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Dislocation Theory of Yielding and Strain Ageing of Iron

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ABSTRACT. A theory of yielding and strain ageing of iron, based on the segregation of carbon atoms to form atmospheres round dislocations, is developed. The form of an atmosphere is discussed and the force needed to release a dislocation from its atmosphere is roughly estimated and found to be reasonable. The dependence on temperature of the yield point is explained on the assumption that thermal fluctuations enable small dislocation loops to break away ; these loops subsequently extend and cause yielding to develop catastrophically by helping other dislocations to break away. The predicted form of the relation between yield point and temperature agrees closely with experiment.

Strain ageing is interpreted as the migration of carbon atoms to free dislocations. The rate of ageing depends upon the concentration of carbon in solution and the estimated initial rate agrees with experiment on the assumption that about 0·003% by weight of carbon is present in solution.

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§1. INTRODUCTION

IT was shown recently (Cottrell 1948) that the relief of stresses round a dislocation by foreign atoms in solid solution can cause an equilibrium "atmosphere" to form, in which large solute atoms are gathered in the dilated part of the dislocation field and small ones in the compressed part. An atom in substitutional solution relieves hydrostatic stresses and its interaction energy with a positive edge dislocation has been given as

$$V = \frac{4}{3} G \epsilon r_a^3 \lambda \frac{1+\nu}{1-\nu} \frac{\sin\alpha}{r}, \qquad \ldots\ldots(1)$$

where r and α are its coordinates relative to the dislocation, α being measured from the slip direction, G and ν are the rigidity modulus and Poisson's ratio respectively, λ is the slip distance in the dislocation, and r_a and $r_a(1+\epsilon)$ are the respective atomic radii of solvent and solute. This formula fails at the origin because the elastic continuum theory is not valid in this region. To a first approximation, the dilatation is zero at the centre of a dislocation and it is reasonable to assume that $V=0$ when $r<r_0$, where $r_0 \sim \lambda$, and is otherwise given by equation (1).

In a dilute atmosphere the concentration of solute will be

$$n(r, \alpha) = n_0 \exp\{-V(r, \alpha)/kT\}, \qquad \ldots\ldots(2)$$

where n_0 is the average concentration expressed as atoms per unit volume, but in a dense atmosphere substantial relaxation of the hydrostatic stresses is possible and an upper (*saturation*) limit of concentration is set by the condition that the local dilatations produced by the solute atoms and the field of the dislocation are everywhere equal and of the same sign.

Dislocations surrounded by atmospheres can produce plastic flow in two ways. If the applied force is small, the dislocations cannot escape from their atmospheres and the solute atoms must migrate with the dislocations. By applying a sufficiently large force, however, the dislocations can be torn from their atmospheres, in which case they then become highly mobile and able to produce rapid flow under smaller forces; a sharp *upper yield point*, followed by flow at a *lower yield point*, occurs. A specimen which is unloaded in this overstrained condition contains free dislocations and, on immediate reloading, shows no yield point, but if it is rested for a sufficient time at not too low a temperature before reloading, solute atoms migrate to the dislocations to form new atmospheres and the yield point returns (*strain ageing*). This type of hardening, caused by the assembly of mobile solute atoms round dislocations, is to be distinguished from the type of hardening considered· previously in which dislocations are impeded by the strain fields of immobile irregularities, such as particles of precipitate. It is to be expected that, under suitable conditions, both types of hardening will be present simultaneously in the same material.

The yield phenomenon is particularly marked in mild steel and soft iron, where it is known to be associated with the presence of small amounts of carbon or nitrogen (Edwards, Phillips and Jones 1940, Snoek 1941b, Low and Gensamer 1944). Accordingly, it has been suggested (Cottrell 1948, Nabarro 1948) that the yield point of iron is due to carbon or nitrogen atmospheres. In the present paper this theory is examined by attempting to estimate the yield point and rate of strain ageing associated with carbon atmospheres; carbon is considered rather than nitrogen because it is more commonly present in iron, but the following discussion can be applied with very little modification to the case of nitrogen.

§2. CARBON ATMOSPHERES IN α-IRON

X-ray studies of martensite show that when carbon dissolves interstitially in α-iron it causes a marked lattice expansion and distorts the structure from cubic to tetragonal. These large strains must lead to a strong interaction between a carbon atom and a dislocation, which is consistent with the pronounced yield point of iron. In the case of a (positive) edge dislocation a carbon atom can relieve hydrostatic stresses by entering the expanded region below the dislocation centre; also, as was first pointed out by Nabarro (1948), an additional interaction can be expected because the non-symmetrical character of the distortion produced by carbon should enable shear stresses to be relieved. For a screw dislocation, Frank (1948) has suggested that the interaction must be due solely to the relief of shear stresses because hydrostatic stresses do not exist round a pure screw dislocation, so that in this case atmosphere formation can only occur with those solutes which, like carbon, produce a non-symmetrical distortion. The conditions for a given solute to be able to produce the yield phenomenon may thus be more stringent than was previously suggested, and a high mobility and a large degree of misfit, although necessary, may not in themselves be sufficient. However, considering the atomic structure in a screw dislocation in an actual metal lattice,

it seems likely that the stress field will contain a small hydrostatic component, so that a weak interaction may exist with atoms which cause only spherically symmetrical strains.

