Probability and Random Processes (15B11MA301)

Lecture-37

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Topics to Be covered

Power Spectral Density Function

- **Examples**
- **Practice questions**

Fourier Transform of Some Important Functions

	$X(\tau)$	$X(\omega) = F[X(\tau)]$
1.	$\alpha\delta(\tau)$	α
2.	$\frac{\alpha}{2\pi}(1)$	$\alpha\delta(\omega)$, $[F(\alpha) = 2\pi\alpha S(\omega)]$
3.	$u(\tau)$	$\pi\delta(\omega) + \frac{1}{i\omega}$
4.	$e^{-i\omega_0\tau}$	$2\pi\delta(\omega + \omega_0)$
5.	$e^{i\omega_0\tau}$	$2\pi\delta(\omega - \omega_0)$
6.	$\cos\omega_0\tau$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
7.	$\sin\omega_0\tau$	$-\pi i[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
8.	$e^{-\alpha\tau}u(\tau)$, $\alpha > 0$	$\frac{1}{\alpha + i\omega}$
9.	$\tau e^{-\alpha\tau}u(\tau)$, $\alpha > 0$	$\frac{1}{(\alpha + i\omega)^2}$
10.	$\tau^2 e^{-\alpha\tau}u(\tau)$, $\alpha > 0$	$\frac{2}{(\alpha + i\omega)^3}$
11.	$e^{-\alpha\|\tau\|}$, $\alpha > 0$	$\frac{2\alpha}{\alpha^2 + \omega^2}$
12.	$e^{-\alpha\tau^2}$	$\sqrt{\frac{\pi}{\alpha}}e^{\frac{-\omega^2}{4\alpha}}$

Example 1

Find Fourier Transform of $cosp\tau$ and $sinp\tau$.

Solution:

$$F^{-1}[\delta(\omega+p)+\delta(\omega-p)]=\frac{1}{2\pi}\int_{-\infty}^{\infty}[\delta(\omega+p)+\delta(\omega-p)]e^{i\omega\tau}d\omega$$

$$=\frac{1}{2\pi}\left[\int_{-\infty}^{\infty}\delta(\omega+p)e^{i\omega\tau}d\omega+\int_{-\infty}^{\infty}\delta(\omega-p)e^{i\omega\tau}d\omega\right]$$

$$=\frac{1}{2\pi}\left[e^{i\tau(-p)}+e^{+i\tau p}\right]$$

$$=\frac{1}{\pi}\left(\frac{e^{i\tau p}+e^{-i\tau p}}{2}\right)=\frac{\cos p\tau}{\pi}$$

$$\pi F^{-1}[\delta(\omega+p)+\delta(\omega-p)]=\cos p\tau$$

$$F^{-1}\{\pi[\delta(\omega+p)+\delta(\omega-p)]\}=\cos p\tau$$

$$\therefore \quad F(\cos p\tau)=\pi[\delta(\omega+p)+\delta(\omega-p)]$$

$$F^{-1}[\delta(\omega+p)-\delta(\omega-p)]=\frac{1}{2\pi}\int_{-\infty}^{\infty}[\delta(\omega+p)-\delta(\omega-p)]e^{i\omega\tau}\,d\omega$$

$$=\frac{1}{2\pi}\left[\int_{-\infty}^{\infty}\delta(\omega+p)e^{i\omega\tau}\,d\omega-\int_{-\infty}^{\infty}\delta(\omega-p)e^{i\omega\tau}\,d\omega\right]$$

$$=\frac{1}{2\pi}\left[e^{i\tau(-p)}-e^{+i\tau p}\right]=\frac{-i}{\pi}\left(\frac{e^{i\tau p}-e^{-i\tau p}}{2i}\right)$$

$$=\frac{-i\sin p\tau}{\pi}=\frac{1}{\pi i}\sin p\tau$$

$$\pi i F^{-1}[\delta(\omega+p)-\delta(\omega-p)]=\sin p\tau$$

$$F^{-1}\{\pi i[\delta(\omega+p)-\delta(\omega-p)]\}=\sin p\tau$$

$$F(\sin p\tau)=\pi i[\delta(\omega+p)-\delta(\omega-p)]$$

Example 2

Find the autocorrelation function whose spectral density is given by

$$S(\omega) = \begin{cases} \pi, & |\omega| \le 1 \\ 0, & otherwise \end{cases}$$

Solution:

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{i\omega\tau}\,d\omega = \frac{1}{2\pi}\int_{-1}^{1} \pi e^{i\omega\tau}\,d\omega$$

$$= \frac{\pi}{2\pi}\int_{-1}^{1} (\cos\omega\tau + i\sin\omega\tau)\,d\omega$$

$$= \frac{1}{2}\int_{-1}^{1} \cos\omega\tau\,d\omega + i\frac{1}{2}\int_{-1}^{1} \sin\omega\tau\,d\omega \qquad (\because \sin\omega\tau \text{ is an odd function})$$

$$= \frac{2}{2}\int_{0}^{1} \cos\omega\tau\,d\omega = \left[\frac{\sin\omega\tau}{\tau}\right]_0^1 = \frac{\sin\tau}{\tau}$$

