EXERCISE 13 Gamov's theory of α decay

Radioactive nuclei can decay to lighter nuclei by emitting α-particles (i.e. Helium nuclei, $Z=2$)

$$ {}^{A}_{Z}X \rightarrow {}^{A-4}_{Z-2}Y + {}^{4}_{2}He $$

(here A is the atomic weight and Z the atomic number).
Geiger and Nuttal (1911) found an empirical relation between the half-like $t_{1/2}$ of α-decay and the energy E of the emitted α particle, of the form

$$ \log_{10} t_{1/2} \simeq -a + b\frac{Z}{\sqrt{E}} \qquad a,b>0 $$

with a,b constants, and Z the atomic number of the daughter nucleus. Shortly after, in 1928, Gamow gave a theoretical explanation of such relation in terms of a quantum mechanical model where the α-particle is described as a particle contained in the parent nucleus and subject to the potential ($r \geq 0$)

$$ V(r) = \begin{cases} -V_0 & \text{for } 0 < r \leq R \\ \dfrac{2Ze^2}{r}\dfrac{1}{4\pi\varepsilon_0} & \text{for } r \geq R \end{cases} $$

where R is the radius of the parent nucleus, whose value is well reproduced by the formula $R \simeq 1.1 \times A^{1/3}$ fm.
Using Gamow's model, derive the Geiger-Nuttal relation.

Solution :

Let us first notice that the potential is a central one, i.e. it depends only on the radius r. It is thus convenient to work in spherical coordinates. Schroedinger's equation reads

$$\left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)-\frac{\hbar^2}{2mr^2}\left(\frac{\partial^2}{\partial\theta^2}+\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)+V(r)\right]\psi(r,\theta,\varphi)$$

$$= E\,\psi(r,\theta,\varphi)$$

where $-\hbar^2\left(\frac{\partial^2}{\partial\theta^2}+\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)$ is the $|\vec{L}|^2$ differential operator. Since $|\vec{L}|^2$ has positive eigenvalues, the state with lowest energy will have zero angular momentum. Furthermore, a non-vanishing centrifugal term will make the energy barrier higher and thus leads to a smaller decay rate. Hence, we focus on eigenstates with zero angular momentum and take $\psi(r,\theta,\varphi)$ constant in θ,φ :

$$\left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)+V(r)\right]\psi(r) = E\,\psi(r)$$

We have thus reduced ourselves to a 1D problem. In order to simplify the form of the differential operator, it is useful to set

$$\psi(r) = \frac{u(r)}{r}$$

and solve for $u(r)$.

One has

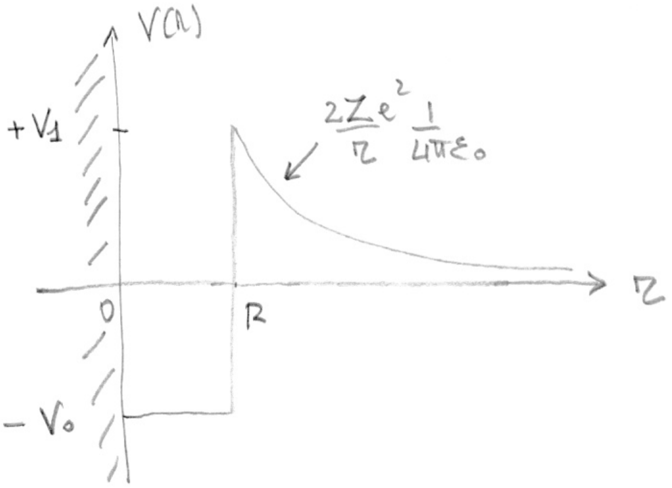
$$\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} = \left(\frac{\partial}{\partial r} + \frac{1}{r}\right)^2$$

$$\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\frac{u}{r} = \frac{u'}{r} - \frac{u}{r^2} + \frac{u}{r^2} = \frac{u'}{r}$$

Therefore, Schroedinger's equation for $u(r)$ is

$$\boxed{-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial r^2}u(r) + V(r)\,u(r) = E\,u(r)}$$

that is, the 1-dimensional Schroedinger equation.
The potential r has the form



The region at $r<0$ is unphysical; in terms of the 1D problem one can think that this region is not accessible because there the potential is infinite (wall).
The nuclear interactions inside the nucleus ($r<R$) are modelled by Gamow as a square potential well. These interactions decay very rapidly outside the nucleus, where the Coulomb interaction is an excellent approximation to the total potential.

