Department of Mathematics
OddSemester 2016-2017

Probability and Random Processes	**15B11MA301**
Probability Theory and Random Processes	**10B11MA411**
Tutorial Sheet 2	**B.Tech. Core**

Random Variables (one dimensional)

1. A fair coin is tossed 3 times and let X be the difference of the number of heads and the number of tails. Find (a) the probability mass function, (b) the cumulative distribution function of X.

2. A random variable X has the probability distribution, defined as

X:	1	2	3	4	5	6
P (X):	0.04	0.15	0.37	0.26	0.11	0.07

Find (i) P (X Odd/X<5) (II) P (X<5/X Odd) (iii) P (X=4/X is not equal to 3)

3. The diameter of an electric cable, say *X*, is assumed to be a continuous random variable with probability function $f(x)=kx(1-x),\ \ 0\le x\le 1$
   (i) Find *k*
   (ii) Determine a number 'b' such that $P(X<b)=P(X>b)$

4. A continuous random variable X is defined as $f(x)=\begin{cases} ax+bx^2 & 0<x<1 \\ 0 & Otherwise \end{cases}$ where *a* and *b* are constants. If E[X]= 0.6, find (i) $P\left\{X<\frac{1}{2}\right\}$ (ii) Var (*X*).

$\hookrightarrow \int x^2 f(x) - \mu^2$

5. Sketch the graph of the function

$$f(x)=\begin{cases} \frac{1}{2}-\frac{1}{4}|x-3|, & 1\le x\le 5 \\ 0, & otherwise \end{cases}$$

and show that it is a pdf. Also, find its cdf.

6. A petrol pump is supplied with petrol once a day. If its daily volume X of sales in thousands of liters is distributed by $f(x)=5(1-x)^4,\ \ 0\le x\le 1$, what must be the capacity of its tank in order that the probability that its supply will be exhausted in a given day shall be 0.01?

7. The cumulative distribution function of a random variable X is given by $F(x)=1-e^{-2x^2},\ x\ge 0$. Find (a) P(0 < X < 3) (b) P(X >1) (c) P(X =5).

8. A player tosses 3 fair coins. He wins Rs.1500 if three heads appear Rs.1000 if two heads appear and Rs.500 if one head appear. On the other hand he loses Rs.2000 if three tails appear. Find the value of the game to the player. Is the game favorable to the player?

9. A box contains 10 items of which 4 are defective. A person draws 3 items form the box. Determine the expected number of defective items he has drawn.

10. A system can function for a random amount of time X. If the density of X is given (in units of months) by $f(x)=\begin{cases} Cxe^{-x/2} & x>0 \\ 0 & x\le 0 \end{cases}$, where C is any constant. What is the probability that the system functions for at least 5 months?

Q1]

No. of Heads	No. of Tails	Difference
0	3	3
1	2	1
2	1	1
3	0	3

P (3 heads or 3 tails) $= 2 \times \left(\frac{1}{2}\right)^3 = \frac{1}{4}$

$$\text{pmf} = \begin{cases} 1/4 & x = 3 \\ 3/4 & x = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{cdf} = \begin{cases} 0 & x < 1 \\ 3/4 & 1 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$

Q2] i) $P(x \text{ odd} / x < 5) = \dfrac{0.04 + 0.37}{0.04 + 0.15 + 0.37 + 0.26} = \dfrac{0.41}{0.82} = \dfrac{1}{2}$

ii) $P(x < 5 / x \text{ odd}) = \dfrac{0.04 + 0.37}{0.04 + 0.37 + 0.11} = \dfrac{0.41}{0.52} = \dfrac{41}{52}$

iii) $P(x = 4 / x \text{ not equal } 3) = \dfrac{0.26}{0.04 + 0.15 + 0.26 + 0.11 + 0.07} = \dfrac{0.26}{0.63}$

Q3] $f(x) = k\,x(1-x) \qquad 0 \le x \le 1$

i) $$1 = k\int_0^1 x(1-x)\,dx = k\left|\frac{x^2}{2} - \frac{x^3}{3}\right|_0^1$$

$$= k\left|\frac{1}{2} - \frac{1}{3}\right| = \frac{k}{6}$$

$$\boxed{k = 6}$$

$$\Rightarrow f(x) = 6x(1-x)$$

ii) $$\int_0^b 6x(1-x) = 6\int_1^b x(1-x)$$

$$\left|\frac{x^2}{2} - \frac{x^3}{3}\right|_0^b = \left|\frac{x^2}{2} - \frac{x^3}{3}\right|_b^1$$

$$\frac{b^2}{2} - \frac{b^3}{3} = \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{b^2}{2} - \frac{b^3}{3}\right)$$