The interstitial nature of the solution of carbon in iron and the non-symmetrical distortion prevent the use of equation (1) for calculating the interaction energy. Since the axis of the tetragonal distortion produced by a carbon atom is confined to one of the $\langle 100 \rangle$ directions of the body-centred cubic cell and a dislocation is confined to a slip-plane, with its direction of displacement lying in a slip direction, the system is not free to arrange that the tetragonal axis coincides with the direction of greatest tension in the dislocation field. For a complete analysis, therefore, the orientation and form of a dislocation in the iron lattice must be determined. The problem is much simplified if only the hydrostatic interaction is considered because hydrostatic stress is a scalar and the question of orientation is then avoided. In order merely to examine whether the anchoring force is sufficient to account for the yield point this simplification is justified and, accordingly, the present treatment deals only with the hydrostatic interaction with an edge dislocation. It is hoped to discuss the shear interaction in a later paper.

The interaction energy is determined by evaluating the work done by forces from the dislocation field when a lattice expansion occurs equivalent to the expansion caused by the introduction of a solute atom (Cottrell 1948). Assuming that each carbon atom alters the edges of the cell in which it is from a, a, a, to c, a_1, a_1, the mean changes in the lattice parameter produced by dissolving n carbon atoms in a crystal of N iron atoms are given by $\Delta c = 2(c-a)n/N$ and $\Delta a = 2(a_1-a)n/N$. Using the lattice parameter measurements on martensite of Lipson and Parker (1944) and extrapolating to $n/N = 0.5$, the volume change caused by the introduction of a carbon atom is given as $\Delta v = 0.78 \times 10^{-23}$ cm^3. Hence, if the stresses due to the dislocation in the surrounding medium remain constant when this local volume change occurs, the interaction energy is

$$V = p\Delta v = -q\Theta\Delta v \qquad \ldots\ldots(3)$$

where p and Θ are the local pressure and dilatation, respectively, caused by the dislocation, and q is the bulk modulus. Using Koehler's (1941) formulae for the stresses round a positive edge dislocation

$$\Theta = -\frac{\lambda}{2\pi}\frac{1-2\nu}{1-\nu}\frac{\sin\alpha}{r},$$

and assuming that $\Theta = 0$ when $r < r_0$ we have

$$V = A\frac{\sin\alpha}{r} = \Delta v \cdot \frac{G\lambda}{3\pi}\frac{1+\nu}{1-\nu}\frac{\sin\alpha}{r} \quad r \geqslant r_0; \qquad V = 0 \quad r < r_0.$$

$$\ldots\ldots(4)$$

The assumption that the dislocation stresses, and hence Θ, remain unchanged when the carbon atom is introduced implies that the stress field is effectively constant over regions large compared with the size of the carbon atom, a reasonable approximation in regions remote from the dislocation centre. Near the centre, however, the dislocation stresses are rapidly varying functions of position and must be greatly altered by the introduction of the carbon atom. In the part of the crystal below the dislocation (i.e. where Θ is positive), which is the region of interest for the case of carbon in iron, the (negative) pressure must relax as the local volume change Δv takes place so that the interaction energy will be less

than is given by equation (4). Very near the dislocation centre equation (4) will grossly overestimate the interaction and can only be used to give a rough indication of the general order of magnitude.

Taking $G = 8 \cdot 28 \times 10^{11}$ dyne.cm^{-2}, $\lambda = 2r_a = 2 \cdot 48 \times 10^{-8}$ cm., and $\nu = 0 \cdot 28$, the constant A in equation (4) is 3×10^{-20} dyne.cm^2. At room temperature $A/kT = 76$ A. and, with such a value, the Maxwell–Boltzmann formula (2) gives impossibly large values of n/n_0 in the lower half crystal within about 10 A. of the dislocation centre. The concentration in this region must be limited by other factors. An assembly of carbon atoms can never completely relieve all the dislocation stresses, but, as the atmosphere builds up, the progressive relief of stress will lead to a condition of saturation in which the addition of more atoms to the atmosphere no longer reduces the energy. An estimate of the number of atoms required for saturation can be made by equating the volume changes produced by the atoms and by the dislocation field. Taking a section consisting of one atomic plane threaded by the dislocation line, the radius R of a semi-circular annulus, in which the total volume change produced by the dislocation is equal to that produced by introducing one carbon atom, is determined by

$$\Delta v = \lambda \int_{r_0}^{R} \int_{\pi}^{2\pi} \Theta r \, dr \, d\alpha.$$

Using the above value for Δv and assuming $r_0 \sim 2$ A., R is about 10 A. It is thus reasonable to assume that in the equilibrium state only one carbon atom per atom plane is contained within 10 A. of the dislocation centre. Since the greatest tensile strain must occur immediately below the dislocation centre and in the atomic plane which constitutes the lower side of the slip plane, the most favourable position for the central carbon atom will be at $\alpha = 3\pi/2$ and $r \sim 2$ A., and it seems likely that the central part of the atmosphere consists of a line of carbon atoms, parallel to the dislocation line, in this position. In the annulus between 10–20 A. the total volume change is again about the same as that produced by one carbon atom per atom plane, but this volume change is spread more uniformly over a much larger region of the crystal and, in this case, the carbon atoms could hardly be arranged along a line. The atmosphere is pictured, therefore, as a central row of carbon atoms, situated just below the dislocation centre, which is surrounded in the lower half-crystal by a dilute distribution of the Maxwell–Boltzmann type.

