B: Let $<$ be a total ordering on A. Then it is a well-ordering iff there does not exist any function (an infinite descending chain) $f : \omega \to A$ with $f(n^+) < f(n)$ for every $n \in \omega$.

Pf: $a) \Leftrightarrow$ b)

($\Rightarrow$)

Assume statement a). i.e. A is a finite set.

Assume for the sake of contradiction the negation of statement b). i.e. There exists a total ordering on A that is not a well-ordering.

By Theorem 7B, this implies that there exists an infinite descending chain on A which contradicts the fact that A is a finite set.

Therefore, statement c) holds and every total ordering on A is also a well-ordering on A.

($\Leftarrow$)

Assume statement b). i.e. Every total ordering on A is a well-ordering on A.

Assume for the sake of contradiction the negation of statement a). i.e. A is an infinite set.

Let R be a total ordering which is transitive and satisfies trichotomy. Therefore R is also a well-ordering by the assumption b).

Clearly, R^{-1} is also a total ordering, as R^{-1} is still transitive and satisfies trichotomy. Therefore, R^{-1} is also a well-ordering by the assumption b).

We know $\langle A, R\rangle$ must have an initial segment that is isomorphic to ω since A is infinite. This initial segment does not have a greatest element.

Therefore, the inverse of this initial segment is an initial segment of $\langle A, R^{-1} \rangle$ which has no least element.

Thus, R^{-1} which is a well-ordering, is not a well-ordering. This is a contradiction.

Therefore, our assumption was wrong and A is finite.

Pf: $c) \Leftrightarrow a)$

($\Rightarrow$)

Assume statement c). i.e. If R is a well-ordering on A, then R^{-1} is also a well-ordering on A.

Assume for the sake of contradiction the negation of statement a). i.e. A is an infinite set.

We know $\langle A, R \rangle$ must have an initial segment that is isomorphic to ω since A is infinite. This initial segment does not have a greatest element.

Therefore, the inverse of this initial segment is an initial segment of $\langle A, R^{-1} \rangle$ which has no least element.

Thus, R^{-1} which is a well-ordering, is not a well-ordering. This is a contradiction.

Therefore, our assumption was wrong and A is finite.
($\Leftarrow$)

Assume statement a). i.e. A is a finite set.

Assume for the sake of contradiction the negation of statement c). i.e. There exists a well-ordering R on A and R^{-1} is not a well-ordering on A.

Since R^{-1} is not a well-ordering, Theorem 7B says that there exists an infinite descending chain on A which contradicts the fact that A is a finite set.

Therefore, our assumption of the negation of statement c) was wrong. Therefore, statement c) holds.

Pf: $c) \Leftrightarrow b)$

Since we have already shown that $c) \Leftrightarrow a)$ and $a) \Leftrightarrow b)$, then by transitivity of the biconditional we have that $c) \Leftrightarrow b)$.

Therefore, we have shown equivalences between all 3 statements, as required.

Question 2

Prove the statement that every countably infinite partially ordered set $\langle A, R\rangle$ is isomorphic to one of its proper subsets $\langle B, R \restriction B\rangle$ or provide a counterexample to it.

We will provide a counter example.

Consider $\langle \omega^+, \in\rangle$.

Assume for the sake of contradiction that $\langle \omega^+, \in\rangle$ is isomorphic to one of its proper subsets $\langle B, \in\restriction B\rangle$ where $B \subseteq \omega^+$ with $B \neq \omega^+$ and $f : \omega^+ \to B$ is an isomorphism.

Since $f : \omega^+ \to B$ is bijective, we know that $card(\omega^+) = \aleph_0 = card(B)$.

So we know that B is also countably infinite.

Now, in the Yang 11 slides we proved Lemma 7.4.

