Exercise 1

Suppose that the langauge L has the equals sign and one binary predicate P. Let $M = (\mathbb{N}, I)$, where $I(P) =<_{\mathbb{N}}$. Sometimes we write this as $M = (\mathbb{N}, <)$. Say that a subset S of $\mathbb{N}$ is cofinite iff $\mathbb{N} - S$ is finite. Show that every finite and every cofinite subset of $\mathbb{N}$ is representable.

First we will show that every natural number is representable.

Proposition: For every $n \in \mathbb{N}$, the set $\{n\}$ is representable by a formula ϕ_n with free variable $\mathbf{v_1}$.

We will use induction on n.

Base Case: $n = 0$

We want to show the set $\{0\}$ is representable by a formula ϕ_0 with free variable $\mathbf{v_1}$.

Note: We will use infix notation.

Let ϕ_0 be the following formula representing $\{0\}$ with free variable $\mathbf{v_1}$.

$$\forall \mathbf{v_3}(\mathbf{v_1} \neq \mathbf{v_3} \rightarrow \mathbf{v_1 P v_3})$$

IH: The set $\{n\}$ is representable by a formula ϕ_n with free variable $\mathbf{v_1}$.

Show: The set $\{n+1\}$ is representable by a formula ϕ_{n+1} with free variable $\mathbf{v_1}$.

By **IH** we know that ϕ_n is a formula that represents $\{n\}$ with free variable $\mathbf{v_1}$. Consider a relabelling of ϕ_n with free variable $\mathbf{v_2}$ that still represents $\{n\}$. Note, we may need to relabel every variable in ϕ_n to avoid issues involving scope.

Let ϕ'_n be the relabelling of the variables in ϕ_n such that ϕ'_n represents $\{n\}$ and has free variable $\mathbf{v_2}$.

Let ϕ_{n+1} be the following formula representing $\{n+1\}$ with free variable $\mathbf{v_1}$.

$$\exists \mathbf{v_2}(\phi'_n \wedge \mathbf{v_2 P v_1} \wedge \forall \mathbf{v_3}((\mathbf{v_3} \neq \mathbf{v_1} \wedge \mathbf{v_2 P v_3}) \rightarrow \mathbf{v_1 P v_3}))$$

Therefore, by induction we have proven the **Proposition** that for every $n \in \mathbb{N}$, the set $\{n\}$ is representable by a formula ϕ_n with free variable $\mathbf{v_1}$.

Show every finite subset of $\mathbb{N}$ is representable

Let $S \subseteq \mathbb{N}$ be finite.

If $S = \emptyset$, then let ϕ_S be the following formula representing S with free variable $\mathbf{v_1}$.

$$\mathbf{v_1} \neq \mathbf{v_1}$$

If $S \neq \emptyset$, then $S = \{s_1, ..., s_n\} \subseteq \mathbb{N}$.

By our earlier **proposition** we know that for each $s_i \in S$ where $i \in \{1, ..., n\}$, there is a formula ϕ_{s_i} that represents the set $\{s_i\}$ with free variable $\mathbf{v_1}$.

We know $S = \bigcup_{i=1}^{n} \{s_i\}$.

Hence, let ϕ_S be the following formula representing S with free variable $\mathbf{v_1}$.

$$\bigvee_{i=1}^{n} \phi_{s_i}$$

Since $S \subseteq \mathbb{N}$ was an arbitrary finite set and is representable, we conclude that every finite subset of $\mathbb{N}$ is representable, as required.

Show every cofinite subset of $\mathbb{N}$ is representable

Let $S \subseteq \mathbb{N}$ be cofinite.

Hence, $S - \mathbb{N} \subseteq \mathbb{N}$ is finite.

We've shown earlier that every finite subset of $\mathbb{N}$ is representable.

Hence, $S - \mathbb{N}$ is representable by some formula ϕ.

Now, consider the formula $\sim \phi$.

Since ϕ represents the elements in $S - \mathbb{N}$, we have that $\sim \phi$ represents the elements that are not in $S - \mathbb{N}$.

i.e $\sim \phi$ represents the elements in S.

Hence, $\sim \phi$ represents S.

Since $S \subseteq \mathbb{N}$ was an arbitrary cofinite set and is representable, we conclude that every cofinite subset of $\mathbb{N}$ is representable, as required.

Exercise 2

Let L be a first-order language with the equals sign and one binary function symbol $\circ$. Show that the muiltiplication relation $\{\langle n, m, k\rangle \in \mathbb{R}^3 : n \times m = k\}$ is not definable in $(\mathbb{R}, +)$. Show that the addition relation $\{\langle n, m, k\rangle \in \mathbb{R}^3 : n + m = k\}$ is not definable in $(\mathbb{R}, \times)$.

Show multiplication not definable in $(\mathbb{R}, +)$

Assume for the sake of contradiction that $S = \{\langle n, m, k\rangle \in \mathbb{R}^3 : n \times m = k\}$ is definable in $(\mathbb{R}, +)$.

Now, consider the map $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = 2x$. We will show that h is an automorphism from $(\mathbb{R}, +)$ onto $(\mathbb{R}, +)$.