Alpha particles were used already by Rutherford in his famous experiment where he studied the inner structure of the atom. They continued to be used as projectiles until the invention of cyclotrons by Lawrence in late 1930's (cyclotrons produce particles of higher energies).

A good source of α-particles is $^{212}_{84}Po$ (polonium), used by Rutherford. The emitted α-particles have, in this case, an energy $E = 8.8$ MeV and the reaction is $^{212}_{84}Po \rightarrow {}^{208}_{82}Pb + {}^{4}_{2}He$.

The fact that the energy of the α-particle is fixed in terms of the properties of the parent (or daughter) nucleus is a simple consequence of momentum and energy conservation:

$$M_1c^2 = M_2c^2 + \frac{p^2}{2M_2} + m_\alpha c^2 + \frac{p^2}{2m_\alpha}$$

$$= M_2c^2 + E\left(1 + \frac{m_\alpha}{M_2}\right) + m_\alpha c^2$$

$$E = [M_1 - (M_2 + m_\alpha)]c^2 \frac{1}{1 + \frac{m_\alpha}{M_2}} \equiv Q \frac{1}{1 + \frac{m_\alpha}{M_2}}$$

where E and p are the energy and the momentum of the α-particle, M_1, M_2 and m_α are the mass of the parent nucleus, daughter nucleus and α particle respectively.

Also, it is common practice to define the Q-value as $Q = [M_1 - (M_2 + m_\alpha)]c^2$. It corresponds to the available energy from the difference of masses. Since $m_\alpha \ll M_2$, the energy of the α-particle is equal to the Q value with good approximation.

Using α-particles with $E = 8.8$ MeV from ^{212}Po Rutherford studied the potential energy experienced by an α-particle outside the nucleus of uranium, $^{238}_{92}U$. He found that the potential followed Coulomb's law, at least for distances greater than $\bar{r} = 3 \times 10^{-14}$ m, where $V(\bar{r}) = \frac{2Ze^2}{4\pi\varepsilon_0} \frac{1}{\bar{r}} = 8.8$ MeV and $\bar{r}$ is the shortest distance from the uranium center probed by the α-particle (here $Z = 92$ is uranium's atomic number).

Since nucleons are bound inside the nucleus by nuclear forces, it is clear that the potential must change and become a strongly attractive one at shorter distances.

A puzzle that confused physicists at that time was the following: uranium-238 is itself an α-particle emitter, according to the reaction $^{238}_{92}U \rightarrow {}^{234}_{90}Th + {}^{4}_{2}He$. The α-particle in this case is emitted with energy $E = 4.2$ MeV. The decay was explained by thinking the α-particle as trapped, with that energy, inside the uranium nucleus. But this raised a paradox, because Rutherford's study indicated that it had to pass an energy barrier (V_1) that was at least 8.8 MeV high, i.e. more than twice higher than the available energy. This was classically impossible.

Gamow had the intuition to explain the decay as the effect of quantum tunneling (a similar proposal came from Condon and Gurney at the same time).

Now we know that the nuclear radius has a value $R \simeq 1.1\, A^{1/3}$ fm. For $^{238}_{92}U$ this formula gives

$$R_U \simeq 1.1\,(238)^{1/3}\ \text{fm} \simeq 6.8\ \text{fm}$$

We can estimate the height of the potential barrier (V_1) as the value of the potential at this distance, $R = R_U$

$$V_1 = \frac{2Ze^2}{R}\frac{1}{4\pi\varepsilon_0} = \frac{2Z}{R}\hbar c\left(\frac{e^2}{4\pi\varepsilon_0 \hbar c}\right) = \frac{2\cdot 90}{6.8\ \text{fm}}\ \frac{197\ \text{MeV}\cdot\text{fm}}{137}$$

