Exercise 1

Required: Do Theorem 2.2.12, Inductive Step $\exists$.

Inductive Step 8. $\exists$
IH: For any assignments s and s', if s and s' agree on $free(\phi)$, then $Val_{M,s}(\phi) = Val_{M,s'}(\phi)$.
Show: For any assignments s and s', if s and s' agree on $free(\exists x\phi)$, then $Val_{M,s}(\exists x\phi) = Val_{M,s'}(\exists x\phi)$.

Proof. Choose any assignments s and s', and assume that s and s' agree on $free(\exists x\phi)$.

Assume for sake of contradiction that $Val_{M,s}(\exists x\phi) \neq Val_{M,s'}(\exists x\phi)$. Without loss of generality, assume $Val_{M,s}(\exists x\phi) = 1$ and $Val_{M,s'}(\exists x\phi) = 0$.

Hence, $\max\{Val_{M,s_x^d}(\phi) : d \in D\} = 1$.

In particular, for some $d^* \in D$, we have that, $Val_{M,s_x^{d^*}}(\phi) = 1$. Now, we will prove the following claim.

Claim: $s_x^{d^*}$ and $s_x'^{d^*}$ agree on $free(\phi)$.

To prove the claim, suppose $y \in free(\phi)$. We have two cases to consider. Either y and x are the same variable, or x and y are distinct variables.

If x and y are the same variable, then we have that $s_x^{d^*}(y) = s_x^{d^*}(x) = d^*$ and $s_x'^{d^*}(y) = s_x'^{d^*}(x) = d^*$. Hence, $s_x^{d^*}(y) = s_x'^{d^*}(y)$.

If x and y are distinct variables, then $y \in free(\phi) - \{x\} = free(\exists x\phi)$. And we assumed that s and s' agree on $free(\exists x\phi)$. Hence, $s(y) = s'(y)$. Hence, $s_x^{d^*}(y) = s(y) = s'(y) = s_x'^{d^*}(y)$.

This proves the claim.

Since $s_x^{d^*}$ and $s_x'^{d^*}$ agree on $free(\phi)$, by **IH** we have that, $Val_{M,s_x^{d^*}}(\phi) = Val_{M,s_x'^{d^*}}(\phi)$.

Since $Val_{M,s_x^{d^*}}(\phi) = 1$, and $Val_{M,s_x^{d^*}}(\phi) = Val_{M,s_x'^{d^*}}(\phi)$, this implies that $Val_{M,s_x'^{d^*}}(\phi) = 1$.

But we assumed that $Val_{M,s'}(\exists x\phi) = 0$. This implies that $\max\{Val_{M,s_x'^d}(\phi) : d \in D\} = 0$. This implies that $Val_{M,s_x'^{d^*}}(\phi) = 0$.

Hence, we have that $Val_{M,s_x'^{d^*}}(\phi) = 1$ and $Val_{M,s_x'^{d^*}}(\phi) = 0$ which is a contradiction.

Therefore, our initial assumption was wrong. Therefore, $Val_{M,s}(\exists x\phi) = Val_{M,s'}(\exists x\phi)$, completing the proof, as required. $\square$

Exercise 2

Theorem 2.5.14: Suppose that the language L' is an expansion of the language L and that $L \subseteq Form_L$ and $\phi \in Form_L$ (in which case $\Gamma \subseteq Form_{L'}$ and $\phi \in Form_{L'}$). Then $\Gamma \models_L \phi$ iff $\Gamma \models_{L'} \phi$.

Proof. $(\Rightarrow)$

Assume $\Gamma \models_L \phi$. Assume for the sake of contradiction that $\Gamma \not\models_{L'} \phi$.

Hence, there exists a model $M' = \langle D', I' \rangle$ for L' and a variable assignment s such that $M' \models \Gamma[s]$ and $M' \not\models \phi[s]$.

Equivalently, for each $\sigma \in \Gamma$, $M' \models \sigma[s]$ and $M' \not\models \phi[s]$.

Equivalently, for each $\sigma \in \Gamma, Val_{M',s}(\sigma) = 1$ and $Val_{M',s}(\phi) = 0$.

Now, define a model $M = \langle D, I \rangle$ for L by,

$D = D'$
$I(c) = I'(c)$ if c is a constant symbol of L
$I(f) = I'(f)$ if f is an n-ary function symbol of L
$I(R) = I'(R)$ if R is an n-ary relation symbol of L

So M' is an expansion of M. By Theorem 2.4.5, we have that,

1) $Val_{M,s}(\sigma) = Val_{M',s}(\sigma) = 1$ for each $\sigma \in \Gamma$

2) $Val_{M,s}(\phi) = Val_{M',s}(\phi) = 0$.