First we will show h is a homomorphism. We only have one function symbol to deal with.

Note: We will be using informal infix notation.

Show: $h(x + y) = h(x) + h(y)$ for all $x, y \in \mathbb{R}$

Let $x, y \in \mathbb{R}$. We have that,

$$h(x + y) = 2(x + y) = 2x + 2y = h(x) + h(y)$$

Therefore, h is a homomorphism from $(\mathbb{R}, +)$ into $(\mathbb{R}, +)$.

One-to-One: Assume $h(x) = h(y)$. Then, $2x = 2y$. Dividing by 2 we get $x = y$ which shows h is one-to-one.

Onto: Assume $y \in \mathbb{R}$. Then let $x = \frac{y}{2}$. Then, $h(x) = h\left(\frac{y}{2}\right) = 2\left(\frac{2}{y}\right) = y$ which shows h is onto.

Therefore, h is an automorphism from $(\mathbb{R}, +)$ onto $(\mathbb{R}, +)$.

By the Automorphism Theorem (8.4.3), we know that $S = \{\langle n, m, k\rangle \in \mathbb{R}^3 : n \times m = k\}$ must be closed under h. i.e. if $\langle d_1, d_2, d_3\rangle \in S$, then $\langle h(d_1), h(d_2), h(d_3)\rangle \in S$.

Let $d_1 = 2$ and $d_2 = 3$ and $d_3 = 6$. We know that $\langle d_1, d_2, d_3\rangle = \langle 2, 3, 6\rangle \in S$ since $2 \times 3 = 6$.

Hence, $\langle h(d_1), h(d_2), h(d_3)\rangle = \langle h(2), h(3), h(6)\rangle = \langle 4, 6, 12\rangle \in S$. But we know that $4 \times 6 = 24 \neq 12$. Therefore, $\langle 4, 6, 12\rangle \notin S$.

So we have $\langle 4, 6, 12\rangle \in S$ and $\langle 4, 6, 12\rangle \notin S$ which is a contradiction.

Therefore, our initial assumption was wrong and $S = \{\langle n, m, k\rangle \in \mathbb{R}^3 : n \times m = k\}$ is not definable in $(\mathbb{R}, +)$.

Show addition is not definable in $(\mathbb{R}, \times)$

Assume for the sake of contradiction that $T = \{\langle n, m, k\rangle \in \mathbb{R}^3 : n + m = k\}$ is definable in $(\mathbb{R}, \times)$.

Now, consider the map $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = x^3$. We will show that h is an automorphism from $(\mathbb{R}, \times)$ onto $(\mathbb{R}, \times)$.

First we will show h is a homomorphism. We only have one function symbol to deal with.

Note: We will be using informal infix notation.

Show: $h(x \times y) = h(x) \times h(y)$ for all $x, y \in \mathbb{R}$

Let $x, y \in \mathbb{R}$. We have that,

$$h(x \times y) = (x \times y)^3 = x^3 \times y^3 = h(x) \times h(y)$$

Therefore, h is a homomorphism from $(\mathbb{R}, \times)$ into $(\mathbb{R}, \times)$.

One-to-One: Assume $h(x) = h(y)$. Then, $x^3 = y^3$. Taking cube roots we get, $x = y$ which shows h is one-to-one.

Onto: Assume $y \in \mathbb{R}$. Then let $x = y^{\frac{1}{3}}$. Then, $h(x) = h\big(y^{\frac{1}{3}}\big) = \big(y^{\frac{1}{3}}\big)^3 = y$ which shows h is onto.

Therefore, h is an automorphism from $(\mathbb{R}, \times)$ onto $(\mathbb{R}, \times)$.

By the Automorphism Theorem (8.4.3), we know that $T = \{\langle n, m, k\rangle \in \mathbb{R}^3 : n + m = k\}$ must be closed under h. i.e. if $\langle d_1, d_2, d_3\rangle \in T$, then $\langle h(d_1), h(d_2), h(d_3)\rangle \in T$.

Let $d_1 = 1$ and $d_2 = 2$ and $d_3 = 3$. We know that $\langle d_1, d_2, d_3\rangle = \langle 1, 2, 3\rangle \in T$ since $1+2 = 3$.

Hence, $\langle h(d_1), h(d_2), h(d_3)\rangle = \langle h(1), h(2), h(3)\rangle = \langle 1, 8, 27\rangle \in T$. But we know that $1+8 = 9 \neq 27$. Therefore, $\langle 1, 8, 27\rangle \notin T$.

So we have $\langle 1, 8, 27\rangle \in T$ and $\langle 1, 8, 27\rangle \notin T$ which is a contradiction.

Therefore, our initial assumption was wrong and $T = \{\langle n, m, k\rangle \in \mathbb{R}^3 : n + m = k\}$ is not definable in $(\mathbb{R}, \times)$.

Exercise 3

Let L be a first-order language with the equals sign and one binary function symbol $\circ$. Show that there are 32 subsets of $\mathbb{R}$ that are representable in $(\mathbb{R}, \times)$. You might use the following fact: If $b, c > 0$ and $b \neq 1$, then there is a d such that $b^d = c$.

