Department of Mathematics

Odd Semester 2016-17

Probability and Random Processes **15B11MA301**

Probability Theory and Random Processes **10B11MA411**

Tutorial Sheet 5 **B.Tech. Core**

Continuous Distributions

1. A man and a Woman agree to meet at a certain place between 10 a.m. and 11 a.m. They agree that the one arriving first will have to wait 15 minutes for the other to arrive. Assuming that the arrival times are independent and uniformly distributed, find the probability that they meet. (Ans. 7/16)

2. If the random variable a is uniformly distributed in the interval (1, 7), what is the probability that the roots of the equation $x^2+2ax+(2a+3)=0$ are real. (Ans. 2/3)

3. A straight line of length 4 units is given. Two points are taken at random on this line. Find the probability that the distance between them is greater than 3 units. (Ans. 1/16)

4. The daily consumption of milk in excess of 20000 gallons is approximately exponentially distributed with λ = 1/3000. The city has a daily stock of 35000 gallons. What is the probability that of 2 days selected at random, the stock is insufficient for both days. (Ans. e^{-10})

5. The length of the shower on a tropical island during rainy season has an exponential distribution with parameter 2, time being measured in minutes. What is the probability that a shower will last more than 3 minutes? If a shower has already lasted for 2 minutes, what is the probability that it will last for at least one more minute? (Ans. (i) 0.0025, (ii) 0.1353)

6. Suppose that X has an exponential distribution with parameter λ. Compute the probability that X exceeds twice its expected value. (Ans. $1/e^2$)

7. If the service life, in hours, of a semiconductor is a RV having a Weibull distribution with the parameters α = 0.0375 and β = 0.55, (i) How long can such a semiconductor be expected to last? (ii) What is the probability that such a semiconductor will still be in operating condition after 4000h? (Ans.0.0276)

8. If the life in years of a certain type of taxi has a Weibull distribution with the parameter $\beta=2$, find the value of the parameter α, given the probability that the life of the taxi exceeds 6 years is $e^{-0.36}$. For these value of α and β, find the mean and variance. (Ans. α =0.01, $E(X)=5\sqrt{\pi}, Var(X)=100(1-\pi/4)$)

9. The life of a bacteria (in days) has a gamma distribution with shape parameter k=2 and scale parameter λ =3. What is the probability that this bacteria will live between 1 to 2 months?

10. The daily consumption of milk in a town in excess of 2000L is approximately distributed as an Erlang variate with parameters λ = 1/1000 and k = 2. The town has a daily stock of 3000L. What is the probability that the stock is insufficient on a particular day? (Ans. $2/e$)

11. In a normal population with mean 12 and standard deviation 4, it is known that 750 observations exceed 15. Find the total number of observations in the population. (Ans. $\approx$ 3310)

12. At a certain examination 10% of the students who appeared for the paper in Advanced Mathematics got less than 30 marks and 97% of the students got less than 62 marks. Assuming the distribution is normal, find the mean and the SD of the distribution. (Ans. $\mu=43.04, \sigma=10.03$)

13. It is given that X and Y are independent normal variates and $X \square N(1,4), Y \square N(3,16)$. Find the value of K such that $P(2X+Y\le K)=P(4X-Y\ge 2K)$. (Ans. $K=(5\sqrt{5}+\sqrt{2})/(\sqrt{5}+2\sqrt{2})$)

TUT 5

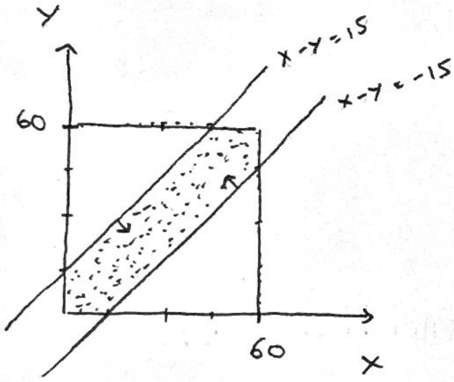
Q1] 10 am - 11 am = 60 mins

Let X : time of arrival of A

Y : time of arrival of B

So, $|X-Y| \le 15$ min

$X, Y \le 60$ min



Area of shaded region $= 60^2 - 45^2$

$= 1575$

Total Area $= 60^2 = 3600$

$P = \dfrac{\text{Shaded Area}}{\text{Total Area}} = \dfrac{1575}{3600} = 7/16$

Q2] X lies b/w (1,7)