Hence, for each $\sigma \in \Gamma$, $Val_{M,s}(\sigma) = 1$ and $Val_{M,s}(\phi) = 0$.

So for each $\sigma \in \Gamma$, $M \models \sigma[s]$ and $M \not\models \phi[s]$.

In other words $M \models \Gamma[s]$ and $M \not\models \phi[s]$.

But we assumed that $\Gamma \models_L \phi$. This implies that given our M and s, if $M \models \Gamma[s]$, then $M \not\models \phi[s]$.

So we have $M \models \Gamma[s]$ and we have that if $M \models \Gamma[s]$, then $M \models \phi[s]$. Therefore, we have that $M \models \phi[s]$.

Hence, we have shown that $M \models \phi[s]$ and $M \not\models \phi[s]$ which is a contradiction. This completes the forward direction.

$(\Leftarrow)$

Assume $\Gamma \models_{L'} \phi$. Assume for the sake of contradiction that $\Gamma \not\models_L \phi$.

Hence, there exists a model $M = \langle D, I \rangle$ for L and a variable assignment s such that $M \models \Gamma[s]$ and $M \not\models \phi[s]$.

Equivalently, for each $\sigma \in \Gamma$, $M \models \sigma[s]$ and $M \not\models \phi[s]$.

Equivalently, for each $\sigma \in \Gamma, Val_{M,s}(\sigma) = 1$ and $Val_{M,s}(\phi) = 0$.

Now, we will define a model $M' = \langle D', I' \rangle$ for L'. Let $d \in D$ be arbitrary. For every $n \geq 1$, let f_n be any n-ary function on D and let R_n be any n-ary relation on D.

$D' = D$
$I'(c) = I(c)$ if c is a constant symbol of L
$I'(c) = d$ if c is a constant symbol of L' but not of L
$I'(f) = I(f)$ if f is an n-ary function symbol of L
$I'(f) = f_n$ if f is an n-ary function symbol of L' but not of L
$I'(R) = I(R)$ if R is an n-ary relation symbol of L
$I'(R) = R_n$ if R is an n-ary relation symbol of L' but not of L

So M' is an expansion of M. By Theorem 2.4.5, we have that,

1) $Val_{M',s}(\sigma) = Val_{M,s}(\sigma) = 1$ for each $\sigma \in \Gamma$

2) $Val_{M',s}(\phi) = Val_{M,s}(\phi) = 0$.

Hence, for each $\sigma \in \Gamma, Val_{M',s}(\sigma) = 1$ and $Val_{M',s}(\phi) = 0$.

So for each $\sigma \in \Gamma$, $M' \models \sigma[s]$ and $M' \not\models \phi[s]$.

Equivalently, $M' \models \Gamma[s]$ and $M' \not\models \phi[s]$.

But we assumed that $\Gamma \models_{L'} \phi$. This implies that given our M' and s, if $M' \models \Gamma[s]$, then $M' \not\models \phi[s]$.

So we have $M' \models \Gamma[s]$ and we have that if $M' \models \Gamma[s]$, then $M' \models \phi[s]$. Therefore, we have that $M' \models \phi[s]$.

Hence, we have shown that $M' \models \phi[s]$ and $M' \not\models \phi[s]$ which is a contradiction. This completes the proof, as required.

$\square$

Exercise 3

Theorem 2.5.19: Suppose that L is a language and that $\Gamma \subseteq Sent_L$ and that $\phi \in Sent_L$. Then $\Gamma \models \phi$ iff every model for L that satisfies Γ also satisfies ϕ.

Proof. Suppose that L is a language and that $\Gamma \subseteq Sent_L$ and that $\phi \in Sent_L$.

($\Rightarrow$)

Assume $\Gamma \models \phi$.
Show: Every model for L that satisfies Γ also satisfies ϕ.

Let M be an arbitrary model for L. Suppose $M \models \Gamma$ (i.e. M satisfies Γ). We want to show that $M \models \phi$ (i.e. M satisfies ϕ).

Let s be an arbitrary assignment. Since $M \models \Gamma$, we have $\forall \gamma \in \Gamma, M \models \gamma$. By Theorem 2.2.15 we have that, $\forall \gamma \in \Gamma, M \models \gamma[s]$ since $\Gamma \subseteq Sent_L$.

