Homework 4 solutions

1. Exercise T3.55.

   Solution.

   (a) The first and second derivatives of f are

   $$f'(x) = e^{-h(x)}, \qquad f''(x) = -h'(x)e^{-h(x)}.$$

   Log-concavity requires $f''(x)f(x) \leq (f'(x))^2$, *i.e.*,

   $$\begin{aligned} -h'(x)e^{-h(x)} \int_{-\infty}^{x} e^{-h(t)}\, dt &\leq e^{-2h(x)} \\ -h'(x) \int_{-\infty}^{x} e^{-h(t)}\, dt &\leq e^{-h(x)}. \end{aligned}$$

   This is obviously true if $h'(x) \geq 0$.

   (b) Taking exponentials and integrating both sides of $-h(t) \leq -h(x) - h'(x)(t-x)$ gives

   $$\begin{aligned} \int_{-\infty}^{x} e^{-h(t)}\, dt &\leq e^{xh'(x)-h(x)} \int_{-\infty}^{x} e^{-th'(x)}\, dt \\ &= e^{xh'(x)-h(x)} e^{-xh'(x)}/(-h'(x)) \\ &= \frac{e^{-h(x)}}{-h'(x)}. \end{aligned}$$

   Therefore

   $$-h'(x) \int_{-\infty}^{x} e^{-h(t)}\, dt \leq e^{-h(x)}.$$

2. Exercise T4.13.

   Solution. We consider the constraints separately. The ith constraint is

   $$\sum_{j=1}^{n} A_{ij}x_j \leq b_i \quad \text{for all } A_{ij} \in [\bar{A}_{ij} - V_{ij}, \bar{A}_{ij} + V_{ij}].$$

   For given x, the expression on the left-hand side of the inequality is maximized by taking $A_{ij} = \bar{A}_{ij} + V_{ij}$ when $x_j \geq 0$ and $A_{ij} = \bar{A}_{ij} - V_{ij}$ when $x_j < 0$. This shows that the ith constraint is equivalent to a nonlinear convex constraint in x:

   $$\sum_{j=1}^{n} \bar{A}_{ij}x_j + V_{ij}|x_j| \leq b_i.$$

The problem is therefore equivalent to

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{A}x + V|x| \preceq b \end{array} \tag{1}$$

where we define $|x|$ as the vector $|x| = (|x_1|, |x_2|, \ldots, |x_n|)$. This in turn is equivalent to the LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{A}x + Vy \preceq b \\ & -y \preceq x \preceq y \end{array} \tag{2}$$

with variables $x \in \mathbf{R}^n$, $y \in \mathbf{R}^n$.

To see the equivalence between (1) and (2), first note that the constraints $-y \preceq x \preceq y$ mean $y \succeq |x|$. Suppose x is feasible in (1). Then clearly x and $y = |x|$ are feasible in (2). Conversely if x and y are feasible in (2), then x is feasible in (1) because V is component-wise nonnegative and $y \succeq |x|$.

3. Exercise A4.39.

Solution.

(a) The limits for $\gamma \to \pm\infty$ are clearer from the expression

$$\log \sum_{i=1}^m p_i e^{z_i} = z_{\max} + \log \sum_{i=1}^m p_i e^{z_i - z_{\max}},$$

where $z_{\max} = \max_{i=1,\ldots,m} z_i$. (This is also useful in computations, to avoid very large numbers when evaluating the function.) For the limit at $+\infty$ we apply this with $z_i = \gamma c_i^T x$:

$$\frac{1}{\gamma} \log \sum_{i=1}^m p_i e^{\gamma y_i} = y_{\max} + \frac{1}{\gamma} \log\Big(\sum_{i: y_i = y_{\max}} p_i + \sum_{i: y_i < y_{\max}} p_i e^{\gamma(y_i - y_{\max})} \Big),$$

where $y_i = c_i^T x$. The second sum on the right-hand side goes to zero as $\gamma \to \infty$. Therefore the limit of the right-hand side is $y_{\max}$. For the limit at $-\infty$ we write

$$\frac{1}{\gamma} \log \sum_{i=1}^m p_i e^{\gamma y_i} = y_{\min} + \frac{1}{\gamma} \log\Big(\sum_{i: y_i = y_{\min}} p_i + \sum_{i: y_i > y_{\min}} p_i e^{\gamma(y_i - y_{\min})} \Big).$$

Now the second sum on the right-hand side goes zero if $\gamma \to -\infty$, and the limit of the right-hand side is $y_{\min}$.

