Question 1

Required: Prove that if $X \subseteq Y$, then $X \cap Y = X$.

Proof. Assume $X \subseteq Y$. To show $X \cap Y = X$, we will show the following claim.

Claim: For every object a, we have $a \in X \cap Y$ iff $a \in X$.

Let a be an object.

$(\Rightarrow)$: Assume $a \in X \cap Y$. Hence, by definition of intersection, we have $a \in X$ and $a \in Y$. In particular, we have that $a \in X$.

$(\Leftarrow)$: Assume $a \in X$. Since we assumed $X \subseteq Y$ and from the fact that $a \in X$, we have that $a \in Y$. Since $a \in X$ and $a \in Y$, by definition of intersection we have that $a \in X \cap Y$.

Therefore, we have shown that for every object a, we have $a \in X \cap Y$ iff $a \in X$. By Extensionality, we have that $X \cap Y = X$, completing the proof, as required. $\square$

Question 2

Required: Prove that if A is a class of sets, such that $\bigcup A$ is a set, then A is a set as well.

Proof. Assume A is a class of sets such that $\bigcup A$ is a set.

Now we will show the following claim.

Claim: $A \subseteq P(\bigcup A)$.

Let $y \in A$. Since we assumed A is a class of sets, we have that y is a set.

Since y is a set, we know that for every $x \in y$, we have $x \in \bigcup A$.

Hence, we have $y \subseteq \bigcup A$.

By the definition of power set, since $y \subseteq \bigcup A$, we have that $y \in P(\bigcup A)$.

Since $y \in A$ was arbitrary, and $y \in P(\bigcup A)$, we conclude that $A \subseteq P(\bigcup A)$.

Since $\bigcup A$ is a set by assumption, by the Power Set Axiom we have that $P(\bigcup A)$ is a set.

Since we've shown that $A \subseteq P(\bigcup A)$, and we've shown that $P(\bigcup A)$ is a set, then by the Axiom of Subsets we conclude that A is also a set. This completes the proof, as required.

$\square$

Question 3

Required: Prove by induction that for every $n \geq 2$, we have $\langle a_1, ...a_n \rangle = \langle b_1, ..., b_n \rangle$ iff $a_i = b_i$, for $i = 1, ..., n$.

Proof. Proof by induction.

Base Case: For $n = 2$, we want to show $\langle a_1, a_2 \rangle = \langle b_1, b_2 \rangle$ iff $a_1 = b_1$ and $a_2 = b_2$.

$(\Leftarrow)$: Assume $a_1 = b_1$ and $a_2 = b_2$.

Hence, we have $\{a_1\} = \{b_1\}$ and $\{a_1, a_2\} = \{b_1, b_2\}$.

Hence, we have $\{\{a_1\}, \{a_1, a_2\}\} = \{\{b_1\}, \{b_1, b_2\}\}$.

Hence, by definition of ordered pairs, we have $\langle a_1, a_2 \rangle = \langle b_1, b_2 \rangle$.

This completes the $(\Leftarrow)$ direction.

$(\Rightarrow)$: Assume $\langle a_1, a_2 \rangle = \langle b_1, b_2 \rangle$. This means that $\{\{a_1\}, \{a_1, a_2\}\} = \{\{b_1\}, \{b_1, b_2\}\}$.

We know that either $a_1 = a_2$ or $a_1 \neq a_2$. We'll consider both cases separately, which are exhaustive.

Case 1: Assume $a_1 = a_2$. Then, $\{\{a_1\}, \{a_1, a_2\}\} = \{\{a_1\}, \{a_1, a_1\}\} = \{\{a_1\}, \{a_1\}\} = \{\{a_1\}\}$.

Since $\{\{a_1\}, \{a_1, a_2\}\} = \{\{b_1\}, \{b_1, b_2\}\}$, we have that $\{\{a_1\}\} = \{\{b_1\}, \{b_1, b_2\}\}$. By extensionality, $\{a_1\} = \{b_1\} = \{b_1, b_2\}$. Hence, $a_1 = b_1 = b_2$. Since $a_1 = a_2$, we have that $a_1 = a_2 = b_1 = b_2$. In particular, $a_1 = b_1$ and $a_2 = b_2$.

Case 2: Assume $a_1 \neq a_2$.

Since $\{\{a_1\}, \{a_1, a_2\}\} = \{\{b_1\}, \{b_1, b_2\}\}$, then by Extensionality, either $\{a_1\} = \{b_1\}$ or $\{a_1\} = \{b_1, b_2\}$.

