$$\frac{1}{v}\frac{\partial \phi}{\partial t} = \partial_p \Sigma_f \phi - \left(-\nabla D \nabla + \Sigma_a\right)\phi + \lambda C = f_\phi(\phi, C, t)$$

$$\frac{\partial C}{\partial t} = \nu_d \Sigma_f \phi - \lambda C = f_C(\phi, C, t)$$

SDIRK $\quad Y_i = y_n + h \sum_{j \le i} a_{ij} F(Y_j, t_j) \qquad Y = \begin{bmatrix} \phi \\ C \end{bmatrix} \quad F = \begin{bmatrix} f_\phi \\ f_C \end{bmatrix}$

$$C_i = C_n + h \sum_{j<i} a_{ij}\left(\Sigma_{d,j}\phi_j - \lambda C_j\right) + h a_{ii}\left(\Sigma_{d,i}\phi_i - \lambda C_i\right)$$

Eliminate $C_i$

$$\boxed{C_i = \frac{1}{1+\lambda h a_{ii}}\left[C_n + h\sum_{j<i} a_{ij}\left(\Sigma_{d,j}\phi_j - \lambda C_j\right) + h a_{ii}\Sigma_{d,i}\phi_i\right]}$$

$$\text{IV}\quad \phi_i = {}^{\text{IV}}\phi_n + h\sum_{j<i} a_{ij}\left\{\left(\Sigma_{p,j} - D_j - A_j\right)\phi_j + \lambda C_j\right\} + h\, a_{ii}\left\{\left(\Sigma_{p,i} - D_i - A_i\right)\phi_i + \lambda C_i\right\}$$

$$\boxed{C_i = \tilde{C}_i + \frac{h a_{ii}\Sigma_{d,i}\phi_i}{1+h\lambda a_{ii}}}$$

$$\boxed{\left[\text{IV} - h a_{ii}\overbrace{\left(\Sigma_{p,i} - D_i - A_i\right)}^{TR} - h a_{ii}\lambda\,\frac{h a_{ii}}{1+\lambda h a_{ii}}\Sigma_{d,i}\right]\phi_i = {}^{\text{IV}}\phi_n + h\sum_{j<i} a_{ij} f_{\phi,j} + \lambda h a_{ii}\tilde{C}_i}$$

$$\left\{\text{IV} - h a_{ii}\left[TR_i + \frac{\lambda h a_{ii}\Sigma_{d,i}}{1+\lambda h a_{ii}}\right]\right\}\phi_i = {}^{\text{IV}}\phi_n + h a_{ii}\lambda\tilde{C}_i + h\sum_{j<i} a_{ij} f_{\phi,j}$$

$$f_{\phi_j} = TR_j\phi_j + \lambda C_j = TR_j\phi_j + \underbrace{\frac{\lambda h a_{jj}\Sigma_{d,j}}{1+\lambda h a_{jj}}\phi_j + \lambda\tilde{C}_j}_{Z_j} \quad !!$$

$$= \lambda C_j$$

IQS shape SDIRK  $\varphi$ = shape

No change on the precursor side:

$$c_i = \underbrace{\frac{1}{1+\lambda h a_{ii}}\left[c_n + h\sum_{j<i} a_{ij}\overbrace{\left(\Sigma_{d,j}\,\rho_j\varphi_j - \lambda c_j\right)}^{f_{c,j}}\right]}_{\tilde{c}_i} + \frac{h a_{ii}}{1+\lambda h a_{ii}}\Sigma_{d,i}\rho_i\varphi_i$$

shape equation: $\frac{1}{v}\frac{\partial\varphi}{\partial t} = \left[\nabla_p\nabla_f - \left(-\nabla\cdot\nabla + \sigma_a + \frac{1}{p}\frac{dp}{dt}\right)\right]\varphi + \frac{\lambda c}{p}$

$$\text{IV}\,\varphi_i = \text{IV}\,\varphi_n + h\sum_{j<i} a_{ij}\left\{\overbrace{\left[TR - A_{iqs}\right]_j\varphi_j + \lambda\frac{c_i}{\rho_j}}^{f_{\phi,j}}\right\} + h\,a_{ii}\left[TR - A_{iqs}\right]_i\varphi_i + h a_{ii}\lambda\frac{c_i}{\rho_i}$$

$$\left[\text{IV} - h a_{ii}\left\{\left[TR - A_{iqs}\right]_i + \frac{\lambda h a_{ii}\Sigma_{d_i}}{1+\lambda h a_{ii}}\right\}\right]\varphi_i = \text{IV}\,\varphi_n + h a_{ii}\left[\frac{\lambda\tilde{c}_i}{\rho_i} + h\sum_{j<i} a_{ij} f_{\phi,j}\right] + h a_{ii}\lambda\left[\frac{\tilde{c}_i}{\rho_i} + \frac{h a_{ii}}{1+\lambda h a_{ii}}\Sigma_{d_i}\varphi_i\right]$$

$$f_{\phi,i} = \left[TR - A_{iqs}\right]_i\varphi_i + \lambda\left[\frac{\tilde{c}_i}{\rho_i} + \frac{h a_{ii}}{1+h\lambda a_{ii}}\Sigma_{d,i}\varphi_i\right]$$

$$= \left[\left(TR - A_{iqs}\right)_i + \frac{\lambda h a_{ii}}{1+\lambda h a_{ii}}\Sigma_{d,i}\right]\varphi_i + \underbrace{\lambda\frac{\tilde{c}_i}{\rho_i}}_{\Im_i}$$