$$V_1 \simeq 38\ \text{MeV}$$

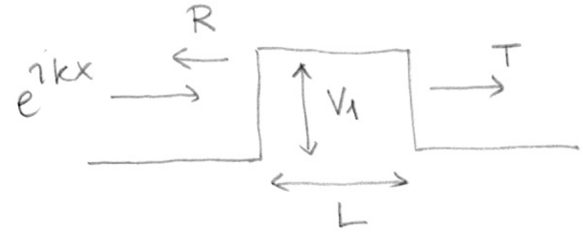
where we used $e^2/4\pi\varepsilon_0 \hbar c = 1/137$ and $Z=90$ for the daughter nucleus.
A more refined estimate is obtained by taking into account the size of the ^4_2He nucleus, so that the minimum distance between the nuclei centers is $R = R_U + R_\alpha \simeq 8.6$ fm ; this gives $V_1 \simeq 30$ MeV.

The height of the potential barrier is approximatively 30 MeV for a wide range of nuclei while the energy of the emitted α-particle varies tipically between 4-9 MeV.
Hence $E \ll V_1$ and α-decay must occur through tunneling.

In order to derive Geiger-Nuttal relation we need to compute the tunneling rate. We have already computed the probability for tunneling in the case of a potential barrier. There we found that for a thick barrier

$$P_{tunneling} = T \simeq e^{-2L\sqrt{2m(V_1-E)}/\hbar}$$

where L was the thickness of the barrier.



We can hope to use the same formula here with some minor modifica= tion. This can work if the presence of the "wall" at $r=0$ does not affect the tunneling process, i.e. if the wall is sufficiently far. We can describe the α-particle inside the parent nucleus as a wave packet of momentum $p = \sqrt{2mE}$ and width Δp. The simplified description we made in terms of stationary waves works if $\Delta p \ll p$; this implies that the spatial spread of the packet is

$$\Delta x \gtrsim \frac{\hbar}{\Delta p} \gg \frac{\hbar}{p} = \frac{\lambda_{deBroglie}}{2\pi}$$

On the other hand, the wall can be considered far if $R \gg \Delta x$. Hence, we can use the formula for $P_{tunneling}$ derived before as long as

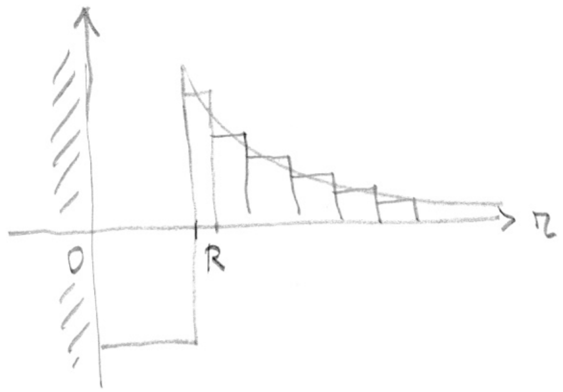
$$\frac{\hbar}{\sqrt{2mE}} \ll \Delta x \ll R$$

For uranium $\quad R \simeq 8.6$ fm $\quad \hbar/p \simeq 1.1$ fm

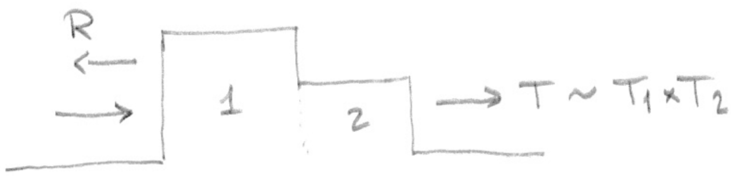
polonium $\quad R \simeq 8.3$ fm $\quad \hbar/p \simeq 0.7$ fm

Although the separation is not huge, this estimate shows that our approximation has a chance to work. Hence, we will make it and move on.