Note: We will be using infix notation.

Let $\phi_{\{0\}}$ be the following formula representing $\{0\}$ with free variable $\mathbf{v_1}$.

$$\forall \mathbf{v_2}(\mathbf{v_1} \circ \mathbf{v_2} = \mathbf{v_1})$$

Let $\phi_{\{1\}}$ be the following formula representing $\{1\}$ with free variable $\mathbf{v_1}$.

$$\forall \mathbf{v_2}(\mathbf{v_1} \circ \mathbf{v_2} = \mathbf{v_2})$$

Let $\phi_{\{\pm 1\}}$ be the following formula representing $\{1, -1\}$ with free variable $\mathbf{v_1}$.

$$\forall \mathbf{v_2}((\mathbf{v_1} \circ \mathbf{v_1}) \circ \mathbf{v_2} = \mathbf{v_2})$$

Let $\phi_{\{-1\}}$ be the following formula representing $\{-1\}$ with free variable $\mathbf{v_1}$.

$$\phi_{\{\pm 1\}} \wedge \sim \phi_{\{1\}}$$

Let $\phi_{\geq 0}$ be the following formula representing $\{x \in \mathbb{R} : x \geq 0\}$ with free variable $\mathbf{v_1}$.

$$\exists \mathbf{v_2}(\mathbf{v_2} \circ \mathbf{v_2} = \mathbf{v_1})$$

Let $\phi_{>0}$ be the following formula representing $\{x \in \mathbb{R} : x > 0\}$ with free variable $\mathbf{v_1}$.

$$\phi_{\geq 0} \wedge \sim \phi_{\{0\}}$$

Let $\phi_{<0}$ be the following formula representing $\{x \in \mathbb{R} : x < 0\}$ with free variable $\mathbf{v_1}$.

$$\sim \phi_{\geq 0}$$

Let ϕ_A be the following formula representing $A = \{x \in \mathbb{R} : x > 0\} \setminus \{1\}$ with free variable $\mathbf{v_1}$.

$$\phi_{>0} \wedge \sim \phi_{\{1\}}$$

Let ϕ_B be the following formula representing $B = \{x \in \mathbb{R} : x < 0\} \setminus \{-1\}$. with free variable $\mathbf{v_1}$.

$$\phi_{<0} \wedge \sim \phi_{\{-1\}}$$

Therefore, we have shown that the sets $\{0\}, \{1\}, \{-1\}, A, B$ are all representable by formulas $\phi_{\{0\}}, \phi_{\{1\}}, \phi_{\{-1\}}, \phi_A, \phi_B$ respectively.

Let $S = \{\{0\}, \{1\}, \{-1\}, A, B\}$. For every $X, Y \in S$ such that $X \neq Y$, we have that $X \cap Y = \emptyset$. i.e. the elements of S are mutually disjoint.

Now we will show that every element of S is minimal. i.e. $X \in S$ is minimal if for every $Y \subseteq X$ such that $Y \neq \emptyset$ and $Y \neq X$, we have that Y is not representable.

Show: For every $X \in S$ and every $Y \subseteq X$ such that $Y \neq \emptyset$ and $Y \neq X$, we have that Y is not representable (i.e. every element of X is minimal).

We will prove this for each $X \in S$.

Case 1: If $X = \{0\}$ or $X = \{1\}$ or $X = \{-1\}$, then the result holds trivially since there is no $Y \subseteq X$ such that $Y \neq \emptyset$ and $Y \neq X$.

Case 2: Consider $X = A = \{x \in \mathbb{R} : x > 0\} \setminus \{1\}$. Assume for the sake of contradiction that X is not minimal.

Hence, there exists a $Y \subseteq X$ such that $Y \neq \emptyset$ and $Y \neq X$ and Y is representable.

Since $Y \subseteq X$ such that $Y \neq \emptyset$ and $Y \neq X$, we know there exists $u, v \in \mathbb{R}$ such that $u \in Y$ and $v \in X \setminus Y$.

We will construct an $h : \mathbb{R} \to \mathbb{R}$ such that $h(u) = v$ where h is an automorphism from $(\mathbb{R}, \times)$ onto $(\mathbb{R}, \times)$.

Since $u \in Y$ and $v \in X \setminus Y$, we know that $u, v > 0$ and $u, v \neq 1$. By the Hint, we know there exists a $d \in \mathbb{R}$ such that $u^d = v$. Note that $d \neq 0$ since if $d = 0$, then $v = u^d = u^0 = 1 \in X \setminus Y$ which would be a contradiction since $1 \notin X \setminus Y$. Hence, $\frac{1}{d} \neq 0$. So we can take powers of $\frac{1}{d}$ without issue.

