Digital Systems
18B11EC213

Module 1: Boolean Function Minimization Techniques and Combinational Circuits-6

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Boolean Algebra

- Boolean algebra was named after George Boole.
- George Boole applied a set of symbols to logical operations.
- Digital electronics applies his set theory and logic to binary switching networks.
- Binary number system is used to represent the two possible states of digital circuits and systems.

The symbols 0 and 1 are used to represent:

True or False

Flow or No Flow

Open or Closed

Voltage1 or Voltage2

etc.

Cont..

- Boolean algebra deals with manipulation of variables and constants.
- Boolean variables, such as X, Y, Z, A, B, C can have values of either 0 or 1.
- 0 and 1 represent two different states of a quantity.

i.e., False (F) or True (T)

Low voltage or high voltage, usually written as L or H

No flow or flow

$$\left.\begin{array}{r} 0\text{ V} \equiv \text{logical } 0 \\ +5\text{ V} \equiv \text{logical } 1 \end{array}\right\}\ +\,\textbf{\textit{ve logic}}$$

or

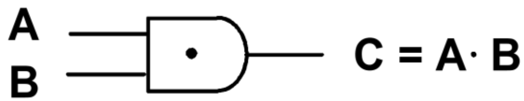
$$\left.\begin{array}{r} 0\text{ V} \equiv \text{logical } 1 \\ +5\text{ V} \equiv \text{logical } 0 \end{array}\right\}\ -\,\textbf{\textit{ve logic}}$$

Cont..

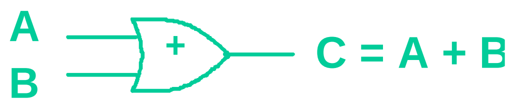
- Basic operations: NOT (Invert)
  AND
  OR
- e.g., (NOT 1) is written as: 1’ or $\overline{1}$
- NOT X : X’ or $\overline{X}$
- X AND Y : X . Y
- X OR Y : X + Y
- 1 OR X : 1 + X
- NOT

Cont..

- AND



- OR



- Characteristics of an inverter (NOT gate):

if X = 0,   X’ = 1

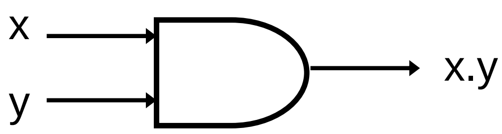
if X = 1, X’ = 0

Truth Table

X	X’
0	1
1	0

Cont..

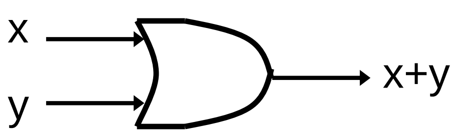
- AND gate



Truth Table

x	y	x . y
0	0	0
0	1	0
1	0	0
1	1	1

- OR gate



Truth Table

x	y	x + y
0	0	0
0	1	1
1	0	1
1	1	1

Boolean Algebra Postulates

A **Boolean algebra** consists of a set of elements *B*, with two binary operations {+} and {.} and a unary operation {'}, such that the following axioms hold:

- The set *B* contains at least two distinct elements x and y.
- **Closure**: For every x, y in *B*,

  x + y is in *B*

  x . y is in *B*
- **Commutative laws**: For every x, y in *B*,

  x + y = y + x

  x . y = y . x

Cont..

- **Associative laws**: For every x, y, z in *B*,

  $(x + y) + z = x + (y + z) = x + y + z$

  $(x \cdot y) \cdot z = x \cdot (y \cdot z) = x \cdot y \cdot z$

- **Identities** (0 and 1):

  $0 + x = x + 0 = x$ for every x in *B*

  $1 \cdot x = x \cdot 1 = x$ for every x in *B*

- **Distributive laws**: For every x, y, z in *B*,

  $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$

  $x + (y \cdot z) = (x + y) \cdot (x + z)$

Cont..

- **Complement**: For every x in *B*, there exists an element x' in *B* such that

  $x + x' = 1$

  $x \cdot x' = 0$

The set *B* = {0, 1} and the logical operations OR, AND and NOT satisfy all the axioms of a Boolean algebra.

- A **Boolean expression** is an algebraic statement containing Boolean variables and operators.

Precedence of Operators

- To lessen the brackets used in writing Boolean expressions, operator precedence can be used.
- Precedence (highest to lowest): ' . +
- Examples:

  a . b + c = (a . b) + c

  b' + c = (b') + c

  a + b' . c = a + ((b') . c)

Truth Table

- A truth table provides a listing of every possible combination of inputs and its corresponding outputs

INPUTS	OUTPUTS
…	…
…	…

- Example: 2 inputs, 2 outputs

x	y	x . y	x + y
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	1

Cont..