$x^2 + 2ax + (2a+3) = 0$

For real roots $D \ge 0$

$B^2 - 4AC \ge 0$

$(2a)^2 - 4(1)(2a+3) \ge 0$

$4(a)^2 - 8a - 12 \ge 0$

$4(a+1)(a-3) \ge 0$

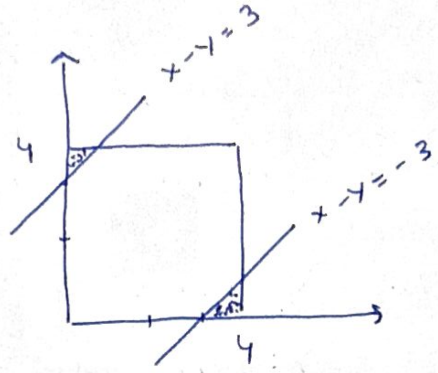
$a \ge -1 \qquad a \ge 3$

$\Rightarrow$ a lies b/w (3,7)

$P = \dfrac{7-3}{7-1} = \dfrac{4}{6} = \dfrac{2}{3}$

Q3] X : point 1

Y : point 2



constraints :

$X, Y \leq 4$

$|X-Y| > 3$

$$P = \frac{\text{Area of Shaded Region}}{\text{Total Area}} = \frac{1^2}{4^2} = \frac{1}{16}$$

Q4] $\lambda = 1/3000$

$$\left.\begin{aligned} pdf &= \lambda e^{-\lambda x} = f(x) \\ cdf &= 1 - e^{-\lambda x} = F(x) \end{aligned}\right\} \text{exponential dist.}$$

Y follows exp dist with $\lambda = 1/3000$

$Y = X - 20,000$

↳ consumption of milk per day

$P(X > 35000) = \underline{P(Y > 15000)} = 1 - P(Y \leq 15,000)$ → $e^{-\lambda k} = e^{-5}$

$= 1 - F(15,000) = e^{-\frac{1}{3000} \times 15000} = e^{-5}$

↑ for one day

⇒ for 2 days

$$P = e^{-5} \times e^{-5} = e^{-10}$$

Q5] $X \sim \exp(2)$

X: length of shower in mins.

$$P(X>3) = 1 - P(X \le 3) = 1 - \left(1 - e^{-(2)(3)}\right)$$

$$= e^{-6} = \underline{\underline{0.002479}}$$

$$P\left(\frac{X>3}{X>2}\right) = P(X>1) = 1 - P(X \le 1)$$

$$= 1 - \left(1 - e^{-2\times 1}\right) = e^{-2} = 0.1353$$

this is due to the fact that exponential distribution is memory less is nature

Q6] $X \sim \exp(\lambda) \Rightarrow E[X] = \frac{1}{\lambda}$

$$P\left[X > 2\left(\frac{1}{\lambda}\right)\right] = 1 - P\left(X \le \frac{2}{\lambda}\right) = e^{-\frac{2}{\lambda}\times\lambda}$$

$$= e^{-2} = \underline{\underline{0.1353}}$$

Q7] $X \sim \text{weibull}(\underset{\alpha}{0.0375}, \underset{\beta}{0.55})$

$$\text{pdf} = \alpha \beta x^{\beta-1} e^{-\alpha x^{\beta}} \qquad E[X] = \frac{1}{(\alpha)^{1/\beta}} \Gamma\left(\frac{1}{\beta}+1\right)$$

a) So, $E[X] = \frac{1}{(0.0375)^{1/0.55}} \Gamma\left(\frac{1}{0.55}+1\right) = 391.33\ \Gamma(2.818)$

$$= 391.33 \times 1.7021 = \underline{\underline{666.09}} \text{ hours}$$

b) $P(X > 4000 \text{ hours}) = 1 - P(X \le 4000)$

$$= 1 - \underbrace{(1 - e^{-\alpha x^{\beta}})}_{\text{cdf of weibull dist.}}$$

$$= 1 - (1 - \exp(-0.0375 \times 4000^{0.55}))$$

$$= e^{-0.0375 \times 4000^{0.55}} = e^{-3.5905}$$

$$= \underline{\underline{0.02758}}$$

Q8) $X \sim$ weibull $\alpha = ?$ $\beta = 2$

X: lifetime of taxi in years

$P(X > 6) = e^{-0.36} = 1 - P(X \le 6)$

$$= 1 - [1 - e^{-\alpha \cdot 6^2}]$$

$$e^{-0.36} = e^{-\alpha \cdot 36}$$

$$0.36 = \alpha \cdot 36$$

$$\boxed{\alpha = \frac{1}{100}}$$

$E[X^n] = \dfrac{1}{(\alpha)^{n/\beta}} \Gamma\left(\dfrac{n}{\beta} + 1\right)$ So, $E[X] = \dfrac{1}{\left(\frac{1}{100}\right)^{1/2}} \Gamma\left(\dfrac{1}{2} + 1\right) = \sqrt{100} \times \dfrac{1}{2} \times \sqrt{\pi}$