Consider as an initial attempt the function $h(x) = x^d$. Note that it is possible that $range(h) \neq \mathbb{R}$ if $d = 2, 4, 6, ...$ etc. And, if $d = e$ or some other troublesome value for d, then $h(x) = x^d$ would result in $range(h) \not\subseteq \mathbb{R}$. i.e. For $d = e$, we have $h(-1) = (-1)^e \in \mathbb{C} \setminus \mathbb{R}$. And if $d < 0$, say $d = \frac{-1}{2}$, then $h(0) = 0^{\frac{-1}{2}}$ is undefined. However, h gives no issues when $x > 0$. So we will define h piecewise as follows.

$$h(x) = \begin{cases} 0 & \text{If } x = 0 \\ x^d & \text{if } x > 0 \\ -|x|^d & \text{if } x < 0 \end{cases}$$

We will show that h is a homomorphism from $(\mathbb{R}, \times)$ into $(\mathbb{R}, \times)$.

Show: For every $x, y \in \mathbb{R}$, we have $h(x \times y) = h(x) \times h(y)$.

Let $x, y \in \mathbb{R}$.

Case i) Either $x = 0$ or $y = 0$.

If $x = 0$, then for any $y \in \mathbb{R}$ we have,

$$h(x \times y) = h(0 \times y) = h(0) = 0 = 0 \times h(y) = h(0) \times h(y) = h(x) \times h(y)$$

If $y = 0$, then for any $x \in \mathbb{R}$ we have,

$$h(x \times y) = h(x \times 0) = h(0) = 0 = h(x) \times 0 = h(x) \times h(0) = h(x) \times h(y)$$

Case ii) Both $x \neq 0$ and $y \neq 0$.

If $x, y > 0$, then

$$\begin{aligned} h(x \times y) &= (x \times y)^d && \text{Since } x \times y > 0 \\ &= x^d \times y^d \\ &= h(x) \times h(y) \end{aligned}$$

If $x, y < 0$, then

$$\begin{aligned} h(x \times y) &= (x \times y)^d && \text{Since } x \times y > 0 \\ &= x^d \times y^d \\ &= h(x) \times h(y) \end{aligned}$$

If $x > 0$ and $y < 0$, then

$$\begin{aligned} h(x \times y) &= -|x \times y|^d && \text{Since } x \times y < 0 \\ &= -(|x|^d \times |y|^d) \\ &= |x|^d \times -|y|^d \\ &= x^d \times -|y|^d && \text{Since } x > 0 \\ &= h(x) \times h(y) \end{aligned}$$

If $x < 0$ and $y > 0$, then

$$\begin{aligned} h(x \times y) &= -|x \times y|^d && \text{Since } x \times y < 0 \\ &= -(|x|^d \times |y|^d) \\ &= -|x|^d \times |y|^d \\ &= -|x|^d \times y^d && \text{Since } y > 0 \\ &= h(x) \times h(y) \end{aligned}$$

Therefore, h is a homomorphism from $(\mathbb{R}, \times)$ into $(\mathbb{R}, \times)$.

One-to-One: Assume $h(x) = h(y)$. If $h(x) = h(y) = 0$, then $x = y = 0$ since h maps no other element to 0. If $h(x) = h(y) > 0$, then clearly we must have $x, y > 0$ since only positive numbers map to positive numbers by h. Hence, $h(x) = x^d = y^d = h(y)$. Taking powers of $\frac{1}{d}$ on both sides we get $x = y$. If $h(x) = h(y) < 0$, then clearly we must have $x, y < 0$ since only negative numbers map to negative numbers by h. Hence, $h(x) = -|x|^d = -|y|^d = h(y)$. Hence, $|x|^d = |y|^d$. Taking powers of $\frac{1}{d}$ on both sides we get $|x| = |y|$. But since $x, y < 0$, we have $-x = |x| = |y| = -y$. Since $-x = -y$, we have $x = y$. This shows h is one-to-one.

Onto: Assume $y \in \mathbb{R}$. If $y = 0$, then let $x = 0$ so that $h(x) = h(0) = 0 = y$. If $y > 0$, then let $x = y^{\frac{1}{d}}$. Hence, $h(x) = h\left(y^{\frac{1}{d}}\right) = \left(y^{\frac{1}{d}}\right)^d = y$. If $y < 0$, then let $x = -|y|^{\frac{1}{d}}$. Hence, $h(x) = h\left(-|y|^{\frac{1}{d}}\right) = -\left||y|^{\frac{1}{d}}\right|^d = -\left(|y|^{\frac{1}{d}}\right)^d = -|y| = -(-y) = y$ since $y < 0$. This shows h is onto.

Therefore, h is an automorphism from $(\mathbb{R}, \times)$ onto $(\mathbb{R}, \times)$.

And note that $h(u) = u^d = v$ where $u \in Y$ and $v \in X \setminus Y$. i.e. $u \in Y$ and $v \notin Y$.

By the Automorphism Theorem (8.4.3), we know that Y must be closed under h. i.e. if $z \in Y$, then $h(z) \in Y$.

We know that $u \in Y$. Hence, $h(u) = u^d = v \in Y$.

But we know that $v \notin Y$.

So we have $v \in Y$ and $v \notin Y$ which is a contradiction. Therefore, our initial assumption was wrong and $X = A = \{x \in \mathbb{R} : x > 0\} \setminus \{1\}$ is minimal.

