Question 1

Let $\vdash^*$ be a deducibility relation with modens ponens as a rule.

(i)

Required: Prove that if Cut and the Deduction Theorem hold, then $\vdash^* (\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma$.

Proof. Assume Cut and the Deduction Theorem hold.

By modus ponens, we know that $\{\alpha, \alpha \to \beta \to \gamma\} \vdash^* \beta \to \gamma$.

Also by modus ponens, we know that $\{\beta, \beta \to \gamma\} \vdash^* \gamma$.

Since $\{\alpha, \alpha \to \beta \to \gamma\} \vdash^* \beta \to \gamma$ and $\{\beta, \beta \to \gamma\} \vdash^* \gamma$, then by the Cut Rule we have that $\{\alpha, \alpha \to \beta \to \gamma, \beta\} \vdash^* \gamma$.

By modus ponens, we know that $\{\alpha, \alpha \to \beta\} \vdash^* \beta$.

Since $\{\alpha, \alpha \to \beta \to \gamma, \beta\} \vdash^* \gamma$ and $\{\alpha, \alpha \to \beta\} \vdash^* \beta$, then by the Cut Rule we have that $\{\alpha, \alpha \to \beta \to \gamma, \alpha \to \beta\} \vdash^* \gamma$.

Since $\{\alpha, \alpha \to \beta \to \gamma, \alpha \to \beta\} \vdash^* \gamma$, by the Deduction Theorem we have that $\{\alpha \to \beta \to \gamma, \alpha \to \beta\} \vdash^* \alpha \to \gamma$.

Since $\{\alpha \to \beta \to \gamma, \alpha \to \beta\} \vdash^* \alpha \to \gamma$, by the Deduction Theorem we have that $\{\alpha \to \beta \to \gamma\} \vdash^* (\alpha \to \beta) \to \alpha \to \gamma$.

Since $\{\alpha \to \beta \to \gamma\} \vdash^* (\alpha \to \beta) \to \alpha \to \gamma$, by the Deduction Theorem we have that $\vdash^* (\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma$.

This completes the proof, as required. $\square$

(ii)

Required: Prove that if the Deduction Theorem and Inconsistency Effect hold, then $\vdash^* \neg\alpha \to \alpha \to \beta$.

Proof. Assume the Deduction Theorem and Inconsistency Effect hold.

We know $\{\alpha, \neg\alpha\} \vdash^* \alpha$ and $\{\alpha, \neg\alpha\} \vdash^* \neg\alpha$. Hence, we have $\{\alpha, \neg\alpha\} \vdash^*$.

Since $\{\alpha, \neg\alpha\} \vdash^*$, by the Inconsistency Effect we have that $\{\alpha, \neg\alpha\} \vdash^* \beta$.

Since $\{\alpha, \neg\alpha\} \vdash^* \beta$, by the Deduction Theorem we have that $\{\neg\alpha\} \vdash^* \alpha \to \beta$.

Since $\{\neg\alpha\} \vdash^* \alpha \to \beta$, by the Deduction Theorem we have that $\vdash^* \neg\alpha \to \alpha \to \beta$.

This completes the proof, as required. $\square$

Question 2

Required: Provide a sentence and prove that it is satisfied by all and only valuations based on structures with domains that have exactly two members.

Let ϕ be the following sentence.

$$\exists x \exists y(x \neq y \wedge \forall z(z = x \vee z = y))$$

For any valuation σ based on a structure $\mathcal{U}$ with domain U, consider the following. We will occasionally add square brackets around metalanguage connectives for increased readability and to avoid ambiguity. And we will assume obvious properties of when existential, disjunctive, and conjunctive formulas are satisfied.