Hence, $M \models \Gamma[s]$.

By Theorem 2.5.18 we know that $\Gamma \models \phi$ iff $\Gamma \models_L \phi$. Since $\Gamma \models \phi$, we have that $\Gamma \models_L \phi$.

By Definition 2.5.11 since $\Gamma \models_L \phi$, we have that, for every model M for L and every assignment s, if $M \models \Gamma[s]$, then $M \models \phi[s]$.

So let M and s be the model and assignment we declared earlier. Since $M \models \Gamma[s]$, we have that $M \models \phi[s]$.

By Theorem 2.2.15 since $M \models \phi[s]$ and $\phi \in Sent_L$, we have that $M \models \phi$ (i.e. M satisfies ϕ) which is what we wanted to prove.

Therefore, every model for L that satisfies Γ also satisfies ϕ.

($\Leftarrow$)

Assume every model for L that satisfies Γ also satisfies ϕ.
Show: $\Gamma \models \phi$.

First we will show that $\Gamma \models_L \phi$. By Definition 2.5.11 $\Gamma \models_L \phi$ if and only if for every model M for L and every assignment s, if $M \models \Gamma[s]$, then $M \models \phi[s]$.

So let M be a model for L and let s be an assignment. Assume $M \models \Gamma[s]$. We want to show $M \models \phi[s]$.

Since $M \models \Gamma[s]$, we have $\forall \gamma \in \Gamma, M \models \gamma[s]$. By Theorem 2.2.15, $\forall \gamma \in \Gamma, M \models \gamma$ since $\Gamma \subseteq Sent_L$.

Since $\forall \gamma \in \Gamma, M \models \gamma$, we have $M \models \Gamma$ (i.e. M satisfies Γ). By our assumption, we then have that $M \models \phi$ (i.e. M satisfies ϕ).

By Theorem 2.2.15, since $M \models \phi$ and $\phi \in Sent_L$, we have that $M \models \phi[s]$.

Hence, we have shown that for every model M for L and every assignment s, if $M \models \Gamma[s]$, then $M \models \phi[s]$.

Therefore, by Definition 2.5.11, we have that $\Gamma \models_L \phi$.

By Theorem 2.5.18 we know that $\Gamma \models \phi$ iff $\Gamma \models_L \phi$. Since $\Gamma \models_L \phi$, we have that $\Gamma \models \phi$ which is what we wanted to prove.

This completes the proof, as required.

$\square$

Exercise 4

Suppose that L is a first-order language with the equals sign and a binary predicate $\mathbf{R}$. Show the following.

1.

Show: $\forall \mathbf{v_1} \exists \mathbf{v_2} \mathbf{R} \mathbf{v_1} \mathbf{v_2}.\ \forall \mathbf{v_1} \forall \mathbf{v_2} \forall \mathbf{v_3}((\mathbf{R}\mathbf{v_1}\mathbf{v_2} \wedge \mathbf{R}\mathbf{v_2}\mathbf{v_3}) \rightarrow \mathbf{R}\mathbf{v_1}\mathbf{v_3}) \not\models \exists \mathbf{v_1} \mathbf{R}\mathbf{v_1}\mathbf{v_1}$.

Consider the model $M = \langle D, I \rangle$ for L defined as follows.

$D = \mathbb{N}$

$I(\mathbf{R}) = \{\langle a, b\rangle : a, b \in \mathbb{N} \text{ and } a < b\}$

Now, consider the following T-biconditionals. Note, we will be using $\forall$ and $\exists$ ambiguously in the metalanguage.

First we have, $M \models \forall \mathbf{v_1} \exists \mathbf{v_2} \mathbf{R}\mathbf{v_1}\mathbf{v_2}$ iff $\forall a \in \mathbb{N}, \exists b \in \mathbb{N}$ such that $a < b$.

Note that $\forall a \in \mathbb{N}, \exists b \in \mathbb{N}$ such that $a < b$ is obviously true in the naturals with the usual ordering.

Hence, $M \models \forall \mathbf{v_1} \exists \mathbf{v_2} \mathbf{R}\mathbf{v_1}\mathbf{v_2}$.

Next, we have $M \models \forall \mathbf{v_1} \forall \mathbf{v_2} \forall \mathbf{v_3}((\mathbf{R}\mathbf{v_1}\mathbf{v_2} \wedge \mathbf{R}\mathbf{v_2}\mathbf{v_3}) \rightarrow \mathbf{R}\mathbf{v_1}\mathbf{v_3})$ iff $\forall a, b, c \in \mathbb{N}$, if $a < b$ and $b < c$, then $a < c$.

