Question 3, Page 73

Enderton provides 3 equivalent definitions of transitive sets on page 71. We will call them definitions 1, 2, 3, 4.

A set A is transitive if one of the following holds.

1. $x \in a \in A \Rightarrow x \in A$
2. $\bigcup A \subseteq A$
3. $a \in A \Rightarrow a \subseteq A$
4. $A \subseteq P(A)$

(a)

Required: Show that if a is a transitive set, then $P(a)$ is a transitive set.

Proof. Assume a is a transitive set.

By definition 4, we have that $a \subseteq P(a)$.

Assume $b \in P(a)$. Then, $b \subseteq a$. Since $b \subseteq a$ and $a \subseteq P(a)$, by transitivity of containment we have that $b \subseteq P(a)$.

Therefore, we have that $b \in P(a) \Rightarrow b \subseteq P(a)$ which satisfies definition 3. Therefore $P(a)$ is transitive. $\square$

(b)

Required: Show that if $P(a)$ is a transitive set, then a is a transitive set.

Proof. Assume $P(a)$ is a transitive set.

By definition 2, we have that $\bigcup P(a) \subseteq P(a)$.

Note that $\bigcup P(a)$ is the union of all the elements of all the subsets of a. Therefore, $\bigcup P(a) = a$.

Therefore, $a \subseteq P(a)$ which satisfies definition 4. Therefore a is a transitive set. $\square$

Question 4, Page 73

Enderton provides 3 equivalent definitions of transitive sets on page 71. We will call them definitions 1, 2, 3, 4.

A set A is transitive if one of the following holds.

1. $x \in a \in A \Rightarrow x \in A$
2. $\bigcup A \subseteq A$
3. $a \in A \Rightarrow a \subseteq A$
4. $A \subseteq P(A)$

Required: Show that if a is a transitive set, then $\bigcup a$ is a transitive set.

Proof. Assume a is a transitive set. By definition 2, we have that $\bigcup a \subseteq a$.

We will show that $\bigcup a$ satisfies definition 3. i.e. $b \in \bigcup a \Rightarrow b \subseteq \bigcup a$.

Now, let $b \in \bigcup a$. Since $\bigcup a \subseteq a$, we have that $b \in a$.

Let $c \in b$.

Since $c \in b$ and $b \in a$, it follows that $c \in \bigcup a$.

Since $c \in b$ was arbitrary, it follows that $b \subseteq \bigcup a$.

Therefore, we have proven that $b \in \bigcup a \Rightarrow b \subseteq \bigcup a$ satisfying definition 3.

Therefore, $\bigcup a$ is transitive. $\square$

Question 5, Page 73

(a)

Required: Assume that every member of A is a transitive set. Show that $\bigcup A$ is a transitive set.

Proof. We will show that $a \in \bigcup A \Rightarrow a \subseteq \bigcup A$. i.e. that $\bigcup A$ satisfies definition 3.

Let $a \in \bigcup A$. This means that $\exists b \in A(a \in b)$.

Since $b \in A$, by assumption we know that b is a transitive set.

Now let $x \in a$. So, we have that $x \in a \in b$.

By definition 1, since $x \in a \in b$ and b is transitive, we have that $x \in b$.

Since $x \in b$ and $b \in A$, we have that $x \in \bigcup A$.

Since $x \in a$ was arbitrary, we have that $a \subseteq \bigcup A$.

Therefore, we have shown that $a \in \bigcup A \Rightarrow a \subseteq \bigcup A$.

Therefore, $\bigcup A$ is transitive. $\square$

(b)

Required: Assume that every member of A is a transitive set. Show that $\bigcap A$ is a transitive set.

Proof. We will show that $a \in \bigcap A \Rightarrow a \subseteq \bigcap A$. i.e. that $\bigcap A$ satisfies definition 3.

Let $a \in \bigcap A$. This means that $\forall b \in A(a \in b)$

For every $b \in A$, we have that b is a transitive set by assumption.

Now let $x \in a$. So, we have that $x \in a \in b$.

By definition 1, since $x \in a \in b$ and b is transitive for every b, we have that $x \in b$ for all b.

Since $x \in b$ and $b \in A$ for each b, we have that $x \in \bigcap A$.

Since $x \in a$ was arbitrary, we have that $a \subseteq \bigcap A$.

Therefore, we have shown that $a \in \bigcap A \Rightarrow a \subseteq \bigcap A$. Therefore, $\bigcap A$ is transitive.

$\square$

Question 8, Page 78

Let $f : A \to A$ be one-to-one and assume that $c \in A - ran(f)$. Define $h : \omega \to A$ by recursion:

$h(0) = c$ and $h(n^+) = f(h(n))$.

Required: Prove that h is one-to-one.

Proof. Let $T = \{n \in \omega | h(n) \text{ is one-to-one}\} = \{n \in \omega | \forall m \in \omega((h(n) = h(m) \Rightarrow n = m)\}$

We will show that $T = \omega$ by induction.