Lemma 7.4: Let $\langle A, <_A\rangle$ and $\langle B, <_B\rangle$ be partially ordered sets and $f : A \to B$ an isomorphism. Then for any $a \in A$, we have that

$$f[seg_A(a)] = seg_B f(a)$$

.
Now, applying Lemma 7.4 to our problem, we get that for $\omega \in \omega^+$,

$$f[seg_{\omega^+}(\omega)] = seg_B f(\omega)$$

Now, we will simplify the left and right sides.

$$\{f(x) \in B | x \in \omega\}] = \{x \in B | x \in f(\omega)\}$$

But, we know that $\{f(x) \in B | x \in \omega\}]$ is an infinite set since f is a bijection, and thus injective. So each $x \in \omega$ produces a unique $f(x)$.

In fact, $card(\{f(x) \in B | x \in \omega\}]) = \aleph_0$.

But since $f(\omega) \in B$, and since every element of B is finite, we have that $\{x \in B | x \in f(\omega)\}$ is a finite set.

So the left hand side is an infinite set, whereas the right hand side is a finite set. This is a contradiction.

Therefore, $\langle \omega^+, \in\rangle$ is not isomorphic to one of its proper subsets.

Question 3

Required: Prove the statement that any partial ordering can be extended to a total ordering [i.e. that if $\langle A, R\rangle$ is an arbitrary poset then there exists $S \subseteq A^2$ such that $R \subseteq S$ and $\langle A, S\rangle$ is a total ordering] or provide a counterexample.

Proof. We will prove the statement. Let $\langle A, R\rangle$ be a partially ordered set.

Now, define the set $B = \{X \subseteq A^2 | R \subseteq X \text{and } X \text{ is a partial ordering on } A\}$.

Now, consider the partially ordered set by STRICT inclusion, $\langle B, \subset\rangle$.

This is a partially ordered set since it is clearly irreflexive and transitive by properties of strict subsets.

Now, consider any chain $C \subseteq B$. Clearly $\bigcup C$ is an upper bound for C since $\forall X \in C$, we have that $X \subset \bigcup C$.

And $\bigcup C \in B$ since $\bigcup C$ inherits its irreflexivity and transitivity from the elements of C.

Therefore, every chain in B has an upper bound in B.

Therefore, by Zorn's Lemma B contains a maximal element. Call this element D.

We will prove that D is a total ordering on A.

We already know that D is a partial ordering since $D \in B$. So all we need to prove is that D satisfies trichotomy.

Assume for the sake of contradiction that $\exists x, y \in A$ such that x and y were not comparable in D.

Well, we can simply add more comparisons in a new relation E such that $D \subset E$. First we will add $\langle x, y\rangle$ to E. We need to add more points to retain transitivity.

If $\langle a, x\rangle \in D$, then add $\langle a, y\rangle$ to E.

If $\langle y, b\rangle \in D$, then add $\langle x, b\rangle$ to E.

If $\langle c, x\rangle \in D$ and $(y, d) \in D$, then add $\langle c, d\rangle$ to E.

Do this process for all points $\langle x, y\rangle$ that are not compared in D.

Thus, E is transitive and is a partial ordering. So $E \in B$.

But, $D \subset E$ which contradicts the fact that D was maximal.

This is a contradiction. Therefore, D must be a total ordering.

Thus, we have proven that there exists a $D \subseteq A^2$ such that $R \subseteq D$ and $\langle A, D \rangle$ is a total ordering, as required. $\square$

Question 4

Required: Prove the well-ordering theorem from Zorn's Lemma without appealing to other forms of the axiom of choice.

Proof. Let A be an arbitrary set. If $A = \emptyset$, then we're done since it is trivially well-ordered.

So, assume $A \neq \emptyset$. Now, consider the following set B.

$$B = \{X \subseteq A^2 | \text{and } X \text{ is well-ordering on } A\}$$

We know $B \neq \emptyset$ since any finite subset of A can be well-ordered by the natural numbers.

Now, consider the partially ordered set by strict inclusion, $\langle B, \subset \rangle$.

This is a partially ordered set since it is clearly irreflexive and transitive by properties of strict subsets.

Now, consider any chain $C \subseteq B$. Clearly $\bigcup C$ is an upper bound for C since $\forall X \in C$, we have that $X \subset \bigcup C$.