$$2\left(\frac{b^2}{2} - \frac{b^3}{3}\right) = \frac{1}{6}$$

$$6b^2 - 4b^3 = 1$$

$$4b^3 - 6b^2 + 1 = 0$$

#

$(b - 1/2)$ is a factor

$$b - 1/2 \;\Big)\; 4b^3 - 6b^2 + 1 \;\Big(\; 4b^2 - 4b - 2$$

$$\pm 4b^3 \mp 2b^2$$

$$-4b^2 + 1$$

$$\mp 4b^2 \pm 2b$$

$$-2b + 1$$

$$\mp 2b \pm 1$$

$$\times$$

$(b - 1/2)(4b^2 - 4b - 2)$

$\hookrightarrow \boxed{b = 1/2}$

OR $$4b^2 - 4b - 2 = 0$$

$$b = \frac{4 \pm \sqrt{16 + 32}}{8}$$

$$= \frac{4 \pm 6.92}{8} = 0.5 \pm 0.866$$

Out of Domain

Q4] $f(x) = \begin{cases} ax + bx^2 & , 0 < x < 1 \\ 0 & , \text{otherwise} \end{cases}$

i) $1 = \int_0^1 ax + bx^2 = \left| \frac{ax^2}{2} + \frac{bx^3}{3} \right|_0^1$

$1 = \frac{a}{2} + \frac{b}{3} = \frac{3a + 2b}{6}$

$6 = 3a + 2b$

$\mu(X) = 0.6 = \int_0^1 x f(x) = \int_0^1 ax^2 + bx^3$

$0.6 = \left| \frac{ax^3}{3} + \frac{bx^4}{4} \right|_0^1 = \frac{a}{3} + \frac{b}{4} = \frac{4a + 3b}{12}$

$7.2 = 4a + 3b$

$8a + 6b = 14.4$
$-9a - 6b = -18$

$-a = -3.6$

$\boxed{a = 3.6}$

$\boxed{b = -2.4}$

$\Longrightarrow f(x) = \begin{cases} 3.6x - 2.4x^2 & , 0 \le x \le 1 \\ 0 & , \text{elsewhere} \end{cases}$

ii) $P(X < 1/2) = \int_0^{1/2} 3.6x - 2.4x^2$

$= \left| \frac{3.6x^2}{2} - \frac{2.4x^3}{3} \right|_0^{1/2}$

$= \frac{3.6}{2} \times \frac{1}{4} - \frac{2.4}{3} \times \frac{1}{8} = 0.45 - 0.1$

$\boxed{P(X < 1/2) = 0.35}$

iii) $Var(x) = \int x^2 f(x) dx - \mu_0^2 = \int_0^1 3.6x^3 - 2.4x^4 - 0.36$

$= \left| \frac{3.6x^4}{4} - \frac{2.4x^5}{5} \right|_0^1 - 0.36 = 0.9 - 0.48 - 0.36 = \underline{0.06}$

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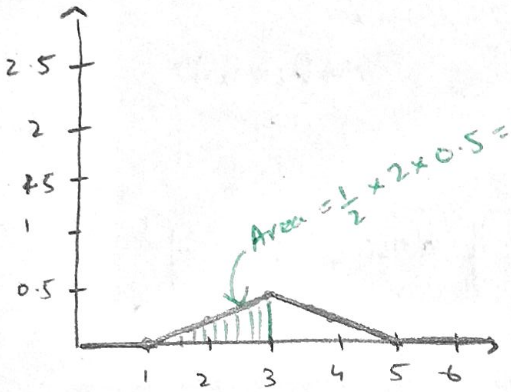

$$f(x) = \begin{cases} \frac{1}{2} - \frac{1}{4}|x-3| & 1 \le x \le 5 \\ 0 & \text{, otherwise} \end{cases}$$

$$f(x) = \begin{cases} \frac{1}{2} - \frac{1}{4}(3-x) & 1 \le x \le 3 \\ \frac{1}{2} - \frac{1}{4}(x-3) & 3 \le x \le 5 \\ 0 & \text{, otherwise} \end{cases}$$

All $f(x) \ge 0$ & continous

$$I_1 = \int_1^x \frac{1}{2} - \frac{1}{4}(3-x) = \left| \frac{1}{2}x - \frac{1}{4}\left(3x - \frac{x^2}{2}\right) \right|_1^x$$

$$= \frac{x}{2} - \frac{3x}{4} + \frac{x^2}{8} - \left(\frac{1}{2} - \frac{1}{4}\left(3 - \frac{1}{2}\right)\right)$$

$$= \frac{x^2}{8} - \frac{x}{4} + \frac{1}{8}$$

$$I_1(x=3) = \frac{9}{8} - \frac{3}{4} + \frac{1}{8} = \frac{10}{8} - 3/4 = 2/4 = \frac{1}{2}$$

