Question 7, Page 26

(a)

Required: For any sets A and B, show that $P(A) \cap P(B) = P(A \cap B)$.

First, let us show that $P(A) \cap P(B) \subseteq P(A \cap B)$.

Consider the following.

$$\begin{aligned} x \in P(A) \cap P(B) &\Rightarrow x \in P(A) \wedge x \in P(B) \\ &\Rightarrow x \subseteq A \wedge x \subseteq B \\ &\Rightarrow (\forall y \in x)(y \in A) \wedge (\forall y \in x)(y \in B) \\ &\Rightarrow (\forall y \in x)(y \in A \wedge y \in B) \\ &\Rightarrow (\forall y \in x)(y \in A \cap B) \\ &\Rightarrow x \subseteq A \cap B \\ &\Rightarrow x \in P(A \cap B) \end{aligned}$$

So, we have that $P(A) \cap P(B) \subseteq P(A \cap B)$.

Now, let us show that $P(A \cap B) \subseteq P(A) \cap P(B)$.

Consider the following.

$$\begin{aligned} x \in P(A \cap B) &\Rightarrow x \subseteq A \cap B \\ &\Rightarrow (\forall y \in x)(y \in A \cap B) \\ &\Rightarrow (\forall y \in x)(y \in A \wedge y \in B) \\ &\Rightarrow (\forall y \in x)(y \in A) \wedge (\forall y \in x)(y \in B) \\ &\Rightarrow x \subseteq A \wedge x \subseteq B \\ &\Rightarrow x \in P(A) \wedge x \in P(B) \\ &\Rightarrow x \in P(A) \cap P(B) \end{aligned}$$

So, we have that $P(A \cap B) \subseteq P(A) \cap P(B)$.

Therefore, $P(A) \cap P(B) = P(A \cap B)$.

(b)

Required: For any sets A and B, show that $P(A) \cup P(B) \subseteq P(A \cup B)$.

Consider the following.

$$\begin{aligned} x \in P(A) \cup P(B) &\Rightarrow x \in P(A) \vee x \in P(B) \\ &\Rightarrow x \subseteq A \vee x \subseteq B \\ &\Rightarrow (\forall y \in x)(y \in A) \vee (\forall y \in x)(x \in B) \\ &\Rightarrow (\forall y \in x)(y \in A \vee y \in B) \\ &\Rightarrow (\forall y \in x)(y \in A \cup B) \\ &\Rightarrow x \subseteq A \cup B \\ &\Rightarrow x \in P(A \cup B) \end{aligned}$$

If $A \subseteq B$ or $B \subseteq A$, then quality holds and $P(A) \cup P(B) = P(A \cup B)$.

Question 8, Page 26

Show that there is no set to which every singleton belongs.

Assume for the sake of contradiction that there is a set A such that A contains all the singletons.

Then, we have that $\bigcup A$ takes the union of all the singleton sets and becomes the set of all sets.

But the set of all sets is a proper class, and not a set. Therefore we have a contradiction.

Therefore, there is no set of all singletons.

Question 10, Page 26

Required: Show that if $a \in B$, then $P(a) \in PP(\bigcup B)$.

If $a \in B$, then for every $x \in a$, $x \in \bigcup B$. Therefore $a \subseteq \bigcup B$.

We must now show that $P(a) \subseteq P(\bigcup B)$.

Let $y \in P(a)$. So $y \subseteq a$. Since $a \subseteq \bigcup B$ and we know that containment is transitive, this implies that $y \subseteq \bigcup B$. Therefore $y \in P(\bigcup B)$.

Therefore, $P(a) \subseteq P(\bigcup B)$.

Therefore, $P(a) \in PP(\bigcup B)$.

Question 24, Page 33

a)

Show that if A is nonempty, then $P(\bigcap A) = \bigcap\{P(X)|X \in A\}$.

We will show that each set is contained in the other.

Let $a \in P(\bigcap A)$. So $a \subseteq \bigcap A$.

So for every $x \in a$ we have that $x \in \bigcap A$.

So for every $X \in A$ and for every $x \in a$ we have that $x \in X$.

So for every $X \in A$, we have $a \subseteq X$.

So for every $X \in A$, we have $a \in P(X)$.

i.e. $a \in \bigcap\{P(X)|X \in A\}$.

Therefore, $P(\bigcap A) \subseteq \bigcap\{P(X)|X \in A\}$.