Note that $\forall a, b, c \in \mathbb{N}$, if $a < b$ and $b < c$, then $a < c$ is obviously true since we know that the usual ordering of the naturals is transitive.

Hence, $M \models \forall \mathbf{v_1} \forall \mathbf{v_2} \forall \mathbf{v_3}((\mathbf{R}\mathbf{v_1}\mathbf{v_2} \wedge \mathbf{R}\mathbf{v_2}\mathbf{v_3}) \rightarrow \mathbf{R}\mathbf{v_1}\mathbf{v_3})$.

Finally, we have that $M \models \exists \mathbf{v_1} \mathbf{R}\mathbf{v_1}\mathbf{v_1}$ iff $\exists a \in \mathbb{N}$ such that $a < a$.

Notice, that $\exists a \in \mathbb{N}$ such that $a < a$ is obviously false since there is no natural number less than itself in the usual ordering.

Hence, $M \not\models \exists \mathbf{v_1} \mathbf{R}\mathbf{v_1}\mathbf{v_1}$.

Therefore, $\forall \mathbf{v_1} \exists \mathbf{v_2} \mathbf{R} \mathbf{v_1} \mathbf{v_2}.\ \forall \mathbf{v_1} \forall \mathbf{v_2} \forall \mathbf{v_3}((\mathbf{R}\mathbf{v_1}\mathbf{v_2} \wedge \mathbf{R}\mathbf{v_2}\mathbf{v_3}) \rightarrow \mathbf{R}\mathbf{v_1}\mathbf{v_3}) \not\models \exists \mathbf{v_1} \mathbf{R}\mathbf{v_1}\mathbf{v_1}$, as required.

2.

Show: $\forall \mathbf{v_1 R v_1 v_1}$. $\forall \mathbf{v_1} \forall \mathbf{v_2} (\mathbf{R v_1 v_2} \to \mathbf{R v_2 v_1})$. $\forall \mathbf{v_1} \forall \mathbf{v_2} \exists \mathbf{v_3} (\mathbf{R v_1 v_3} \wedge \mathbf{R v_3 v_2}) \not\models \forall \mathbf{v_1} \forall \mathbf{v_2} \mathbf{R v_1 v_2}$

Consider the model $M = \langle D, I \rangle$ for L defined as follows.

$D = \{1, 2, 3\}$

$I(\mathbf{R}) = \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,3\rangle, \langle 3,2\rangle\}$

Now, consider the following T-biconditionals. Note, we will be using $\forall$ and $\exists$ ambiguously in the metalanguage.

First, we have $M \models \forall \mathbf{v_1 R v_1 v_1}$ iff $\forall a \in D, \langle a, a\rangle \in I(\mathbf{R})$.

We know that $\forall a \in D, \langle a, a\rangle \in I(\mathbf{R})$ is true because,

For $a = 1$, we have $\langle 1,1\rangle \in I(\mathbf{R})$
For $a = 2$, we have $\langle 2,2\rangle \in I(\mathbf{R})$
For $a = 3$, we have $\langle 3,3\rangle \in I(\mathbf{R})$

Hence, $M \models \forall \mathbf{v_1 R v_1 v_1}$.

Next, we have $M \models \forall \mathbf{v_1} \forall \mathbf{v_2} (\mathbf{R v_1 v_2} \to \mathbf{R v_2 v_1})$ iff $\forall a, b \in D$, if $\langle a, b\rangle \in I(\mathbf{R})$, then $\langle b, a\rangle \in I(\mathbf{R})$.

Notice that $\forall a, b \in D$, if $\langle a, b\rangle \in I(\mathbf{R})$, then $\langle b, a\rangle \in I(\mathbf{R})$ is true because,

