Exercise 1

$$h(d) = \begin{cases} d & \text{if } d \in \text{Standard} \\ S(d) & \text{if } d \in \text{Nonstandard} \end{cases}$$

Required: Show that h is an isomorphism from M^* onto M^*.

Lemma 1: If $d \in$ Standard, then $S(d) \in$ Standard.

Let $d \in$ Standard. Hence, $d = S^n(p)$ for some $n \in \mathbb{N}$. Hence, $S(d) = S(S^n(p)) = S^{n+1}(p) \in$ Standard, proving **Lemma 1**.

Lemma 2: If $d \in$ Nonstandard, then $S(d) \in$ Nonstandard.

Let $d \in$ Nonstandard. Assume for sake of contradiction that $S(d) \notin$ Nonstandard. Then, $S(d) \in$ Standard and $S(d) = S^n(p)$ for some $n \in \mathbb{N}$.

If $n = 0$ then $S(d) = S^n(p) = S^0(p) = p$. Hence, $S(d) = p$. We know that for any $x \in \mathbb{N}$, we have $x + 1 \neq 0$. Hence, $M \models \forall \mathbf{v_1}(\mathbf{f}\mathbf{v_1} \neq \mathbf{c})$. Since $M \equiv M^*$, we know $M^* \models \forall \mathbf{v_1}(\mathbf{f}\mathbf{v_1} \neq \mathbf{c})$. Hence, we must have $S(d) \neq p$ which contradicts $S(d) = p$. Hence, $n \neq 0$. Hence, $n \geq 1$. Hence, $n - 1 \geq 0$.

So we have $S(d) = S^n(p) = S(S^{n-1}(p))$. By Fact 3 on the assignment handout, since $S(d) = S(S^{n-1}(p))$ we have that $d = S^{n-1}(p) \in$ Standard where $n - 1 \geq 0$. So we have $d \in$ Standard which contradicts the fact that $d \in$ Nonstandard.

Therefore, $S(d) \in$ Nonstandard, proving **Lemma 2**.

Lemma 3: For any $x \in$ Standard and for any $y \in$ Nonstandard, we have $x \prec y$.

Let $x \in$ Standard and let $y \in$ Nonstandard.

We know that for $m, n \in \mathbb{N}$, either $m = n$ or $m < n$ or $n < m$.

Hence, $M \models \forall \mathbf{v_1} \forall \mathbf{v_2}(\mathbf{v_1} = \mathbf{v_2} \vee \mathbf{R}\mathbf{v_1}\mathbf{v_2} \vee \mathbf{R}\mathbf{v_2}\mathbf{v_1})$

Since $M \equiv M^*$, we have $M^* \models \forall \mathbf{v_1} \forall \mathbf{v_2}(\mathbf{v_1} = \mathbf{v_2} \vee \mathbf{R}\mathbf{v_1}\mathbf{v_2} \vee \mathbf{R}\mathbf{v_2}\mathbf{v_1})$

Hence, either $x = y$ or $x \prec y$ or $y \prec x$.

We know $x = y$ is impossible since $x \in$ Standard and $y \in$ Nonstandard and Standard $\cap$ Nonstandard $= \emptyset$. Hence, $x \neq y$.

Assume for the sake of contradiction that $y \prec x$. Since $x \in$ Standard, we know $x = S^n(p)$ for some $n \in \mathbb{N}$.

Note, $\{y \in D^\sharp : y \prec x\} = \{y \in D^\sharp : y \prec S^n(p)\} = \{S^0(p), S(p), ..., S^{n-1}(p)\}$.

Since we assumed $y \prec x$, we know that $y = S^m(p)$ for some $m \in \mathbb{N}$ such that $0 \leq m \leq n-1$.

But this implies that $y = S^m(p) \in$ Standard which contradicts the fact that $y \in$ Nonstandard.

Hence, $y \not\prec x$.