It should be noticed that an extremely small amount of carbon is required to provide atmospheres for all dislocations in a crystal; the actual value depends,

Table 1

Density of dislocation (lines/cm²)			10^{12}	10^8
% of C by weight absorbed ⎰ Full yield point			$> 10^{-2}$	$> 10^{-6}$
in dislocations for ⎱ No yield point			$< 10^{-3}$	$< 10^{-7}$

of course, on the density of dislocations. Table 1 shows the results of the assumption that a full yield point is obtained when every dislocation has one carbon atom per atom plane and that the yield point is absent at below one-tenth of this amount of carbon.

§3. FORCE REQUIRED TO PULL A DISLOCATION FROM ITS ATMOSPHERE

It is clear that the force required to pull a dislocation from its atmosphere is determined almost completely by the interaction with the carbon atoms of

the central row, since (i) these atoms, being so near to the dislocation, interact strongly with it, and (ii) the restraining force due to their interaction must reach its maximum at a very small displacement of the dislocation from its equilibrium position, and at such a small displacement the contribution to the restraining force from carbon atoms further out will still be comparatively small. From the point of view of calculating the yield point this is unfortunate, since it means that the important contribution of force is from atoms near the dislocation centre, where the interaction energy cannot be accurately evaluated, so that only a rough estimate is possible.

In Figure 1 consider a positive edge dislocation lying along the Z axis (the X axis is the slip direction) and a line of carbon atoms CC', lying parallel to the dislocation, in the most favourable position at a distance ρ below it. The interaction energy per atom plane, when the dislocation has been displaced a distance x in the slip direction is

$$V(x) = -A \sin\phi/r = -A\rho/(x^2+\rho^2) \qquad \ldots\ldots(5)$$

and the corresponding force is $F(x) = \partial V/\partial x = 2A\rho x/(x^2+\rho^2)^2$ which has a maximum of $3\sqrt{3}A/8\rho^2$ dynes per atom plane at $x = \rho/\sqrt{3}$.

The critical force for releasing a dislocation is thus $F_0 = 3\sqrt{3}A/8\lambda\rho^2$ dynes/cm. and, by using the relation $\tau = F/\lambda$ (Mott and Nabarro 1948) where τ is resolved shear stress, this can be converted to an equivalent critical shear stress $\tau_0 = 3\sqrt{3}A/8\lambda^2\rho^2$ dyne.cm^{-2} on the slip plane. Assuming that in the case of a tensile test the active slip planes are inclined at approximately $\pi/4$ to the tensile axis, the critical tensile stress is

$$\sigma_0 = 3\sqrt{3}A/4\lambda^2\rho^2 \text{ dyne.cm}^{-2}. \qquad \ldots\ldots(6)$$

Table 2 gives the observed values of the yield point of annealed ingot iron at various temperatures, as derived from the graphical results of McAdam and Mebs (1943). The yield shows a strong temperature dependence and it is clear

Table 2

Temperature (° K.) ..	85	144	195	303	373
Yield point σ (dynes/cm$^2 \times 10^9$) .	6·07	4·14	2·83	1·45	1·10
σ/σ_0	0·485	0·330	0·225	0·115	0·090

that σ_0 in equation (6) must correspond with the zero point yield stress. If a curve is fitted to these results and extrapolated to $0°$K., the experimental value of σ_0 is about $1·25 \times 10^{10}$ dynes/cm^2. Using the precious values of A and λ the theoretical value of σ_0 can be made to agree with this if ρ is taken to be 7 A., which shows that the restraining force is strong enough to account for the yield point. If ρ is taken to be 2 A., as suggested in §2, the estimated value of σ_0 is an order of magnitude higher; however, it must be remembered that the interaction energy is seriously over-estimated near the dislocation centre and the theory can only give a rough indication of the yield stress. Comparison with the experimental value of σ_0 suggests that a reasonable value of A near the dislocation centre is 3×10^{-21} dyne.cm^2.

Figure 2 shows the variation of interaction energy and stress with position of the dislocation for the cases $\rho = 2$ A., $A = 3 \times 10^{-20}$ dyne.cm^2, and $\rho = 2$ A., $A = 3 \times 10^{-21}$ dyne.cm^2.

§4. EFFECT OF THERMAL FLUCTUATIONS ON THE YIELD POINT

At temperatures above $0°$ K., thermal fluctuations help the external forces to tear a dislocation from its atmosphere and yielding occurs before σ reaches σ_0. When a stress $\sigma = p\sigma_0$ is applied the dislocation moves forward to the position x_1 of stable equilibrium (Figure 2) and can only be released by this stress after it has been taken further forward to the position x_2 of unstable equilibrium, beyond which the external force exceeds the restraining force.