For $a = 1, b = 1$, we have that $\langle 1,1\rangle \in I(\mathbf{R})$ and $\langle 1,1\rangle \in I(\mathbf{R})$.
For $a = 2, b = 2$, we have that $\langle 2,2\rangle \in I(\mathbf{R})$ and $\langle 2,2\rangle \in I(\mathbf{R})$.
For $a = 3, b = 3$, we have that $\langle 3,3\rangle \in I(\mathbf{R})$ and $\langle 3,3\rangle \in I(\mathbf{R})$.
For $a = 1, b = 2$, we have that $\langle 1,2\rangle \in I(\mathbf{R})$ and $\langle 2,1\rangle \in I(\mathbf{R})$.
For $a = 2, b = 1$, we have that $\langle 2,1\rangle \in I(\mathbf{R})$ and $\langle 1,2\rangle \in I(\mathbf{R})$.
For $a = 2, b = 3$, we have that $\langle 2,3\rangle \in I(\mathbf{R})$ and $\langle 3,2\rangle \in I(\mathbf{R})$.
For $a = 3, b = 2$, we have that $\langle 3,2\rangle \in I(\mathbf{R})$ and $\langle 2,3\rangle \in I(\mathbf{R})$.
For $a = 1, b = 3$, we have that $\langle 1,3\rangle \notin I(\mathbf{R})$.
For $a = 3, b = 1$, we have that $\langle 3,1\rangle \notin I(\mathbf{R})$.

Note, the last two cases above are vacuously true.

Hence, $M \models \forall \mathbf{v_1} \forall \mathbf{v_2} (\mathbf{R v_1 v_2} \to \mathbf{R v_2 v_1})$.

Next, we have that $M \models \forall \mathbf{v_1} \forall \mathbf{v_2} \exists \mathbf{v_3} (\mathbf{R v_1 v_3} \wedge \mathbf{R v_3 v_2})$ iff $\forall a, b \in D, \exists c \in D$ such that $\langle a, c\rangle \in I(\mathbf{R})$ and $\langle c, b\rangle \in I(\mathbf{R})$.

And $\forall a, b \in D, \exists c \in D$ such that $\langle a, c\rangle \in I(\mathbf{R})$ and $\langle c, b\rangle \in I(\mathbf{R})$ is true because,

For $a = 1, b = 1$, take $c = 1$ so that $\langle 1, 1\rangle \in I(\mathbf{R})$ and $\langle 1, 1\rangle \in I(\mathbf{R})$.
For $a = 2, b = 2$, take $c = 2$ so that $\langle 2, 2\rangle \in I(\mathbf{R})$ and $\langle 2, 2\rangle \in I(\mathbf{R})$.
For $a = 3, b = 3$, take $c = 3$ so that $\langle 3, 3\rangle \in I(\mathbf{R})$ and $\langle 3, 3\rangle \in I(\mathbf{R})$.
For $a = 1, b = 2$, take $c = 2$ so that $\langle 1, 2\rangle \in I(\mathbf{R})$ and $\langle 2, 2\rangle \in I(\mathbf{R})$.
For $a = 2, b = 1$, take $c = 2$ so that $\langle 2, 2\rangle \in I(\mathbf{R})$ and $\langle 2, 1\rangle \in I(\mathbf{R})$.
For $a = 2, b = 3$, take $c = 3$ so that $\langle 2, 3\rangle \in I(\mathbf{R})$ and $\langle 3, 3\rangle \in I(\mathbf{R})$.
For $a = 3, b = 2$, take $c = 3$ so that $\langle 3, 3\rangle \in I(\mathbf{R})$ and $\langle 3, 2\rangle \in I(\mathbf{R})$.
For $a = 1, b = 3$, take $c = 2$ so that $\langle 1, 2\rangle \in I(\mathbf{R})$ and $\langle 2, 3\rangle \in I(\mathbf{R})$.
For $a = 3, b = 1$, take $c = 2$ so that $\langle 3, 2\rangle \in I(\mathbf{R})$ and $\langle 2, 1\rangle \in I(\mathbf{R})$.

Hence, $M \models \forall \mathbf{v_1} \forall \mathbf{v_2} \exists \mathbf{v_3} (\mathbf{R v_1 v_3} \wedge \mathbf{R v_3 v_2})$.

Finally, we have that $M \models \forall \mathbf{v_1} \forall \mathbf{v_2} \mathbf{R v_1 v_2}$ iff $\forall a, b \in D, \langle a, b\rangle \in I(\mathbf{R})$.

However, it is clear that $\forall a, b \in D, \langle a, b\rangle \in I(\mathbf{R})$ is not true since we can take $a = 1, b = 3$ and clearly $\langle 1, 3\rangle \notin I(\mathbf{R})$.

Hence, $M \not\models \forall \mathbf{v_1} \forall \mathbf{v_2} \mathbf{R v_1 v_2}$.