Since we have $x \neq y$ and $y \not\prec x$, we must have that $x \prec y$ which proves **Lemma 3**.

Now we will show h is a homomorphism from M^* into M^*.

Show: $h(p) = p$

We know that $p = S^0(p) \in$ Standard. Hence, by definition of h we have $h(p) = p$.

Show: $h(S(d)) = S(h(d))$ for each $d \in D^\sharp$.

Let $d \in D^\sharp$.

If $d \in$ Standard, then $d = S^n(p)$ for some $n \in \mathbb{N}$. Hence,

$$\begin{aligned}
h(S(d)) &= h(S(S^n(p))) && \text{Since } d = S^n(p) \\
&= h(S^{n+1}(p)) && \\
&= S^{n+1}(p) && \text{Since } S^{n+1}(p) \in \text{Standard} \\
&= S(S^n(p)) && \\
&= S(h(S^n(p)) && \text{Since } S^n(p) \in \text{Standard} \\
&= S(h(d)) && \text{Since } d = S^n(p)
\end{aligned}$$

If $d \in$ Nonstandard, then we have the following.

$$\begin{aligned}
h(S(d)) &= S(S(d)) && \text{Since } S(d) \in \text{Nonstandard by } \textbf{Lemma 2} \\
&= S(h(d)) && \text{Since } d \in \text{Nonstandard}
\end{aligned}$$

Therefore, we have shown that $h(S(d)) = S(h(d))$ for each $d \in D^\sharp$.

Show: $d_1 \prec d_2$ iff $h(d_1) \prec h(d_2)$ for each $d_1, d_2 \in D^\sharp$.

Let $d_1, d_2 \in D^\sharp$.

Case 1: If $d_1, d_2 \in$ Standard, then $h(d_1) = d_1$ and $h(d_2) = d_2$. Trivially we have

$$d_1 \prec d_2 \text{ iff } d_1 \prec d_2$$

Since $h(d_1) = d_1$ and $h(d_2) = d_2$, we have

$$d_1 \prec d_2 \text{ iff } h(d_1) \prec h(d_2)$$

Case 2: If $d_1, d_2 \in$ Nonstandard, then $h(d_1) = S(d_1)$ and $h(d_2) = S(d_2)$.

We know that for $m, n \in \mathbb{N}$, we have $m < n$ iff $m + 1 < n + 1$.

Hence, $M \models \forall \mathbf{v_1} \forall \mathbf{v_2}(\mathbf{Rv_1v_2} \leftrightarrow \mathbf{Rfv_1fv_2})$.

Since $M \equiv M^*$, we have $M^* \models \forall \mathbf{v_1} \forall \mathbf{v_2}(\mathbf{Rv_1v_2} \leftrightarrow \mathbf{Rfv_1fv_2})$.

Hence we have,

$$d_1 \prec d_2 \text{ iff } S(d_1) \prec S(d_2)$$

But we know that $h(d_1) = S(d_1)$ and $h(d_2) = S(d_2)$. Hence,

$$d_1 \prec d_2 \text{ iff } h(d_1) \prec h(d_2)$$

Case 3: If $d_1 \in$ Standard and $d_2 \in$ Nonstandard, then we know $h(d_1) = d_1 \in$ Standard and we know $h(d_2) = S(d_2) \in$ Nonstandard by **Lemma 2**.

Hence, $d_1 \prec d_2$ by **Lemma 3**. And, $h(d_1) \prec h(d_2)$ by **Lemma 3**.

Therefore, trivially we have

$$d_1 \prec d_2 \text{ iff } h(d_1) \prec h(d_2)$$

Case 4: If $d_1 \in$ Nonstandard and $d_2 \in$ Standard, then we know $h(d_1) = S(d_1) \in$ Nontandard by **Lemma 2** and $h(d_2) = d_2$.

Hence, $d_2 \prec d_1$ by **Lemma 3** and $h(d_2) \prec h(d_1)$ by **Lemma 3**.