$$\begin{aligned}
& \sigma \models \phi \\
\text{iff} \quad & \phi^\sigma = \top \\
\text{iff} \quad & (\exists x \exists y(x \neq y \wedge \forall z(z = x \vee z = y)))^\sigma = \top \\
\text{iff} \quad & \text{For some } u \in U,\ \exists y(x \neq y \wedge \forall z(z = x \vee z = y))^{\sigma(x/u)} = \top \\
\text{iff} \quad & \text{For some } u, v \in U,\ (x \neq y \wedge \forall z(z = x \vee z = y))^{\sigma(x/u)(y/v)} = \top \\
\text{iff} \quad & \text{For some } u, v \in U,\ [(x \neq y)^{\sigma(x/u)(y/v)} = \top \text{ and } (\forall z(z = x \vee z = y))^{\sigma(x/u)(y/v)} = \top] \\
\text{iff} \quad & \text{For some } u, v \in U,\ [(x \neq y)^{\sigma(x/u)(y/v)} = \top \text{ and for all } w \in U,\ ((z = x \vee z = y))^{\sigma(x/u)(y/v)(z/w)} = \top] \\
\text{iff} \quad & \text{For some } u, v \in U,\ \Big[(x = y)^{\sigma(x/u)(y/v)} = \bot \text{ and for all } w \in U,\ [(z = x)^{\sigma(x/u)(y/v)(z/w)} = \top \\
& \qquad \text{or } (z = y)^{\sigma(x/u)(y/v)(z/w)} = \top]\Big] \\
\text{iff} \quad & \text{For some } u, v \in U,\ \Big[\langle x^{\sigma(x/u)(y/v)}, y^{\sigma(x/u)(y/v)}\rangle \notin =^{\sigma(x/u)(y/v)} \text{ and for all } w \in U, \\
& [\langle z^{\sigma(x/u)(y/v)(z/w)}, x^{\sigma(x/u)(y/v)(z/w)}\rangle \in =^{\sigma(x/u)(y/v)(z/w)} \text{ or } \langle z^{\sigma(x/u)(y/v)(z/w)}, y^{\sigma(x/u)(y/v)(z/w)}\rangle \in =^{\sigma(x/u)(y/v)(z/w)}]\Big] \\
\text{iff} \quad & \text{For some } u, v \in U,\ \Big[\langle x^{\sigma(x/u)(y/v)}, y^{\sigma(x/u)(y/v)}\rangle \notin id_U \text{ and for all } w \in U, \\
& \quad [\langle z^{\sigma(x/u)(y/v)(z/w)}, x^{\sigma(x/u)(y/v)(z/w)}\rangle \in id_U \text{ or } \langle z^{\sigma(x/u)(y/v)(z/w)}, y^{\sigma(x/u)(y/v)(z/w)}\rangle \in id_U]\Big] \\
\text{iff} \quad & \text{For some } u, v \in U,\ \Big[\langle u, v\rangle \notin id_U \text{ and for all } w \in U,\ [\langle w, u\rangle \in id_U \text{ or } \langle w, v\rangle \in id_U]\Big]
\end{aligned}$$

Now we will prove the following claim.

Claim: For every valuation σ based on a structure $\mathcal{U}$ with domain U, we have $\sigma \models \phi$ if and only if U has exactly two members.

Proof. Let σ be a valuation based on a structure $\mathcal{U}$ with domain U.

($\Rightarrow$): Assume $\sigma \models \phi$. Hence, by our earlier biconditional we have that for some $u, v \in U$, $\Big[\langle u, v\rangle \notin id_U$ and for all $w \in U$, $[\langle w, u\rangle \in id_U$ or $\langle w, v\rangle \in id_U]\Big]$.

Since $u, v \in U$ with $\langle u, v\rangle \notin id_U$, we know that $u \neq v$. Hence, U has at least 2 elements. Let $w \in U$ be arbitrary. Assume for the sake of contradiction that $w \neq u$ and $w \neq v$. Hence, $\langle w, u\rangle \notin id_U$ and $\langle w, v\rangle \notin id_U$. But then this contradicts the fact that for all $w \in U$, $[\langle w, u\rangle \in id_U$ or $\langle w, v\rangle \in id_U]$.

Hence, we must have $w = u$ or $w = v$. Since $w \in U$ was arbitrary, we have that every element of U is either u or v where $u \neq v$. Hence, U has exactly 2 elements.

$(\Leftarrow)$: Assume U has exactly two members. Hence, there exists $u, v \in U$ such that $u \neq v$. Hence, $\langle u, v\rangle \notin id_U$.

