Question 1

(a)

$(x + y) + z$ in Polish (prefix) notation is $++xyz$

$x + (y + z)$ in Polish (prefix) notation is $+x+yz$

$(x + (y \cdot z)) \cdot ((y \cdot x)+z)$ in Polish (prefix) notation is $\cdot + x \cdot yz + \cdot yxz$

(b) Section 1.5, Exercise 1 (page 21)

(a) $(\forall x)(\forall y)(x + y = 2)$ has no free variables. Therefore it is a sentence.

(b) $(x + y < x) \vee (\forall z)(z < 0)$ has free variables x and y. Therefore it is NOT a sentence.

(c) $((\forall y)(y < x)) \vee ((\forall x)(x < y))$ has free variable x in the first part of the formula, before the disjunction, and has free variable y in the second part of the formula, after the disjunction. Therefore it is NOT a sentence.

(c) Section 1.8, Exercise 1 (page 36)

(a) $u :\equiv \cos x, t$ is $\sin y$.

So, $u_t^x :\equiv \cos(\sin y)$

(b) $u :\equiv y, t$ is Sy.

So, $u_t^x :\equiv y$

(c) $u :\equiv \natural(x, y, z), t$ is $423 - w$.

So, $u_t^x :\equiv \natural(423 - w, y, z)$

(d) Section 1.8, Exercise 2 (page 36)

(a) $\phi :\equiv \forall x(x = y \to Sx = Sy), t$ is $S0$

So, $\phi_t^x :\equiv \forall x(x = y \to Sx = Sy)$.

So, t is substitutable since x is not free in ϕ, as per definition 1.8.3, 4(a) on page 35.

(b) $\phi :\equiv \forall y(x = y \to Sx = Sy), t$ is Sy

So, $\phi_t^x :\equiv \forall y(Sy = y \to SSy = Sy)$.

So, t is NOT substitutable since y occurs in t, and x is free in ϕ, as per definition 1.8.3, 4(a) and 4(b) on page 35. By substituting t, we bound Sy under the universal quantifier.

(c) $\phi :\equiv x = y \to (\forall x)(Sx = Sy), t$ is Sy

So, $\phi_t^x :\equiv Sy = y \to (\forall x)(Sx = Sy), t$ is Sy

So t is substitutable in ϕ since the first occurence of x is free and the second occurence of x is bounded. So, we may substitute t in the first occurrence, as per definition 1.8.3, 4(a) on page 35.

(e)

Let $A, B, C \in \{T, F\}$ be propositional variables.

We will verify that $P :\equiv (\neg(A \land B) \lor C) \leftrightarrow (A \to (B \to C))$ is a propositional tautology via truth table.

A	B	C	$\neg(A \land B)$	$\neg(A \land B) \lor C$	$B \to C$	$A \to (B \to C)$	P
T	T	T	F	T	T	T	T
T	T	F	F	F	F	F	T
T	F	T	T	T	T	T	T
T	F	F	T	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	T	T	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

We have that P is true for every truth value assignment. Therefore P is a propositional tautology.

Question 2

(a) Section 1.4, Exercise 6 (page 18)

Required to Prove: If s and t are distinct terms of a language $\mathcal{L}$, then s is not an initial segment of t.

Proof. We will use induction on the length of s.

Base Case: s is a constant

Consider $t :\equiv su$, where s is a constant and u is a string. Since su begins with a constant, t must begin with a constant. But, in order for t to be a term, t must only be a constant symbol. But this means that t is a term of length 1 which implies that $t = s$ and u is the empty string. So s is not a PROPER initial segment of t.

Base Case: s is a variable

Consider $t :\equiv su$, where s is a variable and u is a string. Since su begins with a variable, t must begin with a variable. But, in order for t to be a term, t must only be a variable symbol. But this means that t is a term of length 1 which implies that $t = s$ and u is the empty string. So s is not a PROPER initial segment of t.