$$= \underline{\underline{5\sqrt{\pi}}}$$

$E[X^2] = (100)^{2/2}\, \Gamma 2 = 100$ $\quad$ $Var(X) = 100 - (5\sqrt{\pi})^2$

$$= \underline{\underline{100 - 25\pi}}$$

Q9] x: life of Bacteria in days

$x \sim$ gamma $(n=2, \lambda=3)$

$$P(1\times 30 < X < 2\times 30) = P(30 < X < 60)$$

$$P = \int_{30}^{60} \frac{e^{-3x} \cdot x^1 \cdot 3^2}{\Gamma(2)} = 9\int_{30}^{60} e^{-3x} \cdot x$$

$$= 9\left[\left(\frac{e^{-3x}}{-3}\right)(x) - \left(\frac{e^{-3x}}{9}\right)(1)\right]_{30}^{60}$$

$$= 9\left[\left(\frac{60e^{-180}}{-3} - \frac{e^{-180}}{9}\right) - \left(\frac{30e^{-90}}{-3} - \frac{e^{-90}}{9}\right)\right]$$

$$= 9\left[e^{-180}\left(-20 - \frac{1}{9}\right) - e^{-90}\left(-10 - \frac{1}{9}\right)\right]$$

$$= -1.215 \times 10^{-76} + 7.456 \times 10^{-38}$$ ✓

~ 0, so bacteria is most likely to be dead by 1 month

Q10]

X : daily ~~stock~~ consumption of milk in L

Y : daily consumption of milk in excess of 2000 L

$$Y = X - 2000$$

$$Y \sim \text{gamma}\left(k = 2,\ \lambda = \frac{1}{1000}\right)$$

$$P(X > 3000) = P(Y > 1000) = 1 - P(Y \le 1000)$$

$$= \int_{1000}^{\infty} \frac{e^{-\frac{1}{1000} x} \cdot x^1 \left(\frac{1}{1000}\right)^2}{\Gamma(2)}$$

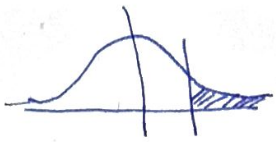
$$= \left(\frac{1}{1000}\right)^2 \left[\frac{e^{-\lambda x}}{(-\lambda)} \cdot x - \frac{e^{-\lambda x}}{\lambda^2}\right]_{1000}^{\infty}$$

$$= \left(\frac{1}{1000}\right)^2 \left[\frac{e^{-1000\lambda}}{\lambda^2} + \frac{1000\, e^{-1000\lambda}}{\cancel{1000}\ \lambda}\right]$$

$$= \left(\frac{1}{1000}\right)^2 \left[\frac{e^{-1}}{\left(\frac{1}{1000}\right)^2} + (1000)^2 e^{-1}\right]$$

$$= e^{-1} + e^{-1} = \underline{\underline{2/e}}$$

Q ii] $X \sim \text{Normal}(\mu = 12, \sigma = 4)$



$$P(X > 15) = P(Z > 0.75) = \cancel{1 - P(Z \le 0.75)}$$

$$= 0.5 - P(Z \le 0.75) = 0.5 - 0.2734$$

$$= 0.2266$$

$$\text{Total Observations} = \frac{750}{0.2266} = 3309.79$$

$$\approx \underline{\underline{3310}} \text{ people}$$

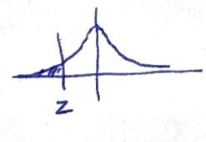
Q12] $X \sim$ normal distribution

X : no. of marks.

$10\% \; (0.1) \longrightarrow \; < 30M$

$97\% (0.97) \longrightarrow \; < 62M$

$$P(x < 30) = 0.1 = P\left(z < \frac{30-\mu}{\sigma}\right)$$



looking for area $(0.5 - 0.1)$

$$\frac{30-\mu}{\sigma} = -1.28 \quad - (1)$$

$$P(x < 62) = 0.97 = P\left(z < \frac{62-\mu}{\sigma}\right)$$

looking for area $(0.97 - 0.5)$

$$\frac{62-\mu}{\sigma} = 1.88 \quad - (2)$$

Solving eq^n's

$$\sigma = \frac{62-30}{1.88+1.28} = \underline{\underline{10.12}}$$

$$\mu = 30 + (1.28)(10.12) = \underline{\underline{42.953}}$$

Q13)

$$X \sim N(1, 2^2)$$

$$Y \sim N(3, 4^2)$$

$$2X + Y \sim N(5, 4\sqrt{2})$$

$$4X - Y \sim N(1, 4\sqrt{5})$$

$$P(2X+Y \leq K) = P(4X - Y \geq 2K)$$

$$\frac{K-5}{\cancel{4}\sqrt{2}} = \frac{1-2K}{\cancel{4}\sqrt{5}}$$

$$\sqrt{5}K - 5\sqrt{5} = \sqrt{2} - 2\sqrt{2}K$$

$$K(\sqrt{5} + 2\sqrt{2}) = \sqrt{2} + 5\sqrt{5}$$

$$K = \frac{\sqrt{2} + 5\sqrt{5}}{\sqrt{5} + 2\sqrt{2}}$$