The limit at zero can be found from the linear approximations $\exp(u) \approx 1 + u$ and $\log(1+u) \approx u$ for small u:

$$\begin{aligned} \frac{1}{\gamma} \log \sum_i p_i e^{\gamma c_i^T x} &\approx \frac{1}{\gamma} (\log \sum_i p_i (1 + \gamma c_i^T x)) \\ &= \frac{1}{\gamma} \log(1 + \gamma \bar{c}^T x) \\ &\approx \bar{c}^T x. \end{aligned}$$

(b) This follows from convexity of the function $e^{\gamma t}$:

$$e^{\gamma \bar{c}^T x} = e^{\gamma \sum_i p_i c_i^T x} \leq \sum_{i=1}^m p_i e^{\gamma c_i^T x}.$$

Hence

$$\gamma \bar{c}^T x \leq \log \sum_{i=1}^m p_i e^{\gamma y_i}$$

and dividing by γ gives the two inequalities.

(c) The gradient and Hessian of f_γ at a point $\hat{y}$ are given by

$$\nabla f_\gamma(\hat{y}) = \frac{1}{\mathbf{1}^T z} z, \qquad \nabla^2 f_\gamma(\hat{y}) = \frac{\gamma}{\mathbf{1}^T z}(\mathbf{diag}(z) - \frac{1}{\mathbf{1}^T z} z z^T),$$

where $z = (p_1 e^{\gamma \hat{y}_1}, \ldots, p_m e^{\gamma \hat{y}_m})$. If $\hat{y} = (\bar{c}^T x)\mathbf{1}$, we have $z = e^{\gamma \bar{c}^T x} p$ and $\mathbf{1}^T z = e^{\gamma \bar{c}^T x}$, and we find the expressions for gradient and Hessian in the assignment.

Substituting these in the quadratic approximation, we get

$$\begin{aligned}
& f_\gamma(\hat{y}) + \nabla f_\gamma(\hat{y})^T (Cx - \hat{y}) + \frac{1}{2}(Cx - \hat{y})^T \nabla f_\gamma(\hat{y})(Cx - \hat{y}) \\
&= \bar{c}^T x + p^T (Cx - \hat{y}) + \frac{\gamma}{2}(Cx - \hat{y})^T (\mathbf{diag}(p) - pp^T)(Cx - \hat{y}) \\
&= \bar{c}^T x + \frac{\gamma}{2}(Cx - \hat{y})^T \, \mathbf{diag}(p)(Cx - \hat{y}) \\
&= \bar{c}^T x + \frac{\gamma}{2} \sum_{i=1}^m p_i (c_i^T x - \bar{c}^T x)^2 \\
&= \bar{c}^T x + \frac{\gamma}{2} x^T \Sigma x.
\end{aligned}$$

The second step follows because $p^T(Cx - \hat{y}) = \sum_i p_i (c_i^T x - \bar{c}^T x) = 0$.

4. Exercise T4.25.

Solution. We first note that the problem is homogeneous in a and b, so we can replace the strict inequalities $a^T x + b > 0$ and $a^T x + b < 0$ with $a^T x + b \geq 1$ and $a^T x + b \leq -1$, respectively. The variables a and b must satisfy

$$\inf_{\|u\|_2 \leq 1} a^T (P_i u + q_i) \geq 1 - b, \quad i = 1, \ldots, K,$$

and

$$\sup_{\|u\|_2 \leq 1} a^T (P_i u + q_i) \leq -1 - b, \quad i = K+1, \ldots, K+L.$$

The left-hand sides can be expressed as

$$\inf_{\|u\|_2 \leq 1} a^T (P_i u + q_i) = -\|P_i^T a\|_2 + q_i^T a$$

and

$$\sup_{\|u\|_2 \le 1} a^T(P_i u + q_i) = \|P_i^T a\|_2 + q_i^T a.$$

We therefore obtain a set of second-order cone constraints in a, b:

$$\|P_i^T a\|_2 \le q_i^T a + b - 1, \quad i = 1, \ldots, L, \quad \|P_i^T a\|_2 \le -q_i^T a - b - 1, \quad i = K+1, \ldots, K+L.$$

5. Exercise A4.21(a,b)

Solution.