Case 3: Consider $X = B = \{x \in \mathbb{R} : x < 0\} \setminus \{-1\}$.

Assume for the sake of contradiction that X is not minimal.

Hence, there exists a $Y \subseteq X$ such that $Y \neq \emptyset$ and $Y \neq X$ and Y is representable.

Since $Y \subseteq X$ such that $Y \neq \emptyset$ and $Y \neq X$, we know there exists $u, v \in \mathbb{R}$ such that $u \in Y$ and $v \in X \setminus Y$.

We will construct an $h : \mathbb{R} \to \mathbb{R}$ such that $h(u) = v$ where h is an automorphism from $(\mathbb{R}, \times)$ onto $(\mathbb{R}, \times)$.

Since $u \in Y$ and $v \in X \setminus Y$, we know that $u, v < 0$ and $u, v \neq -1$. Hence, $|u|, |v| > 0$ and $|u|, |v| \neq 1$. By the Hint, we know there exists a $d \in \mathbb{R}$ such that $|u|^d = |v|$. Note that $d \neq 0$ since if $d = 0$, then $|v| = |u|^d = |u|^0 = 1$ which implies that $v = \pm 1 \in X \setminus Y$ which would be a contradiction since $\pm 1 \notin X \setminus Y$. Hence, $\frac{1}{d} \neq 0$. So we can take powers of $\frac{1}{d}$ without issue.

Consider the following function h. Note, $d \in \mathbb{R}$ is defined as satisfying $|u|^d = |v|$.

$$h(x) = \begin{cases} 0 & \text{If } x = 0 \\ x^d & \text{if } x > 0 \\ -|x|^d & \text{if } x < 0 \end{cases}$$

Note, the h we defined above is nearly identical to the h in **Case 2** apart from a possibly different value of d. So showing h is an automorphism in **Case 2** is identical to how we showed h was an automorphism in **Case 1** with possibly a different d which would not affect any of the work done. The various subcases we went through in **Case 1** would be repeated verbatim. So we will not repeat the work here.

Hence, the h we defined in **Case 2** is an automorphism from $(\mathbb{R}, \times)$ onto $(\mathbb{R}, \times)$.

By the Automorphism Theorem (8.4.3), we know that Y must be closed under h. i.e. if $z \in Y$, then $h(z) \in Y$.

We know that $u \in Y$. By the Automorphism Theorem (8.4.3), we have $h(u) \in Y$.

Notice,

$$\begin{aligned} h(u) &= -|u|^d && \text{Since } u < 0 \\ &= -|v| && \text{Since } |u|^d = |v| \\ &= -(-v) && \text{Since } v < 0 \text{ implies } |v| = -v \\ &= v \in Y \end{aligned}$$

But we know that $v \in X \setminus Y$. Hence, $v \notin Y$.

So we have $v \in Y$ and $v \notin Y$ which is a contradiction.

Therefore, our initial assumption was wrong and $X = B = \{x \in \mathbb{R} : x < 0\} \setminus \{-1\}$ is minimal.

Therefore, we have shown that every element in $S = \{\{0\}, \{1\}, \{-1\}, A, B\}$ is minimal.

Other Possible Representable Subsets:

Note, perhaps there is a set $Z \subseteq \mathbb{R}$ such that $Z \neq \emptyset$ and $Z \neq \mathbb{R}$ and $\forall X \in S, Z \not\subseteq X$ and Z is representable. Assume for the sake of contradiction that such a Z exists and is representable.

Case 1: If Z equals a union of elements of S then we have no issue. For example if $Z = \{0\} \cup \{1\}$, then we can simply take the disjunction of the formulas that represents those elements in S to represent Z. i.e. in our example we can take $\phi_{\{0\}} \vee \phi_{\{1\}}$. We'll include these cases among the 32 representable sets of $\mathbb{R}$ in $(\mathbb{R}, \times)$. So **Case 1** causes no issues.

Case 2: Now suppose Z does not equal a union of some elements of S. For instance, we could have $Z = \{x \in \mathbb{R} : -2 \leq x \leq 2\}$ which does not equal a union of some elements of S.

But consider an arbitrary Z that does not equal a union of elements of S and assume this Z is representable by a formula ϕ_Z. Note, we're still assuming $Z \neq \emptyset$ and $Z \neq \mathbb{R}$ and $\forall X \in S, Z \not\subseteq X$.

We know $Z \neq \{0\} \cup \{1\}$ and $Z \neq \{0\} \cup \{-1\}$ and $Z \neq \{1\} \cup \{-1\}$.

Hence, $Z \cap A \neq \emptyset$ or $Z \cap B \neq \emptyset$. Without loss of generality, suppose $Z \cap A \neq \emptyset$.

We know $Z \neq \emptyset$ and we know $Z \neq A$ since $Z \not\subseteq A$.

Hence, $A \setminus Z \subseteq A$ and $A \setminus Z \neq \emptyset$ and $A \setminus Z \neq A$.

But then $A \setminus Z$ is representable by $\phi_A \wedge \sim \phi_Z$ which contradicts the fact that A is minimal.

