Sezione d'urto di Rutherford

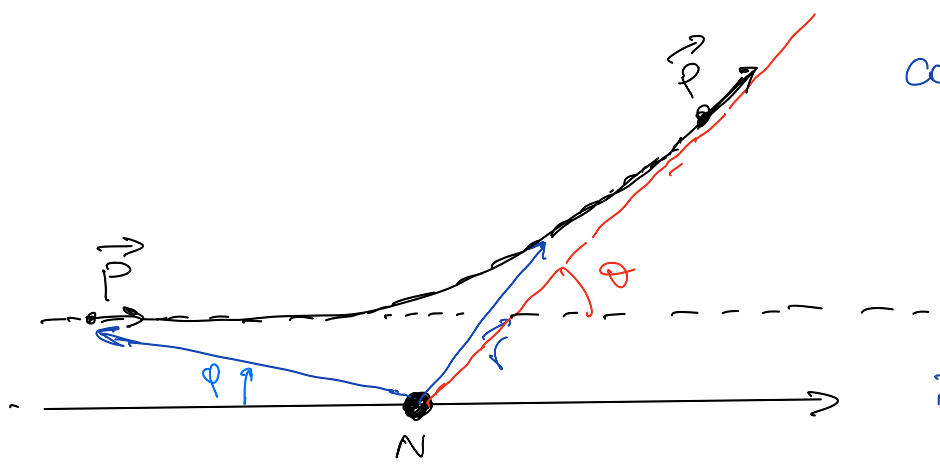
Interazione fra α e nucleo carico
Campo centrale di Coulomb generato dal nucleo.

- trovare legge del moto.
- trovare traiettoria del proiettile deviato.
- relazione $b = b(\theta)$
- calcolare $\frac{d\sigma}{d\Omega} = \left|\frac{db}{d\theta}\right| \frac{b}{\sin\theta}$

$$U(r) = \frac{A}{r} = \frac{e^2}{4\pi\varepsilon_0}\frac{Z_p \cdot Z_N}{r} = \alpha \frac{Z_p Z_N}{r}$$

Moto nel campo centrale $\Rightarrow$ si conserva
- energia E
- mom. angolare $\vec{\mu}$

$\Rightarrow$ moto piano.



Coordinate r, φ

$\vec{r}$: rispetto al centro del campo.

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\varphi}^2) - U(r)$$

$$\frac{\partial L}{\partial \varphi} = 0 \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = 0 \Rightarrow \frac{\partial L}{\partial \varphi} \equiv P_\varphi = \text{cost.}$$

$$P_\varphi \equiv \frac{\partial L}{\partial \dot{\varphi}} = \frac{m}{\cancel{2}} \cancel{2} r^2 \dot{\varphi} = m r^2 \dot{\varphi} \equiv \mu = \text{cost.} \qquad \vec{\mu} = \vec{r} \times \vec{p}$$

$\vec{\mu} \perp \vec{r}$ $\quad \mu \perp \vec{p} \Rightarrow$ moto piano

Si conserva anche l'energia.

$$E = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) + U(r) = \frac{m}{2}\dot{r}^2 + \frac{\overbrace{m^2 r^4 \dot{\varphi}^2}^{\mu^2}}{2mr^2} + U(r)$$

$$= \frac{m}{2}\dot{r}^2 + \frac{\mu^2}{2mr^2} + U(r)$$

E si conserva $\Rightarrow$ $E = \text{cost.}$

$$\frac{m\dot{r}^2}{2} = E - U(r) - \frac{\mu^2}{2mr^2} \Rightarrow \dot{r}^2 = \frac{2}{m}(E - U(r)) - \frac{\mu^2}{m^2r^2}$$

$$\frac{dr}{dt} = \sqrt{\frac{2}{m}(E-U(r)) - \frac{\mu^2}{m^2r^2}} \Rightarrow dt = \frac{dr}{\sqrt{\frac{2}{m}(E-U(r)) - \frac{\mu^2}{m^2r^2}}}$$