(a) First consider the conditions

$$y \le \sqrt{z_1 z_2}, \qquad y, z_1, z_2 \ge 0.$$

This is equivalent to

$$y \ge 0, \qquad \left\| \begin{bmatrix} 2y \\ z_1 - z_2 \end{bmatrix} \right\|_2 \le z_1 + z_2.$$

The equivalence can be seen by expanding the norm inequality, which gives

$$4y^2 + (z_1 - z_2)^2 \le (z_1 + z_2)^2, \qquad z_1 + z_2 \ge 0.$$

This simplifies to $y^2 \le z_1 z_2$ and $z_1, z_2 \ge 0$.
Next consider the second constraint in the problem statement,

$$y \le (z_1 z_2 \cdots z_n)^{1/n}, \qquad y, z_1, \ldots, z_n \ge 0.$$

First suppose $n = 2^l$. Introduce variables y_{ij} for $i = 1, \ldots, l-1$, and $j = 1, \ldots, 2^i$, and write the constraint as the following set of inequalities:

$$z \succeq 0,$$

$$y_{l-1,j} \le (z_{2j-1} z_{2j})^{1/2}, \qquad y_{l-1,j} \ge 0, \qquad j = 1, \ldots, 2^{l-1}$$

$$y_{ij} \le (y_{i+1,2j-1} y_{i+1,2j})^{1/2}, \qquad y_{ij} \ge 0, \qquad i = 1, \ldots, l-2, \qquad j = 1, \ldots, 2^i$$

$$y \le (y_{11} y_{12})^{1/2}, \qquad y \ge 0.$$

Then apply the second order cone formulation as in the case $n = 2$.
For general n we write the constraint as

$$y \le (y^{m-n} z_1 \cdots z_n)^{1/m}, \qquad y \ge 0, \qquad z \succeq 0,$$

where m is the smallest power of two that is greater than or equal to n, and then use the previous formulation.

(b) For the first function, write the constraint as

$$y \le (t^s)^{1/r}, \qquad y \ge x, \qquad y \ge 0, \qquad t \ge 0$$

where $\alpha = r/s$ and s and $r \ge s$ are integers, and then apply part (a) with $z_k = t$ for $k = 1, \ldots, s$ and $z_k = 1$ for $k = s+1, \ldots, r$.
For the second function, we express the constraint as

$$1 \le (x^r t^s)^{1/(r+s)}, \qquad x \ge 0, \qquad t \ge 0,$$

where $\alpha = -r/s$ and r and s are integers, and then apply the formulation of (a).

6. Exercise A15.6.

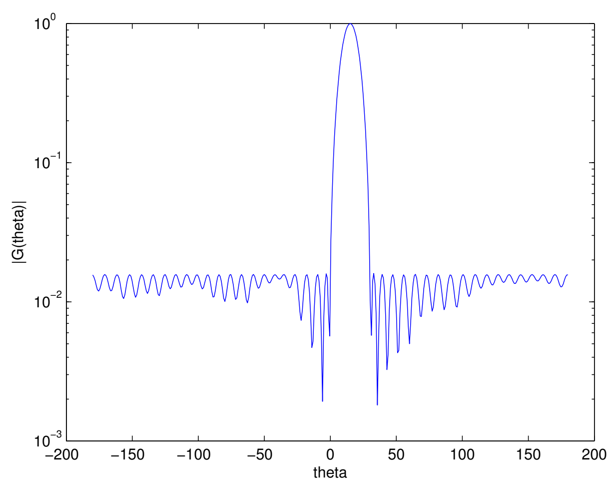
(a) The problem can be expressed as the SOCP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \|A_k x\|_2 \le t, \quad k \in \mathcal{I} \\ & Bx = d \end{array}$$

with variables x, t, where $\mathcal{I} = \{k \mid |\theta_k - \theta^{\text{tar}}| \ge \Delta\}$ and

$$\begin{aligned}
x &= \begin{bmatrix} w_{\text{re}} \\ w_{\text{im}} \end{bmatrix} \in \mathbf{R}^{2n} \\
A_k &= \begin{bmatrix} \cos\gamma_1(\theta_k) & \cdots & \cos\gamma_n(\theta_k) & -\sin\gamma_1(\theta_k) & \cdots & -\sin\gamma_n(\theta_k) \\ \sin\gamma_1(\theta_k) & \cdots & \sin\gamma_n(\theta_k) & \cos\gamma_1(\theta_k) & \cdots & \cos\gamma_n(\theta_k) \end{bmatrix} \\
B &= \begin{bmatrix} \cos\gamma_1(\theta^{\text{tar}}) & \cdots & \cos\gamma_n(\theta^{\text{tar}}) & -\sin\gamma_1(\theta^{\text{tar}}) & \cdots & -\sin\gamma_n(\theta^{\text{tar}}) \\ \sin\gamma_1(\theta^{\text{tar}}) & \cdots & \sin\gamma_n(\theta^{\text{tar}}) & \cos\gamma_1(\theta^{\text{tar}}) & \cdots & \cos\gamma_n(\theta^{\text{tar}}) \end{bmatrix} \\
d &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\end{aligned}$$

(b) The figure below shows the output of the antenna array for different values of θ.