First consider whether or not $0 \in T$.

$$h(0) = h(m) \Rightarrow c = h(m)$$

If $m = 0$, then clearly $0 \in T$.

If $m \neq 0$, then $m = q^+$ for some $q \in \omega$. Then, we have that,

$$c = h(m) = h(q^+) = f(h(q))$$

This says that $c \in ran(f)$. But, we know that $c \in A - ran(f)$ which means that $c \notin ran(f)$.

This is a contradiction. So, $m = 0$ which implies that $0 \in T$.

Now, assume that $n \in T$, where $n \in \omega$. We will show that $n^+ \in T$.

Consider $h(n^+) = h(m)$. If $m = 0$, then since $0 \in T$, this would mean that $n^+ = 0$ which is impossible by definition of successor. So $m \neq 0$ which means that $m = q^+$ for some $q \in \omega$.

Now,

$$\begin{aligned} h(n^+) = h(m) &\Rightarrow h(n^+) = h(q^+) \\ &\Rightarrow f(h(n)) = f(h(q)) && \text{By definition of } h \\ &\Rightarrow h(n) = h(q) && \text{Since } f \text{ is one-to-one} \\ &\Rightarrow n = q && \text{Since } n \in T \end{aligned}$$

Since $n = q$, we have that $\{n\} = \{q\}$ which implies that $n \cup \{n\} = q \cup \{q\}$ which is the same as saying that $n^+ = q^+ = m$.

Therefore, $n^+ \in T$.

By induction, we have that $T = \omega$.

Therefore, h is one-to-one, as required. $\square$

Question 9, Page 78

Let $f : B \to B$ and $A \subseteq B$.

Let $C^* = \{X | A \subseteq X \subseteq B \wedge F[X] \subseteq X\}$

Also, $h(0) = A$ and $h(n^+) = h(n) \cup f[h(n)]$

$C_* = \bigcup_{i \in \omega} h(i)$

Required: Show $C^* = C_*$

Proof. We will first show that $C^* \subseteq C_*$.

Since h starts with $h(0) = A$ and since $f : B \to B$, we have that $A \subseteq C_* \subseteq B$.

Now consider the following.

$$\begin{aligned}
f[C^*] &= f[\bigcup_{i \in \omega} h(i)] \\
&= \bigcup_{i \in \omega} f[h(i)] && \text{By Theorem 3K} \\
&\subseteq \bigcup_{i \in \omega} h(i^+) && \text{Since } h(i^+) = h(i) \cup f[h(i)] \Rightarrow f[h(i)] \subseteq h(i^+) \\
&\subseteq \bigcup_{i \in \omega} h(i) && \text{Since } \omega \text{ is an inductive set} \\
&= C_*
\end{aligned}$$

Thus, $f[C^*] \subseteq C_*$

Since $A \subseteq C_* \subseteq B$ and $f[C^*] \subseteq C_*$ and C^* is an intersection of sets containing C_*, we have that $C^* \subseteq C_*$.

Now we will show that $C_* \subseteq C^*$.

Let $T = \{i \in \omega | h(i) \subseteq C^*\}$.

We will prove that $T = \omega$ which would show that $C_* = \bigcup_{i \in \omega} h(i) \subseteq C^*$.

By definition, $h(0) = A \subseteq C^*$. So $0 \in T$.

Assume that $i \in T$ where $i \in \omega$. i.e. $h(i) \subseteq C^*$.

We must now show that $h(i^+) \subseteq C^*$.

Notice that $h(i^+) = h(i) \cup f[h(i)]$. By assumption $h(i) \subseteq C^*$. So we need that $f[h(i)] \subseteq C^*$ in order for $h(i^+) \subseteq C^*$.

Since $h(i) \subseteq C^*$, we have that $h(i) \subseteq X$ for each X such that $A \subseteq X \subseteq B \wedge f[X] \subseteq X$.

Since $h(i) \subseteq X$, by applying f we get that $f[h(i)] \subseteq f[X]$. Furthermore, $f[X] \subseteq X$. So, $f[h(i)] \subseteq f[X] \subseteq X$.

Therefore $f[h(i)] \subseteq C^*$.

Therefore, $h(i^+) \subseteq C^*$.

Therefore, $i^+ \in T$ which means that $T = \omega$ by induction.

Therefore, $C_* \subseteq C^*$.

Thus, we have proven that $C^* = C_*$. This completes the proof. $\square$

Question 19, Page 88

Required: Prove that if m is a natural number and d is a nonzero natural number, then there exists numbers q and r such that $m = (d \cdot q) + r$ where $r \in d$.

Proof. We will prove that $T = \{m \in \omega | (\exists q \in \omega)(\exists r \in \omega)(m = (d \cdot q) + r \wedge r \in d)\}$ is inductive.

First, notice that $0 = d \cdot 0 + 0$ and that $r = 0 \in d \neq 0$. Therefore, $0 \in T$.

Now, assume that $m \in T$ where $m \in \omega$. i.e. there exists $q, r \in \omega$ such that $m = (d \cdot q) + r$ and $r \in d$.