And $\bigcup C \in B$ since $\bigcup C$ inherits its irreflexivity and transitivity from the elements of C.

Therefore, every chain in B has an upper bound in B.

Therefore, by Zorn's Lemma B contains a maximal element. Call this element D.

By definition, we know that D is a well-ordering on some subset $X \subseteq A$.

Now, we must prove that $X = A$ to show that all of A is well-orderable.

So, assume for the sake of contradiction that $X \neq A$.

Therefore, $\exists a \in A(a \notin X)$.

Now, define a new set $Y = X \cup \{a\}$.

We know that D can be written as a (possibly infinite) ascending chain with a least element x_1.

i.e. $x_1 < x_2 < ....$

where $x_1, x_2, ... \in X$.

We will construct a new well-ordering on Y.

Now, we can simply define a well-ordering on Y to be all of the comparisons in D while also making a the new least element.

i.e. Let $E = D \cup \{\langle a, x\rangle | x \in X \subset A\}$.

So a new (possibly infinite) ascending chain with a least element is

$a < x_1 < x_2 < ...$

where $a, d_1, d_2 \in E$.

So E is a well-ordering. This implies that $E \in B$.

But, $D \subset E$.

This contradicts the fact that D was the maximal element in $\langle B, \subset\rangle$.

Therefore, $X = A$. So E is a well-ordering on all of A.

Therefore, A is well-orderable, completing the proof, as required. $\square$

Question 5

Part 1

Show that any set of mutually disjoint open intervals on $\mathbb{R}$ is countable (i.e. show that if for every $X \in \{U_i | i \in I\}$ there exist $u, v \in \mathbb{R}$ such that $y \in X \leftrightarrow u < y < v$ and furthermore $U_i \cap U_j = \emptyset$ for every $i, j \in I$ where $i \neq j$, then I is countable).

Proof. Let I be an index set as above and let $S = \{U_i | i \in I\}$ where each U_i is an open interval on $\mathbb{R}$ and $U_i \cap U_j = \emptyset$ for every $i, j \in I$ with $i \neq j$.

We will show that I is countable.

Since we know that the rationals are dense in the reals, we know that each open interval contains a rational number.

Since the open intervals are disjoint we know that there are no rationals in common between any of the intervals.

Since each interval is infinite in size, we must use the axiom of choice to select one rational number from each interval.

i.e. Consider the following function $f : S \to \mathbb{Q}$ defined by $f(U_i) = x_i$ where each $x_i \in U_i$ and $i \in I$.

Notice that our choice of x_i comes from the axiom of choice, and from the discussion above about the rationals being dense in the reals we know that f is well-defined.

Clearly f is an injective function since each U_i is disjoint.

Therefore, $S \preccurlyeq \mathbb{Q}$.

Therefore, $card(S) \leq card(\mathbb{Q}) = \aleph_0$.

Since the elements of S are indexed by the elements of I, we have that $card(S) = card(I)$.

Therefore, $card(I) \leq \aleph_0$ as well. So I must either be a finite set, or countably infinite.

Regardless, I is a countable set, completing the proof, as required.

$\square$

Part 2

By a similar reasoning, show that if A is a collection of circular discs in the plane no two of which intersect, then A is countable.

Proof. Let I be an index set and let $D = \{U_i | i \in I\}$ where $U_i = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = r_i\}$ where $r_i \neq r_j$ and $U_i \cap U_j$ for every $i, j \in I$ where $i \neq j$.

In other words, D is a set of disjoint circular discs in $\mathbb{R}^2$.

We will show that I is countable.

Since there are infinitely many points in each disc, and we know that $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$, we know that there exists a point $(x_i, y_i) \in \mathbb{Q}^2$ in each disk U_i.

Since the discs are disjoint, we know that there are no two ordered pairs of rationals common to any two discs.

Since each disc contains an infinite number of points, we must use the axiom of choice to select one ordered pair of rational numbers from each interval.

i.e. Consider the following function $f : D \to \mathbb{Q}^2$ defined by $f(U_i) = (x_i, y_i)$ where each $(x_i, y_i) \in U_i$ and $i \in I$.