Example 3

Find the power density spectral of a stationary process {X(t)} with

$$R_{XX}(\tau) = 6 + e^{-2|\tau|}.$$

Solution:

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau = \int_{-\infty}^{\infty} (6 + e^{-2|\tau|}) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} 6e^{-i\omega\tau} d\tau + \int_{-\infty}^{\infty} e^{-2|\tau|} e^{-i\omega\tau} d\tau = F(6) + F(e^{-2|\tau|})$$

$$= 2\pi(6)\delta(\omega) + \frac{2 \times 2}{2^2 + \omega^2} \Rightarrow 12\pi\delta(\omega) + \frac{4}{4 + \omega^2}.$$

Example 4

Given the power spectral density $S_{XX}(\omega) = \frac{1}{\omega^2+4}$, find the average power of the process.

Solution:

$$\text{Given: } S_{XX}(\omega) = \frac{1}{\omega^2+4}$$

$$\therefore \quad \text{ACF} = R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega) e^{i\omega\tau} d\omega$$

$$= F^{-1}[S_{XX}(\omega)] = F^{-1}\left(\frac{1}{\omega^2+4}\right)$$

$$R_{XX}(\tau) = \frac{1}{4}e^{-2|\tau|}, \text{ using } F(e^{-2|\tau|}) = \frac{2\alpha}{\alpha^2+\omega^2}$$

$\therefore$ The average power of the process

$$R_{XX}(0) = \frac{1}{4}e^{-2\times 0} = \frac{1}{4}$$

Example 5

The power spectral density of a zero mean process {X(t)} is given by

$$S(\omega) = \begin{cases} 1, & |\omega| \le \omega_0 \\ 0, & otherwise \end{cases}$$

Find $\mathrm{R}(\tau)$ and also prove that X(t) and $X(t + \frac{\pi}{\omega_0})$ are uncorrelated.

Solution:

Given $E[X(t)] = 0$.

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{i\omega\tau}d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{i\omega\tau}d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} (\cos\omega\tau + i\sin\omega\tau)d\omega = \frac{2}{2\pi}\int_{0}^{\omega_0} \cos\omega\tau\, d\omega + 0$$

$$= \frac{1}{\pi}\left[\frac{\sin\omega\tau}{\tau}\right]_0^{\omega_0} = \frac{1}{\pi}\left(\frac{\sin\omega_0\tau}{\tau} - 0\right)$$

$$R(\tau) = \frac{\sin\omega_0\tau}{\pi\tau}$$

To show that $X(t)$ and $X\left(t+\frac{\pi}{\omega_0}\right)$ are uncorrelated, we have to show that

$$C\left[X(t)X\left(t+\frac{\pi}{\omega_0}\right)\right] = 0$$

$$C\left[X(t)X\left(t+\frac{\pi}{\omega_0}\right)\right] = E\left[X(t)\,X\left(t+\frac{\pi}{\omega_0}\right)\right] - E[X(t)]E\left[X\left(t+\frac{\pi}{\omega_0}\right)\right]$$

$$= R_{XX}\left(\frac{\pi}{\omega_0}\right) - 0 = R_{XX}\left(\frac{\pi}{\omega_0}\right) \qquad [\because\ E[X(t)]=0]$$

But $$R_{XX}(\tau) = \frac{\sin \omega_0 \tau}{\pi \tau}$$

$$\therefore \quad R_{XX}\left(\frac{\pi}{\omega_0}\right) = \frac{\sin \omega_0\left(\frac{\pi}{\omega_0}\right)}{\pi\left(\frac{\pi}{\omega_0}\right)} = \frac{\sin \pi}{\left(\frac{\pi^2}{\omega_0}\right)} = 0$$

$$\therefore \quad C\left[X(t)X\left(t+\frac{\pi}{\omega_0}\right)\right] = 0$$

$\therefore$ $X(t)$ and $X\left(t+\frac{\pi}{\omega_0}\right)$ are uncorrelated.

Practice Questions

Q.1: Given the power spectral density $S_{XX}(\omega) = \begin{cases} \omega^2 + 1, & |\omega| \leq 1 \\ 0\,, & |\omega| > 1 \end{cases}$

find the autocorrelation of the process {X(t)}.

Ans: $\frac{2}{\pi\tau^3}(\tau^2 \sin\tau + \tau\cos\tau - \sin\tau)$

Q.2: Find the power spectral density of a WSS process with autocorrelation function $R(\tau) = e^{-\alpha\tau^2}, \alpha > 0$.

Ans: $\sqrt{\frac{\pi}{\alpha}}\, e^{\frac{-\omega^2}{4\alpha}}$

THANK YOU