Now, we know that for all $m, n \in \mathbb{N}$, exactly one of $m = n$ or $m < n$ or $n < m$ holds. i.e. we have trichotomy.

Let ϕ be the following formula symbolizing trichotomy.

$\forall \mathbf{v_1} \forall \mathbf{v_2}((\mathbf{v_1} = \mathbf{v_2} \wedge \sim \mathbf{Rv_1v_2} \wedge \sim \mathbf{Rv_2v_1}) \vee (\mathbf{v_1} \neq \mathbf{v_2} \wedge \mathbf{Rv_1v_2} \wedge \sim \mathbf{Rv_2v_1})$
$\vee (\mathbf{v_1} \neq \mathbf{v_2} \wedge \sim \mathbf{Rv_1v_2} \wedge \mathbf{Rv_2v_1}))$

We know $M \models \phi$. Since $M^* \equiv M$, we have $M^* \models \phi$.

Hence, we know exactly one of $d_1 = d_2$ or $d_1 \prec d_2$ or $d_2 \prec d_1$ holds. And exactly one of $h(d_1) = h(d_2)$ or $h(d_1) \prec h(d_2)$ or $h(d_2) \prec h(d_1)$ holds.

Since $d_2 \prec d_1$ and $h(d_2) \prec h(d_1)$, we know that we must have $d_1 \not\prec d_2$ and $h(d_1) \not\prec h(d_2)$.

Hence, trivially we have

$$d_1 \not\prec d_2 \text{ iff } h(d_1) \not\prec h(d_2)$$

Equivalently,

$$d_1 \prec d_2 \text{ iff } h(d_1) \prec h(d_2)$$

Therefore, h is a homomorphism from M^* into M^*.

Show h is One-to-One: Assume $h(d_1) = h(d_2)$.

Case 1: Consider $h(d_1) = h(d_2) \in$ Standard.

If $d_1 \in$ Nonstandard, then $h(d_1) = S(d_1) \in$ Nonstandard by **Lemma 2** which would contradict $h(d_1) \in$ Standard. Hence, $d_1 \in$ Standard.

If $d_2 \in$ Nonstandard, then $h(d_2) = S(d_2) \in$ Nonstandard by **Lemma 2** which would contradict $h(d_2) \in$ Standard. Hence, $d_2 \in$ Standard.

Since $d_1, d_2 \in$ Standard, we have $d_1 = h(d_1) = h(d_2) = d_2$. Hence, $d_1 = d_2$.

Case 2: Consider $h(d_1) = h(d_2) \in$ Nonstandard.

If $d_1 \in$ Standard, then $h(d_1) = d_1 \in$ Standard which would contradict $h(d_1) \in$ Nonstandard. Hence, $d_1 \in$ Nonstandard.

If $d_2 \in$ Standard, then $h(d_2) = d_2 \in$ Standard which would contradict $h(d_2) \in$ Nonstandard. Hence, $d_2 \in$ Nonstandard.

Since $d_1, d_2 \in$ Nonstandard, we have $S(d_1) = h(d_1) = h(d_2) = S(d_2)$. Hence, $S(d_1) = S(d_2)$. By Fact 3 on the assignment handout, since $S(d_1) = S(d_2)$, we have $d_1 = d_2$.

Therefore, in either case h is one-to-one.

Show h is Onto: Let $d' \in D^\sharp$.

Case 1: If $d' \in$ Standard, then let $d = d'$ so that $h(d) = h(d') = d'$.

Case 2: Now, consider $d' \in$ Nonstandard.

We know that for $n \in \mathbb{N}$, if $n \neq 0$, then there exists an $m \in \mathbb{N}$ such that $m + 1 = n$.

Hence, $M \models \forall \mathbf{v_1}(\mathbf{v_1} \neq \mathbf{c} \rightarrow \exists \mathbf{v_2}(\mathbf{f v_2} = \mathbf{v_1}))$.

