Question 1

Part a)

Required: Show that a language B is Turing-recognizable if and only if $B \leq_m A_{TM}$.

Proof. $(\Rightarrow)$: Assume B is a Turing-recognizable language. Hence, there exists a TM M that recognizes B.

First we will define a mapping reduction f from B to A_{TM}.

The mapping reduction f will be defined based on the TM F given below.

$F =$ "On input w:

1. Output $\langle M, w \rangle$."

Hence, $f(w) = \langle M, w \rangle$, where M is the TM that recognizes B.

Clearly, $w \in B$ if and only if $\langle M, w \rangle \in A_{TM}$.

Hence, $w \in B$ if and only if $f(w) \in A_{TM}$. Therefore, $B \leq_m A_{TM}$.

$(\Leftarrow)$: Assume $B \leq_m A_{TM}$. Hence, there exists some computable function $f : \Sigma^* \to \Sigma^*$ such that for every $w \in \Sigma^*$, we have that $w \in B$ if and only if $f(w) \in A_{TM}$.

We know from lecture and the textbook that A_{TM} is recognizable. Let M be the TM that recognizes A_{TM}.

Consider the following TM M' that recognizes B.

$M' =$ "On input w:

1. Compute $f(w)$.

2. Run M on input $f(w)$. If M accepts $f(w)$, then accept. If M rejects $f(w)$, then reject."

To show that M' recognizes B, consider the following. Since f is a mapping reduction from B to A_{TM}, we know $w \in B$ if and only if $f(w) \in A_{TM}$.

If $w \in B$, then $f(w) \in A_{TM}$. So M will accept $f(w)$, and hence M' will accept w.

If $w \notin B$, then $f(w) \notin A_{TM}$. So M will not accept $f(w)$, and hence M' will not accept w. Therefore, M' recognizes B.

Since M' recognizes B, we have shown that B is recognizable. $\square$

Part b)

Required: Show that a language A is decidable if and only if $A \leq_m 0^*1^*$ where 0^*1^* is the language containing strings of an arbitrary number of 0s, followed by an arbitrary number of 1s

Proof. ($\Rightarrow$): Assume A is decidable. Hence, there is a TM M that decides A. Consider the following mapping reduction f from A to 0^*1^* that is given by the following TM F.

$F =$ "On input w:

1. Run M on w.
2. If M accepts w, output 01. If M rejects w, output 10."

Hence, $f : \Sigma^* \to \Sigma^*$ is the following function.

$$f(w) = \begin{cases} 01 & \text{if } M \text{ accepts } w \\ 10 & \text{if } M \text{ rejects } w \end{cases}$$

Notice, $01 \in 0^*1^*$ and $10 \notin 0^*1^*$. Hence, trivially we get $w \in B$ if and only if $f(w) \in 0^*1^*$. So f is a mapping reduction. Hence, $A \leq_m 0^*1^*$.

($\Leftarrow$): Assume $A \leq_m 0^*1^*$. First we will show that 0^*1^* is decidable. Consider the following TM M that decides 0^*1^*.

$M =$ "On input w:

1. Scan w. If a 1 appears before a 0, then reject. Otherwise, accept."

To show that M decides 0^*1^*, first consider $w \in 0^*1^*$. Hence, w does not contain a 1 that appears before a 0. Hence, M will accept. If $w \notin 0^*1^*$, then w must contain a 1 that appears before a 0. Hence, M will reject. Hence, M is indeed a decider for 0^*1^*.

Since $A \leq_m 0^*1^*$, we know there exists some computable function $f : \Sigma^* \to \Sigma^*$ such that for every $w \in \Sigma^*$, we have that $w \in A$ if and only if $f(w) \in 0^*1^*$.

Consider the following TM M' that decides A.