Example: 3 inputs, 2 outputs

x	y	z	y + z	x.(y + z)
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	1	0
1	0	0	0	0
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

Proof Using Truth Table

Prove that $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$

- Construct truth table for LHS and RHS of above equality.

x	y	z	y + z	x.(y + z)	x.y	x.z	(x.y)+(x.z)
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	1	0	1	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

From the truth table, LHS = RHS.

Duality Principle

- Every valid Boolean expression (equality) remains valid if the operators and identity elements are interchanged, as follows:

  $+ \leftrightarrow .$

  $1 \leftrightarrow 0$

- Example: Given the expression

$$a + (b \,.\, c) = (a + b) \,.\, (a + c)$$

then its dual expression is

$$a \,.\, (b + c) = (a \,.\, b) + (a \,.\, c)$$

Cont..

- If $(x + y + z)' = x'.y'.z'$ is valid, then its dual form is also valid, i.e.,

$$(x.y.z)' = x' + y' + z'$$

Basic Theorems of Boolean Algebra

- Apart from the axioms/postulates, there are other useful Boolean theorems.

1. **Idempotency**

   (a) $x + x = x$ (b) $x \cdot x = x$

   Proof of (a):

$$
\begin{aligned}
x + x &= (x + x) \cdot 1 && \text{(identity)} \\
&= (x + x) \cdot (x + x') && \text{(complementarity)} \\
&= x + x \cdot x' && \text{(distributivity)} \\
&= x + 0 && \text{(complementarity)} \\
&= x && \text{(identity)}
\end{aligned}
$$

Cont..

2. **Null elements** for + and . operators

   (a) $x + 1 = 1$ (b) $x \,.\, 0 = 0$

3. **Involution** $(x')' = x$

4. **Absorption**

   (a) $x + x \cdot y = x$ (b) $x \cdot (x + y) = x$

5. **Distributive**

   (a) $x + x' \cdot y = x + y$ (b) $x \cdot (x' + y) = x \cdot y$

Cont..

6. **DeMorgan's theorem**

   (a) $(x + y)' = x' \cdot y'$

   (b) $(x \cdot y)' = x' + y'$

7. **Consensus**

   (a) $x \cdot y + x' \cdot z + y \cdot z = x \cdot y + x' \cdot z$

   (b) $(x+y) \cdot (x'+z) \cdot (y+z) = (x+y) \cdot (x'+z)$

Cont..

- These theorems can be proved using the truth table method.

Exercise: Prove the DeMorgan's theorem using the truth table.

- The theorems can also be proved by algebraic manipulation using axioms/postulates or other basic theorems.

Boolean Functions

- **Boolean function** is an expression formed with binary variables, the two binary operators: OR and AND, and the unary operator: NOT, parenthesis and the equal sign.
- Its result is also a binary value.
- We usually use **.** for AND, **+** for OR, and **'** or **¬** for NOT

Cont..

- Examples: Boolean functions

  $F1 = x.y.z'$

  $F2 = x + y'.z$

  $F3 = (x'.y'.z) + (x'.y.z) + (x.y')$

  $F4 = x.y' + x'.z$

x	y	z	F1	F2	F3	F4
0	0	0	0	0	0	0
0	0	1	0	1	1	1
0	1	0	0	0	0	0
0	1	1	0	0	1	1
1	0	0	0	1	1	1
1	0	1	0	1	1	1
1	1	0	1	1	0	0
1	1	1	0	1	0	0

From the truth table, F3 = F4

We can also prove by algebraic manipulation that F3 = F4

Complement of Functions

- Given a function F, the complement of this function F', is obtained by interchanging 1 with 0 in the function's output values.

Example: Boolean function $F1 = xyz'$

Complement of F1:

$$F1' = (x.y.z')'$$
$$= x' + y' + (z')' \quad \text{DeMorgan}$$
$$= x' + y' + z \quad \text{Involution}$$

x	y	z	**F1**	**F1'**
0	0	0	**0**	**1**
0	0	1	**0**	**1**
0	1	0	**0**	**1**
0	1	1	**0**	**1**
1	0	0	**0**	**1**
1	0	1	**0**	**1**
1	1	0	**1**	**0**
1	1	1	**0**	**1**

References

- M. M. Mano, *Digital Logic and Computer Design*, 5th ed., Pearson Prentice Hall, 2013.
- R. P. Jain, *Modern Digital Electronics*, 4th ed., Tata McGraw-Hill Education, 2009.