Department of Mathematics

Odd Semester 2017

Probability and Random Processes **15B11MA301**

Probability and Random Processes **10B11MA411**

Tutorial Sheet 7 **B.Tech. Core**

Random Process

1. Define a random process and classify them with suitable examples.

2. In an experiment of two fair dice, the process {X(t)} is defined as $X(t) = \sin \pi t$, if the experiment shows a prime sum and X(t) = 2t + 1, otherwise. Find the mean of the process. Is the process stationary? [Ans: not stationary]

3. Let $X(t) = A\cos\lambda t + B\sin\lambda t$, with random variable A taking values 1 and 3 with equal probabilities and random variable B taking values -1 and 1 with probabilities ¼ and ¾ respectively. Test the process {X(t)} for stationarity. [Ans: not stationary]

4. Test the random processes {X(t)} and {Y(t)} for WSS when:

(i) $X(t) = \cos(\lambda t + Y)$, where λ is a constant and Y is uniform in $(0, 2\pi)$
[Ans: WSS]

(ii) $Y(t) = X\sin(\lambda t)$, where λ is a constant and X is uniform in $(-1, 1)$.
[Ans: not WSS]

5. Find auto correlation functions of the processes {X(t)} and {Y(t)} such that $X(t) = A\cos\lambda t + B\sin\lambda t$ and $Y(t) = B\cos\lambda t - A\sin\lambda t$, where A and B are uncorrelated random variables taking value -4 and 4 with equal probabilities. Prove that {X(t)} and {Y(t)} are jointly WSS.

6. Given a random variable Y with characteristic function $\phi(2w) = E(e^{iwY})$ and a random process defined by $X(t) = \cos(3t + 2Y)$. Find the condition under which the process {X(t)} is WSS. [Ans: $\phi(4) = \phi(8) = 0$]

7. If {X(t)} is a WSS process with E{X(t)} = 2 and $R_{XX}(\tau) = 4 + e^{-|\tau|/10}$, find the variance of X(1), X(2) and X(3). Also compute the second order moment about origin of X(1) + X(2) + X(3).

8. Define a Random walk and prove that the limiting form of a random walk is Wiener process.

...en we do any random experiment then we find lot of outcome. A variable which repersent this outcome is called Random Variable, whereas a function or rule which assigns a time function to every outcome of a random experiment is called a random process.

Mean of random process $\{X(t)\}$

$$\mu(t) = E\{X(t)\}$$

Autocorrelation $R_{xx}(t_1, t_2) = R(t_1, t_2) = E\{X(t_1)\, X(t_2)\}$

Autocovariance $C_{xx}(t_1, t_2) = C(t_1, t_2)$

$$= E[\{X(t_1) - \mu(t_1)\}\{X(t_2) - \mu(t_2)\}]$$

$$= R(t_1, t_2) - \mu(t_1)\,\mu(t_2)$$

Correlation coefficient $\rho_{xx}(t_1, t_2) = \rho(t_1, t_2)$

$$= \frac{C(t_1, t_2)}{\sqrt{C(t_1, t_1) \times C(t_2 \times t_2)}}$$

Cross - correlation of 2 processes $\{X(t)\}$ & $\{Y(t)\}$ (jointly)

$$R_{xy}(t_1, t_2) = E\{X(t_1)\, Y(t_2)\}$$

Cross - covariance $= C_{xy}(t_1, t_2) = R_{xy}(t_1, t_2) - \mu_x(t_1)\,\mu_y(t_2)$

Cross - correlation coefficient $\rho_{xy}(t_1, t_2)$

$$= \frac{C_{xy}(t_1, t_2)}{\sqrt{C_{xx}(t_1, t_1) \times C_{yy}(t_2, t_2)}}$$

Strongly stationary Process:- (SSS Process)

(strict sense stationary process)

If SSS then densities of $X(t)$ & $X(t+h)$ are the same i.e. $f(x, t) = f(x, t+h)$

$\Rightarrow f(x, t)$ is independent of t.

$\Rightarrow E\{X(t)\}$ is independent of time

$E\{X(t)\} = $ Constt.

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Wide sense stationary Process (WSS) :-

$E[X(t)] = \text{constt}$

$R(t_1, t_2) = \text{function of } (t_1 - t_2)$

$\{X(t)\}$ need not be SSS.