Now, consider an arbitrary $w \in U$. If $w \neq u$ and $w \neq v$, then U would have at least 3 members which would be a contradiction. Hence, we must have $w = u$ or $w = v$. Hence, we have $\langle w, u\rangle \in id_U$ or $\langle w, v\rangle \in id_U$. Since $w \in U$ was arbitrary, we have that for all $w \in U$ $[\langle w, u\rangle \in id_U$ or $\langle w, v\rangle \in id_U]$

Combining the above two observations, we get that for some $u, v \in U$, $\Big[\langle u, v\rangle \notin id_U$ and for all $w \in U$, $[\langle w, u\rangle \in id_U$ or $\langle w, v\rangle \in id_U]\Big]$. By our earlier biconditional we get that $\sigma \models \phi$.

This completes the proof of the **Claim**. And our **Claim** demonstrates that our sentence ϕ is satisfied by all and only valuations based on structures with domains that have exactly two members, as required. $\square$

Question 3

Let $\mathcal{L}$ be a first-order language with one constant symbol c, one function symbol f, and one two-place relation symbol A. Let $\mathcal{U}$ be the structure such that the domain $U = \{1, 2, 3\}$ and the interpretation function is such that:

$c^{\mathcal{U}} = 3$

$f^{\mathcal{U}}(1) = 2,\ f^{\mathcal{U}}(2) = 3,\ f^{\mathcal{U}}(3) = 2$

$A^{\mathcal{U}} = \{\langle 1, 2\rangle, \langle 2, 3\rangle, \langle 3, 3\rangle\}$

(a)

Let σ be a valuation based on $\mathcal{U}$ such that $\sigma(v) = 1$ for all variables v. Determine whether the following holds and prove why this is so:

$$\sigma \models \exists x(A(f(z), c) \to \forall y(A(y, x) \lor A(f(y), x)))$$

Consider the following. We will occasionally add square brackets around metalanguage connectives for increased readability and to avoid ambiguity. And we will assume obvious properties of when existential and disjunctive formulas are satisfied.

$$\begin{aligned}
&\sigma \models \exists x(A(f(z), c) \to \forall y(A(y, x) \lor A(f(y), x))) \\
\text{iff}\quad &(\exists x(A(f(z), c) \to \forall y(A(y, x) \lor A(f(y), x))))^{\sigma} = \top \\
\text{iff}\quad &\text{For some } u \in U,\ (A(f(z), c) \to \forall y(A(y, x) \lor A(f(y), x)))^{\sigma(x/u)} = \top \\
\text{iff}\quad &\text{For some } u \in U,\ [\ (A(f(z), c))^{\sigma(x/u)} = \bot \quad \text{or} \quad (\forall y(A(y, x) \lor A(f(y), x)))^{\sigma(x/u)} = \top] \\
\text{iff}\quad &\text{For some } u \in U,\ \Big[\ \langle f^{\sigma(x/u)}(z^{\sigma(x/u)}), c^{\sigma(x/u)}\rangle \notin A^{\sigma(x/u)} \quad \text{or} \\
&[\text{for all } v \in U,\ (A(y, x) \lor A(f(y), x)))^{\sigma(x/u)(y/v)} = \top]\Big] \\
\text{iff}\quad &\text{For some } u \in U,\ \Big[\ \langle f^{\sigma(x/u)}(z^{\sigma(x/u)}), c^{\sigma(x/u)}\rangle \notin A^{\sigma(x/u)} \quad \text{or} \\
&[\text{for all } v \in U,\ \langle y^{\sigma(x/u)(y/v)}, x^{\sigma(x/u)(y/v)}\rangle \in A^{\sigma(x/u)(y/v)} \quad \text{or} \quad \langle f^{\sigma(x/u)(y/v)}(y^{\sigma(x/u)(y/v)}), x^{\sigma(x/u)(y/v)}\rangle \in A^{\sigma(x/u)(y/v)}]\Big] \\
\text{iff}\quad &\text{For some } u \in U,\ \Big[\ \langle f^{\mathcal{U}}(1), c^{\mathcal{U}}\rangle \notin A^{\mathcal{U}} \quad \text{or} \quad [\text{for all } v \in U,\ \langle v, u\rangle \in A^{\mathcal{U}} \quad \text{or} \quad \langle f^{\mathcal{U}}(v), u\rangle \in A^{\mathcal{U}}]\Big] \\
\text{iff}\quad &\text{For some } u \in U,\ \Big[\ \langle 2, 3\rangle \notin A^{\mathcal{U}} \quad \text{or} \quad [\text{for all } v \in U,\ \langle v, u\rangle \in A^{\mathcal{U}} \quad \text{or} \quad \langle f^{\mathcal{U}}(v), u\rangle \in A^{\mathcal{U}}]\Big]
\end{aligned}$$

Now, let $u = 3 \in U$. Consider an arbitary $v \in U$. We know that either $v = 1$ or $v = 2$ or $v = 3$. We will consider these three cases separately.

