Weak Form for the Self-Adjoint S_N Formalism

1 Self-Adjoint S_N

$$
\begin{aligned}
\text{LHS}: &= (-\vec{\Omega}\cdot\vec{\nabla}+\sigma_a)(\vec{\Omega}\cdot\vec{\nabla}+\sigma_a)\psi && (1)\\
&= \underbrace{-\vec{\Omega}\cdot\vec{\nabla}\vec{\Omega}\cdot\vec{\nabla}\psi}_{①} + \underbrace{\left[\sigma_a\vec{\Omega}\cdot\vec{\nabla}\psi - \vec{\Omega}\cdot\vec{\nabla}(\sigma_a\psi)\right]}_{②} + \underbrace{\sigma_a^2\psi}_{③} && (2)\\
\text{RHS}: &= (-\vec{\Omega}\cdot\vec{\nabla}+\sigma_a)Q && (3)\\
&= \underbrace{-\vec{\Omega}\cdot\vec{\nabla}Q}_{④} + \underbrace{\sigma_a Q}_{⑤} && (4)
\end{aligned}
$$

Note that for radiative transfer problem, σ_a is embedded in the source Q:

$$Q = \sigma_a B \tag{5}$$

where B is the Planckian source determined by the temperature.

2 Weak form

Multiply every term in LHS and RHS by basis function b_i and integrate over the problem volume:

LHS:

$$
\begin{aligned}
&\boxed{\int ① b_i dV}\\
&= \int -b_i\vec{\Omega}\cdot\vec{\nabla}\vec{\Omega}\cdot\vec{\nabla}\psi dV && (6)\\
&= \underbrace{-\oint(\vec{\Omega}\cdot\vec{\nabla}\psi)\ b_i\ \vec{\Omega}\cdot\vec{n}dA}_{*} + \int(\vec{\Omega}\cdot\vec{\nabla}\psi)(\vec{\Omega}\cdot\vec{\nabla}b_i)dV && (7)
\end{aligned}
$$

$$
\begin{aligned}
&\boxed{\int ② b_i dV}\\
&= \int \left[\sigma_a\vec{\nabla}\cdot(\vec{\Omega}\psi) - \vec{\nabla}\cdot(\sigma_a\,\vec{\Omega}\psi)\right]\ b_i\ dV && (8)\\
&= \int(-\vec{\Omega}\psi\cdot\vec{\nabla}\sigma_a)\ b_i\ dV = \underline{\underline{\int(-\vec{\Omega}\psi b_i\cdot\vec{\nabla}\sigma_a)dV}} && (9)\\
&= \underline{-\oint\sigma_a\ \psi b_i\ \vec{\Omega}\cdot\vec{n}dA + \int\sigma_a\vec{\nabla}\cdot(\vec{\Omega}\psi b_i)dV} && (10)
\end{aligned}
$$

$$\boxed{\int ③ b_i dV}$$

$$= \int \sigma_a^2 \psi b_i dV \tag{11}$$

RHS:

$$\boxed{\int ④ b_i dV}$$

$$= \int (-\vec{\Omega} \cdot \vec{\nabla} Q) b_i dV \tag{12}$$

$$= \underline{- \oint b_i Q \vec{\Omega} \cdot \vec{n} dA + \int Q \vec{\Omega} \cdot \vec{\nabla} b_i dV} \tag{13}$$

$$\text{or} \tag{14}$$

$$= - \int \vec{\Omega} \cdot \vec{\nabla}(\sigma_a B)\ b_i\ dV \tag{15}$$

$$= \underline{\underline{- \int \vec{\Omega}(\vec{\nabla}\sigma_a B + \vec{\nabla} B \sigma_a)\ b\ dV}} \tag{16}$$

$$\boxed{\int ⑤ b_i dV}$$

$$= \int \sigma_a\ Q\ b_i\ dV = \int \sigma_a^2\ B\ b_i\ dV \tag{17}$$

Depending on the order of finite element chosen to represent the solution and material property (opacity (cross-section), source), two different sets of weak form are proposed:

If we choose to use piece-wise constant (0^{th} order) finite element, then it is suggested to use the terms *underlined* whenever possible (as seen in Eq. (10) and Eq. (13)).

If we choose to use piece-wise linear (1^{st} order) finite element, then it is suggested to use the terms $\underline{\underline{double - lined}}$ whenever possible (as seen in Eq. (9) and Eq. (16)).