Now for the other containment.

Let $a \in \bigcap\{P(X)|X \in A\}$.

So for every $X \in A$, we have $a \in P(X)$.

So for every $X \in A$, we have $a \subseteq X$.

So for every $X \in A$ and for every $x \in a$, we have $x \in X$.

So for every $x \in a$, we have $x \in \bigcap A$.

So $a \subseteq \bigcap A$.

So $a \in P(\bigcap A)$.

Therefore $\bigcap\{P(X)|X \in A\} \subseteq P(\bigcap A)$.

Therefore, we have shown that $P(\bigcap A) = \bigcap\{P(X)|X \in A\}$.

b)

Show that $\bigcup\{P(X)|X \in A\} \subseteq P(\bigcup A)$. Under what conditions does equality hold?

Let $x \in \bigcup\{P(X)|X \in A\}$.

i.e. There exists an $X \in A$ such that $x \in P(X)$.

So, there exists an $X \in A$ such that $x \subseteq X$.

Since $X \in A$, we have that $X \subseteq \bigcup A$.

Since $x \subseteq X$ and $X \subseteq \bigcup A$, we have that $x \subseteq \bigcup A$ by transitivity of containment.

Finally, $x \in P(\bigcup A)$.

Therefore, $\bigcup\{P(X)|X \in A\} \subseteq P(\bigcup A)$.

In order for equality to hold the other containment must hold.

i.e. $P(\bigcup A) \subseteq \bigcup\{P(X)|X \in A\}$

So consider $x \in P(\bigcup A)$.

So, $x \subseteq \bigcup A$.

Now, we need $x \in \bigcup\{P(X)|X \in A\}$.

So we need $x \in P(X)$ for some $X \in A$.

So we need $x \subseteq X$ for some $X \in A$.

So, we have that $x \subseteq \bigcup A$ and we have that $x \subseteq X$ for some $X \in A$.

But since $X \in A$, we need $\bigcup A \subseteq X$.

Therefore the condition for equality is the following.

We need $\bigcup A \subseteq X$ for some $X \in A$.

Question 25, Page 33

Is $A \cup \bigcup B$ always the same as $\bigcup\{A \cup X | X \in B\}$? If not, then under what conditions does equality hold?

We will show that $A \cup \bigcup B = \bigcup\{A \cup X | X \in B\}$. i.e. equality holds when B is NON-EMPTY.

We will show that each set is contained in the other.

Let $x \in A \cup \bigcup B$. We now have to consider two cases.

Case 1: If $x \in A$, then there exists an $X \in B$ such that $x \in A \cup X$. So, $x \in \bigcup\{A \cup X | X \in B\}$.

Case 2: If $x \in \bigcup B$, then there exists an $X \in B$ such that $x \in X$. So $x \in A \cup X$. So, $x \in \bigcup\{A \cup X | X \in B\}$.

Therefore, $A \cup \bigcup B \subseteq \bigcup\{A \cup X | X \in B\}$.

For the other containment, let $x \in \bigcup\{A \cup X | X \in B\}$.

So, there exists an $X \in B$ such that $x \in A \cup X$. We now have to consider two cases.

Case 1: If $x \in A$, then clearly $x \in A \cup \bigcup B$.

Case 2: If $x \in X$, then clearly $x \in \bigcup B$. So clearly $x \in A \cup \bigcup B$.

Therefore, $\bigcup\{A \cup X | X \in B\} \subseteq A \cup \bigcup B$.

Therefore, we have shown that $A \cup \bigcup B = \bigcup\{A \cup X | X \in B\}$ when B is NON-EMPTY.

Question 35, Page 34

Assume that $P(A) = P(B)$. Show that $A = B$.

We will show that each set is contained in the other.

$A \subseteq B$:

Let $a \in A$. Then $\{a\} \subseteq A$. So $\{a\} \in P(A)$. Since $P(A) = P(B)$, we have that $\{a\} \in P(B)$. So $\{a\} \subseteq B$. And finally, $a \in B$. Therefore $A \subseteq B$.

$B \subseteq A$:

Let $b \in B$. Then $\{b\} \subseteq B$. So $\{b\} \in P(B)$. Since $P(A) = P(B)$, we have that $\{b\} \in P(A)$. So $\{b\} \subseteq A$. And finally, $b \in A$. Therefore $B \subseteq A$.

Since $A \subseteq B$ and $B \subseteq A$, we have that $A = B$, as required.