The activation energy per atom plane for the release, by thermal fluctuations, of a dislocation which remains straight is thus the difference $V(x_2) - V(x_1)$ in
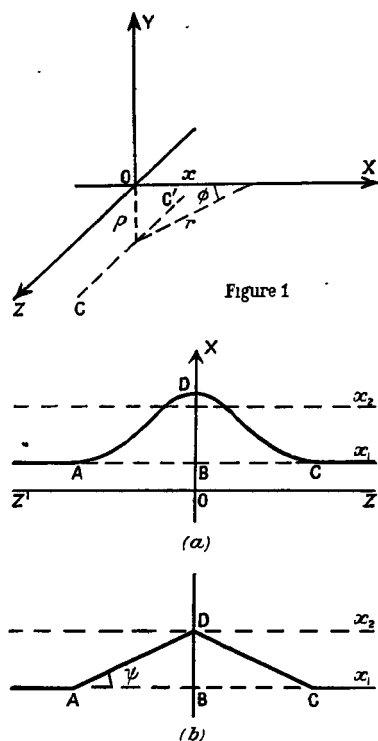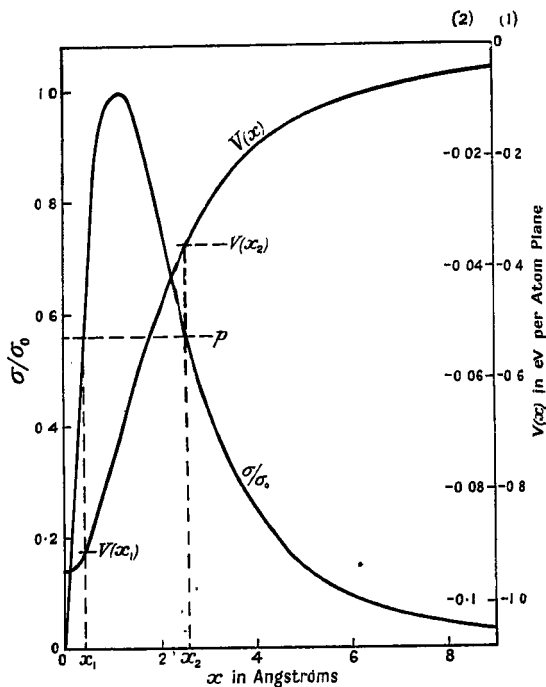
Figure 1

(a)

(b)

Figure 3.

Figure 2. Variation of interaction energy (V) and applied stress (σ/σ_0) with dislocation position.

Scale (1) $A = 3 \times 10^{-20}$ erg.cm. $\rho = 2$ A.
Scale (2) $A = 3 \times 10^{-21}$ erg.cm. $\rho = 2$ A.

interaction energy minus the work done by the external force when the dislocation moves from x_1 to x_2. In calculating the activation energy which thermal fluctuations must supply in order to release the whole dislocation, however, an energy contribution caused by the change in form of the dislocation must be considered.

Let the plane of Figure 3 (a) represent the slip plane. When the applied stress is zero the equilibrium position of the dislocation is ZZ', vertically above the line of the atmosphere. Under the stress σ the dislocation moves forward to the line x_1. To produce yielding it is not necessary for the fluctuation to

move the entire dislocation to x_2 as this would require an unnecessarily large energy. If a loop ADC of the dislocation is thrown forward, and this loop is sufficiently large, the force pulling the looped part forward can exceed the restraining force and the dislocation will then be torn from its atmosphere, the centre of the loop moving forward and the sides moving laterally towards the ends of the dislocation. This process depends upon the ability of the dislocation to be bent and extended. The theory of flexible dislocations has been developed by Mott and Nabarro (1948) who have shown that a dislocation behaves like a one-dimensional soap film; it possesses an energy WL, where L is its length and $W = 1 - 5$ ev. per atom plane, which it tries to minimize by becoming as short as possible. Thus to form the loop ADC, an activation energy has to be supplied which is made up of (i) the energy of interaction with the atmosphere, (ii) the length energy associated with the increase in length, $ADC - ABC$, and (iii) the energy gain due to the work done by the external forces.

To calculate the exact size and shape of the smallest stable loop is difficult, but it is clear that an essential condition is that part of the dislocation has to be brought by the fluctuation to at least the line x_2, since otherwise there will be no part of the looped dislocation on which the net force acts in the forward direction. Thus consider the simpler model shown in Figure 3 (*b*). Here a triangular loop is formed, of depth BD, reaching to the line x_2 at its apex, and the value of the angle ψ is chosen to make the activation energy a minimum. This loop is, of course, too small to be stable; however, the true activation energy cannot be very much greater than is given by this model, for calculation shows that about one-half of the energy of the triangular loop arises from the length energy term and much larger loops could be formed with only a small additional increase in the length of the dislocation.