Since $M \equiv M^*$, we have $M^* \models \forall \mathbf{v_1}(\mathbf{v_1} \neq \mathbf{c} \rightarrow \exists \mathbf{v_2}(\mathbf{f v_2} = \mathbf{v_1}))$.

Hence, for all $x \in D^\sharp$, if $x \neq p$, then there exists $y \in D^\sharp$ such that $S(y) = x$.

If $d' = p$, then $d' = p = S^0(p) \in$ Standard which would contradict the fact that $d' \in$ Nonstandard. Hence, $d' \neq p$.

Since $d' \neq p$, there exists $d \in D^\sharp$ such that $S(d) = d'$.

Now, if $d \in$ Standard, then $S(d) = d' \in$ Standard by **Lemma 1** which would contradict the fact that $d' \in$ Nonstandard.

Hence, $d \in$ Nonstandard.

Hence, $h(d) = S(d) = d'$.

Therefore, h is onto.

Since we've shown h is a homomorphism from M^* into M^* and is one-to-one and onto, we conclude that h is an isomorphism from M^* onto M^*, as required. Notice that this shows that h is an automorphism from M^* onto M^*.

Exercise 2

Suppose that $A \subseteq D^\sharp$ is definable in M^*. Show the following: there is an object $d \in D^\sharp$ such that, for every $d' \in D^\sharp$, if $d \prec d'$, then $d' \in A$ iff $S(d') \in A$.

Let $d = q$. We know from Fact 8 on the assignment handout that for every $n \geq 0, S^n(p) \neq q$. Hence, $d = q \neq S^n(p)$ for every $n \geq 0$. Hence, $d = q \notin$ Standard. Hence, $d = q \in$ Nonstandard.

Let $d' \in D^\sharp$ and assume $d \prec d'$.

Show: $d' \in A$ iff $S(d') \in A$.

We know that $d \prec d'$. We want to first show that $d' \in$ Nonstandard. Assume for the sake of contradiction that $d' \in$ Standard. Hence, $d' = S^m(p)$ for some $m \in \mathbb{N}$.

By Fact 7 on the assignment sheet we know that for every $n \geq 0$, we have $S^n(p) \prec q$. Hence, $S^m(p) \prec q$. Notice, $S^m(p) = d'$ and $q = d$. Hence, $d' \prec d$.

So we have $d \prec d'$ and $d' \prec d$ which contradicts Fact 5 on the assignment handout which says that $\prec$ is antisymmetric.

Hence, our initial assumption was wrong and $d' \in$ Nonstandard. Hence, $h(d') = S(d')$.

($\Rightarrow$): Assume $d' \in A$.

By the Automorphism Theorem, we know that A is closed under h. i.e. if $x \in A$, then $h(x) \in A$.

Since $d' \in A$, we have that $h(d') \in A$. Since $h(d') = S(d')$, we have that $S(d') \in A$.

($\Leftarrow$): Assume $S(d') \in A$.

Since $h(d') = S(d')$, we have that $h(d') \in A$.

We know that h is an automorphism from M^* onto M^*. Hence, h^{-1} is an automorphism from M^* onto M^*.

By the Automorphism Theorem, we know that h^{-1} is closed under A. i.e. if $x \in A$, then $h^{-1}(x) \in A$.

Since $h(d') \in A$, we have that $h^{-1}(h(d')) \in A$. And we know $h^{-1}(h(d')) = d'$.

Therefore, $d' \in A$. This completes the proof, as required.

Exercise 3

Suppose that ϕ is a formula of L with at most one free variable, $\mathbf{v_1}$. Show that $M \models \exists\mathbf{v_2}\forall\mathbf{v_1}(\mathbf{Rv_2v_1} \rightarrow (\phi \leftrightarrow sub(\phi, \mathbf{fv_1}, \mathbf{v_1})))$.

Let $A \subseteq D^\sharp$ be the set that is defined by the formula ϕ.