Therefore, $\forall \mathbf{v_1} \mathbf{R v_1 v_1}$. $\forall \mathbf{v_1} \forall \mathbf{v_2} (\mathbf{R v_1 v_2} \rightarrow \mathbf{R v_2 v_1})$. $\forall \mathbf{v_1} \forall \mathbf{v_2} \exists \mathbf{v_3} (\mathbf{R v_1 v_3} \wedge \mathbf{R v_3 v_2}) \not\models \forall \mathbf{v_1} \forall \mathbf{v_2} \mathbf{R v_1 v_2}$.

Question 5

First we will restate the following given claims to reference in our proofs later.

We have a function $f : D \to D$ such that $d \triangleright f(d)$.

$(7) \forall d \in D, \exists d' \in D$ such that $d \triangleright d'$

$(8) \forall d, d', d''$, if $d \triangleright d'$ and $d' \triangleright d''$, then $d \triangleright d''$

$(9) \forall d \in D, d \not\triangleright d$

$(10) f^0(d) = d$

$(11) f^{n+1}(d) = f(f^n(d))$

Proof of Claim (12)

Claim (12): Fix any $d \in D$. Then $\forall n \geq 1, d \triangleright f^n(d)$.

Proof. Fix $d \in D$. We will use induction on n.

Base Case: For $n = 1$, we know that $d \triangleright f(d)$ by definition of f.
IH: $d \triangleright f^n(d)$ for some $n \geq 1$.
Show: $d \triangleright f^{n+1}(d)$.

By definition of f, we know that $f^n(d) \triangleright f(f^n(d))$. And $f^{n+1}(d) = f(f^n(d))$ by (11). Hence, we have that $f^n(d) \triangleright f^{n+1}(d)$.

Since $d \triangleright f^n(d)$ by **IH** and $f^n(d) \triangleright f^{n+1}(d)$, therefore, by (8) we have that $d \triangleright f^{n+1}(d)$. This completes the proof, as required. $\square$

Lemma

We will use the following lemma to complete our proof of Claim (13).

Lemma: For a fixed $m \in \mathbb{N}$, we have that $f^{m+n} = f^n(f^m(d))$ for all $n \geq 1$.

Proof. Fix $m \in \mathbb{N}$.

Base Case: For $n = 1$, we have that $f^{m+1}(d) = f(f^m(d))$ by (11).
IH: $f^{m+n} = f^n(f^m(d))$
Show: $f^{m+(n+1)}(d) = f^{n+1}(f^m(d))$

Notice that,

$$\begin{aligned} f^{m+(n+1)}(d) &= f^{(m+n)+1}(d) \\ &= f(f^{m+n}(d)) && \text{By (11)} \\ &= f(f^n(f^m(d))) && \text{By } \textbf{IH} \\ &= f^{n+1}(f^m(d)) && \text{By (11)} \end{aligned}$$

This completes the proof of the Lemma. $\square$

Proof of Claim (13)

Claim (13): $\forall d \in D, \forall i, j \in \mathbb{N}$, if $i \neq j$, then $f^i(d) \neq f^j(d)$.

Proof. Let $d \in D$. Let $i, j \in \mathbb{N}$. Assume $i \neq j$. And assume for the sake of contradiction that $f^i(d) = f^j(d)$.

WLOG, assume $i > j$. So, $\exists k \geq 1$ such that $i = j + k$.

And, $f^j(d) \triangleright f^k(f^j(d))$ by Claim (12). But clearly we have that $f^k(f^j(d)) = f^{j+k}(d) = f^i(d)$ by our Lemma. Hence, $f^j(d) \triangleright f^i(d)$.

But we assumed that $f^i(d) = f^j(d)$. Hence, we have that $f^j(d) \triangleright f^j(d)$ which contradicts (9).

Therefore, our assumption was wrong and $f^i(d) = f^j(d)$, completing the proof, as required. $\square$

Extra: Why does Claim (13) mean D is infinite?

Let $d \in D$ be fixed. By Claim (13), for every $i, j \in \mathbb{N}$ such that $i \neq j$, we have that $f^i(d) \neq f^j(d)$.

Hence, $d = f^0(d) \neq f^1(d) \neq f^2(d) \neq f^3(d) \neq f^4(d) \neq ....$

This shows that for every natural number, there is a distinct element of D. Hence, we have at least as many distinct elements of D as there are natural numbers.

In other words, the cardinality of D is greater than or equal to the cardinality of $\mathbb{N}$. And $\mathbb{N}$ is an infinite set.

Therefore, D is an infinite set.