It is assumed that the interaction energy of an element at any point on the portions AD and DC is the same as that of an element with that point as centre and lying parallel to ZZ′; this assumption is consistent with the results of the analysis by Mott and Nabarro (1948) of the strains round a zigzag dislocation. Accordingly, when an element dz of the dislocation moves forward from x_1 to x, with the formation of the loop, its interaction energy changes by

$$\{V(x) - V(x_1)\}\, dz = -\cot\psi\{V(x) - V(x_1)\}\, dx$$

and the contribution of the interaction with the atmosphere to the activation energy is

$$2\int_{x_2}^{x_1} -\cot\psi\{V(x) - V(x_1)\}\, dx.$$

Taking $V(x) = -A\rho/\lambda(x^2 + \rho^2)$ erg/cm. this becomes

$$-\left[\frac{2A}{\lambda}\{\tan^{-1}(x_2/\rho) - \tan^{-1}(x_1/\rho)\} + 2(x_2 - x_1)V(x_1)\right]\cot\psi. \qquad \ldots\ldots(7)$$

The increase in length energy is

$$W(ADC - ABC) = 2W(x_2 - x_1)(\operatorname{cosec}\psi - \cot\psi) \qquad \ldots\ldots(8)$$

and, if the external force per unit length on the dislocation is F, the work it does when the loop is formed is

$$F(x_2 - x_1)^2 \cot\psi. \qquad \ldots\ldots(9)$$

Subtracting (9) from the sum of (7) and (8) gives the activation energy, U, which is of the form $\alpha \, \text{cosec} \, \psi - \beta \, \cot \psi$; minimizing U with respect to ψ defines ψ as $\cos^{-1}(\beta/\alpha)$ and substituting this value then gives the activation energy in terms of W and σ/σ_0. It can be readily shown that U is of the form

$$U = D(\sigma/\sigma_0)[AE(\sigma/\sigma_0)\{2W\rho - AE(\sigma/\sigma_0)\}]^{\frac{1}{2}},$$

where D and E do not depend on the physical parameters A, W and ρ. Thus, if $2W\rho \gg AE$, $U/(2AW\rho)^{\frac{1}{2}}$ is essentially a function of σ/σ_0 only, so that altering A, W and ρ only alters the curve of U against σ/σ_0 by a scale factor, the form of the curve remaining unchanged. For the range of values of A, W, ρ and σ/σ_0 considered, AE is never greater than about $W\rho/2$ and in most cases is much less than this, so that the form of the activation energy curve is almost independent of the parameters. Figure 4 shows the curve for various values of A, W and ρ.
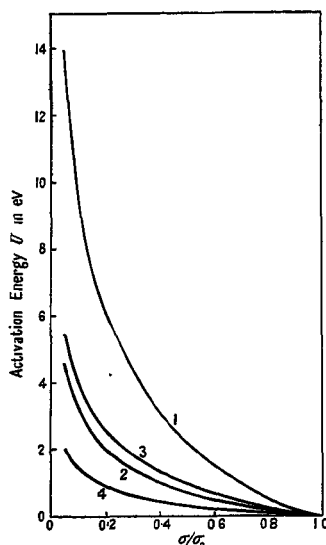

Figure 4. Activation energy for the formation of a loop.

Curve	1	2	3	4
W (ev.) per atom plane	5	5	1	1
A (erg cm.)	3×10^{-20}	3×10^{-21}	3×10^{-20}	3×10^{-21}
ρ(A.)	2	2	2	2

It is to be expected that, as the stress is increased towards σ_0, the first successful attempt to form a loop occurs on a dislocation in a highly stressed region of the specimen. The first stable loop to be formed is subsequently extended by the applied force until the entire dislocation to which it belongs is released from its atmosphere. This dislocation accelerates and enables other dislocations to break away by setting up an elastic disturbance in their midst; in this way yielding develops catastrophically throughout the entire specimen. This description agrees with the well-known observations that yielding starts in highly stressed regions and is propagated, in the form of Luder's bands, from these regions along the specimen. The elementary yielding process—the release of a dislocation from its atmosphere—occurs in both the start and the propagation of yielding; the difference between the upper and lower yield points is that, in the case of the

lower yield, the release of dislocations (at the edges of a Luder's band) is helped by the elastic disturbance caused by nearby plastic flow, whereas in the upper yield this h lp is not available.

With the above picture the effects of temperature and testing rate on the yield point can be examined. When the stress in the neighbourhood of a dislocation lies in the range σ/σ_0 to $(\sigma+d\sigma)/\sigma_0$ the time t_1 before a successful fluctuation occurs is proportional to $\exp\{U(\sigma/\sigma_0)/kT\}$. On the other hand, the time t_2 spent in this stress range is inversely proportional to the rate of stressing, $d\sigma/dt$. Yielding should occur when $ts \sim t_2$, i.e. when the quantity

$$S = (d\sigma/dt)\exp\{-U(\sigma/\sigma_0)/kT\} \qquad \ldots\ldots(10)$$

reaches a characteristic, fixed value. Thus, in experiments at constant testing rate, the yield point should vary with temperature in such a manner that U/kT remains constant, and similarly, at constant temperature U should depend logarithmically on the testing rate. From McAdam and Mebs' results (Table 2) the temperature dependence can be examined. If the experimental and theoretical values of σ/σ_0 are equated at one arbitrarily chosen temperature, the value of U and hence U/kT can be found from Figure 4. Keeping U/kT constant the theoretical σ/σ_0 values can be obtained at various temperatures from Figure 4 and compared with the experimental values. Figure 5 shows the results obtained
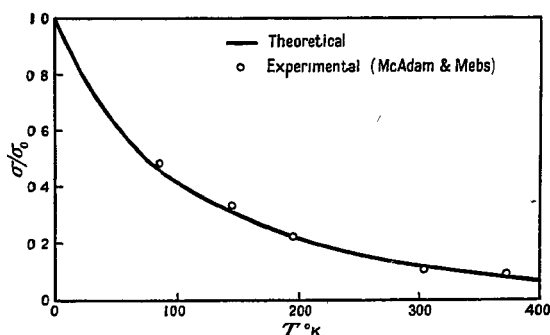