Q.2:-

$S = \{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (1,5)\ (1,6)$
$(2,1)\ (2,2)\ (2,3)\ (2,4)\ (2,5)\ (2,6)$
$(3,1)\ (3,2)\ (3,3)\ (3,4)\ (3,5)\ (3,6)$
$(4,1)\ (4,2)\ (4,3)\ (4,4)\ (4,5)\ (4,6)$
$(5,1)\ (5,2)\ (5,3)\ (5,4)\ (5,5)\ (5,6)$
$(6,1)\ (6,2)\ (6,3)\ (6,4)\ (6,5)\ (6,6)\}$

prime sum
2 - (1,1)
3 - (1,2) (2,1)
5 - (1,4) (2,3) (3,2) (4,1)
7 - (1,6) (2,5) (3,4) (4,3) (5,2) (6,1)
11 - (5,6) (6,5)

$P(\text{prime sum}) = \frac{15}{36} = \frac{5}{12}$ $\qquad E(X(t)) = \sum x_i p_i$

$P(\text{not prime sum}) = \frac{21}{36} = \frac{7}{12}$

$E\{X(t)\} = p\{X(t) = \sin \pi t\} \sin \pi t + p\{X(t) = 2t+1\}\, 2t+1$

$= \frac{5}{12} \sin \pi t + \frac{7}{12}(2t+1)$

It depends upon t so it is not stationary.

Q.3:- $\{X(t)\} = A \cos \lambda t + B \sin \lambda t$

with R.V A taking values 1 & 3 equal probability & R.V B taking values -1 & 1 with Prob. $\frac{1}{4}$ & $\frac{3}{4}$

$E\{X(t)\} = E\{A \cos \lambda t + B \sin \lambda t\}$

$= \cos \lambda t\, E(A) + \sin \lambda t\, E(B)$

$E(A) = 1 \times \frac{1}{2} + \frac{1}{2} \times 3 = 2$

$= -1 \times \frac{1}{4} + 1 \times \frac{3}{4} = \frac{3-1}{4} = \frac{1}{2}$

$E\{X(t)\} = 2\cos\lambda t + \frac{1}{2}\sin\lambda t$

$E\{X(t)\}$ depends on t, so not stationary.

Q.4:- i) $X(t) = \cos(\lambda t + Y)$, $Y \sim (0, 2\pi)$

p.d.f of this $Y \sim f_Y(y) = \frac{1}{2\pi}$, $0 \leq y \leq 2\pi$

$E(X(t)) = E[\cos(\lambda t + Y)]$

$= E[\cos\lambda t \cos Y - \sin\lambda t \sin Y]$

$= \cos\lambda t \; E(\cos Y) - \sin\lambda t \; E(\sin Y)$

$E(\cos Y) = \int_0^{2\pi} \frac{1}{2\pi} \cdot \cos Y \, dY$

$= \frac{1}{2\pi}[-\sin Y]_0^{2\pi} = 0$

$E(\sin Y) = \int_0^{2\pi} \frac{1}{2\pi} \sin Y \, dY$

$= \frac{1}{2\pi}[\cos Y]_0^{2\pi} = \frac{1}{2\pi}[1 - 1] = 0$

$E(X(t)) = \cos\lambda t \; E(\cos Y) - \sin\lambda t \; E(\sin Y)$

$= 0$

or $E\{X(t)\} = \int_0^{2\pi} \frac{1}{2\pi} \cos(\lambda t + Y) \, dy$

$= \frac{1}{2\pi}[\sin(\lambda t + Y)]_0^{2\pi}$

$= \frac{1}{2\pi}[\sin(2\pi + \lambda t) - \sin(\lambda t)]$

$= \frac{1}{2\pi}[\sin\lambda t - \sin\lambda t]$

$= 0 =$ Constt

$E\{X(t_1)\,X(t_2)\} = E\{\cos(\lambda t_1 + Y)\,\cos(\lambda t_2 + Y)\}$

$$= \frac{E}{2}\{\cos(\lambda t_1 + Y + \lambda t_2 + Y) + \cos(\lambda t_1 + Y - \lambda t_2 - Y)\}$$

$$= \frac{E}{2}\{\cos(\lambda(t_1+t_2)+2Y) + \cos\lambda(t_1-t_2)\}$$

$$= \frac{1}{2}\times\frac{1}{2\pi}\int_0^{2\pi} \cos[\lambda(t_1+t_2)+2Y] + \cos\lambda(t_1-t_2)]\,dy$$