If $\{a_1\} = \{b_1, b_2\}$, then $a_1 = b_1 = b_2$.

Hence, we have $\{\{b_1\}, \{b_1, b_2\}\} = \{\{a_1\}, \{a_1, a_1\}\} = \{\{a_1\}, \{a_1\}\} = \{\{a_1\}\}$.

Since $\{\{a_1\}, \{a_1, a_2\}\} = \{\{b_1\}, \{b_1, b_2\}\}$, we have $\{\{a_1\}, \{a_1, a_2\}\} = \{\{a_1\}\}$. Hence, we must have $\{a_1, a_2\} = \{a_1\}$. Hence, $a_2 = a_1$. But this contradicts the fact that $a_1 \neq a_2$.

Therefore, we must have $\{a_1\} \neq \{b_1, b_2\}$. Since either $\{a_1\} = \{b_1\}$ or $\{a_1\} = \{b_1, b_2\}$, we must have that $\{a_1\} = \{b_1\}$.

Therefore, $a_1 = b_1$. Now, all that we have left to show is that $a_2 = b_2$.

Since $\{\{a_1\}, \{a_1, a_2\}\} = \{\{b_1\}, \{b_1, b_2\}\}$, then by Extensionality, either $\{a_1, a_2\} = \{b_1\}$ or $\{a_1, a_2\} = \{b_1, b_2\}$.

If $\{a_1, a_2\} = \{b_1\}$, then $a_1 = a_2 = b_1$. But then $a_1 = a_2$ contradicts $a_1 \neq a_2$.

Hence, we must have $\{a_1, a_2\} \neq \{b_1\}$. Since either $\{a_1, a_2\} = \{b_1\}$ or $\{a_1, a_2\} = \{b_1, b_2\}$, we must have $\{a_1, a_2\} = \{b_1, b_2\}$.

Since $\{a_1, a_2\} = \{b_1, b_2\}$, then either $b_2 = a_1$ or $b_2 = a_2$.

If $b_2 = a_1$, then since we've shown earlier that $a_1 = b_1$, we have $\{b_1, b_2\} = \{a_1, a_1\} = \{a_1\}$.

Since $\{a_1, a_2\} = \{b_1, b_2\}$, we have $\{a_1, a_2\} = \{a_1\}$. Hence, $a_1 = a_2$ which contradicts $a_1 \neq a_2$.

Therefore, $b_2 \neq a_1$. Since either $b_2 = a_1$ or $b_2 = a_2$, we must have that $b_2 = a_2$. Rearranging, we get $a_2 = b_2$.

Hence, we've shown $a_1 = b_1$ and $a_2 = b_2$ in **Case 2**.

In both **Case 1** and **Case 2** we've shown that $a_1 = b_1$ and $a_2 = b_2$.

This completes the $(\Rightarrow)$ direction.

This proves the **Base Case**.

Inductive Hypothesis: Assume $\langle a_1, ...a_n\rangle = \langle b_1, ..., b_n\rangle$ iff $a_i = b_i$, for $i = 1, ..., n$.

Want to Show: $\langle a_1, ...a_n, a_{n+1}\rangle = \langle b_1, ..., b_n, b_{n+1}\rangle$ iff $a_i = b_i$, for $i = 1, ..., n, n+1$.

Consider the following biconditional proof.

$$
\begin{aligned}
\langle a_1, ...a_n, a_{n+1}\rangle = \langle b_1, ..., b_n, b_{n+1}\rangle \quad &\textbf{iff} \quad \langle\langle a_1, ...a_n\rangle, a_{n+1}\rangle = \langle\langle b_1, ..., b_n\rangle, b_{n+1}\rangle && \text{By def of } n\text{-tuples} \\
&\textbf{iff} \quad \langle a_1, ...a_n\rangle = \langle b_1, ..., b_n\rangle \ \text{ and } \ a_{n+1} = b_{n+1} && \text{By } \textbf{Base Case} \\
&\textbf{iff} \quad a_i = b_i \ \text{ for } \ i = 1, ...n \ \text{ and } \ a_{n+1} = b_{n+1} && \text{By } \textbf{Ind. Hyp} \\
&\textbf{iff} \quad a_i = b_i \ \text{ for } \ i = 1, ...n, n+1
\end{aligned}
$$

Therefore, by induction we have proven the claim, as required. $\square$

Question 4

Required: Prove $f^{-1} \circ f = id_{dom(f)}$.

