LECTURE-2

MATHEMATICS -2 (15B11MA211)

DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENT-PARTICULAR INTEGRAL

Differential Equation CO[C105.5]

Table of Contents

R.K. Jain and S.R.K. Iyenger, "Advanced Engineering Mathematics" fifth edition, Narosa publishing house, 2016.

Particular Integral

Consider the linear non homogeneous differential equation with constant coefficients

$L(y) = (a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\, a_n)y = F(D)y = r(x)$

Where $F(D) = (a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\, a_0)$ $and\ a_0, a_1,$ $\ldots\ldots\ldots\, a_n$ are constants.

Then Particular integral is

$$\boldsymbol{y_p(\mathbf{x}) = [\mathbf{F(D)}]^{-1}\, r(x) = \frac{1}{F(D)} r(x),}$$

Rules for Finding the Particular Integral

Case 1. When $r(x)= \boldsymbol{e^{\alpha x}}$

If $r(x)= \boldsymbol{e^{\alpha x}}$; then the particular Integral is given as

$$y_p(\mathrm{x}) = \frac{\mathbf{1}}{\boldsymbol{F(D)}}\boldsymbol{e^{\alpha x}} = \frac{\mathbf{1}}{\boldsymbol{F(\alpha)}}\boldsymbol{e^{\alpha x}}, \text{ provided } \boldsymbol{F(\alpha\,)} \neq \mathbf{0}$$

$De^{\alpha x}= \alpha e^{\alpha x}$, $D^2\ e^{\alpha x} =\alpha^2 e^{\alpha x}$ .... $D^n\ e^{\alpha x} =\alpha^n e^{\alpha x}$

$\Longrightarrow \boldsymbol{F(D)\ e^{\alpha x}}= \boldsymbol{F}(\alpha)\ \boldsymbol{e^{\alpha x}}$

Rules for Finding the PI Cont….

Operating $[\mathbf{F(D)}]^{-1}$ on both sides of the above equation

and dividing by F(α), we get

PI = y_p(x) = $\boldsymbol{\frac{1}{F(D)} e^{\alpha x} = \frac{1}{F(\alpha)} e^{\alpha x}}$,

provided $\boldsymbol{F(\alpha\,) \neq 0}$

Example: Find the particular integral of the following differential equation $y'' - 2y' - 3y = 3e^{2x}$

Solution: Given differential equation is $(D^2 - 2D - 3)y = 3e^{2x}$ a non homogeneous linear differential equation. The particular integral is given as

$$\text{PI} = y_p(x) = \frac{1}{F(D)} e^{2x} = \frac{1}{(D^2 - 2D - 3)} 3e^{2x} = -e^{2x}$$

If $\boldsymbol{F(\alpha) = 0}$ then above rule fails and we have

PI = y_p(x) = $\boldsymbol{\frac{1}{F'(D)} e^{\alpha x} = x \frac{1}{F'(\alpha)} e^{\alpha x}}$,

provided $\boldsymbol{F'(\alpha) \neq 0}$

If $\boldsymbol{F'(\alpha) = 0}$ then we have

PI = y_p(x) = $\boldsymbol{\frac{1}{F''(D)} e^{\alpha x} = x^2 \frac{1}{F''(\alpha)} e^{\alpha x}}$,

provided $\boldsymbol{F''(\alpha) \neq 0}$

Example: Find the general solution of the following differential equation

$$y'' - 2y' + y = e^{x}$$

Solution: Given differential equation is $(D^2 - 2D + 1)y = e^{x}$

a non homogeneous linear differential equation.

C.F.= $(C_1 + C_2 x)e^{x}$

The particular integral is given as

$$\text{PI} = y_p(x) = \frac{1}{(D^2 - 2D + 1)} e^{x} = \frac{x}{2D - 2)} e^{x} = \frac{x^2}{2} e^{x}$$

Solution $y = (C_1 + C_2 x)e^{x} + \frac{x^2}{2} e^{x}$

Case 2. When $r(x) = sin(\alpha x+\beta)$ or $cos(\alpha x+\beta)$

then the particular Integral is given as

$$y_p(\mathrm{x}) = \frac{\mathbf{1}}{\boldsymbol{F}(D^2)} sin(\alpha x+\beta) \text{ or } cos(\alpha x+\beta) =$$

$$\frac{\mathbf{1}}{\boldsymbol{F}(-\alpha^2)} sin(\alpha x+\beta) \text{ or } cos(\alpha x+\beta)$$

Replace D^2 by $-\alpha^2$, provided $F(-\alpha^2) \neq 0$

If $F(-\alpha^2) = 0$ then above rule fails and we have

PI = y_p(x) =$\frac{\mathbf{1}}{\boldsymbol{F\prime}(D^2)} sin(\alpha x + \beta)/\, cos(\alpha x + \beta) = \boldsymbol{x} \frac{\mathbf{1}}{\boldsymbol{F}(-\alpha^2)} sin(\alpha x + \beta)/\, cos(\alpha x + \beta)$, provided $F'(-\alpha^2) \neq 0$