Notice that our choice of (x_i, y_i) comes from the axiom of choice, and from the discussion above about $\mathbb{Q}^2$ being dense in $\mathbb{R}^2$ we know that f is well-defined.

Clearly f is an injective function since each U_i is disjoint.

Therefore, $D \preccurlyeq \mathbb{Q}^2$.

Therefore, $card(D) \leq card(\mathbb{Q}^2) = \aleph_0$.

Since the elements of D are indexed by the elements of I, we have that $card(D) = card(I)$.

Therefore, $card(I) \leq \aleph_0$ as well. So I must either be a finite set, or countably infinite.

Regardless, I is a countable set, completing the proof, as required. $\square$

Question 6

Define a permutation of K to be any one-to-one function from K onto K. If $\kappa = card(K)$, then $\kappa!$ is $\{f | f \text{is a permutation of } K\}$. Show that if K is infinite, then

$$\kappa! = 2^\kappa$$

First we will show that $2^\kappa = \kappa^\kappa$.

Since κ is infinite, we have that $2 \leq \kappa$. By Theorem 6L, we have that $2^\kappa \leq \kappa^\kappa$

Now we will show that $\kappa^\kappa \leq 2^\kappa$. Consider the following.

$$\begin{aligned}
\kappa &\leq 2^\kappa && \text{Obvious fact of cardinals} \\
\kappa^\kappa &\leq (2^\kappa)^\kappa && \text{By Theorem 6L, since } \kappa \leq 2^\kappa \\
&= 2^{\kappa \cdot \kappa} && \text{By Theorem 6I} \\
&= 2^\kappa && \text{By absorption law since } \kappa \text{ infinite so that } \kappa \cdot \kappa = \max(\kappa, \kappa) = \kappa
\end{aligned}$$

Therefore, $k^\kappa \leq 2^\kappa$.

Since $2^\kappa \leq \kappa^\kappa$ and $k^\kappa \leq 2^\kappa$, by Cantor-Shroder-Bernstein we have that $2^\kappa = \kappa^\kappa$.

Now we will show that $\kappa! = 2^\kappa$.

First we will show that $\kappa! \leq 2^\kappa$.

First consider the function $g : \{f | f \text{is a permutation of } K\} \to {}^K K$ defined by $g(f) = f$.

i.e. g takes a permutation (which is a bijective function) to itself.

Clearly g is one-to-one since if $g(i) = g(j)$ then $i = j$.

Therefore, $\{f | f \text{is a permutation of } K\} \preccurlyeq {}^K K$. Therefore, $\kappa! \leq \kappa^\kappa = 2^\kappa$.

Now we will show that $2^\kappa \leq \kappa!$

Notice that $card(P(K)) = 2^\kappa$.

So consider the function $h : P(K) \to \{f | f \text{is a permutation of } K\}$ defined by $h(A) = f$ where f is a permutation whose ONLY FIXED POINTS are elements of A.

i.e. $\forall a \in A(f(a) = a)$ and $\forall k \in K(f(k) = k \Rightarrow k \in A)$.

So h is a function that takes any subset $A \subseteq K$ and maps it to a permutation which ONLY keeps the elements of A fixed.

Clearly h is one-to-one because if $h(A) = i$ and $h(B) = j$, and $h(A) = h(B)$ then $i = j$. So i and j are the same function and have the same fixed-point mappings. Therefore, $A = B$.

Therefore, $P(K) \preccurlyeq \{f | f \text{is a permutation of } K\}$. Therefore, $2^\kappa \leq \kappa!$

Since $\kappa! \leq 2^\kappa$ and $2^\kappa \leq \kappa!$, by Cantor-Shroder-Bernstein we have that $\kappa! = 2^\kappa$.

Question 7

Let $A \neq \emptyset$ and let $Pt(A)$ be the set of all partitions of A. Define a relation $\preccurlyeq$ on $Pt(A)$ by

$$S_1 \preccurlyeq S_2$$

if and only if $\forall C \in S_1$, $\exists D \in S_2$ such that $C \subseteq D$.