Question 1, Page 38

Let $< x.y.z >^* = \{\{x\}, \{x, y\}, \{x, y, z\}\}$.

We will show that the above definition is not successful by providing x, y, z, u, v, w such that $< x, y, z >^* = < u, v, w >^*$ but either $y \neq v$ or $z \neq w$ or both.

Let $x = \emptyset, y = \{\emptyset\}, z = \{\emptyset\}, u = \emptyset, v = \{\emptyset\}, w = \emptyset$.

$< x, y, z >^* = \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset\}\}\} = \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

$< u, v, w >^* = \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \emptyset\}\} = \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

So clearly $< x, y, z >^* = < u, v, w >^*$ but $z \neq w$.

Question 2, Page 38

a)

Required: Show that $A \times (B \cup C) = (A \times B) \cup (A \times C)$

We will show that each set is contained in the other.

Let $< x, y > \in A \times (B \cup C)$. So $x \in A$ and $y \in B \cup C$. We now have to consider two cases.

If $y \in B$, then $< x, y > \in A \times B$.

If $y \in C$, then $< x, y > \in A \times C$.

So we have that $(< x, y > \in A \times B) \vee (< x, y > \in A \times C)$.

Therefore, $< x, y > \in (A \times B) \cup (A \times C)$.

Therefore, $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.

Now we will show the other containment.

Let $< x, y > \in (A \times B) \cup (A \times C)$. We now have to consider two cases.

If $< x, y > \in (A \times B)$, then $x \in A$ and $y \in B$. So $(y \in B) \vee (y \in C)$. So $y \in B \cup C$. Therefore $< x, y > \in A \times (B \cup C)$.

If $< x, y > \in (A \times C)$, then $x \in A$ and $y \in C$. So $(y \in B) \vee (y \in C)$. So $y \in B \cup C$. Therefore $< x, y > \in A \times (B \cup C)$.

Therefore $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$.

Therefore, we have shown that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

b)

Required: Show that if $A \times B = A \times C$ and $A \neq \emptyset$, then $B = C$.

Assume $A \times B = A \times C$ and $A \neq \emptyset$.

Let $< x, y > \in A \times B$. Since $A \times B = A \times C$, there exists a $< u, v > \in A \times C$ such that $< x, y > = < u, v >$.

Enderton has proved that $< u, v > = < x, y >$ iff $u = x$ and $v = y$ on page 36.

Therefore $y = v$. Therefore $y \in C$.

Therefore $B \subseteq C$.

Let $< x, y > \in A \times C$. Since $A \times B = A \times C$, there exists a $< u, v > \in A \times B$ such that $< x, y > = < u, v >$.

Enderton has proved that $< u, v > = < x, y >$ iff $u = x$ and $v = y$ on page 36.

Therefore $y = v$. Therefore $y \in B$.

Therefore $C \subseteq B$.

Question 3, Page 38

Show that $A \times \bigcup B = \bigcup\{A \times X | X \in B\}$.

We will show that each set is contained in the other.

Let $< a, b > \in A \times \bigcup B$. So we have that $a \in A$ and $b \in \bigcup B$.

So, there exists some $X \in B$ such that $b \in X$.

Since $a \in A$ and $b \in X$, we have that $< a, b > \in A \times X$.

Therefore, $< a, b > \in \bigcup\{A \times X | X \in B\}$.

Therefore, $A \times \bigcup B \subseteq \bigcup\{A \times X | X \in B\}$.

Now we will show the other containment.

Let $< a, b > \in \bigcup\{A \times X | X \in B\}$.

Then, there exists an $X \in B$ such that $< a, b > \in A \times X$.

So, $a \in A$ and $b \in X$.

But if $b \in X$ and $X \in B$, then $b \in \bigcup B$.

Since $a \in A$ and $b \in \bigcup B$ we have that $< a, b > \in A \times \bigcup B$.

Therefore, $\bigcup\{A \times X | X \in B\} \subseteq A \times \bigcup B$.

Therefore, $A \times \bigcup B = \bigcup\{A \times X | X \in B\}$.

Question 4, Page 38

Required: Show that there is no set to which every ordered pair belongs.

Suppose for sake of contradiction that there is a set to which every ordered pair belongs.

Call this set A.

So, for every set a, the ordered pair $< a, a > \in A$.