If$F'(-\alpha^2) = 0$ then PI = y_p(x) = $\frac{\mathbf{1}}{\boldsymbol{F}''(D^2)} sin(\alpha x + \beta)/\, cos(\alpha x + \beta)$

$$= \boldsymbol{x^2} \frac{\mathbf{1}}{\boldsymbol{F}''(-\alpha^2)} sin(\alpha x + \beta)/\, cos(\alpha x + \beta),$$

provided $\boldsymbol{F}''(-\alpha^2) \neq \mathbf{0}$

Example: Find the particular integral of the following differential equation

$$(D^2+4)y = sinx + cos2x$$

Solution: The particular integral is given as

$$\text{PI} = y_p(\text{x}) = \frac{1}{(D^2+4)}(sinx + cos2x)$$

$$= \frac{1}{(-1+4)} sinx + \frac{1}{(D^2+4)} cos2x$$

$$= \frac{1}{3} sinx + \frac{x}{2D} cos2x$$

$$= \frac{1}{3} sinx + \frac{x}{4} sin2x$$

Rules for Finding the Particular Integral

❑Case 3: When $r(X) = X^{\alpha}, \alpha > 0$

The particular Integral of $F(D)y = \boldsymbol{X^{\alpha}, \alpha > 0}$, is given as

$$\boldsymbol{y_p(\mathrm{x}) = [\mathrm{F(D)}]^{-1}\, X^{\alpha}} \,,$$

Expand $[\mathbf{F(D)}]^{-1}$ as an infinite series in ascending powers of D

and then operate on $\boldsymbol{X^{\alpha}}$.

Example: Find the particular Integral of the following differential equation:

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4.$$

Solution: Given differential equation is $(D^2 + D)y = x^2 + 2x + 4$. The particular integral is given as

$$y_p(x) = \frac{1}{F(D)}(x^2 + 2x + 4) = \frac{1}{(D^2+D)}(x^2 + 2x + 4)$$

$$= \frac{1}{D(D+1)}(x^2 + 2x + 4)$$

$$= \frac{1}{(D)}(1 + D)^{-1}(x^2 + 2x + 4)$$

$$= \frac{1}{D}(1 - D + D^2 - \cdots)(x^2 + 2x + 4)$$

$$= \frac{1}{D}[(x^2 + 2x + 4) - (2x+2) + 2]$$

$$= \frac{x^3}{3} + 4x$$

Rules for Finding the Particular Integral (Contd..)

❑Case 4: When $r(X) = e^{\alpha X} h(X), \alpha > 0$

The particular Integral of $F(D)y = e^{\alpha X} h(X), \alpha > 0$, is given as

$$y_p(\mathrm{x}) = [\mathrm{F(D)}]^{-1} e^{\alpha X} h(X)$$

$$= e^{\alpha X}[\mathrm{F}(\mathrm{D} + \alpha)]^{-1} h(X)$$

$$= e^{\alpha X} \frac{1}{\mathrm{F}(\mathrm{D}+\alpha)} h(X)$$

Example: Find the general solution of the following differential equation: $16\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + y = 48xe^{\left(-\frac{x}{4}\right)}$

Solution: **Given differential equation is**

$$(16D^2 + 8D + 1)y = 48xe^{\left(-\frac{x}{4}\right)}.$$

The auxiliary equation is given as

$$16m^2 + 8m + 1 = 0.$$

On solving, $m = -\frac{1}{4}, -\frac{1}{4}.$

Hence the complimentary function is given by CF = (A + Bx) $e^{\left(-\frac{x}{4}\right)}$**.**

Now the particular integral is given as

$$y_p(x) = \frac{1}{F(D)} 48xe^{\left(-\frac{x}{4}\right)} = 48 \frac{1}{(16D^2 + 8D + 1)} \left(xe^{\left(-\frac{x}{4}\right)}\right)$$

$$= 48e^{\left(-\frac{x}{4}\right)} \frac{1}{\left(16(D - \frac{1}{4})^2 + 8(D - \frac{1}{4}) + 1\right)} (x)$$

$$= 48e^{\left(-\frac{x}{4}\right)} \frac{1}{(16D^2)} (x) \quad = 3e^{\left(-\frac{x}{4}\right)} \frac{x^3}{6}$$

The general solution is therefore,

y(x) = CF+PI

$$= (A + B\,x)\, e^{\left(-\frac{x}{4}\right)} + \ 3e^{\left(-\frac{x}{4}\right)} \frac{x^3}{6}$$

Rules for Finding the Particular Integral (Contd..)