$M' =$ "On input w:

1. Compute $f(w)$.

2. Run M on input $f(w)$. If M accepts $f(w)$, then accept. If M rejects $f(w)$, then reject."

To see that M' decides A, consider the following. Since f is a mapping reduction from A to 0^*1^*, we know that $w \in A$ if and only if $f(w) \in 0^*1^*$. So if $w \in A$, we have that $f(w) \in 0^*1^*$, and so M accepts $f(w)$. Hence, M' will accept w. If $w \notin A$, then $f(w) \notin 0^*1^*$. Hence, M rejects $f(w)$ since M decides 0^*1^*. Hence, M' will also reject w. This shows that M' is a decider for A.

Therefore, A is decidable. $\square$

Question 2

Part a)

Required: Let $A = \{\langle M \rangle : M \text{ is a TM and } |L(M)| = 5\}$. Is A Turing recognizable? Is A Turing co-recognizable.

We will show that A is neither Turing-recognizable nor Turing co-recognizable.

First we will define a mapping reduction f from A_{TM} to A.

The mapping reduction f will be defined based on the TM F given below.

$F =$ "On input $\langle M, w \rangle$:

1. Construct the following TM M_w. Let w_1, w_2, w_3, w_4 be the first four distinct strings of Σ^* that appear in standard string order that are also distinct from w.
   $M_w =$ 'On input s:
   1. If $s = w_i$ for $i \in \{1, 2, 3, 4\}$, then accept.
   2. If $s = w$, then run M on w and output what M outputs.'

2. Output $\langle M_w \rangle$".

And if F is given a string that is not an encoding of a TM M and a string w, then output $\langle M_{reject} \rangle$ where M_{reject} rejects every string so that $|L(M_{reject})| = 0$.

Hence,

$$f(x) = \begin{cases} \langle M_w \rangle & \text{if } x = \langle M, w \rangle \text{ for some TM } M \text{ and string } w \\ \langle M_{reject} \rangle & \text{otherwise} \end{cases}$$

We have that $\langle M, w \rangle \in A_{TM}$ if and only if $f(\langle M, w \rangle) \in A$.

So we have a mapping reduction f from A_{TM} to A. Hence, $A_{TM} \leq_m A$.

Since $A_{TM} \leq_m A$, we also know that $\overline{A_{TM}} \leq_m \overline{A}$. Since we know that $\overline{A_{TM}}$ is not recognizable from the textbook, we have that $\overline{A}$ is not recognizable.

Since $\overline{A}$ is not recognizable, we have that A is not co-recognizable.

Next we will define a mapping reduction g from A_{TM} to $\overline{A}$.

The mapping reduction g will be defined based on the TM G given below.

$G =$ "On input $\langle M, w \rangle$:

1. Construct the following TM M_w. Let w_1, w_2, w_3, w_4, w_5 be the first five distinct strings of Σ^* that appear in standard string order that are also distinct from w.

   M_w = 'On input s:
   1. If $s = w_i$ for $i \in \{1, 2, 3, 4, 5\}$, then accept.
   2. If $s = w$, then run M on w and output what M outputs.'

2. Output $\langle M_w \rangle$".

And if G is given a string that is not an encoding of a TM M and a string w, then output $\langle M_5 \rangle$ where M_5 is the TM that accepts exactly the first 5 strings of Σ^* in standard string order.

Hence,

$$g(x) = \begin{cases} \langle M_w \rangle & \text{if } x = \langle M, w \rangle \text{ for some TM } M \text{ and string } w \\ \langle M_5 \rangle & \text{otherwise} \end{cases}$$

We have that $\langle M, w \rangle \in A_{TM}$ if and only if $f(\langle M, w \rangle) \in \overline{A}$.

So we have a mapping reduction g from A_{TM} to $\overline{A}$. Hence, $A_{TM} \leq_m \overline{A}$.

Since $A_{TM} \leq_m \overline{A}$, we also have $\overline{A_{TM}} \leq_m \overline{\overline{A}}$. Since $\overline{\overline{A}} = A$, we have that $\overline{A_{TM}} \leq_m A$. Since we know that $\overline{A_{TM}}$ is not recognizable from the textbook, we have that A is not recognizable.

Conclusion: A is not Turing-recognizable and A is not Turing co-recognizable.