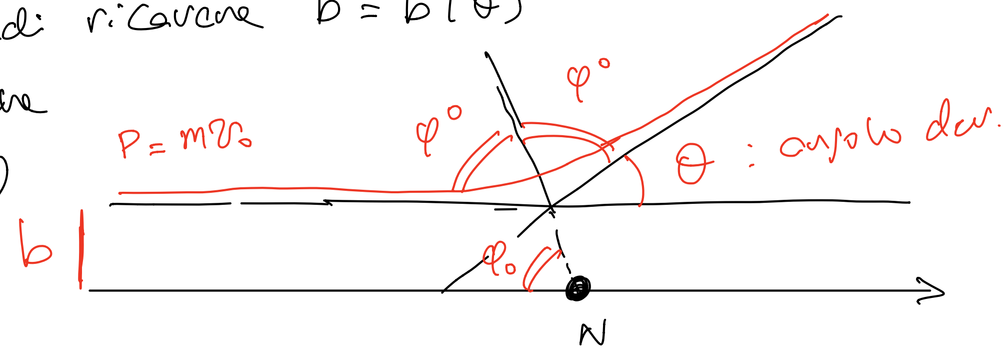
$$\mu = mr^2\dot{\varphi} = \text{cost} \Rightarrow \mu = mr^2\frac{d\varphi}{dt}.$$

$$\Rightarrow d\varphi = \frac{\mu}{mr^2}dt$$

$$d\varphi = \frac{\frac{\mu}{mr^2}\,dr}{\sqrt{\frac{2}{m}(E-U(r)) - \frac{\mu^2}{m^2r^2}}} = \frac{\frac{\mu}{r^2}}{\sqrt{2m(E-U) - \frac{\mu^2}{r^2}}}dr$$

Questa eq. ci permette di ricavare la traiettoria del proiettile e quindi ricavare $b = b(\theta)$

notiamo che $(r = \infty, \varphi = 0)$



$\theta + 2\varphi_0 = \pi$ $\qquad$ φ_0: angolo tra all'infinito e punto di minimo approccio

$$\int_{\varphi_{r=\infty}}^{\varphi_{r_{min}}} d\varphi = \int_{r_\infty}^{r_{min}} \frac{\mu/r^2}{\sqrt{(- -)}}dr.$$

$$\Rightarrow \varphi_{r_{min}} = \varphi_0 = \underbrace{\varphi(r=\infty)}_{0} + \int_{r_\infty}^{r_{min}} \frac{dr}{\sqrt{(-)}}$$

φ_0 calcolabile.

$\Rightarrow$ $\theta = \pi - 2\varphi_0$ $\qquad$ possiamo calcolare θ in funzione di b

stato iniziale @ $r=\infty$.

$$E = E_0 = \frac{1}{2} m v_\infty^2 = \frac{1}{2} m v_0^2 = K. \qquad \text{per } r \to \infty.$$

$$v_0 = \sqrt{2K/m}$$

$$M = |\vec{r}_\infty \times \vec{p}_\infty| = b m v_0$$

$$\varphi_0 = \int_{r_{min}}^{\infty} \frac{\frac{b m v_0}{r^2}}{\sqrt{2m\left(\frac{1}{2} m v_0^2 - U(r)\right) - \frac{b^2 m^2 v_0^2}{r^2}}} \, dr$$

tiriamo fuori $m^2 v_0^2$

$$2m\left(\frac{1}{2} m v_0^2 - U\right) - \frac{b^2 m^2 v_0^2}{r^2} = m^2 v_0^2 - 2mU - m^2 v_0^2 \frac{b^2}{r^2}$$

$$= m^2 v_0^2 \left(1 - \frac{b^2}{r^2} - \frac{2U}{m v_0^2}\right) = m^2 v_0^2 \left(1 - \frac{b^2}{r^2} - \frac{U}{E_0}\right)$$

$$\varphi_0 = \int_{r_{min}}^{\infty} \frac{b/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{U}{E_0}}} \, dr$$