Figure 5. Variation of yield point with temperature.
The theoretical curve is fitted to the experimental value of σ/σ_0 at 195° K.

when the σ/σ_0 values are equated at 195° K. and the activation energy curve 4 of Figure 4 is used. Since the activation energy curves have almost identical forms essentially the same results are obtained when the other curves are used. The good fit obtained over a wide range shows that the activation energy curves

Table 3

Activation energy curve	..	1	2	3	4
W (ev. per atom plane)..	..	5	5	1	1
A (dyne.cm² $\times 10^{-20}$) ..	..	3	0·3	3	0·3
U/kT	..	330	107	140	47

have the correct shape to account for the experimental observations. The values of U/kT given by equating the σ/σ_0 values are shown in Table 3. Since U/kT in practice could hardly exceed about 50, it appears that the most reasonable values of W and A are those of curve 4.

In the present theory, the large temperature dependence of the yield point of iron is explained by the narrowness of the potential energy trough associated with an atmosphere of solute atoms (cf. Figure 2), so that only a small displacement of the dislocation, and hence a small activation energy, is required to overcome the restraining force. In a precipitation-hardened alloy, on the other hand, the existence of wide potential energy troughs causes a large activation energy for moving a dislocation, thus giving a small temperature dependence of yield strength (Mott and Nabarro 1948).

§ 5. RATE OF FORMATION OF THE ATMOSPHERE

In the present theory, strain ageing is regarded as the process whereby carbon atoms in solution migrate towards free dislocations, at rest in an overstrained crystal, to form new atmospheres. To examine this process consider a free dislocation at rest in a region where the solute is initially randomly dispersed, with concentration n_0 atoms per cm^3. Moving by thermal agitation the solute atoms acquire a drift velocity relative to the dislocation (Cottrell 1948) given by

$$\mathbf{v} = -(D/kT)\nabla V \qquad \ldots\ldots(11)$$

where D is the diffusion coefficient and V is given by equation (4). As the atmosphere builds up to equilibrium, both saturation and concentration differences become important in modifying the flow velocity, but in the early stages of atmosphere formation they are negligible. As it is extremely difficult to take account of these factors quantitatively, only the initial stages of ageing, where the drift flow is dominant, will be considered.

Equation (11) shows that the atoms move normally to the equipotential surfaces of V. It is thus convenient to choose as coordinate lines for the problem an orthogonal, curvilinear set consisting of the families made up from the lines of constant V and the lines of flow; the problem is, of course, two-dimensional. Since $V = A\sin\alpha/r$, the first coordinate may be defined by the parameter $\eta = r/\sin\alpha$, so that the equipotential lines are those on which η is constant. The flow lines, conjugate to these, are defined by the coordinate $\xi = r/\cos\alpha$. Using the standard transformation formulae, the elements of length along the ξ and η lines are, respectively,

$$\frac{\xi^2}{\xi^2+\eta^2}\,d\eta \quad \text{and} \quad \frac{\eta^2}{\xi^2+\eta^2}\,d\xi \qquad \ldots\ldots(12)$$

and ∇V is

$$-A\frac{\xi^2+\eta^2}{\xi^2\eta^2}\,\mathbf{i}, \qquad \ldots\ldots(13)$$

where $\mathbf{i}$ is the unit vector in the η direction. As shown in Figure 6, the ξ, η coordinate system consists of circles centred on the X, Y axes and passing through the origin.

A carbon atom will migrate towards the most favourable position below the dislocation, moving along a line of constant ξ and eventually come to rest at a position where its η coordinate has a value, $-\eta_0$, of a few Angströms. From (11), (12) and (13), the time required for an atom to move an elementary distance in the η direction is

$$dt = \frac{kT}{AD}\frac{\xi^4\eta^2}{(\xi^2+\eta^2)^2}\,d\eta,$$

and hence the time to move from $(\xi, +\eta_0)$ to $(\xi, -\eta_0)$ is

$$t = \frac{kT}{AD}\,\xi^4 \left(\int_{+\eta_0}^{+\infty} \{\eta^2\,d\eta/(\xi^2+\eta^2)^2\} + \int_{-\infty}^{-\eta_0} \{\eta^2\,d\eta/(\xi^2+\eta^2)^2\} \right)$$

$$= (kT\xi^3/AD)[\pi/2 - \tan^{-1}(\eta_0/\xi) + \tfrac{1}{2}\sin\{2\tan^{-1}(\eta_0/\xi)\}].$$