Case: s is a function term

Assume $s :\equiv ft_1...t_n$ for an n-ary function symbol f, and assume as the inductive hypothesis, that the terms $t_1, ..., t_n$ are such that no t_i is an initial segment of any term.

Assume that $t :\equiv su$ where u is a non-empty string. i.e. s is an initial segment of t.

Since su begins with the function symbol f, it follows that t must begin with the function symbol f, and for t to be a term it must be that $t :\equiv fh_1...h_n$. Assume by inductive hypothesis that the terms $h_1, ...,h_n$ are such that no h_i is an initial segment of any term.

It follows that,

$$t :\equiv su$$
$$fh_1...h_n :\equiv ft_1...t_nu$$

Clearly we have the same function symbol f on both sides.

We also have the terms $h_1, ..., h_n$ and $t_1, ..., t_n$ on both sides with no h_i and t_i being the initial segment of each other.

It follows, that $h_i = t_i$ for $i \in \{1, ..., n\}$.

But, then that leaves us with $u :\equiv$ empty string. But we assumed that u is a nonempty string.

This is a contradiction. Therefore our assumption was wrong. Therefore s is not an initial segment of t.

By induction on the length of s, this completes the proof, as required. $\square$

(b) Section 1.4, Exercise 7 (page 18)

Required: Prove the unique readability of terms.

Proof. By the syntax of first order logic, we know that constant terms, variable terms, and function terms cannot be written in terms of each other. So, we only need to check the uniqueness of each case separately.

We will prove the unique readability of terms by induction on the length of t.

Let t be a term.

Base Case: t is a constant

Assume that $t :\equiv c$ and $t :\equiv c'$ where c, c' are constant symbols. Since terms that are constants are of length 1, and using the result in question 6 (the previous question), the constants c, c' are not initial segments of each other. Therefore $c = c'$. Thus, we have uniqueness.

Base Case: t is a variable

Assume that $t :\equiv v$ and $t :\equiv v'$ where v, v' are variable symbols. Since terms that are variables are of length 1, and using the result in question 6 (the previous question), the variables c, c' are not initial segments of each other. Therefore $c = c'$. Thus, we have uniqueness.

Case: t is a complex term (function term)

Assume that $t :\equiv ft_1...t_n$ and $t :\equiv f't'_1...t'_n$, where and f_1 and f'_1 are function symbols, and as the inductive hypothesis assume that $t_1, ..., t_n$ and $t'_1, ..., t'_n$ are unique terms.

Since both versions of t begin with a function symbol, it must be the case that f and f' are the same function.

Now, by using the result in question 6 (the previous question), we have the terms $t_1, ..., t_n$ and $t'_1, ..., t'_n$ with each t_i and t'_i not being the initial segments of any other term.

Thus, we have that $t_i = t'_i$ for all $i \in \{1, .., n\}$. But, this gives us that $ft_1...t_n = f't'_1...t'_n$. Thus, we have uniqueness. By induction on the length of t, this completes the proof, as required. $\square$

Question 3

(a)

Required: Write a variable-free term t, other than $SS0$, such that $\mathbb{N} \models (t = SS0)$.

Let $t :\equiv +0SS0$. In unofficial notation, $t :\equiv 0{+}SS0$

We will show that this is a suitable choice of t. We will use some unofficial notation below.

Now, let s be any variable assignment function, and let $\bar{s}$ be the corresponding term assignment function. To show that $\mathbb{N} \models (t = SS0)[s]$, notice that,

$$\begin{aligned}\bar{s}(+0SS0) &= +(\bar{s}(0), \bar{s}(SS0)) \\ &= +(0, SS\bar{s}(0)) \\ &= +(0, SS0) \\ &= +(0, 2) \\ &= 2\end{aligned}$$

Now, notice that,

$$\begin{aligned}\bar{s}(SS0) &= SS\bar{s}(0) \\ &= SS0 \\ &= 2\end{aligned}$$

Therefore, we have that, $\mathbb{N} \models (t = SS0)$, where $t :\equiv +0SS0$

(b)

Required: Write a sentence ϕ_1 without using $+$ such that $\mathbb{Z} \models \phi_1$ and $\mathbb{N} \not\models \phi_1$. Also, write a sentence ϕ_2 without using S such that $\mathbb{Z} \models \phi_1$ and $\mathbb{N} \not\models \phi_1$.