Nel nostro caso $U(r) = \frac{\alpha Z_p Z_N}{r} = \frac{A}{r}$

$$Z_p = 2 \quad Z_N > 0 \Rightarrow A > 0$$

$$\varphi_0 = \int_{r_{min}}^{\infty} \frac{b/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{A}{E_0}\frac{1}{r}}} \, dr =$$

$$= \arccos \frac{\frac{A}{2E_0 b}}{\sqrt{1 + \left(\frac{A}{2E_0 b}\right)^2}}$$

$$\varphi_0 = \arccos \frac{B}{\sqrt{1+B^2}} \qquad B = \frac{A}{2E_0 b}$$

$$\cos\varphi_0 = \frac{B}{\sqrt{1+B^2}} \Rightarrow \cos^2\varphi_0 = \frac{B^2}{1+B^2}$$

$$\sin^2\varphi_0 = 1 - \cos^2\varphi_0 = \frac{1 + \cancel{B^2} - \cancel{B^2}}{1+B^2} = \frac{1}{1+B^2}$$

$$\frac{\cos^2\varphi_0}{\sin^2\varphi_0} = B^2 = \frac{A^2}{(2E_0 b)^2} \Rightarrow (2E_0 b)^2 = \frac{A^2}{(\operatorname{ctg}\varphi_0)^2}$$

$$b^2 = \frac{A^2}{4E_0^2}\operatorname{tg}^2\varphi_0$$

$$\theta = \pi - 2\varphi_0 = |\pi - 2\varphi_0|$$

$$\varphi_0 = \left|\frac{\pi}{2} - \frac{\theta}{2}\right|$$

$$\operatorname{tg}\varphi_0 = \operatorname{tg}\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \operatorname{ctg}\frac{\theta}{2} = \frac{1}{\operatorname{tg}\frac{\theta}{2}}$$

$$b = \frac{A}{2E_0}\frac{1}{\operatorname{tg}\frac{\theta}{2}}$$

$$A = \alpha Z_p Z_N. = \frac{Z_p Z_N}{137}$$

$$2E_0 = m v_0^2$$

$$b = \frac{A}{2E_0}\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}.$$

$$b = b(\theta).$$

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

$$\frac{d}{dx}\frac{1}{\operatorname{tg}x} = -\frac{1}{\sin^2 x}.$$

$$\frac{db}{d\theta} = \frac{A}{2E_0}\frac{d}{d\theta}\frac{1}{\operatorname{tg}\frac{\theta}{2}}$$

$$\frac{d}{d\theta}\frac{1}{\operatorname{tg}\frac{\theta}{2}} = -\frac{1}{\sin^2\frac{\theta}{2}}\ \frac{1}{2}$$

$$\left|\frac{db}{d\theta}\right| = \frac{A}{2E_0}\frac{1}{2}\frac{1}{\sin^2\frac{\theta}{2}}$$

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \overbrace{\frac{A}{2E_0}\frac{\cancel{\cos\frac{\theta}{2}}}{\sin\frac{\theta}{2}}}^{b}\ \frac{1}{2\sin\frac{\theta}{2}\cancel{\cos\frac{\theta}{2}}}\ \frac{A}{4E_0}\frac{1}{\sin^2\frac{\theta}{2}}$$

$$\sin\theta = \sin\left(2\frac{\theta}{2}\right)$$

$$= \frac{A^2}{16E_0^2}\frac{1}{\sin^4\frac{\theta}{2}} = \left(\frac{A}{4E_0}\right)^2\frac{1}{\sin^4\frac{\theta}{2}}$$

$$\frac{A}{4E_0} = \frac{\alpha\cdot\cancel{Z}\cdot Z_N}{\cancel{4}\,\cancel{\frac{1}{2}}\,m v_0^2} = \frac{\alpha Z_N}{m v_0^2} = \frac{\alpha Z_N}{2K}$$

$$K = 5\ \text{MeV} \Rightarrow \frac{A}{4E_0} \simeq \frac{1}{137\times 2\times 5}\ Z_N\ \text{MeV}^{-1} = \frac{Z_N}{1370}\ \text{MeV}^{-1}$$

$$\frac{d\sigma}{d\Omega} \propto \frac{1}{\sin^4\frac{\theta}{2}}$$