The textbook and the lecture notes defines f^{-1} when f is injective. So we'll assume that f is injective.

We will first show that $f^{-1} \circ f \subseteq id_{dom(f)}$ and $id_{dom(f)} \subseteq f^{-1} \circ f$.

To show $f^{-1} \circ f \subseteq id_{dom(f)}$, let $\langle x, y \rangle \in f^{-1} \circ f$ be arbitrary. Hence, $x \in dom(f)$.

Since $\langle x, y \rangle \in f^{-1} \circ f$, then by definition of composition there exists some z such that $\langle x, z \rangle \in f$ and $\langle z, y \rangle \in f^{-1}$.

Since $\langle z, y \rangle \in f^{-1}$, we have that $\langle y, z \rangle \in f$.

So we have $\langle x, z \rangle \in f$ and $\langle y, z \rangle \in f$. Assume for the sake of contradiction that $x \neq y$. Then, since f is injective, we have that $fx \neq fy$. But we know that $fx = z = fy$ which is a contradiction. Hence, we must have that $x = y$.

Since $x = y$, we have that $\langle x, y \rangle = \langle x, x \rangle \in id_{dom(f)}$. Therefore, we've shown $f^{-1} \circ f \subseteq id_{dom(f)}$.

To show $id_{dom(f)} \subseteq f^{-1} \circ f$, let $\langle x, x \rangle \in id_{dom(f)}$.

Hence, $x \in dom(f)$. Hence, $\langle x, fx \rangle \in f$. Furthermore, $\langle fx, x \rangle \in f^{-1}$. By definition of composition, since $\langle x, fx \rangle \in f$ and $\langle fx, x \rangle \in f^{-1}$ we have that $\langle x, x \rangle \in f^{-1} \circ f$. Therefore, we have shown $id_{dom(f)} \subseteq f^{-1} \circ f$.

Since we have shown $f^{-1} \circ f \subseteq id_{dom(f)}$ and $id_{dom(f)} \subseteq f^{-1} \circ f$, we have demonstrated the following by definition of $\subseteq$.

1. For every object a, if $a \in f^{-1} \circ f$, then $a \in id_{dom(f)}$.

2. For every object a, if $a \in id_{dom(f)}$, then $a \in f^{-1} \circ f$.

Hence, combining 1 and 2 we've shown that for every object a, we have $a \in f^{-1} \circ f$ if and only if $a \in id_{dom(f)}$.

Therefore, by Extensionality we have that $f^{-1} \circ f = id_{dom(f)}$, completing the proof, as required.

Question 5

Let S be a sharp total order on some set A. Prove: S^b is a blunt total order.

Proof. We know S is a sharp total order on some set A. Hence, S satisfies trichotomy and transitivity.

We want to show $S^b = S \cup id_A$ is a blunt total order. i.e. We want to show S^b satisfies connectedness, weak anti-symmetry and transitivity.

Show S^b satisfies Connectedness

Def Connectedness: For every $x, y \in A$, we have $\langle x, y\rangle \in S^b$ or $\langle y, x\rangle \in S^b$.

Let $x, y \in A$. We know either $x = y$ or $x \neq y$.

Case 1: Consider the case where $x = y$. Then $\langle x, y\rangle \in id_A$. Since $S^b = S \cup id_A$, we have $\langle x, y\rangle \in S^b$. This implies that $\langle x, y\rangle \in S^b$ or $\langle y, x\rangle \in S^b$ since we are proving an 'or' statement.

Case 2: Consider the case where $x \neq y$. Then since S satisfies trichotomy we know we must have exactly one of $\langle x, y\rangle \in S$ or $\langle y, x\rangle \in S$ or $x = y$. Since $x \neq y$, we must have exactly one of $\langle x, y\rangle \in S$ or $\langle y, x\rangle \in S$. Since $S^b = S \cup id_A$, we have exactly one of $\langle x, y\rangle \in S^b$ or $\langle y, x\rangle \in S^b$.

In either case, we have $\langle x, y\rangle \in S^b$ or $\langle y, x\rangle \in S^b$. Therefore, S^b is connected.

Show S^b satisfies weak anti-symmetry

Def Weak Anti-Symmetry: For $x, y \in A$, if $\langle x, y\rangle \in S^b$ and $\langle y, x\rangle \in S^b$, then $x = y$.

We will show the contrapositive instead.