Case 1: If $v = 1$, then we know that $f^{\mathcal{U}}(v) = f^{\mathcal{U}}(1) = 2$. Hence, $\langle f^{\mathcal{U}}(v), u\rangle = \langle 2, 3\rangle \in A^{\mathcal{U}}$. Hence, trivially we have shown that $\langle v, u\rangle \in A^{\mathcal{U}}$ or $\langle f^{\mathcal{U}}(v), u\rangle \in A^{\mathcal{U}}$.

Case 2: If $v = 2$, then we know that $\langle v, u\rangle = \langle 2, 3\rangle \in A^{\mathcal{U}}$. Hence, trivially we have shown that $\langle v, u\rangle \in A^{\mathcal{U}}$ or $\langle f^{\mathcal{U}}(v), u\rangle \in A^{\mathcal{U}}$.

Case 3: If $v = 3$, then we know that $\langle v, u\rangle = \langle 3, 3\rangle \in A^{\mathcal{U}}$. Hence, trivially we have shown that $\langle v, u\rangle \in A^{\mathcal{U}}$ or $\langle f^{\mathcal{U}}(v), u\rangle \in A^{\mathcal{U}}$.

In all three cases we have shown that $\langle v, u\rangle \in A^{\mathcal{U}}$ or $\langle f^{\mathcal{U}}(v), u\rangle \in A^{\mathcal{U}}$. Hence, we have shown that for all $v \in U$, $\langle v, u\rangle \in A^{\mathcal{U}}$ or $\langle f^{\mathcal{U}}(v), u\rangle \in A^{\mathcal{U}}$.

Hence, we have shown that $\langle 2, 3\rangle \notin A^{\mathcal{U}}$ or [for all $v \in U$, $\langle v, u\rangle \in A^{\mathcal{U}}$ or $\langle f^{\mathcal{U}}(v), u\rangle \in A^{\mathcal{U}}$].

Since we have shown the above for $u = 3$, we have shown that,

For some $u \in U$, $\Big[\langle 2, 3\rangle \notin A^{\mathcal{U}}$ or [for all $v \in U$, $\langle v, u\rangle \in A^{\mathcal{U}}$ or $\langle f^{\mathcal{U}}(v), u\rangle \in A^{\mathcal{U}}]\Big]$.

By our earlier biconditional, we then have that $\sigma \models \exists x(A(f(z), c) \rightarrow \forall y(A(y, x) \vee A(f(y), x)))$, as required.

Part b) of Question 3 is on the next page.

(b)

Required: Give a different structure and variable assignment in which the formula is not satisfied.

Consider the following structure $\mathcal{U}$ with domain $U = \{1, 2\}$ and the interpretation function is such that:

$c^{\mathcal{U}} = 1$

$f^{\mathcal{U}}(1) = 1,\ f^{\mathcal{U}}(2) = 2$

$A^{\mathcal{U}} = \{\langle 1, 1\rangle\}$

Let σ be a valuation based on $\mathcal{U}$ such that $\sigma(v) = 1$ for all variables v.

Consider the following. We will occasionally add square brackets around metalanguage connectives for increased readability and to avoid ambiguity. And we will assume obvious properties of when existential and disjunctive formulas are satisfied.