But, $< a, a > = \{\{a\}, \{a, a\}\} = \{\{a\}, \{a\}\} = \{\{a\}\}$.

So the set A contains all the sets containing a singleton set.

Let $B \subseteq A$ be such that it contains all the sets containing a singleton set.

i.e. $\{\{a\}\} \in B$.

So $\{a\} \in \bigcup B$.

So $a \in \bigcup\bigcup B$.

Since a was arbitrary, the set $\bigcup\bigcup B$ is the set of all sets. But we know this is a proper class, and not a set.

This is a contradiction.

Therefore, there is no set to which every ordered pair belongs.

Question 14, Page 53

Assume that f and g are functions.

a)

Required: Show that $f \cap g$ is a function.

We must show that $f \cap g$ is single-valued in order to be a function.

i.e. for every $x \in dom(f \cap g)$, there is a unique $y \in range(f \cap g)$ such that $< x, .y >\in f \cap g$.

Let $< x, y >\in f \cap g$ and $< x, z >\in f \cap g$. This implies that $< x, y >\in f$ and $< x, z >\in f$.

But we assumed that f is a function. This means that f is single-valued.

i.e. $< x, y >=< x, z >$ which implies that $y = z$.

Therefore $f \cap g$ is single-valued, and is therefore a function.

b)

Required: Show that $f \cup g$ is a function iff $f(x) = g(x)$ for every $x \in (dom f) \cap (dom g)$

($\Rightarrow$):

Assume $f \cup g$ is a function.

Let $x \in (dom f) \cap (dom g)$.

So $x \in (dom f)$ and $x \in (dom g)$.

Therefore $< x, f(x) >\in f$ and $< x, g(x) >\in g$.

Therefore $< x, f(x) >\in f \cup g$ and $< x, g(x) >\in f \cup g$.

Since we assume that $f \cup g$ is a function, it is single-valued.

Therefore $< x, f(x) >=< x, g(x) >$ which implies that $f(x) = g(x)$.

($\Leftarrow$):

Assume that $f(x) = g(x)$ for every $x \in (dom f) \cap (dom g)$

We must show that $f \cup g$ is single-valued in order to be a function.

Let $< x, y > \in f \cup g$ and $< x, z > \in f \cup g$.

We now have to consider 4 cases.

Case 1: If $< x, y > \in f$ and $< x, z > \in f$ then since f is a function and single-valued, $< x, y > = < x, z >$ which implies that $y = z$.

Case 2: If $< x, y > \in g$ and $< x, z > \in g$ then since g is a function and single-valued, $< x, y > = < x, z >$ which implies that $y = z$.

Case 3: If $< x, y > \in f$ and $< x, z > \in g$ then by assumption we know that $f(x) = g(x)$. So clearly we have that $f(x) = y = z = g(x)$.

Case 4: If $< x, y > \in g$ and $< x, z > \in f$ then by assumption we know that $f(x) = g(x)$. So clearly we have that $g(x) = y = z = f(x)$.

Therefore, $f \cup g$ is a function.

Question 16, Page 53

Required: Show that there is no set to which every function belongs.

Assume for sake of contradiction that there is a set to which every function belongs.

Call this set A.

Let x be some arbitrary set.

Therefore the relation $f = \{< a, a >\}$ is an identity function as it is single-valued. Therefore $f \in A$.

Let $B \subseteq A$ be such that it contains all the identity functions for each set a.

i.e. $f = \{< a, a >\} \in B$.

But, $\{< a, a >\} = \{\{\{a\}, \{a, a\}\}\} = \{\{\{a\}, \{a\}\}\} = \{\{\{a\}\}\}$.

So we have that $\{\{\{a\}\}\} \in B$.

So, $\{\{a\}\} \in \bigcup B$.

So, $\{a\} \in \bigcup\bigcup B$.

So, $a \in \bigcup\bigcup\bigcup B$.

So, since a was arbitrary, the set $\bigcup\bigcup\bigcup B$ is the set of all sets.

But we know that the set of all sets is a proper class, and not a set.

We have a contradiction. Therefore, there is no set of all functions.

Question 29, Page 54

Assume that $f : A \to B$ and define a function $G : B \to P(A)$ by $G(b) = \{x \in A | f(x) = b\}$.

Required: Show that if f maps A onto B, then G is one-to-one. Does the converse hold?

Assume f maps A onto B.

Let $G(u) = G(v)$.