Therefore, Z is not representable in **Case 2**. Hence, any **nonempty proper subset** of $\mathbb{R}$ that is not contained in any element of S and that is not a union of elements of S is NOT representable.

We will now show the following claim that generalizes **Case 1**.

Claim: If $T \subseteq S$, then $\bigcup_{X \in T} X$ is representable. Informally, this says that unions of elements of S are representable.

Let $T \subseteq S$.

If $T = \emptyset$, then the formula $\mathbf{v_1} \neq \mathbf{v_1}$ represents $\bigcup_{X \in T} X = \emptyset$.

If $T \neq \emptyset$, then T contains between 1 and 5 elements of S.

Then the following formula represents $\bigcup_{X \in T} X$.

$$\bigvee_{X \in T} \phi_X$$

This proves the **Claim**. For instance, if $T = \{\{0\}, \{1\}, \{-1\}\}$, then $\phi_{\{0\}} \vee \phi_{\{1\}} \vee \phi_{\{-1\}}$ represents $\bigcup_{X \in T} X = \{0, 1, -1\}$.

Note: For $S \subseteq S$, we have $\bigcup_{X \in S} X = \mathbb{R}$ is representable by $\phi_{\{0\}} \vee \phi_{\{1\}} \vee \phi_{\{-1\}} \vee \phi_A \vee \phi_B$.

Note: For $Y \in S$, we know $T = \{Y\} \subseteq S$. Hence, ϕ_Y represents $\bigcup_{X \in T} X = Y$. So each individual element of S is shown to be representable by the **Claim**. So this **Claim** counts our original 5 representable sets we began with.

Counting the Number of Representable Subsets

We know every element of $S = \{\{0\}, \{1\}, \{-1\}, A, B\}$ is representable.

Since we have shown every element of S is minimal, we know that for each $X \in S$, no **nonempty proper subset** of X is representable.

From our **Other Possible Representable Subsets Case 2** section we know that any **nonempty proper subset** of $\mathbb{R}$ that is not contained in any element of S and that is not a union of elements of S is NOT representable.

From our **Claim** we have shown that unions of elements of S are representable (including the individual elements of S, the empty set and $\mathbb{R}$).

Therefore, all the representable subsets of $\mathbb{R}$ in $(\mathbb{R}, \times)$ are described in our **Claim**. Our **Claim** says that for each $T \subseteq S$, we have $\bigcup_{X \in T} X$ is representable. Hence, we just have to count the number of subsets $T \subseteq S$.

Now, we know that there are $2^{card(S)}$ many subsets of S. And we know that $2^{card(S)} = 2^5 = 32$.

Therefore, there are exactly 32 subsets of $\mathbb{R}$ that are representable in $(\mathbb{R}, \times)$, as required.

Exercise 4

Let L be a first-order language with the equals sign and one binary function symbol $\circ$. Show that the addition relation, $S = \{\langle n, m, k\rangle \in \mathbb{N}^3 : n + m = k\}$ is not definable in $(\mathbb{N}, \times)$.

Assume for the sake of contradiction that $S = \{\langle n, m, k\rangle \in \mathbb{N}^3 : n + m = k\}$ is definable in $(\mathbb{N}, \times)$.

Let $\mathbb{P} \subseteq \mathbb{N}$ be the set of prime numbers.

Now, consider the map $g : \mathbb{P} \to \mathbb{P}$ defined as follows.

$$g(p) = \begin{cases} p & \text{if } p \in \mathbb{P} \setminus \{2, 3\} \\ 3 & \text{if } p = 2 \\ 2 & \text{if } p = 3 \end{cases}$$

By the Fundamental Theorem of Arithmetic, we know every $n \in \mathbb{N}$ such that $n > 1$ can be written as a unique product of primes (up to ordering of primes). i.e. We can write $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ where each $p_i \in \mathbb{P}$ and $\alpha_i \geq 1$ for $i \in \{1, ..., k\}$.

Now, consider the map $h : \mathbb{N} \to \mathbb{N}$ defined as follows.

$$h(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ g(p_1)^{\alpha_1} \times \cdots \times g(p_k)^{\alpha_k} & \text{if } n > 1 \text{ and } n = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \end{cases}$$

Note on Factorizations: When considering an $n > 1$, we will allow for multiple "representations" of the prime factorizations for n. i.e. if $n = 540$, then we can represent the factorizations of n as $n = 2^2 \times 3^3 \times 5$ or $n = 3 \times 2^2 \times 3^2 \times 5$ or $n = 2 \times 2 \times 3 \times 3 \times 3 \times 5$ etc. This will not affect the value of $h(n)$, but this will be important when showing h is one-to-one.

We want to show that h is an automorphism from $(\mathbb{N}, \times)$ onto $(\mathbb{N}, \times)$.

First we will show that h is a homomorphism. We only have one function symbol to deal with.

Note: We will be using informal infix notation.

Show: $h(m \times n) = h(m) \times h(n)$ for all $m, n \in \mathbb{N}$

Let $m, n \in \mathbb{N}$. We will consider 3 cases.