Part b)

Required: Let $B = \{\langle M \rangle : M \text{ is a TM and } |L(M)| \geq 5\}$. Is A Turing recognizable? Is A Turing co-recognizable.

We will show that B is Turing recognizable, but not Turing co-recognizable.

Def: An ordering of strings is in Standard String Order (SSO) if they are first ordered by length, and strings of the same length are ordered by some alphabetical ordering.

Note: Standard String Order (SSO) is also known as lexicographical ordering. In Tutorials, we used the term Standard String Order (SSO).

First consider an enumeration of Σ^* in Standard String Order (SSO).

Consider the following TM N that recognizes B.

N = "On input $\langle M \rangle$:

1. For each $w \in \Sigma^*$ in standard string order: simulate M on w. If M accepts w, keep track of this result.

2. If w has accepted 5 distinct strings w, then accept."

And if N is given an input that is not an encoding of a TM M, then simply reject.

To show that N is a recognizer of B, let $x \in B$. Hence, $x = \langle M \rangle$ such that $|L(M)| \geq 5$. Hence, N will simulate M on w for each $w \in \Sigma^*$ in standard string order. Since $|L(M)| \geq 5$, eventually M will accept 5 strings, and hence N will accept x.

If $x \notin B$ and x is not an encoding of a TM, then N will not accept x. If $x \notin B$ and $x = \langle M \rangle$ for some TM M, then M accepts at most 4 strings. Hence, N will not accept x.

Therefore, N recognizes B. Therefore, B is Turing-recognizable.

Now we will show that B is not Turing co-recognizable. In lecture we proved Rice's Theorem.

Rice's Theorem: Suppose P is a language of TM descriptions such that

(i) P is non-trivial: it contains some but not all TM descriptions.

(ii) If $L(M_1) = L(M_2)$ for TMs M_1 and M_2, then $\langle M_1 \rangle \in P$ if and only if $\langle M_2 \rangle \in P$.

Then P is not decidable.

Clearly B is a language of TM descriptions. And clearly B does not contain all TM descriptions. For instance $\langle M_{reject} \rangle \notin B$ where M_{reject} is the TM that rejects all strings since $|L(M_{reject})| = 0$. And clearly B contains some TM descriptions. For instance, $\langle M_5 \rangle \in B$, where M_5 is the TM that accepts exactly the first 5 strings of Σ^* in standard string order.

And, if $L(M_1) = L(M_2)$ for TMs M_1 and M_2, then $|L(M_1)| = |L(M_2)|$. Hence, trivially we have $\langle M_1 \rangle \in B$ if and only if $\langle M_2 \rangle \in B$.

Hence, B is not decidable by Rice's Theorem.

By Theorem 4.22 in the textbook, we know that a language is decidable if and only if it is recognizable and co-recognizable.

Hence, a language is not decidable if and only if it is not recognizable or not co-recognizable.

Since B is not decidable, we know B is not recognizable or B is not co-recognizable.

Since we have shown that B is recognizable, we must have that B is not co-recognizable.

Conclusion: B is Turing-recognizable, but B is NOT Turing co-recognizable.

Question 3

Required: Show that for every language A, there exists a language B such that $B \not\leq_m A$.

Proof. Assume for the sake of contradiction that there exists a language A such that for every language B, we have that $B \leq_m A$.

Let Σ be the finite alphabet that we are working with.

We know that every language is a subset of Σ^* which is an infinite set.

Let Σ_i be the set of all strings that are of length $i \in \mathbb{N}$. Hence, $\Sigma^* = \bigcup_{i=0}^{\infty} \Sigma_i$. Since each Σ_i is finite, each Σ_i is countable. Hence, $\Sigma^* = \bigcup_{i=0}^{\infty} \Sigma_i$ is a countable union of countable sets. Hence, Σ^* is countably infinite.

Since Σ^* is countably infinite, we know that its powerset, $\mathcal{P}(\Sigma^*)$ is uncountable.

