Question 1

Required: For each Γ and ϕ, decide if ϕ is a propositional consequence of Γ

(a)

Γ is $\{(\forall x P(x)) \to Q(y), (\forall x P(x)) \vee (\forall x R(x)), \exists x \neg R(x))\}$, and ϕ is $Q(y)$

Let $A :\equiv (\forall x P(x))$ and $B :\equiv Q(y)$ and $C :\equiv (\forall x R(x))$. So $\neg C :\equiv \exists x \neg R(x))$

We have that ϕ is a propositional consequence of Γ if and only if the propositional formula,

$((A \to B) \wedge (A \vee C) \wedge \neg C) \to B$ is a tautology.

It is trivial to check by truth table, that this is indeed a tautology.

Therefore ϕ IS a propositional consequence of Γ.

(b)

Γ is $\{x = y \wedge Q(y), (Q(y) \vee x + y < z\}$, and ϕ is $x + y < z$

Let $A :\equiv x = y$ and $B :\equiv Q(y)$ and $C :\equiv x + y < z$

We have that ϕ is a propositional consequence of Γ if and only if the propositional formula,

$((A \wedge B) \wedge (B \vee C)) \to C$ is a tautology.

However, consider the following truth value assignment.

A	B	C	$A \wedge B$	$B \vee C$	$((A \wedge B) \wedge (B \vee C)) \to C$
T	T	F	T	T	F

Thus, there is a truth value assignment where $((A \wedge B) \wedge (B \vee C)) \to C$ is false, so it is not a tautology.

Therefore, ϕ IS NOT a propositional consequence of Γ.

(c)

Γ is $\{P(x, y, x), x < y \vee M(w, p), (\neg P(x, y, x) \wedge \neg x < y)\}$, and ϕ is $\neg M(w, p)$

Let $A :\equiv P(x, y, x)$ and $B :\equiv x < y$ and $C :\equiv M(w, p)$

We have that ϕ is a propositional consequence of Γ if and only if the propositional formula,

$(A \wedge (B \vee C) \wedge (\neg A \wedge \neg B)) \to \neg C$ is a tautology.

It is trivial to check by truth table, that this is indeed a tautology.

Therefore ϕ IS a propositional consequence of Γ.

Question 2

Required to Prove: For variables $x_1, ..., x_n, y_1, ..., y_n$ and n-ary function symbol f, (E3) is valid.

i.e. $[(x_1 = y_1) \wedge ... \wedge (x_n = y_n)] \to ((R(x_1, ..., x_n) \to R(y_1, ..., y_n))$ is valid.

Proof. Fix a structure $\mathfrak{A}$ and a variable assignment function $s : \text{Vars} \to A$, where A is the universe of $\mathfrak{A}$.

We must show that $\mathfrak{A} \models [(x_1 = y_1) \wedge ... \wedge (x_n = y_n)] \to ((R(x_1, ..., x_n) \to R(y_1, ..., y_n))[s]$.

If $\mathfrak{A} \not\models [(x_1 = y_1) \wedge ... \wedge (x_n = y_n)][s]$, then we are done.

So assume that $\mathfrak{A} \models [(x_1 = y_1) \wedge ... \wedge (x_n = y_n)][s]$. Call this assumption 1.

We must now show that $\mathfrak{A} \models (R(x_1, ..., x_n) \to R(y_1, ..., y_n))[s]$.

If $\mathfrak{A} \not\models R(x_1, ..., x_n)[s]$, then we are done.

So assume that $\mathfrak{A} \models R(x_1, ..., x_n)[s]$. Call this assumption 2.

We must now show that $\mathfrak{A} \models R(y_1, ..., y_n)[s]$.

From assumption 1 and by definition of satisfaction (1.7.4) and $\wedge$, we get that,

$\mathfrak{A} \models (x_i = y_i)[s]$ for all $i \in \{1, .., n\}$.

Consider $\bar{s}$ to the be corresponding term assignment function to s. We now have that,

$\bar{s}(x_i) = \bar{s}(y_i)$ for all $i \in \{1, .., n\}$.

From assumption 2, we have that $((\bar{s}(x_1), ..., (\bar{s}(x_n)) \in R^{\mathfrak{A}}$.

Since $\bar{s}(x_i) = \bar{s}(y_i)$ for all $i \in \{1, .., n\}$, we have that $((\bar{s}(y_1), ..., (\bar{s}(y_n)) \in R^{\mathfrak{A}}$.

Therefore, $\mathfrak{A} \models R(y_1, ..., y_n)[s]$, by definition of satisfaction (1.7.4).

Thus, we have proven that $\mathfrak{A} \models ((R(x_1, ..., x_n) \to R(y_1, ..., y_n))[s]$.

Thus, we have proven that $\mathfrak{A} \models [(x_1 = y_1) \wedge ... \wedge (x_n = y_n)] \to ((R(x_1, ..., x_n) \to R(y_1, ..., y_n))[s]$.