If $\eta_0/\xi \ll 1$ this can be approximated to

$$t = \pi k T\xi^3/2AD \qquad\qquad \dots\dots(14)$$

since $\tan^{-1}(\eta_0/\xi) \sim \tfrac{1}{2}\sin\{2\tan^{-1}(\eta_0/\xi)\} \sim \eta_0/\xi$. After the time t, all lines of flow for which $\xi^3 \leqslant 2ADt/\pi kT$ will have been drained of carbon atoms and will take no further part in the ageing process. Taking the smallest value of ξ compatible with the above approximation to be 10 A., and using the value of §2 for A/kT,
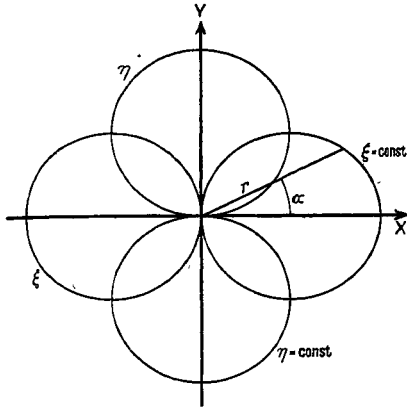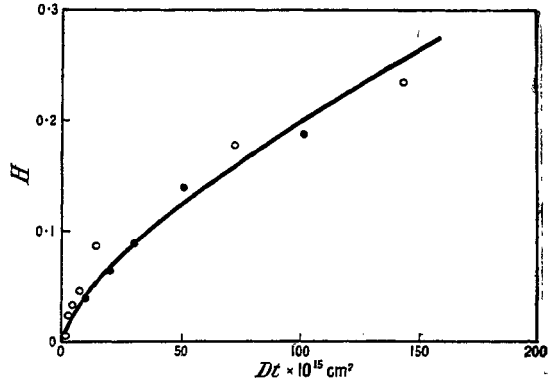

Figure 6.

Figure 7.   Strain ageing of a low-carbon steel.
(From results of Davenport and Bain.)
○ = Ageing at 20° c.
● = Ageing at 40° c.
▬▬ = Equation (16) assuming a carbon concen-
tration of 0 003 % by weight.

it can be seen that this conclusion should be valid for ageing times such that $Dt > 2 \times 10^{-15}$ cm². As Figure 7 shows, strain ageing does not become appreciable until much longer ageing times than this have elapsed.

Consider flow along a filament of width $d\xi$ which is still supplying atoms at the time t. The time for an atom to move an elementary distance along this filament, in the region where $\eta = -\eta_0$, is

$$dt = \frac{kT}{AD}\,\frac{\xi^4\eta_0^2}{(\xi^2+\eta_0^2)^2}\,d\eta$$

and the area of the corresponding element of the filament is

$$\frac{\xi^2\eta_0^2}{(\xi^2+\eta_0^2)^2}\,d\eta\,d\xi = \frac{AD}{kT}\,\frac{d\xi}{\xi^2}\,dt.$$

Using equation (14) to define the limits of integration, the total area which can supply carbon atoms to the dislocation in the interval of time from t to $t+dt$ is

$$\frac{AD}{kT}\left(\int_{+(2ADt/\pi kT)^{\frac{1}{3}}}^{+\infty} \xi^{-2}\,d\xi + \int_{-\infty}^{-(2ADt/\pi kT)^{\frac{1}{3}}} \xi^{-2}\,d\xi \right)dt = 2\left(\frac{\pi}{2}\right)^{\frac{1}{3}}\left(\frac{AD}{kT}\right)^{\frac{2}{3}}\frac{dt}{t^{\frac{1}{3}}}. \quad \dots\dots(15)$$

An important feature is that, in the region of the dislocation field where the relation $V = A \sin \alpha / r$ holds, and outside the regions defined by equation (14) from which the carbon atoms have been removed, the concentration of carbon remains constant during the drift flow. The hydrodynamical equation of continuity, $\nabla . (\rho \mathbf{v}) = -\partial \rho / \partial t$, here takes the form $\nabla . \{ n(D/kT)\nabla V \} = \partial n / \partial t$ where $n = n(r, \alpha, t)$ is the concentration of solute. Since $n = n_0 = \text{constant}$ at $t = 0$, and since $\nabla^2 V = (\partial^2/\partial r^2 + \partial/r\partial r + \partial^2/r^2\partial\alpha^2)(A \sin \alpha / r) = 0$ then $\partial n/\partial t = 0$ and the condition $n = n_0$ is perpetuated.

If unit length of the dislocation is considered, the number of carbon atoms which arrive within a time t is, from equation (15),

$$N(t) = n_0 2 \left(\frac{\pi}{2}\right)^{\frac{1}{3}} \left(\frac{AD}{kT}\right)^{\frac{2}{3}} \int_0^t t^{-\frac{1}{3}} dt = n_0 3 \left(\frac{\pi}{2}\right)^{\frac{1}{3}} \left(\frac{ADt}{kT}\right)^{\frac{2}{3}}.$$

The total number, N_s, of carbon atoms per unit length of the dislocation required to form an atmosphere of one atom per atom plane is $1/\lambda$, so that the degree of formation of the atmosphere at time t is

$$N(t)/N_s = n_0 \lambda 3 (\pi/2)^{\frac{1}{3}} (ADt/kT)^{\frac{2}{3}}. \qquad \ldots \ldots (16)$$

§6. RATE OF STRAIN AGEING

An experimental study of strain ageing has been made by Davenport and Bain (1935) who measured the increase in hardness of a low-carbon steel with time of ageing at various temperatures. Nabarro (1948) has shown that several features of strain ageing, shown by their results, are explicable on the basis of the present theory and provide evidence in its support. To compare the ageing rates found by Davenport and Bain with the rate predicted in the previous section the quantity $H = (H_t - H_0)/(H_m - H_0)$ which is a measure of the degree of strain age-hardening, is first derived from the graphical results given in their Figure 11 (upper block); in this expression, H_t is the hardness after ageing for a time t, H_0 is the value before ageing and H_m is the maximum produced by prolonged ageing. To assemble results from different temperatures in the same diagram it is convenient to plot H against Dt rather than the time t itself, where $D = 5.2 \times 10^{-4} e^{-9000/T}$ (Snoek 1941a, Polder 1945) is the diffusion coefficient of carbon in α-iron, since all the results then lie on the same curve.