Contrapositive: For $x, y \in A$, if $x \neq y$, then $\langle x, y\rangle \notin S^b$ or $\langle y, x\rangle \notin S^b$.

Let $x, y \in A$. Assume $x \neq y$.

We know S satisfies trichotomy. Hence, exactly one of $\langle x, y\rangle \in S$ or $\langle y, x\rangle \in S$ or $x = y$ holds.

Since $x \neq y$, we know exactly one of $\langle x, y\rangle \in S$ or $\langle y, x\rangle \in S$ holds. We'll consider these two cases separately now.

Case 1: If $\langle x, y\rangle \in S$ holds, then we know that $\langle y, x\rangle \notin S$. Furthermore, since $x \neq y$, we have that $\langle y, x\rangle \notin id_A$. Since $S^b = S \cup id_A$ with $\langle y, x\rangle \notin S$ and $\langle y, x\rangle \notin id_A$, we have that $\langle y, x\rangle \notin S^b$. This implies that $\langle x, y\rangle \notin S^b$ or $\langle y, x\rangle \notin S^b$ since we are proving an 'or' statement.

Case 2: If $\langle y, x\rangle \in S$ holds, then we know that $\langle x, y\rangle \notin S$. Furthermore, since $x \neq y$, we have that $\langle x, y\rangle \notin id_A$. Since $S^b = S \cup id_A$ with $\langle x, y\rangle \notin S$ and $\langle x, y\rangle \notin id_A$, we have that $\langle x, y\rangle \notin S^b$. This implies that $\langle x, y\rangle \notin S^b$ or $\langle y, x\rangle \notin S^b$ since we are proving an 'or' statement.

In either case, we have shown that $\langle x, y\rangle \notin S^b$ or $\langle y, x\rangle \notin S^b$. This proves that S^b satisfies weak-antisymmetry.

Show S^b satisfies transitivity

Def Transitivity: For every $x, y, z \in A$, if $\langle x, y\rangle \in S^b$ and $\langle y, z\rangle \in S^b$, then $\langle x, z\rangle \in S^b$.

Let $x, y, z \in A$. Assume $\langle x, y\rangle \in S^b$ and $\langle y, z\rangle \in S^b$.

Since $S^b = S \cup id_A$, we'll consider 4 separate cases.

Case 1: Consider $\langle x, y\rangle \in S$ and $\langle y, z\rangle \in S$. Since S satisfies transitivity, we have $\langle x, z\rangle \in S$. Since $S^b = S \cup id_A$, we have that $\langle x, z\rangle \in S^b$.

Case 2: Consider $\langle x, y\rangle \in id_A$ and $\langle y, z\rangle \in id_A$. Hence, we have $x = y$ and $y = z$. Hence, $x = y = z$. Hence, we have $\langle x, z\rangle = \langle x, x\rangle \in id_A$. Since $S^b = S \cup id_A$, we have that $\langle x, z\rangle \in S^b$.

Case 3: Consider $\langle x, y\rangle \in S$ and $\langle y, z\rangle \in id_A$. Hence, we have $y = z$. Hence, we have that $\langle x, z\rangle = \langle x, y\rangle \in S$. Since $S^b = S \cup id_A$, we have that $\langle x, z\rangle \in S^b$.

Case 4: Consider $\langle x, y\rangle \in id_A$ and $\langle y, z\rangle \in S$. Hence, we have $x = y$. Hence, we have that $\langle x, z\rangle = \langle y, z\rangle \in S$. Since $S^b = S \cup id_A$, we have that $\langle x, z\rangle \in S^b$.

In all 4 cases above, we have that $\langle x, z\rangle \in S^b$. Therefore, S^b satisfies transitivity.

Since we have shown that S^b satisfies connectedness, weak anti-symmetry, and transitivity, we have proven that S^b is a blunt total order, as required. $\square$

Question 6

Prove: If $\alpha = s_1 ... s_l$ is a formula, and $k < l$, then $w(s_1...s_k) \geq 0$.

Proof. Proof by induction on the $deg(\alpha)$.

Base Case: Consider α to be some propositional symbol. i.e. $deg(\alpha) = 0$.

We know in this case, α has no nonempty proper substring. Hence, every nonempty proper substring of α vacuously has the property of having weight greater than or equal to 0.

Inductive Hypothesis: Assume every formula $\alpha = s_1...s_l$ such that $deg(\alpha) < n$ satisfies the property that if $k < l$, then $w(s_1...s_k) \geq 0$.

We will show that our property also holds for α where $deg(\alpha) = n$. We have two cases to consider.