$$= \frac{1}{4\pi}\left[\frac{\sin(\lambda(t_1+t_2)+2Y)}{2} + \cos\lambda(t_1-t_2)\cdot Y\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\cancel{\frac{\sin\lambda(t_1+t_2)+4\pi}{2}} + 2\pi\cos\lambda(t_1-t_2) - \cancel{\frac{\sin\lambda(t_1+t_2)}{2}}\right]$$

$$= \frac{1}{2}\cos\lambda(t_1-t_2)$$ which is a function of $t_1 - t_2$

$\therefore$ $\{X(t)\}$ is WSS process.

ii) $Y(t) = X\sin(\lambda t)$, $X \sim (-1,1)$

$X \sim f_X(x) = \frac{1}{2}$, $-1 \le x \le 1$

$$E\{Y(t)\} = \int_{-1}^{1} x\sin(\lambda t)\times\frac{1}{2}\,dx$$

$$= \frac{1}{2}\sin(\lambda t)\cdot\left[\frac{x^2}{2}\right]_{-1}^{1}$$

$= 0$ Constt.

$$R(t_1,t_2) = E\{Y(t_1)\,Y(t_2)\}$$

$$= E\{X^2\sin(\lambda t_1)\,\sin(\lambda t_2)\}$$

$$= E\left\{\frac{X^2\cos(\lambda(t_1-t_2)) - \cos(\lambda(t_1+t_2))}{2}\right\}$$

$$= \int_{-1}^{1}\frac{\cos(\lambda(t_1-t_2)) - \cos(\lambda(t_1+t_2))}{2}\cdot\frac{x^2}{2}\,dx$$

$$= \frac{\cos(\lambda(t_1-t_2)) - \cos(\lambda(t_1+t_2))}{4\times 3} \cdot x^3 \Big|_{-1}^{1}$$

$$= \frac{\cos(\lambda(t_1-t_2)) - \cos(\lambda(t_1+t_2))}{6}$$

Not a function of $t_1 - t_2$

So not WSS Process.

(5) Auto correlation function $R_{xy}(t_1, t_2)$

$X(t) = A\cos\lambda t + B\sin\lambda t$

$Y(t) = B\cos\lambda t - A\sin\lambda t$

A & B are uncorrelated r.v

i.e $E(AB) = E(A)\,E(B)$

$E(A) = -4\times\frac{1}{2} + 4\times\frac{1}{2} = 0$

$E(B) = 0$

I.e. $E(AB) = 0$

$E(A^2) = 16\times\frac{1}{2} + 16\times\frac{1}{2} = 16$ Var(A).

$E(B^2) = 16 = Var(B)$

$R_{xx}(t_1, t_2) = E\{X(t_1)\,X(t_2)\} = E\{(A\cos\lambda t_1 + B\sin\lambda t_1)(A\cos\lambda t_2 + B\sin\lambda t_2)\}$

$= E(A^2)\cos\lambda t_1\cos\lambda t_2 + E(BA)\sin\lambda t_1\cos\lambda t_2 + E(AB)\cos\lambda t_1\cos\lambda t_2 + E(B^2)\sin\lambda t_1\sin\lambda t_2$

$= 16(\cos\lambda t_1\cos\lambda t_2 + \sin\lambda t_1\sin\lambda t_2)$

$= 16\cos(\lambda(t_1-t_2))$

$R_{yy}(t_1, t_2) = E(Y(t_1)\,Y(t_2))$

$= E[(B\cos\lambda t_1 - A\sin\lambda t_1)(B\cos\lambda t_2 - A\sin\lambda t_2)]$

$= 16[\cos\lambda t_1\cos\lambda t_2 + \sin\lambda t_1\sin\lambda t_2]$

$= 16\cos[\lambda(t_1-t_2)]$

$R_{xy}(t_1, t_2) = E\{X(t_1)\, Y(t_2)\}$

$= E\{(A\cos\lambda t_1 + B\sin\lambda t_1)(B\cos\lambda t_2 - A\sin\lambda t_2)\}$

$= E\{AB\cos\lambda t_1\cos\lambda t_2 - A^2\cos\lambda t_1\sin\lambda t_2 + B^2\sin\lambda t_1\cos\lambda t_2$
$- AB\sin\lambda t_1\sin\lambda t_2\}$

$= E(B^2)\sin\lambda t_1\cos\lambda t_2 - E(A^2)\cos\lambda t_1\sin\lambda t_2$

$= 16\,[\cos\lambda t_2\sin\lambda t_1 - \cos\lambda t_1\sin\lambda t_2]$

$= 16\sin[\lambda(t_1 - t_2)]$

is a function of $t_1 - t_2$

$\therefore$ jointly WSS.

