Homework 4

Submit answers for problems 1–6. Problems 7 and 8 are included as practice problems.

1. Exercise T3.55.

2. Exercise T4.13.

3. Exercise A4.39.

4. Exercise T4.25.

5. Exercise A4.21(a,b)

6. Exercise A15.6.

7. Exercise T4.26(a).

   Solution. The problem is equivalent to

$$
\begin{array}{ll}
\text{minimize} & \mathbf{1}^T t \\
\text{subject to} & t_i(a_i^T x - b_i) \geq 1, \quad i = 1, \ldots, m \\
& t \succeq 0.
\end{array}
$$

   Writing the hyperbolic constraints as second order cone constraints yields an SOCP

$$
\begin{array}{ll}
\text{minimize} & \mathbf{1}^T t \\
\text{subject to} & \left\| \begin{bmatrix} 2 \\ a_i^T x - b_i - t_i \end{bmatrix} \right\|_2 \leq a_i^T x - b_i + t_i, \quad i = 1, \ldots, m \\
& t \succeq 0.
\end{array}
$$

8. Exercise T4.27.

   Solution. To show the equivalence with the problem in the hint, we assume $x \succeq 0$ is fixed, and optimize over v and w. This is a quadratic problem with equality constraints:

$$
\begin{array}{ll}
\text{minimize} & \begin{bmatrix} v \\ w \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & \mathbf{diag}(x)^{-1} \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} \\
\text{subject to} & \begin{bmatrix} I & B \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = Ax + b.
\end{array}
$$

The optimality conditions (from lecture 4, page 13) are

$$\left[\begin{array}{c} v \\ \mathbf{diag}(x)^{-1}w \end{array}\right] = \left[\begin{array}{c} I \\ B^T \end{array}\right]\nu, \qquad \left[\begin{array}{cc} I & B \end{array}\right]\left[\begin{array}{c} v \\ w \end{array}\right] = Ax+b.$$

In other words, v and w are optimal if and only if there exists a ν such that

$$v = \nu, \qquad w = \mathbf{diag}(x)B^T\nu, \qquad v + Bw = Ax + b.$$

Substituting the expressions for v and w from the first two equations in the third equation, we find that ν must satisfy

$$(I + B\,\mathbf{diag}(x)B^T)\nu = Ax + b.$$

Since the matrix on the left is invertible for $x \succeq 0$, we can solve for ν. Therefore the optimal v, w for the optimization problem in the hint are

$$v = \nu = (I + B\,\mathbf{diag}(x)B^T)^{-1}(Ax+b)$$

and

$$w = \mathbf{diag}(x)B^T\nu = \mathbf{diag}(x)B^T(I + B\,\mathbf{diag}(x)B^T)^{-1}(Ax+b).$$

Substituting these expressions for v and w in the objective of the problem in the hint, we obtain

$$v^Tv + w^T\,\mathbf{diag}(x)^{-1}w = (Ax+b)^T(I + B\,\mathbf{diag}(x)B^T)^{-1}(Ax+b).$$

This shows that the problem is equivalent to the problem in the hint.

As in exercise 4.26 we now introduce hyperbolic constraints and formulate the problem in the hint as

$$\begin{array}{ll} \text{minimize} & t + \mathbf{1}^T s \\ \text{subject to} & v^T v \leq t \\ & w_i^2 \leq s_i x_i, \quad i = 1, \ldots, n \\ & x \succeq 0 \end{array}$$

with variables $t \in \mathbf{R}$, $s, x, w \in \mathbf{R}^n$, $v \in \mathbf{R}^m$. Converting the hyperbolic constraints into second order cone constraints results in the SOCP

$$\begin{array}{ll} \text{minimize} & t + \mathbf{1}^T s \\ \text{subject to} & \left\| \left[\begin{array}{c} 2v \\ 1-t \end{array}\right] \right\|_2 \leq 1 + t \\ & \left\| \left[\begin{array}{c} 2w_i \\ s_i - x_i \end{array}\right] \right\|_2 \leq s_i + x_i, \quad i = 1, \ldots, n \\ & x \succeq 0. \end{array}$$