(a)

Show that $\preccurlyeq$ is an order relation.

First, notice that $\preccurlyeq$ is clearly reflexive. For every $S \in Pt(A)$, we have that $S \preccurlyeq S$ since for every $C \in S_1$ we have that $C \subseteq C$.

Also, $\preccurlyeq$ is anti-symmetric. To show this, assume $S_1, S_2 \in Pt(A)$ such that $S_1 \preccurlyeq S_2$ and $S_2 \preccurlyeq S_1$.

This means that $\forall C_i \in S_1$, $\exists D_i \in S_2$ such that $C_i \subseteq D_i$.

Similarly, $\forall D_i \in S_2$, $\exists C_j \in S_1$ such that $D_i \subseteq C_j$.

So, let $C_i \in S_1$ be arbitrary. Let $D_i \in S_2$ such that $C_i \subseteq D_i$.

Now, since $D_i \in S_2$, we know that there exists a $C_j \in S_1$ such that $D_i \subseteq C_j$.
Since $C_i \subseteq D_i$ and $D_i \subseteq C_j$ we have that $C_i \subseteq C_j$.

Since S_1 and S_2 are partitions we know that elements of S_1 are mutually disjoint and elements of S_2 are mutually disjoint. Every $x \in A$ is in exactly one element of S_1 and one element of S_2.

So it must be the case that $C_i = C_j$.

This implies that $C_i \subseteq D_i$ and $D_i \subseteq C_i$ which implies that $C_i = D_i$. Therefore, $S_1 = S_2$, so $\preccurlyeq$ is anti-symmetric.

Finally, is transitive. Assume $S_1, S_2, S_3 \in Pt(A)$ such that $S_1 \preccurlyeq S_2$ and $S_2 \preccurlyeq S_3$. So we have that,

$\forall A \in S_1$, $\exists B \in S_2$ such that $A \subseteq B$.

$\forall B \in S_2$, $\exists C \in S_3$ such that $B \subseteq C$.

Now, let $A \in S_1$ be arbitrary. So, there exists a $B \in S_2$ such that $A \subseteq B$.

Since $B \in S_2$, there exists $C \in S_3$ such that $B \subseteq C$.

But since $A \subseteq B$ and $B \subseteq C$, we have that $A \subseteq C$.

Therefore, $\forall A \in S_1$, $\exists C \in S_3$ such that $A \subseteq C$. This means that $S_1 \preccurlyeq S_3$

Therefore, $\preccurlyeq$ is transitive.

Since $\preccurlyeq$ is symmetric, anti-symmetric and transitive, we have that $\preccurlyeq$ is a partial ordering.

Note, $\preccurlyeq$ is not a total ordering since not all elements of $Pt(A)$ are comparable.

For instance, if $A = \{1, 2, 3, 4\}$ and $S_1 = \{\{1, 2\}, \{3, 4\}\} \in Pt(A)$ and $S_2 = \{\{1, 3\}, \{2, 4\}\} \in Pt(A)$ then we have that $S_1 \not\preccurlyeq S_2$ and $S_2 \not\preccurlyeq S_1$.

(b)

Show that if $U \subseteq Pt(A)$ then U has both an infimum and a supremum.

Note that $J = \{A\}$ is clearly an upper bound for U.

And the set of singletons of elements of A is clearly a lower bound. Call this set K.

But J and K are not necessarily the supremum and infimum.

But a supremum and infimum must clearly exist since $\preccurlyeq$ simply deals with refinements of partitions. So, we can simply refine the upper bound J by breaking up cells into smaller cells until we reach the smallest upper bound for U.

Similarly, we can simply take unions of cells in K in order to get a partition that is the greatest lower bound for U.

Here is a possible construction of the supremum.

For the supremum, first take $\bigcup U$. Now, $\bigcup U$ is simply the set of all the cells of the partitions contained in U.

Now, we will construct new sets V and W.