$$
\begin{aligned}
& \sigma \models \exists x(A(f(z), c) \to \forall y(A(y, x) \lor A(f(y), x))) \\
\text{iff}\quad & (\exists x(A(f(z), c) \to \forall y(A(y, x) \lor A(f(y), x))))^{\sigma} = \top \\
\text{iff}\quad & \text{For some } u \in U,\ (A(f(z), c) \to \forall y(A(y, x) \lor A(f(y), x)))^{\sigma(x/u)} = \top \\
\text{iff}\quad & \text{For some } u \in U,\ [\,(A(f(z), c))^{\sigma(x/u)} = \bot \ \text{ or } \ (\forall y(A(y, x) \lor A(f(y), x)))^{\sigma(x/u)} = \top] \\
\text{iff}\quad & \text{For some } u \in U,\ \Big[\ \langle f^{\sigma(x/u)}(z^{\sigma(x/u)}), c^{\sigma(x/u)}\rangle \notin A^{\sigma(x/u)} \ \text{ or} \\
& \quad [\text{for all } v \in U,\ (A(y, x) \lor A(f(y), x)))^{\sigma(x/u)(y/v)} = \top]\Big] \\
\text{iff}\quad & \text{For some } u \in U,\ \Big[\ \langle f^{\sigma(x/u)}(z^{\sigma(x/u)}), c^{\sigma(x/u)}\rangle \notin A^{\sigma(x/u)} \ \text{ or} \\
& [\text{for all } v \in U,\ \langle y^{\sigma(x/u)(y/v)}, x^{\sigma(x/u)(y/v)}\rangle \in A^{\sigma(x/u)(y/v)} \ \text{ or } \ \langle f^{\sigma(x/u)(y/v)}(y^{\sigma(x/u)(y/v)}), x^{\sigma(x/u)(y/v)}\rangle \in A^{\sigma(x/u)(y/v)}]\Big] \\
\text{iff}\quad & \text{For some } u \in U,\ \Big[\ \langle f^{\mathcal{U}}(1), c^{\mathcal{U}}\rangle \notin A^{\mathcal{U}} \ \text{ or } \ [\text{for all } v \in U,\ \langle v, u\rangle \in A^{\mathcal{U}} \ \text{ or } \ \langle f^{\mathcal{U}}(v), u\rangle \in A^{\mathcal{U}}]\Big] \\
\text{iff}\quad & \text{For some } u \in U,\ \Big[\ \langle 1, 1\rangle \notin A^{\mathcal{U}} \ \text{ or } \ [\text{for all } v \in U,\ \langle v, u\rangle \in A^{\mathcal{U}} \ \text{ or } \ \langle f^{\mathcal{U}}(v), u\rangle \in A^{\mathcal{U}}]\Big]
\end{aligned}
$$

Show: It is not the case that $\Big[\text{For some } u \in U,\ \Big[\ \langle 1, 1\rangle \notin A^{\mathcal{U}} \ \text{ or } \ [\text{for all } v \in U,\ \langle v, u\rangle \in A^{\mathcal{U}} \ \text{ or } \ \langle f^{\mathcal{U}}(v), u\rangle \in A^{\mathcal{U}}]\Big]\Big]$.

Case 1: First consider $u = 1$.

Let $v = 2$. Then we know that $\langle v, u\rangle = \langle 2, 1\rangle \notin A^{\mathcal{U}}$. And we know $\langle f^{\mathcal{U}}(v), u\rangle = \langle f^{\mathcal{U}}(2), 1\rangle = \langle 2, 1\rangle \notin A^{\mathcal{U}}$. Hence, $[\langle v, u\rangle \in A^{\mathcal{U}} \ \text{ or } \ \langle f^{\mathcal{U}}(v), u\rangle \in A^{\mathcal{U}}]$ does not hold. Hence, we have that $[\text{for all } v \in U,\ \langle v, u\rangle \in A^{\mathcal{U}} \ \text{ or } \ \langle f^{\mathcal{U}}(v), u\rangle \in A^{\mathcal{U}}]$ does not hold for $u = 1$.

Case 2: Next consider $u = 2$.

Let $v = 2$. Then we know that $\langle v, u\rangle = \langle 2, 2\rangle \notin A^{\mathcal{U}}$. And we know $\langle f^{\mathcal{U}}(v), u\rangle = \langle f^{\mathcal{U}}(2), 2\rangle = \langle 2, 2\rangle \notin A^{\mathcal{U}}$. Hence, $[\langle v, u\rangle \in A^{\mathcal{U}}$ or $\langle f^{\mathcal{U}}(v), u\rangle \in A^{\mathcal{U}}]$ does not hold. Hence, we have that [for all $v \in U$, $\langle v, u\rangle \in A^{\mathcal{U}}$ or $\langle f^{\mathcal{U}}(v), u\rangle \in A^{\mathcal{U}}$] does not hold for $u = 2$.