Therefore, (E3) is valid. $\square$

Question 3

(a)

The structure $(\mathbb{N}, <)$ DOES NOT SATISFY the axioms for dense linear order without endpoints.

The structure does not satisfy axiom 4, $(\forall x)(\forall y)[x < y \to (\exists z)(x < z \land z < y)]$, since for $x = 1$ and $y = 2$, we have no $z \in \mathbb{N}$ such that $1 < z \land z < 2$.

(b)

The structure $(\mathbb{Z}, <)$ DOES NOT SATISFY the axioms for dense linear order without endpoints.

The structure does not satisfy axiom 4, $(\forall x)(\forall y)[x < y \to (\exists z)(x < z \land z < y)]$, since for $x = 1$ and $y = 2$, we have no $z \in \mathbb{Z}$ such that $1 < z \land z < 2$.

(c)

The structure $(\mathbb{Q}, <)$ SATISFIES the axioms for dense linear order without endpoints.

(d)

The structure $(\mathbb{R}, <)$ SATISFIES the axioms for dense linear order without endpoints.

(e)

The structure $(\mathbb{C}, <)$ DOES NOT SATISFY the axioms for dense linear order without endpoints with the given interpretation of $<$.

The structure does not satisfy axiom 5, $(\forall x)(\exists y)(\exists z)(y < x \land x < z)$.

Consider $x = 0 + 0i$. There is no y such that $y < 0$ since if $y = a + ib$ where $a, b \in \mathbb{R}$, then $a^2 + b^2 \geq 0$. So we have $y \geq 0 = x$. So we cannot have $(y < x \land x < z)$ for any $y, z \in \mathbb{C}$.

Question 4

Note: I am following the typsetting and using conventions of writing deductions as the sample solutions on pages 296-298. Like the book, I will omit the subscripts and superscripts for term substitution for (Q1) and (Q2).

(a)

Required: Show that $\vdash t = t$ for all terms t.

Consider the following. Note, that t is an arbitrary term.

1.	$x = x$	(E1)
2.	$(\forall x)(x = x)$	1, $\vdash \alpha$ iff $\vdash \forall x\alpha$
3.	$(\forall x)(x = x) \rightarrow (t = t)$	(Q1)
4.	$t = t$	2,3, (PC)

This shows that $\vdash t = t$ for all terms t.

Alternatively, we can show the above without using the fact that $\vdash \alpha$ iff $\vdash \forall x\alpha$.

Consider the following. Note that t is an arbitrary term.

1.	$x = x$	(E1)
2.	$[(y = y) \vee \neg(y = y)] \rightarrow (x = x)$	1, (PC)
3.	$[(y = y) \vee \neg(y = y)] \rightarrow (\forall x)(x = x)$	2, (QR)
4.	$(\forall x)(x = x)$	3, (PC)
5.	$(\forall x)(x = x) \rightarrow (t = t)$	(Q1)
6.	$t = t$	4, 5, (PC)

Again, this shows that $\vdash t = t$ for all terms t.

(b)

Required: Show that $\vdash (\forall x)(\exists y)(fx = y)$

Consider the following,

1.	$x = x$	(E1)
2.	$x = x \to fx = fx$	(E2)
3.	$fx = fx$	1,2, (PC)
4.	$fx = fx \to (\exists y)(fx = y)$	(Q2)
5.	$(\exists y)(fx = y)$	3, 4, (PC)
6.	$[(z = z) \vee \neg(z = z)] \to (\exists y)(fx = y)$	5, (PC)
7.	$[(z = z) \vee \neg(z = z)] \to (\forall x)(\exists y)(fx = y)$	6, (QR)
8.	$(\forall x)(\exists y)(fx = y)$	7, (PC)

(c)

Required: Show that $\vdash (\forall x)[(\forall y)(fx = fy)] \to (\exists z)[(\forall y)(z = fy)]$

By the Deduction Theorem, we will take $(\forall x)[(\forall y)(fx = fy)]$ as part of our set of axioms.

i.e. We will show $\{(\forall x)[(\forall y)(fx = fy)]\} \vdash (\exists z)[(\forall y)(z = fy)]$

Consider the following.

1.	$(\forall x)[(\forall y)(fx = fy)]$	
2.	$(\forall x)[(\forall y)(fx = fy)] \to (\forall y)(fx = fy)$	1, (Q1)
3.	$(\forall y)(fx = fy)$	1,2, (PC)
4.	$(\forall y)(fx = fy) \to (\exists z)[(\forall y)(z = fy)]$	3, (Q2)
5.	$(\exists z)[(\forall y)(z = fy)]$	3,4, (PC)

By the Deduction Theorem, this shows that $\vdash (\forall x)[(\forall y)(fx = fy)] \to (\exists z)[(\forall y)(z = fy)]$.