We also know from lecture that there are only countably many Turing machines. Since there are only countably many Turing machines, there are only countably many computable functions.

We assumed every $B \in P(\Sigma^*)$ is such that $B \leq_m A$. Since there are uncountably many sets $B \in P(\Sigma^*)$, and there are only countably many computable functions, there must exist at least two distinct sets $B_1, B_2 \in P(\Sigma^*)$ such that $B_1 \neq B_2$ where the mapping reductions for $B_1 \leq_m A$ and $B_2 \leq_m A$ are given by the same computable function $f : \Sigma^* \to \Sigma^*$.

Hence, for all $w \in \Sigma^*$, we have the following by definition of mapping reducibility for $B_1 \leq_m A$ and $B_2 \leq_m A$.

$$w \in B_1 \Leftrightarrow f(w) \in A \tag{1}$$

$$w \in B_2 \Leftrightarrow f(w) \in A \tag{2}$$

Since $B_1 \neq B_2$, there exists a string in one of these sets, but not the other. Without loss of generality, assume $s \in B_1$ but $s \notin B_2$.

Since $s \in B_1$, by (1) we get that $f(s) \in A$.

Since $s \notin B_2$, by (2) we get that $f(s) \notin A$.

Notice, $f(s) \in A$ and $f(s) \notin A$ is a contradiction. Therefore, our initial assumption was wrong.

Therefore, for every language A, there exists a language B such that $B \not\leq_m A$. This completes the proof, as required. □

Question 4

Required: A useless state in a Turing machine is one that is never entered on any input string. Consider the problem of determining whether a Turing machine has any useless states. Formulate this problem as a language and show that is it undecidable.

Proof. Let $A = \{\langle M\rangle | M \text{ is a TM that has NO useless states}\}$

NOTE: The textbook defines every Turing machine M to have an accept state q_{accept} and a reject state q_{reject} such that $q_{accept} \neq q_{reject}$. This is given on page 168, definition 3.3.
From the textbook we know that $HALT$ is undecidable, where $HALT$ is defined as follows.

$$HALT = \{\langle M, w\rangle : M \text{ is a TM and } M \text{ halts on } w\}$$

We will show that $HALT \leq_m A$ by defining a mapping reduction f given by the following TM F.

$F =$ "On input $\langle M, w\rangle$:

1. Construct the following TM M_w with no useless states except possibly q_{accept}.
   $M_w =$ 'On input s:
   1. If $s \neq w$, then reject.
   2. If $s = w$, run M on w. If M halts on w, then accept.'

2. Output $\langle M_w\rangle$"

And if F is given a string x such that $x \neq \langle M, w\rangle$ for some TM M and string w, then output $\langle M_{reject}\rangle$ where M_{reject} is the TM that rejects all inputs (and hence q_{accept} is useless).

So F determines the following computable function $f : \Sigma^* \to \Sigma^*$.

$$f(x) = \begin{cases} \langle M_w\rangle & \text{if } x = \langle M, w\rangle \text{ for some TM } M \text{ and string } w \\ \langle M_{reject}\rangle & \text{otherwise} \end{cases}$$

And $\langle M, w\rangle \in HALT$ if and only if $f(\langle M, w\rangle)$ has no useless states.

Therefore, $\langle M, w\rangle \in HALT$ if and only if $f(\langle M, w\rangle) \in A$.

Hence, f is a mapping reduction from $HALT$ to A. i.e. $HALT \leq_m A$.

We know $HALT$ and $\overline{HALT}$ are both undecidable from lecture and the textbook. Since $HALT \leq_m A$ and $HALT$ is undecidable, we have that A is undecidable. Since $HALT \leq_m A$, we also have $\overline{HALT} \leq_m \overline{A}$. Since $\overline{HALT} \leq_m \overline{A}$ and $\overline{HALT}$ is undecidable, we also have that $\overline{A}$ is undecidable.

Therefore, since A and $\overline{A}$ are undecidable, we have shown that the problem of determining whether a TM has any useless states is undecidable. $\square$