# Jaypee Institute Of Information Technology
## ODD Semester 2022
## Theoretical Foundation of Computer Science(CI212)
## Tutorial – 6

***Topic: Induction and Recurrence***

**Ques 1:** Prove that $1^2 + 3^2 + 5^2 + \ \ldots + (2n+1)^2 = \frac{(n+1)(2n+1)(3n+1)}{3}$ whenever n is a nonnegative integer.

**Solution:**

"$1^2 + 3^2 + \cdots + (2n+1)^2 = (n+1)(2n+1)(2n+3)/3$." *Basis step:* $P(0)$ is true because $1^2 = 1 = (0+1)(2\cdot 0+1)(2\cdot 0+3)/3$. *Inductive step:* Assume that $P(k)$ is true. Then $1^2 + 3^2 + \cdots + (2k+1)^2 + [2(k+1)+1]^2 = (k+1)(2k+1)(2k+3)/3 + (2k+3)^2 = (2k+3)[(k+1)(2k+1)/3 + (2k+3)] = (2k+3)(2k^2+9k+10)/3 = (2k+3)(2k+5)(k+2)/3 = [(k+1)+1][2(k+1)+1][2(k+1)+3]/3$.

**Ques 2:** Prove that $2^n > n^2$ is n is an integer greater than 4.

**Solution:**

Let $P(n)$ be "$2^n > n^2$." *Basis step:* $P(5)$ is true because $2^5 = 32 > 25 = 5^2$. *Inductive step:* Assume that $P(k)$ is true, that is, $2^k > k^2$. Then $2^{k+1} = 2\cdot 2^k > k^2 + k^2 > k^2 + 4k \geq k^2 + 2k + 1 = (k+1)^2$ because $k > 4$.

**Ques 3:** Give the recursive definition of the following sequence $a_n \ \ n = 1,2,3$

**a)** $a_n = 6n$. **b)** $a_n = 2n + 1$.
**c)** $a_n = 10^n$. **d)** $a_n = 5$.

**Solution:**

a) $a_{n+1} = a_n + 6 \ for\ n > 1 \ \ a_1 = 6$
b) $a_{n+1} = a_n + 2 \ \ for\ n > 1 \ \ a_1 = 3$
c) $a_{n+1} = 10a_n \ \ for\ n > 1 \ \ a_1 = 10$
d) $a_{n+1} = a_n \ \ for\ n > 1 \ \ a_1 = 5$

**Ques 4:** Find the solution for the recurrence relation

$$a_n = \begin{cases} 2 & if\ n = 0, \\ 7 & if\ n = 1, \\ a_{n-1} + 2a_{n-2} & otherwise. \end{cases}$$

**Solution:**

Characteristic equation:

$$r^2 - r - 2 = 0$$

Roots $r = -1,2$

General Solution:

$$c_1 2^n + c_2(-1)^n$$

Solve for c1, c2

$$c_1 + c_2 = 2$$

$$2c_1 - c_2 = 7$$

$$c_1 = 3, c_2 = -1$$

$3 \cdot 2^n - (-1)^n$

**Ques 5:** Find the solution for the recurrence relation

$$\begin{aligned} x_n &= 6x_{n-1} - 9x_{n-2} \\ x_0 &= 2 \\ x_1 &= 3 \end{aligned}$$

*Solution.* The characteristic equation

$$r^2 - 6r + 9 = 0 \iff (r-3)^2 = 0$$

has only one root $r = 3$. Then the general solution is

$$x_n = c_1 3^n + c_2 n 3^n.$$

The initial conditions $x_0 = 2$ and $x_1 = 3$ imply that $c_1 = 2$ and $c_2 = -1$. Thus the solution is

$$x_n = 2 \cdot 3^n - n \cdot 3^n = (2-n)3^n, \ \ n \geq 0.$$

**Ques 6:** Find the solution for the recurrence relation

$$\begin{cases} x_n &= 10x_{n-1} - 25x_{n-2} + 8 \cdot 5^n \\ x_0 &= 6 \\ x_1 &= 10 \end{cases}$$

*Solution.* The characteristic equation is

$$t^2 - 10t + 25 = 0 \Longleftrightarrow (t-5)^2 = 0.$$

We have roots $r_1 = r_2 = 5$, then $r = r_1 = r_2 = 5$. A special solution can be of the type $x_n = An^2 5^n$. Put the solution into the non-homogeneous relation. We have

$$An^2 5^n = 10A(n-1)^2 5^{n-1} - 25A(n-2)^2 5^{n-2} + 8 \cdot 5^n$$

Dividing both sides by $5^{n-2}$,

$$An^2 5^2 = 10A(n-1)^2 5 - 25A(n-2)^2 + 8 \cdot 5^2.$$

Since $An^2 5^2 = 10An^2 5 - 25n^2$, we have

$$10A(-2n+1)5 - 25A(-4n+4) + 8 \cdot 5^2 = 0 \Longrightarrow A = 4.$$

So a nonhomogeneous solution is

$$x_n = 4n^2 5^n.$$

The general solution is

$$x_n = 4n5^n + c_1 5^n + c_2 n 5^n.$$

The initial condition implies $c_1 = 6$ and $c_2 = -8$. Therefore

$$x_n = (4n^2 - 8n + 6)5^n.$$