Next, we have to cope with the fact that the potential barrier does not have a square-shape. To solve this problem we can imagine approximating the Coulomb shape with a sum of rectangles



It can be shown that the probability of passing two square barriers put one after the other is just the product of the individual probabilities, in the limit of thick barriers:



Taking the limit of many barriers one finds that the tunneling probability is simply given by

$$P_{tunneling} \simeq \exp\left[-\frac{2}{\hbar}\int_R^{\bar{r}} dr \sqrt{2m(V(r)-E)}\right]$$

where $\bar{r}$ is the distance at which $V(r) = E$ (inversion point). The integral can be solved exactly for the Coulomb potential:

$$\int_R^{\bar{r}} dr \left(\frac{2Ze^2}{4\pi\varepsilon_0} \frac{1}{r} - E \right)^{1/2} = \left(\frac{2Ze^2}{4\pi\varepsilon_0} \right)^{1/2} \int_R^{\bar{r}} dr \left(\frac{1}{r} - \frac{1}{\bar{r}} \right)^{1/2}$$

$$= \left(\frac{2Ze^2}{4\pi\varepsilon_0} \right)^{1/2} \sqrt{\bar{r}} \left[\arccos\sqrt{\frac{R}{\bar{r}}} - \left(\frac{R}{\bar{r}} - \frac{R^2}{\bar{r}^2} \right)^{1/2} \right]$$

When $E \ll V_1$ it will follow $\bar{r} \gg R$, hence the integral can be approximated as

$$\int_R^{\bar{r}} dr \, \left(V(r) - E \right)^{1/2} \simeq \left(\frac{2Ze^2}{4\pi\varepsilon_0} \right)^{1/2} \sqrt{\bar{r}} \, \frac{\pi}{2}$$

$$= \left(\frac{2Ze^2}{4\pi\varepsilon_0} \right)^{1/2} \left(\frac{2Ze^2}{4\pi\varepsilon_0} \frac{1}{E} \right)^{1/2} \frac{\pi}{2}$$

$$= \frac{Ze^2\pi}{4\pi\varepsilon_0} \frac{1}{\sqrt{E}}$$

$$P_{tunneling} \simeq \exp\left[-\frac{2\sqrt{2M}}{\hbar} \left(\frac{Ze^2\pi}{4\pi\varepsilon_0} \right) \frac{1}{\sqrt{E}} \right]$$

$$= \exp\left[-\frac{BZ}{\sqrt{E}} \right] \qquad \text{where} \quad B = \frac{2e^2\pi}{4\pi\varepsilon_0} \frac{\sqrt{2M}}{\hbar}$$

Let us connect $P_{tunneling}$ to the half-life of the parent nucleus. Radioactive decay follows an exponential law, according to which the number of nuclei decreases as

$$N(t) = N(0)\, e^{-t\xi}$$

where ξ is the decay rate.
This formula is derived as follows:

the rate ξ is the probability of decaying per unit time, hence the number of decays in the time interval dt is

$$dN = -N \xi \, dt$$

which gives $N(t) = N(0) \exp(-t\xi)$ once integrated.

The half-life is defined as the time interval during which the number of nuclei halves; that is

$$\frac{N(t_{1/2})}{N(0)} = \frac{1}{2} \rightarrow \boxed{t_{1/2} = \frac{1}{\xi} \log 2}$$

We then have to connect the rate ξ with the tunneling probability. If the α-particle is described as free inside the nucleus, it will bounce back and forth on the barrier until it finally tunnels. The number of bounces before tunneling, on average, is just $1/P_{tunneling}$. The time that one has to wait, on average, before the α-particle can tunnel is thus the time interval between two bounces

$$\Delta t = \frac{2R}{v} \simeq \frac{2R}{\sqrt{\frac{2E_K}{m_\alpha}}}$$

where $E_K = E + V_0$ is the kinetic energy of the α particle inside the nucleus times the number of bounces. Its inverse gives the decay rate:

$$\frac{1}{W} = \Delta t \frac{1}{P_{tunneling}} = 2R\sqrt{\frac{M_\alpha}{2E_K}}\frac{1}{P_{tunneling}}$$

We thus obtain the formula for the half-life

$$t_{1/2} = \sqrt{\frac{2M_\alpha}{E_K}}\, R \log 2 \; e^{BZ/\sqrt{E}}$$

$$\boxed{\log_{10} t_{1/2} = \log_{10}\left[\sqrt{\frac{2M_\alpha}{E_K}}\, R\log 2\right] + \frac{BZ}{\sqrt{E}}}$$