Note that one can choose to use either the *underlined* or the $\underline{\underline{double - lined}}$, not both. Because they are mathematically equivalent.

3 Boundary condition

For the in-coming directions, we use Dirichlet boundary condition to explicitly specify the value of ψ on the boundary.

For the out-going directions, we use the first order equation:

$$\vec{\Omega} \cdot \vec{\nabla}\psi + \sigma_a \psi = Q \Rightarrow \boxed{\vec{\Omega} \cdot \vec{\nabla}\psi = Q - \sigma_a \psi} \tag{18}$$

The out-going boundary condition Eq. (18) is applied to the $*$ term only, where $\vec{\Omega} \cdot \vec{\nabla}\psi$ is replaced by $Q - \sigma_a \psi$, yielding:

$$* = \oint (\vec{\Omega} \cdot \vec{\nabla}\psi)\ b_i\ \vec{\Omega} \cdot \vec{n} dA = \underbrace{-\oint Q\ b_i\ \vec{\Omega} \cdot \vec{n} dA}_{\text{move to RHS}} + \underbrace{\oint \sigma_a \psi b_i \vec{\Omega} \cdot \vec{n} dA}_{\text{keep on LHS}}, \quad \text{for } \vec{\Omega} \cdot \vec{n} > 0 \tag{19}$$

The $\vec{\Omega} \cdot \vec{n} > 0$ portion of this $*$ term is neglected, it is equivalent to force:

$$* = \oint (\vec{\Omega} \cdot \vec{\nabla}\psi)\ b_i\ \vec{\Omega} \cdot \vec{n} dA = 0, \quad \text{for } \vec{\Omega} \cdot \vec{n} < 0 \tag{20}$$

4 Final form after cancellation of the surface terms

Take the five kernels listed in the above section, using the *underlined* as seen in Eq. (10) and Eq. (13), and also applying the first order boundary condition Eq. (18) for both out-going and in-coming directions for the boundary term ($*$), we get:

$$\begin{aligned}\text{LHS} := &- \cancelto{\text{II}}{\oint Q\ b_i \vec{\Omega} \cdot \vec{n} dA} + \cancelto{\text{I}}{\oint \sigma_a \psi b_i \vec{\Omega} \cdot \vec{n} dA} + \int (\vec{\Omega} \cdot \vec{\nabla}\psi)(\vec{\Omega} \cdot \vec{\nabla} b_i) dV \\ &- \cancelto{\text{I}}{\oint \sigma_a\ \psi b_i\ \vec{\Omega} \cdot \vec{n} dA} + \boxed{\int \sigma_a \vec{\nabla} \cdot (\vec{\Omega}\psi b_i) dV} \\ &+ \int \sigma_a^2 \psi b_i dV\end{aligned} \tag{21}$$

$$\begin{aligned}\text{RHS} := &- \cancelto{\text{II}}{\oint b_i Q \vec{\Omega} \cdot \vec{n} dA} + \int Q \vec{\Omega} \cdot \vec{\nabla} b_i dV \\ &+ \int \sigma_a\ Q\ b_i\ dV\end{aligned} \tag{22}$$

We can see that the "I" terms cancels out each other, and the same for the "II" terms. Finally, expand out the boxed term in LHS:

$$\boxed{\int \sigma_a \vec{\nabla} \cdot (\vec{\Omega}\psi b_i) dV} = \int \sigma_a b_i \vec{\Omega} \cdot \vec{\nabla}\psi dV + \int \sigma_a \psi \vec{\Omega} \cdot \vec{\nabla} b_i \tag{23}$$

Our final form becomes:

$$\begin{aligned}
\text{LHS} :=& \int (\vec{\Omega} \cdot \vec{\nabla}\psi)(\vec{\Omega} \cdot \vec{\nabla} b_i) dV \\
& \int \sigma_a b_i \vec{\Omega} \cdot \vec{\nabla}\psi dV + \int \sigma_a \psi \vec{\Omega} \cdot \vec{\nabla} b_i \\
& + \int \sigma_a^2 \psi b_i dV \qquad (24) \\
\text{RHS} :=& \int Q \vec{\Omega} \cdot \vec{\nabla} b_i dV \\
& + \int \sigma_a \; Q \; b_i \; dV \qquad (25)
\end{aligned}$$

The same as Dr. Ragusa's derivation. However, in my implementation I only used the first order boundary condition for the out-going directions, as explained in the Boundary condition section. Therefore, the "I" and "II" terms doesn't cancel themselves out completely. The in-coming portion of those terms remains.