Case 1: Consider $\alpha = \neg\beta$, where $deg(\beta) = n - 1$ and $\beta = r_1...r_p$.

Consider any proper nonempty substring of α. We will consider 2 exhaustive subcases of all possible nonempty proper substrings of α.

Sub-Case i) The smallest possible nonempty proper substring of α is just $\neg$. And we know $w(\neg) = 0 \geq 0$.

Sub-Case ii) Otherwise, any possible nonempty proper substring of α is of the form $\neg r_1...r_i$ for some $i < p$.

Hence,

$$\begin{aligned} w(\neg r_1...r_i) &= w(\neg) + w(r_1...r_i) \\ &= 0 + w(r_1...r_i) \\ &= w(r_1...r_i) \\ &\geq 0 && \text{By Inductive Hypothesis on } \beta \end{aligned}$$

Hence, every nonempty, proper substring of $\alpha = \neg\beta$ satisfies our required property.

Note: If β were just a propositional symbol, then the only proper nonempty substring of $\alpha = \neg\beta$ is just $\neg$, which is covered in Sub-Case i). Hence, there are no degenerate cases.

Case 2: Consider $\alpha = \rightarrow \beta\gamma$, where deg($\beta$) $< n$ and $deg(\gamma) < n$. Furthermore, assume $\beta = r_1...r_p$ and $\gamma = t_1...t_q$.

Consider any nonempty proper substring of α. We will consider 4 exhaustive cases of all possible nonempty proper substrings of α.

Sub-Case i) The smallest possible nonempty proper substring of α is $\rightarrow$. And we know that $w(\rightarrow) = 1 \geq 0$.

Sub-Case ii) Consider a substring of α of the form $\rightarrow r_1...r_i$ for some $i < p$. Hence,

$$\begin{aligned} w(\rightarrow r_1...r_i) &= w(\rightarrow) + w(r_1...r_i) \\ &= 1 + w(r_1...r_i) \\ &\geq 1 + 0 && \text{By Inductive Hypothesis on } \beta \\ &= 1 \\ &\geq 0 \end{aligned}$$

Sub-Case iii) Consider the substring of α of the form $\rightarrow \beta$. Since $\beta = r_1...r_p$ is a formula, we know r_p must be a propositional symbol since no formula ends with a connective. Hence,

$$\begin{aligned} w(\rightarrow \beta) &= w(\rightarrow r_1...r_p) \\ &= w(\rightarrow) + w(r_1...r_{p-1}) + w(r_p) \\ &= 1 + w(r_1...r_{p-1}) - 1 && \text{Since } r_p \text{ is a propositional symbol} \\ &= w(r_1...r_{p-1}) \\ &\geq 0 && \text{By Inductive Hypothesis on } \beta \end{aligned}$$

Sub-Case iv) Consider the substring of alpha of the form $\rightarrow \beta t_1...t_j$ for some $j < q$. Also, since $\beta = r_1...r_p$ is a formula, we know r_p must be a propositional symbol since no formula ends with a connective. Hence,

$$\begin{aligned} w(\rightarrow \beta t_1...t_j) &= w(\rightarrow) + w(\beta) + w(t_1...t_j) \\ &= w(\rightarrow) + w(r_1...r_p) + w(t_1...t_j) \\ &= w(\rightarrow) + w(r_1...r_{p-1}) + w(r_p) + w(t_1...t_j) \\ &= 1 + w(r_1...r_{p-1}) - 1 + w(t_1...t_j) && \text{Since } r_p \text{ is a propositional symbol} \\ &= w(r_1...r_{p-1}) + w(t_1...t_j) \\ &\geq 0 + w(t_1...t_j) && \text{By Inductive Hypothesis on } \beta \\ &\geq 0 + 0 && \text{By Inductive Hypothesis on } \gamma \\ &= 0 \end{aligned}$$

Hence, in all 4 subcases, our desired property holds. Namely, every nonempty, proper substring of $\alpha = \rightarrow \beta\gamma$ has weight greater than or equal to 0.

Note: If β is just a propositional symbol, then the possible proper nonempty substrings of $\alpha = \rightarrow \beta\gamma$ is still covered by Sub-Cases i),iii),iv). And if γ is just a propositional symbol, then the possible nonempty proper substrings of α is still covered by Sub-Cases i),ii),iii). Hence, there are no degenerate cases.

Therefore, by induction on $deg(\alpha)$, we have proven our claim, as required. $\square$