Combining **Case 1 and 2**, we've shown [for all $v \in U$, $\langle v, u\rangle \in A^{\mathcal{U}}$ or $\langle f^{\mathcal{U}}(v), u\rangle \in A^{\mathcal{U}}$] does not hold for any $u \in U$.

We know that $\langle 1, 1\rangle \in A^{\mathcal{U}}$. Hence, $\langle 1, 1\rangle \notin A^{\mathcal{U}}$ does not hold for any $u \in U$.

Hence, $\Big[\langle 1, 1\rangle \notin A^{\mathcal{U}}$ or [for all $v \in U$, $\langle v, u\rangle \in A^{\mathcal{U}}$ or $\langle f^{\mathcal{U}}(v), u\rangle \in A^{\mathcal{U}}]\Big]$ does not hold for any $u \in U$.

Hence, it is not the case that $\Big[$For some $u \in U$, $\Big[\langle 1, 1\rangle \notin A^{\mathcal{U}}$ or [for all $v \in U$, $\langle v, u\rangle \in A^{\mathcal{U}}$ or $\langle f^{\mathcal{U}}(v), u\rangle \in A^{\mathcal{U}}]\Big]\Big]$.

Looking at our earlier biconditional and taking the contrapositive we conclude that $\sigma \not\models \exists x(A(f(z), c) \rightarrow \forall y(A(y, x) \vee A(f(y), x)))$, as required.

Question 4

(i)

Required to Prove: $\Gamma \vdash^* \phi \Rightarrow \Gamma \models^* \phi$

Proof. Assume $\Gamma \vdash^* \phi$. Hence, we have a deduction $\phi_1, ..., \phi_n$ from Γ where $\phi_n = \phi$.

Show: $\Gamma \models^* \phi_k$ for $k = 1, ..., n$ by strong induction.

Inductive Hypothesis: Assume $\Gamma \models^* \phi_l$ for all $l < k$.

We will show that $\Gamma \models^* \phi_k$. We have three cases to consider.

Case 1: ϕ_k is an axiom. We will first show that Axiom 1, 2, and 3 are valid.

Show: Axiom 1 is valid. i.e. $\models^* \alpha \to \beta \to \alpha$.

Assume for the sake of contradiction that Axiom 1 is not valid. i.e. $\not\models^* \alpha \to \beta \to \alpha$. Hence, there exists a valuation σ such that $(\alpha \to \beta \to \alpha)^\sigma = \bot$. Now consider the following. All the implications below can be an 'if and only if', but it is not relevant to our work.

$$\begin{aligned} &(\alpha \to \beta \to \alpha)^\sigma = \bot \\ \Rightarrow\ &\alpha^\sigma = \top \ \text{ and } \ (\beta \to \alpha))^\sigma = \bot \\ \Rightarrow\ &\alpha^\sigma = \top \ \text{ and } \ \beta^\sigma = \top \ \text{ and } \ \alpha^\sigma = \bot \end{aligned}$$

But notice that $\alpha^\sigma = \top$ and $\alpha^\sigma = \bot$ is a contradiction.

Hence, Axiom 1 is valid.

Show: Axiom 2 is valid. i.e. $\models^* (\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma$.

Assume for the sake of contradiction that Axiom 2 is not valid. i.e. $\not\models^* (\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma$. Hence, there exists a valuation σ such that $((\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma)^\sigma = \bot$.