We will now show that $m^+ \in T$. Consider the following.

$$\begin{aligned} m^+ &= ((d \cdot q) + r)^+ && \text{By successor} \\ &= (d \cdot q) + r^+ && \text{By A1} \end{aligned}$$

We must now show that $r^+ \in d$.

We know that $r \in d \Rightarrow r^+ \in d^+ \Rightarrow r^+ \in d \vee r^+ = d$.

We now consider two cases.

Case 1: If $r^+ \in d$, then we are done.

Case 2: If $r^+ = d$, then consider the following.

$$\begin{aligned} m^+ &= (d \cdot q) + r^+ \\ &= (d \cdot q) + d && \text{Since } r^+ = d \\ &= (d \cdot q^+) && \text{By M2} \\ &= (d \cdot q^+) + 0 && \text{By A1} \end{aligned}$$

Since $d \neq 0$, we know that $0 \in d$.

Therefore, $m^+ \in T$ and by induction, $T = \omega$, as required.

□

Question 20, Page 88

Required: Let A be a nonempty subset of ω such that $\bigcup A = A$. Show that $A = \omega$.

Proof. We will show that $A = \omega$ by induction.

If $0 \in A$, then we are done with the base case.

If $0 \notin A$, then since A is nonempty, there exists an $m \in A$ such that $m \neq 0$.

But, we know that $0 \in m$ for every nonzero $m \in \omega$.

So, $0 \in \bigcup A = A$.

Now, assume that $n \in A$. This is the hypothesis for strong induction.

Since $A = \bigcup A$, we have that $n \in \bigcup A$. It follows that $\exists m \in A(n \in m)$.

Now, consider the following 2 cases for m.

Case 1: If $m = n^+$, then $n^+ \in A$ and we're done.

Case 2: If $m \neq n^+$, then have the following.

If $m \in n^+$, then $m \in n \vee m = n$ which is is impossible because it would violate trichotomy since $n \in m$.

So it must be that $n^+ \in m$. Since $m \in A$, this implies that $n^+ \in \bigcup A$. Since $A = \bigcup A$, we have that $n^+ \in A$.

Therefore, by induction, we have that $A = \omega$, as required. $\square$

Question 21, Page 88

Required: Prove that no natural number is a subset of any of its elements.

Proof. Let $a \in \omega$, If $a = 0$, then clearly $a = \emptyset$ which means a has no elements. Therefore a is not a subset of any of its elements.

Now, consider $a \neq 0$.

Assume for the sake of contradiction that $\exists b \in a (a \subseteq b)$.

But, by $4M$ in Enderton, we know that $a \subseteq b \Leftrightarrow a \in b \vee a = b$.

But, we have that $b \in a$. So we cannot have $a \in b \vee a = b$ since the natural numbers satisfies trichotomy.

This is a contradiction. Therefore no natural number is a subset of any of its elements.

This completes the proof, as required. □

Question 22, Page 88

Required: Show that for any natural numbers m and p, we have $m \in m + p^+$.

Proof. Let $m, p \in \omega$.

Notice, that $p^+ \neq 0$. Then, since every natural number contains all numbers less than it, we have that $0 \in p^+$.

Consider the following.

$$\begin{aligned} 0 &\in p^+ \\ m + 0 &\in m + p^+ && \text{By Theorem 4N} \\ m &\in m + p^+ && \text{Since } m + 0 = m \text{ by A1} \end{aligned}$$

This completes the proof, as required. □

Question 23, Page 88

Required: Assume that m and n are natural numbers with m less than n. Show that there is a p in ω for which $m + p^+ = n$.

Proof. Let $n \in \omega$ be arbitrary and fixed such that $n \neq 0$.

Let $T = \{m \in n | \exists p \in \omega(m + p^+ = n)\}$.

We will show that $T = n$ by induction.

Consider $m = 0$.

Then, $0 + p^+ = n$ means that we can take $p^+ = n$. Since $n \neq 0$, we know that there exists a corresponding $p \in \omega$.

So, $0 \in T$.

Assume $m \in T$. i.e. There exists a $p \in \omega$ such that $m + p^+ = n$.

Now consider m^+. If $m^+ = n \vee n \in m^+$, then we don't have to do anything since $m^+ \notin n$ which would mean that $m^+ \notin T$. So, consider $m^+ \in n$.

Now, consider the following.

$$\begin{aligned}
m + p^+ &= n && \text{By induction hypothesis} \\
(m + p)^+ &= n && \text{By A2} \\
(p + m)^+ &= n && \text{By commutativity} \\
p + m^+ &= n && \text{By A2} \\
m^+ + p &= n && \text{By commutativity}
\end{aligned}$$

Since we know that $m^+ \in n$, we know that $p \neq 0$.

This means that $p = a^+$ for some $a \in \omega$.

Thus, $m^+ + a^+ = n$.

Therefore $m^+ \in T$.

Therefore, $T = n$, completing the proof, as required. $\square$