The following code solves the problem.

```
rand('state',0);
n = 40;
X = 30*[rand(1,n); rand(1,n)];
N = 400;
beamwidth = 15*pi/180;
theta_tar = 15*pi/180;
theta = linspace(theta_tar + beamwidth, 2*pi + theta_tar - beamwidth, N)';
A = exp(i * [cos(theta), sin(theta)] * X);
Atar = exp(i * [cos(theta_tar), sin(theta_tar)] * X)
cvx_begin
    variable w(n) complex
    minimize( max(abs(A*w)) )
    subject to
        Atar*w == 1;
cvx_end
```

7. Exercise T4.26(a).

Solution. The problem is equivalent to

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T t \\ \text{subject to} & t_i(a_i^T x - b_i) \geq 1, \quad i = 1, \ldots, m \\ & t \succeq 0. \end{array}$$

Writing the hyperbolic constraints as second order cone constraints yields an SOCP

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T t \\ \text{subject to} & \left\| \left[\begin{array}{c} 2 \\ a_i^T x - b_i - t_i \end{array} \right] \right\|_2 \leq a_i^T x - b_i + t_i, \quad i = 1, \ldots, m \\ & t \succeq 0. \end{array}$$

8. Exercise T4.27.

Solution. To show the equivalence with the problem in the hint, we assume $x \succeq 0$ is fixed, and optimize over v and w. This is a quadratic problem with equality constraints:

$$\begin{array}{ll} \text{minimize} & \left[\begin{array}{c} v \\ w \end{array} \right]^T \left[\begin{array}{cc} I & 0 \\ 0 & \mathbf{diag}(x)^{-1} \end{array} \right] \left[\begin{array}{c} v \\ w \end{array} \right] \\ \text{subject to} & \left[\begin{array}{cc} I & B \end{array} \right] \left[\begin{array}{c} v \\ w \end{array} \right] = Ax + b. \end{array}$$

The optimality conditions (from lecture 4, page 13) are

$$\left[\begin{array}{c} v \\ \mathbf{diag}(x)^{-1} w \end{array} \right] = \left[\begin{array}{c} I \\ B^T \end{array} \right] \nu, \qquad \left[\begin{array}{cc} I & B \end{array} \right] \left[\begin{array}{c} v \\ w \end{array} \right] = Ax + b.$$

In other words, v and w are optimal if and only if there exists a ν such that

$$v = \nu, \qquad w = \mathbf{diag}(x)B^T\nu, \qquad v + Bw = Ax + b.$$

Substituting the expressions for v and w from the first two equations in the third equation, we find that ν must satisfy

$$(I + B\,\mathbf{diag}(x)B^T)\nu = Ax + b.$$

Since the matrix on the left is invertible for $x \succeq 0$, we can solve for ν. Therefore the optimal v, w for the optimization problem in the hint are

$$v = \nu = (I + B\,\mathbf{diag}(x)B^T)^{-1}(Ax + b)$$

and

$$w = \mathbf{diag}(x)B^T\nu = \mathbf{diag}(x)B^T(I + B\,\mathbf{diag}(x)B^T)^{-1}(Ax + b).$$

Substituting these expressions for v and w in the objective of the problem in the hint, we obtain

$$v^Tv + w^T\,\mathbf{diag}(x)^{-1}w = (Ax + b)^T(I + B\,\mathbf{diag}(x)B^T)^{-1}(Ax + b).$$

This shows that the problem is equivalent to the problem in the hint.

As in exercise 4.26 we now introduce hyperbolic constraints and formulate the problem in the hint as

$$\begin{array}{ll} \mbox{minimize} & t + \mathbf{1}^Ts \\ \mbox{subject to} & v^Tv \leq t \\ & w_i^2 \leq s_ix_i, \quad i = 1, \ldots, n \\ & x \succeq 0 \end{array}$$

with variables $t \in \mathbf{R}$, $s, x, w \in \mathbf{R}^n$, $v \in \mathbf{R}^m$. Converting the hyperbolic constraints into second order cone constraints results in the SOCP

$$\begin{array}{ll} \mbox{minimize} & t + \mathbf{1}^Ts \\ \mbox{subject to} & \left\| \left[\begin{array}{c} 2v \\ 1 - t \end{array} \right] \right\|_2 \leq 1 + t \\ & \left\| \left[\begin{array}{c} 2w_i \\ s_i - x_i \end{array} \right] \right\|_2 \leq s_i + x_i, \quad i = 1, \ldots, n \\ & x \succeq 0. \end{array}$$