Since f is onto, there exists an $x \in A$ such that $f(x) = u$.

So $x \in G(u)$.

Since $G(u) = G(v)$, we have that $x \in G(v)$.

So $f(x) = v$.

This implies that $f(x) = u = v$.

Therefore, G is one-to-one.

But, the converse does not hold. We will show that the converse is false.

i.e. There is a function f such that G is one-to-one AND f does NOT map A onto B.

Let $A = \{\emptyset\}$
Let $B = \{\emptyset, \{\emptyset\}\}$
Let $f = \{< \emptyset, \emptyset >\}$.

It follows that $G = \{< \emptyset, \{\emptyset\} >\}$. So clearly G is one-to-one.

However, f clearly does not map A onto B since for $\{\emptyset\} \in B$, there is no $x \in A$ such that $f(x) = \{\emptyset\}$.

Question 30, Page 54

Assume that $F : P(A) \to P(A)$ and that F has the monotonicity property: $X \subseteq Y \subseteq A \Rightarrow F(X) \subseteq F(Y)$

Define $B = \bigcap\{X \subseteq A | F(X) \subseteq X\}$ and $C = \bigcup\{X \subseteq A | X \subseteq F(X)\}$

For notational convenience, we will write,

$B = \bigcap\{X | X \subseteq A \wedge F(X) \subseteq X\}$ and $C = \bigcup\{X | X \subseteq A \wedge X \subseteq F(X)\}$

a)

Show that $F(B) = B$ and $F(C) = C$.

Consider the following.

$$\begin{aligned}
F(B) &= F(\bigcap\{X | X \subseteq A \wedge F(X) \subseteq X\}) \\
&\subseteq \bigcap\{F(X) | X \subseteq A \wedge F(X) \subseteq X\} && \text{By Theorem 3K on page 50} \\
&\subseteq \bigcap\{X | X \subseteq A \wedge F(X) \subseteq X\} && \text{Since } F(X) \subseteq X \\
&= B && \text{By definition of B}
\end{aligned}$$

Therefore, $F(B) \subseteq B$.

Now, if $F(B) \subseteq B$, then we have that $F(F(B)) \subseteq F(B)$ by the monotonicity property given in the question.

But this implies that $B \subseteq F(B)$.

Therefore, we have shown that $F(B) = B$.

Now, we will show that $F(C) = C$.

Consider the following.

$$\begin{aligned}
F(C) &= F(\bigcup\{X | X \subseteq A \wedge X \subseteq F(X)\}) \\
&\subseteq \bigcup\{F(X) | X \subseteq A \wedge X \subseteq F(X)\} && \text{By Theorem 3K on page 50} \\
&\supseteq \bigcup\{X | X \subseteq A \wedge X \subseteq F(X)\} && \text{Since } X \subseteq F(X) \\
&= C && \text{By definition of C}
\end{aligned}$$

Therefore, $C \subseteq F(C)$.

Now, if $C \subseteq F(C)$, then we have that $F(C) \subseteq F(F(C))$ by the monotonicity property given in the question.

But this implies that $F(C) \subseteq C$.

Therefore, we have shown that $F(C) = C$.

b)

Show that if $F(X) = X$, then $B \subseteq X \subseteq C$.

Assume $F(X) = X$.

Since $F(X) = X$, we have that $F(X) \subseteq X$.

Since we know that one of the conditions of B is that $F(X) \subseteq X$ and since B is defined as an intersection, we have that $B \subseteq X$.

Since $F(X) = X$, we have that $X \subseteq F(X)$.

Since we know that one of the conditions of C is that $X \subseteq F(X)$. Since C is defined as a union, we have that $X \subseteq C$.

Since $B \subseteq X$ and $X \subseteq C$, we have that $B \subseteq X \subseteq C$.

Question 45, Page 64

Assume that $<_A$ and $<_B$ are linear orderings on A and B, respectively. Define the binary relation $<_L$ on the cartesian product $A \times B$ by:

$< a_1, b_1 > <_L < a_2, b_2 >$ iff $a_1 <_A a_2 \vee (a_1 = a_2 \wedge b_1 <_B b_2)$

Show that $<_L$ is a linear ordering on $A \times B$.

i.e. We must show that $<_L$ satisfies transitivity and trichotomy.

Transitivity

Assume $< a_1, b_1 > <_L < a_2, b_2 >$ and $< a_2, b_2 > <_L < a_3, b_3 >$.

