Twin Primes Proof

1 Verification of Prime Constellations

Here, we provide a verification of prime constellations.

1.1 Formal Definition

A prime constellation is defined by a base p and a set of offsets $\mathcal{O}$. In Lean 4, we define the property $P_{\mathcal{O}}(p)$ as:

$$P_{\mathcal{O}}(p) \iff \bigwedge_{s \in \mathcal{O}} \text{is\_prime}(p+s)$$

1.2 Computational Search (algorithm2e)

Algorithm 1: K-Twin Decidability Algorithm

```
input  : Search limit N, Offsets O = {s_1, ..., s_k}
output: List L of valid base primes

L ← ∅;
for p ← 2 to N do
    valid ← true;
    for s ∈ O do
        if ¬ isPrime(p + s) then
            valid ← false;
            break;
    if valid then
        L ← L ∪ {p};
```

1.3 Proof: Exclusion of Case $[0, 2, 4]$

Theorem 1. *For $p > 3$, the constellation $\{p, p+2, p+4\}$ is never prime.*

Proof. Let $p \in \mathbb{N}, p > 3$. By the pigeonhole principle applied to residues modulo 3:

- If $p \equiv 0 \pmod 3$, p is composite (since $p > 3$).
- If $p \equiv 1 \pmod 3$, then $p+2 \equiv 1+2 \equiv 0 \pmod 3$.
- If $p \equiv 2 \pmod 3$, then $p+4 \equiv 2+4 \equiv 0 \pmod 3$.

In all cases, one element is a multiple of 3. Since the elements are > 3, at least one must be composite. □

Definition 1. *A **Twin Prime** is a pair $(p, p+2)$ where both are elements of $\mathbb{P}$.*

Theorem 2. *There is only one prime triplet of the form $(p, p+2, p+4)$.*

Proof. By testing $p = 3$, we find $\{3, 5, 7\}$, which are all prime. As proven via modular arithmetic $\pmod 3$, no other such triplets exist. □