Now consider the following. We will occasionally add square brackets around metalanguage connectives for increased readability and to avoid ambiguity. All the implications below can be an 'if and only if', but it is not relevant to our work.

$$\begin{aligned} &((\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma)^\sigma = \bot \\ \Rightarrow\ &(\alpha \to \beta \to \gamma)^\sigma = \top \ \text{ and } \ ((\alpha \to \beta) \to \alpha \to \gamma)^\sigma = \bot \\ \Rightarrow\ &[\alpha^\sigma = \bot \ \text{ or } \ (\beta \to \gamma)^\sigma = \top] \ \text{ and } \ [(\alpha \to \beta)^\sigma = \top \ \text{ and } \ (\alpha \to \gamma)^\sigma = \bot] \\ \Rightarrow\ &[\alpha^\sigma = \bot \ \text{ or } [\ \beta^\sigma = \bot \ \text{ or } \ \gamma^\sigma = \top]] \ \text{ and } \ \big[[\ \alpha^\sigma = \bot \ \text{ or } \ \beta^\alpha = \top] \ \text{ and } \ [\alpha^\sigma = \top \ \text{ and } \ \gamma^\sigma = \bot]\big] \end{aligned}$$

Notice that towards the end of the above line we have $\alpha^\sigma = \top$. Since $\alpha^\sigma = \top$ and we have $[\alpha^\sigma = \bot \text{ or } \beta^\alpha = \top]$, we must have that $\beta^\alpha = \top$.

We also have that $\gamma^\sigma = \bot$. Since $\gamma^\sigma = \bot$ and $\beta^\alpha = \top$, we have that $[\beta^\sigma = \bot \text{ or } \gamma^\sigma = \top]$ does not hold. Since $[\alpha^\sigma = \bot \text{ or } [\beta^\sigma = \bot \text{ or } \gamma^\sigma = \top]]$ where $[\beta^\sigma = \bot \text{ or } \gamma^\sigma = \top]$ does not hold, we conclude that $\alpha^\sigma = \bot$.

But notice that we have $\alpha^\sigma = \top$ and $\alpha^\sigma = \bot$ which is a contradiction.

Hence, Axiom 2 is valid.

Show: Axiom 3 is valid. i.e. $\models^* ((\alpha \to \beta) \to \alpha) \to \alpha$.

Assume for the sake of contradiction that Axiom 3 is not valid. i.e. $\not\models^* ((\alpha \to \beta) \to \alpha) \to \alpha$. Hence, there exists a valuation σ such that $(((\alpha \to \beta) \to \alpha) \to \alpha)^\sigma = \bot$.

Now consider the following. We will occasionally add square brackets around metalanguage conjuncts and disjuncts for increased readability and to avoid ambiguity. All the implications below can be an 'if and only if', but it is not relevant to our work.

$$\begin{aligned}
&(((\alpha \to \beta) \to \alpha) \to \alpha)^\sigma = \bot \\
\Rightarrow\ &((\alpha \to \beta) \to \alpha)^\sigma = \top \text{ and } \alpha^\sigma = \bot \\
\Rightarrow\ &[(\alpha \to \beta)^\sigma = \bot \text{ or } \alpha^\sigma = \top] \text{ and } \alpha^\sigma = \bot \\
\Rightarrow\ &\big[[\alpha^\sigma = \top \text{ and } \beta^\sigma = \bot] \text{ or } \alpha^\sigma = \top\big] \text{ and } \alpha^\sigma = \bot
\end{aligned}$$

From the above line we have $\alpha^\sigma = \bot$. Since $\alpha^\sigma = \bot$ and since $\big[[\alpha^\sigma = \top \text{ and } \beta^\sigma = \bot] \text{ or } \alpha^\sigma = \top\big]$, we must have $[\alpha^\sigma = \top \text{ and } \beta^\sigma = \bot]$. Since $[\alpha^\sigma = \top \text{ and } \beta^\sigma = \bot]$, we have $\alpha^\sigma = \top$.

But notice that we have $\alpha^\sigma = \top$ and $\alpha^\sigma = \bot$ which is a contradiction.

Hence, Axiom 3 is valid.

So we have shown that Axiom 1,2,3 are all valid.

Since **Case 1** considers ϕ_k to be either Axiom 1,2, or 3, we have that $\models^* \phi_k$. Now, consider any valuation σ such that $\sigma \models^* \Gamma$. Since $\models^* \phi_k$, we have that $\sigma \models^* \phi_k$. Since σ was arbitrary, we have that $\Gamma \models^* \phi_k$. This completes **Case 1**.

Case 2: $\phi_k \in \Gamma$.

Consider any valuation σ such that $\sigma \models^* \Gamma$. Since $\phi_k \in \Gamma$, we have $\sigma \models^* \phi_k$. Since σ was arbitrary, we have that $\Gamma \models^* \phi_k$. This completes **Case 2**.