Case 1: Either $m = 0$ or $n = 0$

If $m = 0$, then

$$h(m \times n) = h(0 \times n) = h(0) = 0 = 0 \times h(n) = h(0) \times h(n) = h(m) \times h(n)$$

If $n = 0$, then

$$h(m \times n) = h(m \times 0) = h(0) = 0 = h(m) \times 0 = h(m) \times h(0) = h(m) \times h(n)$$

Case 2: Either $m = 1$ or $n = 1$

If $m = 1$, then

$$h(m \times n) = h(1 \times n) = h(n) = 1 \times h(n) = h(1) \times h(n) = h(m) \times h(n)$$

If $n = 1$, then

$$h(m \times n) = h(m \times 1) = h(m) = h(m) \times 1 = h(m) \times h(1) = h(m) \times h(n)$$

Case 3: $m, n \neq 0$ and $m, n \neq 1$

Hence, $m, n > 1$. So we can write $m = p_1^{\alpha_1} \times \cdots \times p_k^{\alpha_k}$ where each $p_i \in \mathbb{P}$ and $\alpha_i \geq 1$ for $i \in \{1, ..., k\}$ and we can write $n = q_1^{\beta_1} \times \cdots \times q_l^{\beta_l}$ where each $q_j \in \mathbb{P}$ and $\beta_j \geq 1$ for $j \in \{1, ..., l\}$.

Note, m, n may have primes in common in their prime factorizations. But this won't affect our work below given what we mentioned in **Note on Factorizations**. Now, consider the following. Line 3 follows from what was mentioned in **Note on Factorizations**.

$$\begin{aligned}
h(m \times n) &= h((p_1^{\alpha_1} \times \cdots \times p_k^{\alpha_k}) \times (q_1^{\beta_1} \times \cdots \times q_l^{\beta_l})) \\
&= h(p_1^{\alpha_1} \times \cdots \times p_k^{\alpha_k} \times q_1^{\beta_1} \times \cdots \times q_l^{\beta_l}) \\
&= g(p_1)^{\alpha_1} \times \cdots \times g(p_k)^{\alpha_k} \times g(q_1)^{\beta_1} \times \cdots \times g(q_l)^{\beta_l} \\
&= (g(p_1)^{\alpha_1} \times \cdots \times g(p_k)^{\alpha_k}) \times (g(q_1)^{\beta_1} \times \cdots \times g(q_l)^{\beta_l}) \\
&= h(p_1^{\alpha_1} \times \cdots \times p_k^{\alpha_k}) \times h(q_1^{\beta_1} \times \cdots \times q_l^{\beta_l}) \\
&= h(m) \times h(n)
\end{aligned}$$

Note that **Case 1,2,3** overlap, but they are exhaustive. Hence, for all $m, n \in \mathbb{N}$, we have demonstrated that $h(m \times n) = h(m) \times h(n)$.

Therefore, h is a homomorphism from $(\mathbb{N}, \times)$ into $(\mathbb{N}, \times)$. Now we will show that h is one-to-one and onto. But before we do this, we will show that g is one-to-one and onto.

Show g is one-to-one: Assume $g(p_1) = g(p_2)$. If $g(p_1) = g(p_2) \in \mathbb{P} \setminus \{2, 3\}$, then $p_1 = p_2$ since g is identity on $\mathbb{P} \setminus \{2, 3\}$. If $g(p_1) = g(p_2) = 2$, then $p_1 = p_2 = 3$ by definition of g. If $g(p_1) = g(p_2) = 3$, then $p_1 = p_2 = 2$ by definition of g. Hence, g is one-to-one.

Show g is onto: Assume $q \in \mathbb{P}$. If $q \in \mathbb{P} \setminus \{2, 3\}$, then let $p = q$ so that $g(p) = g(q) = q$. If $q = 2$, then let $p = 3$ so that $g(p) = g(3) = 2 = q$. If $q = 3$, then let $p = 2$ so that $g(p) = g(2) = 3 = q$. Hence, g is onto.

Now we will show h is one-to-one and onto.

Show h is one-to-one:

Assume $h(m) = h(n)$.

Case 1: If $h(m) = h(n) = 0$, then $m = n = 0$ since no other elements map to 0.

Case 2: If $h(m) = h(n) = 1$, then $m = n = 1$ since no other elements map to 1.

Case 3: If $h(m) = h(n) > 1$, then we know $m, n \neq 0$ since if $m = 0$ or $n = 0$, then $h(m) = 0$ or $h(n) = 0$. And, $m, n \neq 1$ since if $m = 1$ or $n = 1$, then $h(m) = 1$ or $h(n) = 1$.

Hence, $m, n > 1$. Hence, we can write $m = p_1^{\alpha_1} \times \cdots \times p_k^{\alpha_k}$ where each $p_i \in \mathbb{P}$ and $\alpha_i \geq 1$ for $i \in \{1, ..., k\}$ and we can write $n = q_1^{\beta_1} \times \cdots \times q_l^{\beta_l}$ where each $q_j \in \mathbb{P}$ and $\beta_j \geq 1$ for $j \in \{1, ..., l\}$.