Consider the following T-biconditional. Note, we will skip the trivial steps of showing a T-Biconditional below. And we will use $\forall$ and $\exists$ ambiguously in the metalanguage.

$$\begin{array}{ll}
& M^* \models \exists\mathbf{v_2}\forall\mathbf{v_1}(\mathbf{Rv_2v_1} \rightarrow (\phi \leftrightarrow sub(\phi, \mathbf{fv_1}, \mathbf{v_1}))) \\
\text{iff} & M^* \models \exists\mathbf{v_2}\forall\mathbf{v_1}(\mathbf{Rv_2v_1} \rightarrow (\phi \leftrightarrow sub(\phi, \mathbf{fv_1}, \mathbf{v_1})))[s] \\
\text{iff} & \exists d \in D^\sharp, \forall d' \in D^\sharp, \text{ if } d \prec d', \text{ then } M^* \models \phi[(s^d_{\mathbf{v_2}})^{d'}_{\mathbf{v_1}}] \text{ iff } M^* \models sub(\phi, \mathbf{fv_1}, \mathbf{v_1})[(s^d_{\mathbf{v_2}})^{d'}_{\mathbf{v_1}}] \\
\text{iff} & \exists d \in D^\sharp, \forall d' \in D^\sharp, \text{ if } d \prec d', \text{ then } M^* \models \phi[(s^d_{\mathbf{v_2}})^{d'}_{\mathbf{v_1}}] \text{ iff } M^* \models \phi[(s^d_{\mathbf{v_2}})^{S(d')}_{\mathbf{v_1}}] \\
\text{iff} & \exists d \in D^\sharp, \forall d' \in D^\sharp, \text{ if } d \prec d', \text{ then } d' \in A \text{ iff } S(d') \in A
\end{array}$$

Note that line 4 follows from the fact that $\mathbf{fv_1}$ is free for $\mathbf{v_1}$ in ϕ and from **Theorem 3.1.11** in the booklet. And line 5 follows from the fact that ϕ defines the set A.

We have shown $\exists d \in D^\sharp, \forall d' \in D^\sharp$, if $d \prec d'$, then $d' \in A$ iff $S(d') \in A$ in Exercise 2.

Therefore, looking at our T-biconditional we have that $M^* \models \exists\mathbf{v_2}\forall\mathbf{v_1}(\mathbf{Rv_2v_1} \rightarrow (\phi \leftrightarrow sub(\phi, \mathbf{fv_1}, \mathbf{v_1})))$.

Since $M \equiv M^*$, we have that $M \models \exists\mathbf{v_2}\forall\mathbf{v_1}(\mathbf{Rv_2v_1} \rightarrow (\phi \leftrightarrow sub(\phi, \mathbf{fv_1}, \mathbf{v_1})))$.

This is what we wanted to show, as required.

Exercise 4

Show that every subset of $\mathbb{N}$ that is definable in M is either finite or cofinite.

Let $A \subseteq \mathbb{N}$ be definable by a formula ϕ.

Consider the following T-biconditional. Note, we will skip the trivial steps of showing a T-Biconditional below. And we will use $\forall$ and $\exists$ ambiguously in the metalanguage. And instead of writing $successor(n)$, we will instead write $n+1$ for notational convenience.