The quantity $\log_{10}\left[\sqrt{\frac{2M_\alpha}{E_K}}\, R\log 2\right]$ varies very slowly with the energy of the emitted α-particle and can be considered as approximately a costant. When expressed in seconds this constant is negative. Hence we obtain the Geiger-Nuttal relation,

$$\boxed{\log_{10} t_{1/2} = -a + \frac{bZ}{\sqrt{E}}}$$

with

$$\boxed{\begin{aligned} a &= -\log_{10}\left[\sqrt{\frac{2M_\alpha}{E_K}}\, R\log 2\right] \\ b &= \left(\frac{2\pi e^2}{4\pi\varepsilon_0}\frac{\sqrt{2M_\alpha}}{\hbar}\right)\log_{10} e \end{aligned}}$$

The value of b can be computed easily to be

$$b = 2\pi \frac{\log_{10} e}{137} \sqrt{2 \times 4 \times 931 \text{ MeV}} \simeq 1.78 \text{ MeV}^{1/2}$$

where we used $m_\alpha c^2 = 4 \times 931$ MeV.

The value of a instead, as predicted by our model, has a mild dependence on E and R. For example, if $t_{1/2}$ is measured in seconds:

for ${}^{212}_{84}Po$ decay : $R \simeq 8.3$ fm, $E_K \sim E \simeq 8.8$ MeV $\rightarrow$ $a \simeq 21.2$

for ${}^{238}_{92}U$ decay : $R \simeq 8.6$ fm, $E_K \sim E \simeq 4.2$ MeV $\rightarrow$ $a \simeq 21.1$

In these cases our result is thus

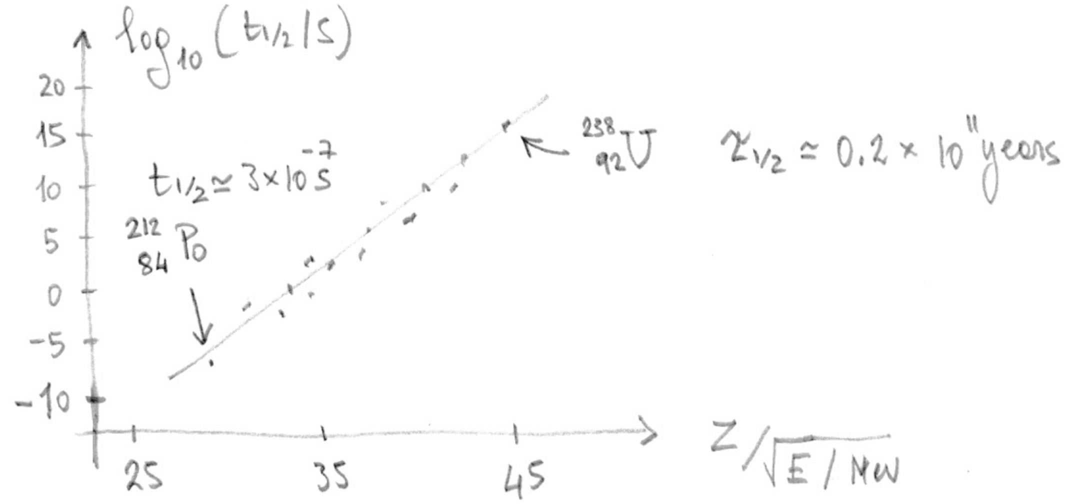
$$\log_{10}(t_{1/2}/s) \simeq -21 + 1.78 \frac{Z}{\sqrt{E/\text{MeV}}}$$

A fit to experimental data gives instead

$$\boxed{\log_{10}(t_{1/2}/s) \simeq -46.83 + \frac{1.454\, Z}{\sqrt{E/\text{MeV}}}}$$

Experimental Fit
see: Blunder, "Gamow Model for α decay: The Geiger-Nuttal law"
from the Wolfram Demonstration Project.

Gamow's model explains well the main dependence of the half-life on E and Z, and also reproduce correctly the order of magnitude of the coefficients in the Geiger-Nuttal relation. On the other hand, it does not provide a precision prediction for $t_{1/2}$. The experimental fit, on the other hand, reproduces well the half-life of several α-decays, which span a vast range of values:



The agreement between data and the predicted dependence of $t_{1/2}$ on E and Z is a clear indication that α-decay is a tunneling effect.