❑**Case 5: When $r(X) = X.h(X)$,**

The particular Integral of $F(D)y = X.h(X)$, is given as

$$y_p(\mathrm{x}) = [\mathrm{F(D)}]^{-1}X.h(X)$$

$$= X\frac{1}{\mathrm{F(D)}}h(X) + \frac{d}{dD}\left(\frac{1}{\mathrm{F(D)}}\right)h(X)$$

$$= X\frac{1}{\mathrm{F(D)}}h(X) - \left(\frac{F'(D)}{[F(D)]^2}\right)h(X)$$

Example: Solve the differential equation $y'' - y = x\sin x$

Solution: Given differential equation is

$$(D^2 - 1)y = x\,sin\,x.$$

The auxiliary equation is given as

$$m^2 - 1 = 0.$$

On solving, $m = -1,\ 1.$

Hence the complimentary function is given by

$$CF = Ae^x + Be^{-x}.$$

Now the particular integral is given as

$$y_p(x) = \frac{1}{F(D)}\, x \sin x = \frac{1}{D^2 - 1}(x \sin x)$$

$$= x\, \frac{1}{(D^2-1)} \sin x - \left(\frac{2D}{[D^2-1]^2}\right) \sin x$$

$$= x\, \frac{1}{-2} \sin x - \left(\frac{2D}{4}\right) \sin x$$

$$= -x\, \frac{\sin x}{2} - \left(\frac{2 \cos x}{4}\right)$$

The general solution is therefore, y(x)= CF+PI

$$y(x) = Ae^{x} + Be^{-x} + \left(-x\, \frac{\sin x}{2} - \left(\frac{2 \cos x}{4}\right)\right)$$

General Rule for Particular Integral

The particular Integral of the differential equation

$F(D)y = (D - \alpha)y = r(x)$ is given as

$$y_p(\mathbf{x}) = \frac{1}{(D-\alpha)} r(x)$$

$$= e^{\alpha x} \int e^{-\alpha x} \, r(x) dx$$

Example: Find the particular integral of the differential equation $y'' + y = sec\ x$

Solution: Given differential equation is $(D^2+1)y = sec\ x$.

The particular integral is given as

$$y_p(x) = \frac{1}{(D^2+1)}\ sec\ x = \frac{1}{(D-i)(D+i)}(sec\ x)$$

$$= \frac{1}{2i}\left[\frac{1}{(D-i)} - \frac{1}{(D+i)}\right] sec\ x$$

Now, $\frac{1}{(D-i)}\ sec\ x = e^{ix}\int e^{-ix}\ sec\ x\ dx$

$$= e^{ix}\int (cos\ x - isin\ x)\ sec\ x\ dx$$

$$= e^{ix}\int (1 - i\ tan\ x)\ dx$$

$$= e^{ix}(x + i\ log\ (cos\ x))$$

Similarly, $\frac{1}{(D+i)}\ sec\ x = e^{-ix}\int e^{ix}\ sec\ x\ dx = e^{-ix}(x - i\ log\ (cos\ x))$

Therefore,

$$\boldsymbol{y_p(x) = \frac{1}{(D^2+1)}\ sec\ x = \frac{1}{2i}\left[\frac{1}{(D-i)} - \frac{1}{(D+i)}\right] sec\ x}$$

$$= \boldsymbol{\frac{1}{2i} e^{ix}(x + i\ log\ (cos\ x))} \text{ - } \boldsymbol{\frac{1}{2i} e^{-ix}(x - i\ log\ (cos\ x))}$$

$$= \quad \boldsymbol{x\ sin\ x + cos\ log\ (cos\ x)}$$

Practice Questions

Q1. Solve the differential equation $(D-2)^2 \mathrm{y} = 8(e^{2x} + sin2x)$

Q2. Find the particular integral of the following equation

$$(D^3+1)y = \cos(2x-1).$$

Practice Questions

1. **Solve the differential equation** $y'' - 2y' + y = x\, e^{x} \sin x$.

   (Ans. $y(x) = (A + Bx)e^{x} - e^{x}(x \sin x + 2 \cos x)$)

2. **Find the particular integral of the following** $\boldsymbol{differential\ equation}$

   $y'' - 3y' + 2y = x\, e^{3x} + \sin 2x$.

1. (Ans. P.I.= $e^{3x}\left(\frac{x}{2} - \frac{3}{4}\right) + \frac{1}{20}(3 \cos 2x - \sin 2x)$)

3. **Solve** $(D^2 - 1)y = x \sin 3x$

1. (Ans. $y(x) = (Ae^{x} + Be^{-x}) - \frac{1}{10}\left(x \sin 3x + \frac{3}{5} \cos 3x\right)$)

THANK YOU