Figure 7 shows the experimental and theoretical variation of H with Dt. The theoretical curve has been derived from equation (16) by assuming that $H = N(t)/N_s$, by using the values of λ and A/kT given in §2, and by taking a value of n_0 corresponding to 0.003% carbon by weight. The assumption $H = N(t)/N_s$ cannot be strictly valid since the first atoms to arrive at a dislocation ought to be more effective in anchoring it than those that arrive later. Equation (16) represents the observed ageing behaviour fairly well up to about $N(t)/N_s = 0.3$ but beyond this range it gives a rate of ageing which is too high; this is to be expected since the neglect of the effects of saturation and concentration differences, and the assumption $H = N(t)/N_s$, will each cause the theoretically estimated rate to be too high in the later stages of ageing.

The carbon concentration determined above by fitting the observed and calculated ageing rates ought to correspond to the value for the residual carbon

in solution in α-iron at room temperature. Most of the carbon in irons and steels of higher carbon content is, of course, precipitated as cementite and only that which remains in solution takes part in strain ageing. Experimental evidence for the solubility of carbon in iron at room temperature is meagre, but suggests that the above value of 0·003% is reasonable. Thus Yensen (1924) and Dijkstra (1947) have given the upper limit of the residual solubility as 0·006% and 0·001% respectively.

§7. OTHER AGEING EFFECTS

The migration of solute atoms into dislocations, causing the latter to become anchored, ought to produce several mechanical and physical effects, which can be divided into two groups: (i) those resulting from the anchoring of the dislocations, and (ii) those resulting from the removal of solute atoms from random solution to form atmospheres.

The most striking effect belonging to the first group is, of course, the return of the yield point. In addition, a reduction of certain "anelastic" effects (Zener 1946) is to be expected, and this is confirmed by experiment. A freshly strained specimen possesses a high internal friction and shows slight plasticity at stresses well below the stress at which extensive plastic flow sets in. Zener has attributed these anelastic effects to local readjustments in position of dislocations in the slip bands of the worked material; this is reasonable since small displacements of free dislocations in a Taylor (1934) array can occur under small stresses, even though a high stress is required to overcome the interaction forces of neighbouring dislocations and so produce extensive flow. Low-temperature annealing, for times which would allow strain ageing to occur, greatly reduces this source of internal friction and causes a true elastic range to return, which is consistent with the suggestion that dislocations become anchored by the ageing process.

Ageing effects belonging to the second group should be provided by properties which are sensitive to the amount of carbon in random solution in α-iron. Thus we may expect both electrical resistance and magnetic ageing effects. Recent work by Mr. A. T. Churchman and one of us (A. H. C.) has shown that a small decrease in electrical resistance of the right order of magnitude does in fact occur during strain ageing. Another effect has been observed by Snoek (1941a), who found that cold working and low-temperature annealing caused the elimination of a certain type of elastic after-effect known to be caused by mobile carbon (or nitrogen) atoms in solution in α-iron. Snoek suggested that this treatment probably caused the solute atoms to become bound by internal stress fields created by the cold working. If we interpret the sources of the stress fields as dislocations, this suggestion becomes quite specific, since then the solute atoms cannot produce internal friction by Snoek's mechanism because they are assembled in dislocations.

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Investigations on the Binding Energy of Heavy Nuclei

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ABSTRACT. The binding energy of a heavy nucleus is one of the quantities which a satisfactory nuclear force theory ought to predict correctly. Approximate calculations of this binding energy are performed, taking as nuclear force the static interaction of Møller and Rosenfeld. The parameters appearing in the interaction have been fixed by data on light nuclei.

The Fermi-gas model is taken as basis for the calculations, a first solution being obtained by a first-order perturbation (or variation) method. For refinement, a second-order perturbation calculation is made, and Svartholm's variation-iteration method is attempted. The first-order results yield about 10% of the required energy, and the second-order about 40%; a reasonable prediction of the size of the nucleus is obtained. The reliability of the perturbation results is uncertain ; Svartholm's method does not appear to be well suited to the investigation of heavy nuclei.

Two alternative modifications are made to the nuclear force to make it yield the correct binding energy according to the perturbation calculation.

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§ 1. INTRODUCTION

IN principle, one of the most important tests of any proposed nuclear force is its compatibility with the observed binding energies of heavy nuclei. By a semi-empirical procedure these can be approximately expressed by a simple function of atomic number Z and mass number A, containing a few parameters. One of these parameters, the coefficient of an energy contribution simply proportional to A, can be taken as representing the major effect of the nuclear forces, and a theory might be tested by the value which it predicts for this parameter. Of the theories which have been put forward, that of Møller and Rosenfeld (Møller and Rosenfeld 1940, Rosenfeld 1945, 1948) is capable of accounting for many of