$V = \{X \in \bigcup U | \exists Y \in \bigcup U(Y \neq X \wedge X \cap Y \neq \emptyset)\}$.

So V is the set of all of all cells in $\bigcup U$ which intersect with some other cell in $\bigcup U$.

$W = \{X \in \bigcup U | \forall Y \in \bigcup U(Y \neq X \Rightarrow X \cap Y = \emptyset\}$.

So W is the set of all disjoint cells in $\bigcup U$.

Now, take $S = \{\bigcup V\} \cup W$. By our construction, S is a partition and is the supremum of U.

A similar construction can be done for the infimum.

Question 8

Required: Prove that the Cantor Normal Form is unique.

Notation: Instead of using $\in$, we will simply use $<$ for notational convenience.

Proof. Assume that a non-zero ordinal number α has two Cantor Normal Form representations. i.e.

$$\alpha = \omega^{\gamma_1} n_1 + ... + \omega^{\gamma_k} n_k = \omega^{\beta_1} m_1 + ... + \omega^{\beta_k} m_k$$

where $n_1, ..., n_k, m_1, ..., m_k$ are nonzero natural numbers and $\gamma_k < \gamma_{k-1} < ... < \gamma_1$ and $\beta_k < \beta_{k-1} < ... < \beta_1$

We will prove that $\gamma_i = \beta_i$ and $n_i = m_i$ for every $i \in \{1, ..., k\}$.

Step 1

Now consider the following.

$$\begin{aligned}
\omega^{\gamma_1} &\leq \alpha \\
&= \omega^{\gamma_1} n_1 + ... + \omega^{\gamma_k} n_k \\
&< \omega^{\gamma_1} n_1 + ... + \omega^{\gamma_1} n_k && \text{Since } \gamma_k < \gamma_{k-1} < ... < \gamma_1 \\
&= \omega^{\gamma_1}(n_1 + ... + n_k) && \text{Left Distributive Law} \\
&< \omega^{\gamma_1} \omega && \text{Since } (n_1 + ... + n_k) \in \omega \\
&= \omega^{\gamma_1^+}
\end{aligned}$$

So, we have that $\omega^{\gamma_1} \leq \alpha < \omega^{\gamma_1^+}$. So γ_1 is the largest ordinal number for which $\omega^{\gamma_1} \leq \alpha$.

Thus, it must be the case that we have uniqueness and that $\gamma_1 = \beta_1$. From now on, we we will just refer to γ_1.

Step 2

Now we will show that $n_1 = m_1$. Assume for the sake of contradiction that $n_1 \neq m_1$. Without loss of generality, assume that $n_1 < m_1$. Then, $n_1^+ \leq m_1$.

First notice that by the first use of the logarithm theorem, we have that $\alpha = \omega^{\gamma_1} n_1 + \rho_1$ where $\rho_1 = \omega^{\gamma_2} n_2 + ... + \omega^{\gamma_k} n_k$.

Importantly, $\rho_1 < \omega^{\gamma_1}$.

Now, consider the following.

$$\begin{aligned}
\alpha &= \omega^{\gamma_1} n_1 + ... + \omega^{\gamma_k} n_k \\
&= \omega^{\gamma_1} n_1 + \rho_1 \\
&< \omega^{\gamma_1} n_1 + \omega^{\gamma_1} && \text{Since } \rho_1 < \omega^{\gamma_1} \\
&= \omega^{\gamma_1} n_1^+ \\
&\leq \omega^{\gamma_1} m_1 && \text{By assumption} \\
&< \omega^{\gamma_1} m_1 + \omega^{\beta_2} m_2 + ... + \omega^{\beta_k} m_k \\
&= \alpha && \text{Since } \gamma_1 = \beta_1
\end{aligned}$$

But the above shows that $\alpha < \alpha$ which is a contradiction.

Therefore our assumption that $n_1 \in m_1$ was wrong. We could make a symmetric argument where $m_1 \in n_1$ to derive a similar contradiction.

Regardless, it must be the case that $n_1 = m_1$.

So, we know that $\omega^{\gamma_1} n_1 = \omega^{\beta_1} m_1$.

