Analysis and Modeling of Social and Information Networks

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Home Work 3

Problem 1

Problem 1. In an Erdős–Rényi graph with N=4000 nodes, the linking probability is p=0.001
a) What is the average degree of a node in this graph?
b) What is the variance in the degrees of the nodes?
c) What is the expected number of nodes with a degree which is at least twice larger than the average degree?

Solution

a) In an Erdős–Rényi graph, each pair of nodes has a probability p of being linked, and the average degree of a node is equal to the expected number of links per node.

Let's calculate the average degree of a node:

$$= px(n-1)$$
$$= 0.001x(4000-1)$$
$$= 3.999$$
$$\approx 4$$

b) The variance in the degrees of the nodes in an Erdős–Rényi graph can be calculated as:

$$\partial^2 = p(1-p)(n-1)$$
$$= 0.001x(1-0.001)x(4000-1)$$
$$= 3.995001$$
$$\approx 4$$

c) solution 1:

In an Erdős–Rényi graph, each pair of nodes has a probability p of being linked, and the average degree of a node is equal to the expected number of links per node.
Let's calculate the expected number of links per node:
The total number of links in the graph is:

$$Nx(N-1)/2$$

The expected number of links per node is the total number of links divided by the number of nodes:

$$= (Nx(N-1)/2)xp/N$$
$$= (N-1)x0.5xp$$

To find the expected number of nodes with a degree that is at least twice larger than the average degree, we need to use the Poisson approximation of the binomial distribution. If the average degree is lambda, the expected number of nodes with a degree of at least 2 lambda is given by the Poisson distribution with parameter 2 lambda.

In this case, lambda = 3.999, so the expected number of nodes with a degree of at least 2 * 3.999 = 7.998.

So, we can solve the equation:

$$\Rightarrow (Nx(N-1)/2)xp/N = 7.998$$
$$\Rightarrow (N-1)x0.5xp = 7.998$$
$$\Rightarrow N-1 = 15.996/0.001$$
$$\Rightarrow N = 15996+1$$
$$\Rightarrow N = 15997$$

Solution 2:
We know that the expected number of nodes with a degree is:

$$p(k) = \binom{n-1}{k} xp^k x(1-p)^{(\text{n-1-k})}$$

where k is the degree [2] and p = 0.001.
From problem 1(a) we found the average degree of a node:

$$= px(n-1)$$
$$= 0.001x(4000-1)$$
$$= 3.999$$
$$\approx 4$$

According to the question,

$$Expectednumberofnodeswithadegree \geq Averagedegree$$
$$\Rightarrow p(k) \geq 2 * \widetilde{(k)}$$
$$\Rightarrow \binom{n-1}{k} xp^k x(1-p)^{(\text{n-1-k})} \geq 2x4$$
$$\Rightarrow \binom{n-1}{1} x0.001^1 x(1-0.001)^{(\text{n-1-1})} \geq 8$$

note: k is the degree of 1

$$\Rightarrow (n-1)x(0.9999)^{(\text{n-2})} \geq 8000$$
$$\Rightarrow \log((n-1)x(0.9999)^{(\text{n-2})}) \geq log(8000)$$

note: take natural log from both side of the equation

$$\Rightarrow \log(n-1) + log((0.9999)^{(\text{n-2})}) \geq 8.987$$
$$\Rightarrow \log(n-1) + (n-2)log(0.9999) \geq 8.987$$
$$\Rightarrow \log(n-1) + (n-2)x0.0001 \geq 8.987$$
$$\Rightarrow \log(n-1) + 0001xn - 0.0002 \geq 8.987$$
$$\Rightarrow \log(n-1) + 0001xn \geq 8.9872$$

note: 0.0001 n is small compared to others, that's why this portion is ignored to reduce the complexity

$$\Rightarrow (n-1) \geq \exp 8.9872$$
$$\Rightarrow n \geq 8000+1$$
$$\Rightarrow n \geq 8001$$

Problem 2

Consider Gn,p, an Erdös–Rényi random graph with n nodes, m edges, and mean degree c:
a) Compute the probability p of creating an edge in Gn,p.
b) Show that in the limit (large n) the expected number of triangles in Gn,p is 1/6 · c3

Solution

a) We know that the clustering coefficient,

$$K_i = (\frac{2m_i}{c_i(c_i - 1)}) \tag{1}$$

where m_i is the number of edges and c_i is the degree of that node.

We also know that

$$E(m_i) = px\frac{c_i(c_i - 1)}{2} \tag{2}$$

And expected clustering coefficient is:

$$E(K_i) = \frac{\widetilde{c}}{n-1} \tag{3}$$

If we take the expected value on both sides of equation 1, we will get the following:

$$E(K_i) = (\frac{E(m_i)}{c_i(c_i - 1)}) \tag{4}$$

let's combine all the values from equations no: 1, 2, and 3 into equation no: 4

$$\Rightarrow \frac{\widetilde{c}}{n-1} = \frac{2xpxc_ix(c_i - 1)}{2xc_ix(c_i - 1)} \Rightarrow p = \frac{\widetilde{c}}{n-1} \Rightarrow \approx p = \frac{\widetilde{c}}{n} \tag{5}$$

b) p = c/n, where c is the mean degree of the graph, As the number of vertices, n, increases one might expect the number of triangles to increase but this is not the case. Although the number of triples of vertices grows as n^3, the probability of an edge between two specific vertices decreases linearly with n and thus the probability of all three edges between the pairs of vertices in a triple of vertices being present goes down as n^{-3}, exactly canceling the rate of growth of triples.
A random graph with n vertices and edge probability(p) = c/n has an expected number of triangles independent of n, namely $c^3/6$. There are $\binom{n}{3}$ triples of vertices. Each triple has probability $(\frac{c}{n})^3$ of being a triangle.
Let $\triangle ijk$ be the indicator variable for the triangle with vertices i, j, and k being present. That is, all three edges (i, j), (j, k), and (i, k) are present. Then the number of triangles is: $x = \sum_{ijk} \triangle_{ijk}$

Even though the events are not statistically independent, by the linearity of expectation, which does not assume independence of the variables, the expected value of a sum of random variables is the sum of the expected values. Thus, the expected number of triangles is:

$$E(x) = E(\sum_{ijk} \triangle_{ijk})$$
$$= \binom{n}{3} x (\frac{c}{n})^3$$
$$\approx (\frac{c}{n})^3$$

References

[1] https://www.cs.cmu.edu/ avrim/598/chap4only.pdf

[2] https://math.stackexchange.com/questions/2854364/expected-number-of-nodes-with-degree-two

[3] https://en.wikipedia.org/wiki/Erd