Case 3: ϕ_k is obtained from modus ponens from two earlier formulas in the deduction.

Hence, there exists $i, j < k$ such that $\Gamma \vdash^* \phi_i$ and $\Gamma \vdash^* \phi_j$ where $\phi_j = \phi_i \to \phi_k$.

Since $\Gamma \vdash^* \phi_i$ and $\Gamma \vdash^* \phi_i \to \phi_k$, then by Inductive Hypothesis we have that $\Gamma \models^* \phi_i$ and $\Gamma \models^* \phi_i \to \phi_k$.

Consider any valuation σ such that $\sigma \models^* \Gamma$. Since $\sigma \models^* \Gamma$ and $\Gamma \models^* \phi_i$, we have that $\sigma \models^* \phi_i$. Since $\sigma \models^* \Gamma$ and $\Gamma \models^* \phi_i \to \phi_k$, we have that $\sigma \models^* \phi_i \to \phi_k$.

Since $\sigma \models^* \phi_i$ and $\sigma \models^* \phi_i \to \phi_k$, we have that $\phi_i^\sigma = \top$ and $(\phi_i \to \phi_k)^\sigma = \top$.

Assume for the sake of contradiction that $\phi_k^\sigma = \bot$. Then we would have $\phi_i^\sigma = \top$ and $\phi_k^\sigma = \bot$ which implies that $(\phi_i \to \phi_k)^\sigma = \bot$. But $(\phi_i \to \phi_k)^\sigma = \bot$ contradicts the fact that $(\phi_i \to \phi_k)^\sigma = \top$.

Hence, we must have that $\phi_k^\sigma = \top$. Hence, $\sigma \models^* \phi_k$.

Since we assumed $\sigma \models \Gamma$ and we showed $\sigma \models^* \phi_k$, where σ was an arbitrary valuation, we have shown that $\Gamma \models^* \phi_k$.

Therefore, in **Case 1,2,3** we have shown that $\Gamma \models^* \phi_k$.

Therefore, by strong induction we have shown that for each $k = 1, ..., n$, we have $\Gamma \models^* \phi_k$.

In particular, we have $\Gamma \models^* \phi_n$. Since $\phi_n = \phi$, we conclude that $\Gamma \models^* \phi$, completing the proof, as required. $\square$

(ii)

Required: Let α be a formula, and let Γ be such that $\Gamma \nvdash^* \alpha$. Further, assume that Γ is maximal with respect to this property, i.e., there is no $\Gamma' \supset \Gamma$ such that $\Gamma' \nvdash^* \alpha$. Show that if $\Gamma \vdash^* \beta$, then $\beta \in \Gamma$.

Proof. Assume $\Gamma \vdash^* \beta$. And assume for the sake of contradiction that $\beta \notin \Gamma$.

Let $\Gamma' = \Gamma \cup \{\beta\}$. Since $\beta \notin \Gamma$, we have that $\Gamma' \supset \Gamma$.

Since Γ is assumed to be maximal with respect to the given property, we have that $\Gamma' \vdash^* \alpha$.

Since $\Gamma' = \Gamma \cup \{\beta\}$ and $\Gamma' \vdash^* \alpha$, we have that $\Gamma \cup \{\beta\} \vdash^* \alpha$.

Recall that our derivability relation $\vdash^*$ has Axiom 1, Axiom 2, and Axiom 3. In the textbook, Machover proves the Deduction Theorem while invoking only Axiom 1 and 2. Hence, the

Deduction Theorem also holds for our derivability relation $\vdash^*$.

Since $\Gamma \cup \{\beta\} \vdash^* \alpha$, by the Deduction Theorem we have that $\Gamma \vdash^* \beta \to \alpha$.

So we have $\Gamma \vdash^* \beta \to \alpha$, and we assumed that $\Gamma \vdash^* \beta$. Hence, by modus ponens we have that $\Gamma \vdash^* \alpha$. But $\Gamma \vdash^* \alpha$ contradicts the fact that $\Gamma \nvdash^* \alpha$.

Therefore, our assumption that $\beta \notin \Gamma$ was wrong.

Therefore, we conclude $\beta \in \Gamma$, completing the proof, as required. $\square$