Let $\phi_1 :\equiv (\exists x)(Sx = 0)$.

It is clear that $\mathbb{Z} \models \phi_1$ since we can instantiate $x = -1$ so that $Sx = 0$.

Similarly, it is clear that $\mathbb{N} \not\models \phi_1$ since the successor of any natural number is greater than or equal to 1.

We will not show the above rigorously since the topic of deductions and instantiations with quantifiers is covered in Chapter 2. Also, the question simply asks to write down a ϕ_1.

Let $\phi_2 :\equiv (\exists x)(\exists y)(x \neq y \land x + y = 0)$.

It is clear that $\mathbb{Z} \models \phi_1$ since we can instantiate $x = -1$ and $y = 1$ so that $x + y = 0$.

Similarly, it is clear that $\mathbb{N} \not\models \phi_1$ since every two distinct natural numbers clearly sum to at least 1.

Again, we will not show the above rigorously since the topic of deductions and instantiations with quantifiers is covered in Chapter 2. Also, the question simply asks to write down a ϕ_2.

(c)

Required: Define an $\mathcal{L}$-structure $\mathfrak{A}$ such that $\mathfrak{A} \models \neg(t_1 = t_2)$ for every two distinct variable-free terms t_1, t_2.

Let the $\mathcal{L}$-structure, $\mathfrak{A} = (A, 0^{\mathfrak{A}}, S^{\mathfrak{A}}, +^{\mathfrak{A}})$.

Let the universe A be the set of all finite strings of symbols that are taken from the set $\{", 0, S, +\}$, where the symbols can be repeated any finite number of times in a string.

i.e. $"S0" \in A$ and $"S"S0"" \in S$

Now, we will provide our interpretations of our structure.

Let $0^{\mathfrak{A}}$ be $0 \in A$

For any $x \in A$, let $S^{\mathfrak{A}} : A \to A$ be $S^{\mathfrak{A}}x = "Sx"$

For any $x, y \in A$, let $+^{\mathfrak{A}} : A^2 \to A$ be $+^{\mathfrak{A}}xy = "+xy"$

Note that the superscripts $\mathfrak{A}$ will not actually appear in the syntax of our formulas and sentences. But they are stated above for the interpretations, i.e. semantics.

We now have that for any two distinct non-variable terms t_1 and t_2, $\mathfrak{A} \models \neg(t_1 = t_2)$.

To see why this is the case, first notice that it cannot be that $t_1 :\equiv 0$ and $t_2 :\equiv 0$, since then t_1 and t_2 would not be distinct.

Similarly, both functions $S, +$ never output the constant symbol 0 for any input, as the interpretations of the function symbols in this structure always puts quotes around the symbols.

So, we only need to check the case where t_1 and t_2 are both terms that start with a function symbol.

Let t_1, t_2 be distinct non-variable, non-constant terms that starts with some function symbol, i.e. starting with S or $+$. Notice, that they could each start with different function symbols, or the same. We just need the terms t_1 and t_2 to be distinct.

Let s be any variable assignment function. Let $\bar{s}$ be the corresponding term assignment function.

It follows that,

$$\bar{s}(t_1) = "t_1"$$

This is true since we know that t_1 begins with an S or a $+$. So quotations are added.

Similarly,

$$\bar{s}(t_2) = "t_2"$$

This is the case since we know that t_1 begins with an S or a $+$. So quotations are added.

Since t_1 is distinct to t_2, it follows that $"t_1" \neq "t_1"$, as we are simply adding quotation marks.

Therefore, we have that, $\mathfrak{A} \models \neg(t_1 = t_1)$, as required.