$$I_2 = \frac{1}{2} + \int_3^x \frac{1}{2} - \frac{1}{4}(x-3) = \frac{1}{2} + \left| \frac{1}{2}x - \frac{1}{4}\left(\frac{x^2}{2} - 3x\right) \right|_3^x$$

$$= \frac{1}{2} - \left(\frac{3}{2} - \frac{1}{4}\left(\frac{9}{2} - 9\right)\right) + \left(\frac{x}{2} - \frac{x^2}{8} + \frac{3x}{4}\right)$$

$$= \frac{5x}{4} - \frac{x^2}{8} - \frac{17}{8}$$

$$\text{Cdf} = \begin{cases} 0 & -\infty < x \le 1 \\ \frac{x^2}{8} - \frac{x}{4} + \frac{1}{8} & , 1 \le x \le 3 \\ \frac{5x}{4} - \frac{x^2}{8} - \frac{17}{8} & , 3 \le x \le 5 \\ 1 & , x \ge 5 \end{cases}$$

Q6] $f(x) = 5(1-x)^4$

$\uparrow$ sales function

resource exhausted when sales > capacity

$$P(x > \text{capacity}) = \int_c^1 f(x)\,dx$$

$$= 5\int_c^1 (1-x)^4\,dx$$

let $1-x = y \Rightarrow x = 1-y$

$-dx = dy$

$$0.01 = 5\int_{1-c}^{0} y^4(-dy) = -5\left|\frac{y^5}{5}\right|_{1-c}^{0} = \cancel{5} \times \frac{(1-c)^5}{\cancel{5}}$$

$$1-c = \sqrt[5]{0.01} = 0.3981$$

$$c = 0.6018$$

Q7] $cdf = F(x) = \begin{cases} 1 - e^{-2x^2} & x \geq 0 \\ \cancel{0}, & \cancel{\text{otherwise}} \end{cases}$

a) $P(0 < X < 3) = F(3) = 1 - e^{-2\times 9} = \underline{\underline{1 - e^{-18}}}$

b) $P(X > 1) = 1 - P(X < 1) = 1 - F(1) = 1 - (1 - e^{-2})$

$= \underline{\underline{e^{-2}}}$

c) Not defined for Pdf

Q8] Coin Toss follows binomial distribution

$$P(3\text{ heads}) = P(3\text{ tails}) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

$$P(2H\text{-}1T) = P(2T\text{-}1H) = 3\times\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^1 = \frac{3}{8}$$

X :	-2000	500	1000	1500
$P(x)$:	$1/8$	$3/8$	$3/8$	$1/8$

$$\mu = \sum x\,P(x) = -\frac{2000}{8} + \frac{1500}{8} + \frac{3000}{8} + \frac{1500}{8}$$

$$= \frac{4000}{8} = \underline{\underline{500}}$$, favourable to play

Q9] $p = \frac{6}{10}$ $\quad q = \frac{4}{10}$

0 defective $= {}^3C_3 \times \left(\frac{6}{10}\right)^3 = \frac{216}{1000}$

1 defective $= {}^3C_2 \times \left(\frac{6}{10}\right)^2\left(\frac{4}{10}\right)^1 = \frac{144}{1000}\times 3 = \frac{432}{1000}$

2 defective $= {}^3C_1 \times \left(\frac{6}{10}\right)^1\left(\frac{4}{10}\right)^2 = \frac{96}{1000}\times 3 = \frac{288}{1000}$

3 defective $= {}^3C_0\left(\frac{4}{10}\right)^3 = \frac{64}{1000}$

$\mu_{\text{defective}}$ ~~P(defective)~~ $= \frac{0\times 216}{1000} + \frac{1\times 432}{1000} + \frac{2\times 288}{1000} + \frac{3\times 64}{1000}$

$$= \frac{1200}{1000} = \frac{6}{5} = \underline{\underline{1.2}}$$

Q10] $f(x) = \begin{cases} ce^{-x/2} & , 0 \le x \le \infty \\ 0 & , \text{otherwise} \end{cases}$

$$1 = c \int_0^\infty xe^{-x/2}$$

$$1 = c \left[\frac{xe^{-x/2}}{-1/2} - \frac{e^{-x/2}}{1/4} \right]$$

$$= c \left[(2x+4)(-e^{-x/2}) \right]_0^\infty$$

$$1 = 4c \Rightarrow \boxed{c = 1/4}$$

$$P(x>5) = \int_5^\infty \frac{1}{4}xe^{-x/2} = -\frac{1}{4} \left| (2x+4)e^{-x/2} \right|_5^\infty$$

$$= -\frac{1}{4} \left| -14 e^{-5/2} \right| = \frac{7}{2} e^{-5/2}$$