We must show that $< a_1, b_1 > <_L < a_3, b_3 >$.

Note, that $a_1 <_A a_2 \vee (a_1 = a_2 \wedge b_1 <_B b_2)$ and $a_2 <_A a_3 \vee (a_2 = a_3 \wedge b_2 <_B b_3)$.

We must now consider 4 cases.

Case 1: If $a_1 <_A a_2$ and $a_2 <_A a_3$, then by transitivity of $<_A$ we have $a_1 <_A a_3$. Therefore, we have that $< a_1, b_1 > <_L < a_3, b_3 >$.

Case 2: If $a_1 <_A a_2$ and $(a_2 = a_3 \wedge b_2 <_B b_3)$, then we have $a_1 <_A a_3$ since $a_2 = a_3$. Therefore, we have that $< a_1, b_1 > <_L < a_3, b_3 >$.

Case 3: If $(a_1 = a_2 \wedge b_1 <_B b_2)$ and $a_2 <_A a_3$, then we have $a_1 <_A a_3$ since $a_1 = a_2$. Therefore, we have that $< a_1, b_1 > <_L < a_3, b_3 >$.

Case 4: If $(a_1 = a_2 \wedge b_1 <_B b_2$ and $(a_2 = a_3 \wedge b_2 <_B b_3$, then by transitivity of $<_B$ we have that $b_1 <_B b_3$. Also, since $a_1 = a_2$ and $a_2 = a_3$, by transitivity of $=$ we have that $a_1 = a_3$. So, we have that $a_1 = a_3 \wedge b_1 <_B b_3$. Therefore, we have that $< a_1, b_1 > <_L < a_3, b_3 >$.

Therefore, $<_L$ satisfies transitivity.

Trichotomy

Consider $< a_1, b_1 >$ and $< a_2, b_2 >$.

We will first show that at least one of $< a_1, b_1 > = < a_2, b_2 >$ or $< a_1, b_1 > <_L < a_2, b_2 >$ or $< a_2, b_2 > <_L < a_1, b_1 >$ always holds.

Consider the following two cases.

Case 1: If $< a_1, b_1 >=< a_2, b_2 >$, then we are done.

Case 2: If $< a_1, b_1 >\neq< a_2, b_2 >$, then either $a_1 \neq a_2 \vee b_1 \neq b_2$. If $a_1 \neq a_2$, then since $<_A$ is a linear ordering, we have either $a_1 <_A a_2$ or $a_2 <_A a_1$. So, we would either have $< a_1, b_1 ><_L< a_2, b_2 >$ or $< a_2, b_2 ><_L< a_1, b_1 >$.

Therefore, at least one of the three cases always holds.

Now we must show that at most one occurs. Consider the following three cases.

Case 1: If $< a_1, b_1 >=< a_2, b_2 >$, then clearly $a_1 = a_2$ and $b_1 = b_2$, and since $<_A$ and $<_B$ are linear orderings, we have that $a_1 \not<_A a_2$ and $a_2 \not<_A a_1$ and $b_1 \not<_A b_2$ and $b_2 \not<_A b_1$. Therefore, we have that $< a_1, b_1 >\not<_L< a_2, b_2 >$ and $< a_2, b_2 >\not<_L< a_1, b_1 >$.

Case 2: If $< a_1, b_1 ><_L< a_2, b_2 >$, then we have that $a_1 <_A a_2 \vee (a_1 = a_2 \wedge b_1 <_B b_2)$. Notice that if $a_1 <_A a_2$, then we have $a_1 \not<_A a_2$ and $a_1 \neq a_2$. So we cannot have $< a_1, b_1 >=< a_2, b_2 >$ or $< a_2, b_2 ><_L< a_1, b_1 >$. If $(a_1 = a_2 \wedge b_1 <_B b_2)$, then we have $a_1 \not<_A a_2$ and $a_2 \not<_A a_1$ and $b_2 \not<_B b_1$. Therefore, we cannot have $< a_1, b_1 >=< a_2, b_2 >$ or $< a_2, b_2 ><_L< a_1, b_1 >$.

Case 3: If $< a_2, b_2 ><_L< a_1, b_1 >$, then we have the same case as Case 2, expect with the roles of $< a_1, b_1 >$ and $< a_2, b_2 >$ reversed.

Therefore, $<_L$ also satisfies trichotomy.

Therefore, $<_L$ is a linear ordering.