(7) $\{X(t)\}$ is a WSS Process i.e. $\tau = t_1 - t_2$
with $E\{X(t)\} = 2$

$R_{xx}(t_1, t_2) = 4 + e^{-|t_1 - t_2|/10}$

$R_x(t_1, t_2) = E[X(t_1)X(t_2)]$

$E\{X^2(t)\} = R_{xx}(t, t) = 4 + e^0 = 5$

$E(X(t_1)\, X(t_2))$

variance $= E\{X^2(t)\} - (E(X(t)))^2$

$= 5 - 4 = 1$

2^{nd} order moment of $X(1) + X(2) + X(3)$

$\boxed{E(X^2(t))}$

i.e. $E\{X(1) + X(2) + X(3)\}^2$

$= E\{X^2(1)\} + E\{X^2(2)\} + E\{X^2(3)\} + 2E\{X(1)X(2)\}$
$+ 2E\{X(2)X(3)\} + 2E\{X(3)X(1)\}$

$= 5 + 5 + 5 + 2[4 + e^{-1/10} + 4 + e^{-1/5} + 4 + e^{-1/10}]$

$= 39 + 4e^{-1/10} + 2e^{-1/5}$ Ans

$\because E\{X(1)X(2)\} = R(1, 2) = 4 + e^{-1/10}$

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(6) Given $X(t) = \cos(3t + 2Y)$

$$\phi(2w) = E(e^{iwY})$$

$$E[X(t)] = E[\cos(3t+2Y)]$$

$$= E[\cos 3t \cos 2Y - \sin 3t \sin 2Y]$$

$$= \cos 3t \, E(\cos 2Y) - \sin 3t \, E(\sin 2Y)$$

Now find $E(\cos 2Y) = ?$

$E(\sin 2Y) = ?$

$$\phi(2w) = E[e^{iwY}] = E[\cos wY + i \sin wY]$$

$$E(\cos wY) + i E(\sin wY) = \phi(2w)$$

Put $w = 2$

$$\phi(4) = E(\cos 2Y) + i E(\sin 2Y)$$

Put $\phi(4) = 0$

$$0 = E(\cos 2Y) + i E(\sin 2Y)$$

$$\Rightarrow E(\cos 2Y) = 0, \quad E(\sin 2Y) = 0$$

$$\boxed{E(X(t)) = 0}$$

$$R(t_1, t_2) = E[X(t_1) X(t_2)]$$

$$= E[\cos(3t_1 + 2Y) \cos(3t_2 + 2Y)]$$

$$= \frac{E}{2}[\cos(3t_1 + 2Y + 3t_2 + 2Y) + \cos(3t_1 + 2Y - 3t_2 - 2Y)]$$

$$= \frac{E}{2}[\cos(3(t_1+t_2) + 4Y) + \cos(3(t_1 - t_2))]$$

$$= \frac{1}{2}\cos 3(t_1 - t_2) + \frac{1}{2} E[\cos 3(t_1+t_2) \cos 4Y - \sin 3(t_1+t_2) \sin 4Y]$$

$$= \frac{1}{2}\cos 3(t_1 - t_2) + \frac{1}{2} E(\cos 4Y) \cos 3(t_1+t_2) + \frac{1}{2} E(\sin 4Y) \sin 3(t_1+t_2)$$

$$= \frac{1}{2}\cos 3(t_1 - t_2) + \frac{1}{2} \times 0 + \frac{1}{2} \times 0$$

$$= \frac{1}{2}\cos 3(t_1 - t_2)$$ is WSS process.

$\because \phi(2w) = E(\cos 2wY) + iE(\sin wY)$

Put $w = 4$ $\quad \phi(8) = E(\cos 4Y) + i E(\sin 4Y)$

$\because \phi(8) = 0$ $\quad E(\cos 4Y) = 0 \quad E(\sin 4Y) = 0$