$$\begin{aligned}
& M \models \exists \mathbf{v_2} \forall \mathbf{v_1} (\mathbf{R v_2 v_1} \rightarrow (\phi \leftrightarrow sub(\phi, \mathbf{f v_1}, \mathbf{v_1}))) \\
\text{iff} \quad & M \models \exists \mathbf{v_2} \forall \mathbf{v_1} (\mathbf{R v_2 v_1} \rightarrow (\phi \leftrightarrow sub(\phi, \mathbf{f v_1}, \mathbf{v_1})))[s] \\
\text{iff} \quad & \exists m \in \mathbb{N}, \forall n \in \mathbb{N}, \text{ if } m < n, \text{ then } M \models \phi[(s^m_{\mathbf{v_2}})^n_{\mathbf{v_1}}] \text{ iff } M \models sub(\phi, \mathbf{f v_1}, \mathbf{v_1})[(s^m_{\mathbf{v_2}})^n_{\mathbf{v_1}}] \\
\text{iff} \quad & \exists m \in \mathbb{N}, \forall n \in \mathbb{N}, \text{ if } m < n, \text{ then } M \models \phi[(s^m_{\mathbf{v_2}})^n_{\mathbf{v_1}}] \text{ iff } M \models \phi[(s^m_{\mathbf{v_2}})^{n+1}_{\mathbf{v_1}}] \\
\text{iff} \quad & \exists m \in \mathbb{N}, \forall n \in \mathbb{N}, \text{ if } m < n, \text{ then } n \in A \text{ iff } n+1 \in A
\end{aligned}$$

Note that line 4 follows from the fact that $\mathbf{f v_1}$ is free for $\mathbf{v_1}$ in ϕ and from **Theorem 3.1.11** in the booklet. And line 5 follows from the fact that ϕ defines the set A.

By Exercise 3 we know that $M \models \exists \mathbf{v_2} \forall \mathbf{v_1} (\mathbf{R v_2 v_1} \rightarrow (\phi \leftrightarrow sub(\phi, \mathbf{f v_1}, \mathbf{v_1})))$.

Therefore, looking at our T-biconditional we have that $\exists m \in \mathbb{N}, \forall n \in \mathbb{N}$, if $m < n$, then $n \in A$ iff $n+1 \in A$.

So consider such an $m \in \mathbb{N}$.

Hence, for all $n \in \mathbb{N}$, if $m < n$, then $n \in A$ iff $n+1 \in A$. Call this **Fact 1**.

Consider $m+1 \in \mathbb{N}$. We have two cases to consider. Either $m+1 \in A$ or $m+1 \notin A$.

Case 1: $m+1 \in A$.

Claim: $m+k \in A$ for all $k \geq 1$.

We will prove this by induction on k.

Base Case: $m+1 \in A$ by assumption.
IH: $m+k \in A$
Show: $m+k+1 \in A$

We know $m < m+k$. Hence, by **Fact 1** we know that $m+k \in A$ iff $m+k+1 \in A$. Since $m+k \in A$ by **IH**, we have that $m+k+1 \in A$, proving the **Claim**.

By our **Claim** we have that $m+k \in A$ for all $k \geq 1$.

But this implies that $m + k \notin \mathbb{N} \setminus A$ for all $k \geq 1$.

Hence, the only possible elements of $\mathbb{N} \setminus A$ are among $0, 1, ..., m$.

Hence, $\mathbb{N} \setminus A \subseteq \{0, 1, ...m\}$. Hence, $\mathbb{N} \setminus A$ is finite.

Therefore, A is cofinite.

Case 2: $m + 1 \notin A$.

Claim: $m + k \notin A$ for all $k \geq 1$.

We will prove this by induction on k.

Base Case: $m + 1 \notin A$ by assumption.
IH: $m + k \notin A$
Show: $m + k + 1 \notin A$

We know $m < m + k$. Hence, by **Fact 1** we know that $m + k \in A$ iff $m + k + 1 \in A$. Since $m + k \notin A$ by **IH**, we have that $m + k + 1 \notin A$, proving the **Claim**.

By our **Claim** we have that $m + k \notin A$ for all $k \geq 1$.

Hence, the only possible elements of A are among $0, 1, ..., m$.

Hence, $A \subseteq \{0, 1, ...m\}$.

Therefore, A is finite.

Hence, in either case we have that A is finite or cofinite.

Since $A \subseteq \mathbb{N}$ was an arbitrary subset that was definable in M and was shown to be finite or cofinite, we conclude that every subset of $\mathbb{N}$ that is definable in M is either finite or cofinite, as required.