Now, by the cancellation laws, we can get rid of this term and define

$$\alpha_2 = \omega^{\gamma_2} n_2 + ... + \omega^{\gamma_k} n_k = \omega^{\beta_2} m_2 + ... + \omega^{\beta_k} m_k$$

We can now repeat Step 1 and Step 2 with α_2 to show that $\omega^{\gamma_2} n_2 = \omega^{\beta_2} m_2$.

By repeated applications of Step 1 and Step 2 we eventually get that $\omega^{\gamma_i} n_i = \omega^{\beta_i} m_i$ for all $i \in \{1, ..., k\}$.

Therefore, the Cantor Normal Form representation of α is unique, completing the proof, as required. $\square$

Question 9

(a)

Required: Show that if $\beta < \gamma$, γ a limit ordinal, then $\omega^\beta \cdot \omega < \omega^\gamma$.

Since γ is a limit ordinal and $\beta < \gamma$, we know that $\beta + 1 < \gamma$.

By Corollary 8P, we have that $\beta + 1 < \gamma \Leftrightarrow \omega^{\beta+1} < \omega^\gamma$.

Therefore, $\omega^{\beta+1} < \omega^\gamma$.

Finally, by Theorem 8R we have that $\omega^{\beta+1} = \omega^\beta \cdot \omega$.

Therefore, $\omega^\beta \cdot \omega < \omega^\gamma$, as required.

(b)

Required: Show that if $\beta < \gamma$, γ a limit ordinal, then $\omega^\beta + \omega^\gamma = \omega^\gamma$.

Since $\beta < \gamma$, by the subtraction theorem we know that there exists an ordinal σ such that $\beta + \sigma = \gamma$.

It follows that,

$$\begin{aligned}
\omega^\gamma &= \omega^{\beta+\sigma} \\
\omega^\gamma &= \omega^\beta \cdot \omega^\sigma && \text{By Theorem 8R}
\end{aligned}$$

Now, consider the following when we add ω^β.

$$\begin{aligned}
\omega^\beta + \omega^\gamma &= \omega^\beta + \omega^\beta \omega^\sigma && \text{Adding } \omega^\beta \text{ to both sides from the left} \\
&= \omega^\beta (1 + \omega^\sigma) && \text{Left Distributive Law} \\
&= \omega^\beta \cdot \omega^\sigma && \text{Since } 1 + \omega^\sigma = \omega^\sigma \\
&= \omega^{\beta+\sigma} && \text{By Theorem 8R} \\
&= \omega^\gamma
\end{aligned}$$

Therefore, $\omega^\beta + \omega^\gamma = \omega^\gamma$, as required.

(c)

Show that if $\alpha < \omega^\lambda$, λ a limit ordinal, then $\alpha + \omega^\lambda = \omega^\lambda$.

Since $\alpha < \omega^\lambda$, by the subtraction theorem we know there exists an ordinal σ such that $\alpha + \sigma = \omega^\lambda$.

Now, we know by the result in part (d) that it cannot be the case that $0 < \sigma < \omega^\lambda$.

Furthermore, $\sigma \neq 0$ since then we would have $\alpha = \omega^\lambda$ which would contradict the fact that $\alpha < \omega^\lambda$.

Similarly, it cannot be the case that $\sigma > \omega^\lambda$ since then $\alpha + \sigma > \omega^\lambda$ which would contradict the fact that $\alpha + \sigma = \omega^\lambda$.

So it must be the case that $\sigma = \omega^\lambda$.

Therefore, $\alpha + \omega^\lambda = \omega^\lambda$, as required.

(d)

Show that ω^β cannot be written as $\omega^\beta = \gamma_1 + \gamma_2$ where $0 < \gamma_2 < \omega^\beta$.

Assume for the sake of contradiction that $\omega^\beta = \gamma_1 + \gamma_2$ and $0 < \gamma_2 < \omega^\beta$.

I have not completely figured out how to solve this problem. But I think it has something to do with how ordinal arithmetic is right continuous and strictly increasing.