We will assume that $p_1 \neq ... \neq p_k$ and $q_1 \neq ... \neq q_l$ in the representations of the prime factorizations of m and n. This assumption will be important for the work below.

Notice that we have the following.

$$\begin{aligned} h(m) &= h(n) \\ h(p_1^{\alpha_1} \times \cdots \times p_k^{\alpha_k}) &= h(q_1^{\beta_1} \times \cdots \times q_l^{\beta_l}) \\ g(p_1)^{\alpha_1} \times \cdots \times g(p_k)^{\alpha_k} &= g(q_1)^{\beta_1} \times \cdots \times g(q_l)^{\beta_l} \end{aligned}$$

Since $p_1 \neq ... \neq p_k$, we have that $g(p_1) \neq ... \neq g(p_k)$ since g is one-to-one.

Since $q_1 \neq ... \neq q_l$, we have that $g(q_1) \neq ... \neq g(q_l)$ since g is one-to-one.

So we know $g(p_1) \neq ... \neq g(p_k)$ and $g(q_1) \neq ... \neq g(q_l)$ and we know by the Fundamental Theorem of Arithmetic the same primes appear on both sides with the same powers. Hence, $k = l$. From now on we will just use the subscript k instead of l. i.e, we have,

$$g(p_1)^{\alpha_1} \times \cdots \times g(p_k)^{\alpha_k} = g(q_1)^{\beta_1} \times \cdots \times g(q_k)^{\beta_k}$$

Hence, we can pair up equal primes on both sides with the same powers.

Equivalently, there is a bijective function $f : \{1, ..., k\} \to \{1, ..., k\}$ (i.e. f permutes the subscripts $1, ..., k$) such that,

$$g(p_i)^{\alpha_i} = g(q_{f(i)})^{\beta_{f(i)}}$$

where we have $g(p_i) = g(q_{f(i)})$ and $\alpha_i = \beta_{f(i)}$.

Since $g(p_i) = g(q_{f(i)})$ and g is one-to-one we have that,

$$p_i = q_{f(i)} \tag{1}$$

Looking at (1), since $\alpha_i = \beta_{f(i)}$ we have that,

$$p_i^{\alpha_i} = q_{f(i)}^{\beta_{f(i)}}$$

Therefore,

$$m = p_1^{\alpha_1} \times \cdots \times p_k^{\alpha_k} = q_{f(1)}^{\beta_{f(1)}} \times \cdots \times q_{f(k)}^{\beta_{f(k)}} = n$$

Hence, h is one-to-one.

Show h is onto: Let $n \in \mathbb{N}$. If $n = 0$, then let $m = 0$ so that $h(m) = h(0) = 0 = n$. If $n = 1$, then let $m = 1$ so that $h(m) = h(1) = 1 = n$.

If $n > 1$, then $n = q_1^{\beta_1} \times \cdots \times q_l^{\beta_l}$ where each $q_j \in \mathbb{P}$ and $\beta_j \geq 1$ for $j \in \{1, ..., l\}$. Since g is onto, we know for $q_j \in \mathbb{P}$ such that $j \in \{1, ..., l\}$, there exists $p_j \in \mathbb{P}$ such that $g(p_j) = q_j$. Now, let $m = p_1^{\beta_1} \times \cdots \times p_j^{\beta_l}$

Hence,

$$\begin{aligned} h(m) &= h(p_1^{\beta_1} \times \cdots \times p_j^{\beta_l}) \\ &= g(p_1)^{\beta_1} \times \cdots \times g(p_l)^{\beta_l} \\ &= q_1^{\beta_1} \times \cdots \times q_l^{\beta_l} \\ &= n \end{aligned}$$

Hence, h is onto.

Since we have shown h is a homomorphism from $(\mathbb{R}, +)$ into $(\mathbb{R}, +)$ and h is one-to-one and onto, we have that h is an automorphism from $(\mathbb{R}, +)$ onto $(\mathbb{R}, +)$.

By the Automorphism Theorem (8.4.3), we know that $S = \{\langle n, m, k \rangle \in \mathbb{N}^3 : n + m = k\}$ must be closed under h. i.e. if $\langle d_1, d_2, d_3 \rangle \in S$, then $\langle h(d_1), h(d_2), h(d_3) \rangle \in S$.

Let $d_1 = 1$ and $d_2 = 2$ and $d_3 = 3$. We know that $\langle d_1, d_2, d_3 \rangle = \langle 1, 2, 3 \rangle \in S$ since $1 + 2 = 3$.

Hence, $\langle h(d_1), h(d_2), h(d_3) \rangle = \langle h(1), h(2), h(3) \rangle = \langle 1, 3, 2 \rangle \in S$,

But we know that $1 + 3 = 4 \neq 2$. Therefore, $\langle 1, 3, 2 \rangle \notin S$.

So we have $\langle 1, 3, 2 \rangle \in S$ and $\langle 1, 3, 2 \rangle \notin S$ which is a contradiction.

Therefore, our initial assumption was wrong and $S = \{\langle n, m, k \rangle \in \mathbb{N}^3 : n + m = k\}$ is not definable in $(\mathbb{N}, \times)$.