Probability and Random Processes (15B11MA301)

Lecture-25

(Content Covered: Reliability of Systems, Serial and Parallel Configuration)

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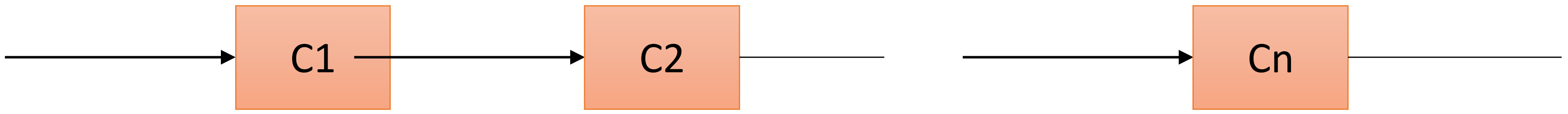
Contents

Reliability of Systems : An Introduction

- A system is generally understood as a set of components assembled to perform a certain function.
- To evaluate the reliability of a complex system, its is required to apply a particular – failure law to the entire system.
- But it will be more proper if we determine an appropriate reliability model for each component and then compute the reliability of the system by applying the relevant rules of probability according to the configuration of the components within the system.
- The configuration of the components can be of either of the two categories viz., (i) Serial Configuration and (ii) Parallel Configuration.

Serial Configuration

- Serial or nonredundant configuration is one in which the components of the system are connected in series (or serially) as shown in figure. Here, each block represents a component.



- In series configuration, all components must function for the system to function.

- The failure of any component causes system failure.

- Consider a system comprising of two components in series C_1 and C_2 .
- Let R_1(t), R_2(t) and R_s(t) be the reliabilities of the components and the system respectively.

Then, R_1= P(C_1) = probability that C_1 functions

R_2= P(C_2) = probability that C_2 functions

Also, R_s = probability that both C_1 and C_2 function

= $P(C_1 \cap C_2)$

= $P(C_1)$. P(C_2) (assuming that C_1 and C_2 function independently)

$\boldsymbol{R_s = R_1 . R_2}$ **…….(1)**

The result obtained for two component series system can be extended for any n independent components in series.

Consider a system comprising of any n components in series $C_1, C_2, \ldots, C_n$ with reliabilities as R_1(t), R_2(t),… R_n(t), and R_s(t) be the reliability of the system.

Then, $\boldsymbol{R_s(t)} = \boldsymbol{R_1(t)} \times \boldsymbol{R_2(t)} \times \cdots \times \boldsymbol{R_n(t)}$

$$\leq \boldsymbol{min}\,\{\boldsymbol{R_1}(\mathbf{t}), \boldsymbol{R_2}(\mathbf{t}), \ldots \boldsymbol{R_n}(\mathbf{t})\} \qquad [\text{since, } 0 < R_i(\text{t}) < 1]$$

i.e. the system reliability will not be greater than the smallest of the component reliabilities.

Some Special Deductions:

1. If each component has a constant failure rate λ_i, then

$$R_s(t) = e^{-\lambda_1 t}\, e^{-\lambda_2 t} \ldots\ldots e^{-\lambda_n t}$$

$$= e^{-(\lambda_1+\lambda_2+\ldots+\lambda_n)t} = e^{-\lambda_s t}$$

It means that the system also has a constant failure rate $\lambda_s = (\lambda_1+\lambda_2 + \ldots + \lambda_n)$

2. If the components follow the Weibull failure law with parameters β_i and θ_i, then

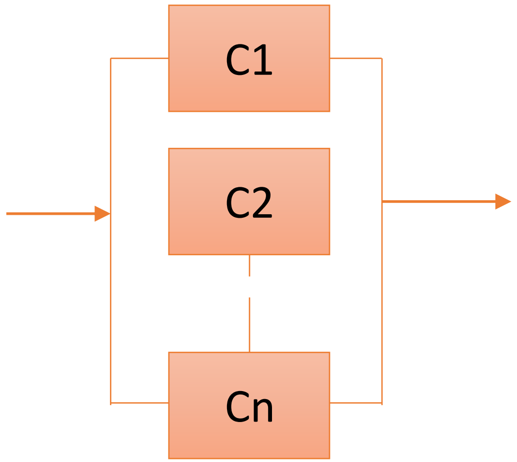
$$R_s(t) = e^{-\left(\frac{t}{\theta_1}\right)^{\beta_1}} \times e^{-\left(\frac{t}{\theta_2}\right)^{\beta_2}} \times \ldots\ldots \times e^{-\left(\frac{t}{\theta_n}\right)^{\beta_n}}$$

$$= \exp\left[-\sum_{i=1}^{n}\left(\frac{t}{\theta_i}\right)^{\beta_i}\right]$$

It means that the system does not follow Weibull failure law, even though every component follows a Weibull failure distribution.

Parallel Configuration

- Parallel or redundant configuration is one in which the components of the system are connected in parallel as shown in figure. Here, each block represents a component.



- In parallel configuration, all components must fail for a system to fail.

- It means that if one or more components function, the system continues to function.

- Consider a system comprising of two components in parallel C_1 and C_2 .
- Let R_1(t), R_2(t) and R_P(t) be the reliabilities of the components and the system respectively.

Then, R_1= P(C_1) = probability that C_1 functions

R_2= P(C_2) = probability that C_2 functions

Also, R_P = probability that (C_1 or C_2 or both function)

$= P(C_1 \cup C_2)$

$= P(C_1) + \mathrm{P}(C_2) - P(C_1 \cap C_2)$

$= P(C_1) + \mathrm{P}(C_2) - P(C_1).P(C_2)$ (as components are independent)

$\boldsymbol{R_P = R_1 + R_2 - R_1 R_2}$

= 1- (1- $\boldsymbol{R_1}$)(1- $\boldsymbol{R_2}$)

The result obtained for two component parallel system can be extended for any n independent components in parallel.

Consider a system comprising of any n components in parallel $C_1, \; C_2, \ldots, C_n$ with reliabilities as R_1(t), R_2(t),… R_n(t), and R_P(t) be the reliability of the system.

Then, $\boldsymbol{R_P(t)}$ **= 1- {(1-** $\boldsymbol{R_1}$**) (1-** $\boldsymbol{R_2}$**)…. (1-** $\boldsymbol{R_n}$**)}**

$$\geq \boldsymbol{max}\,\{\boldsymbol{R_1}, \boldsymbol{R_2}, \ldots \boldsymbol{R_n}\} \qquad [\text{since, } 0 < R_i(\text{t}) < 1]$$

i.e. the system reliability will be greater than the largest of the component reliabilities.

Deduction:

If each component has a constant failure rate λ_i, then

$$R_i(t) = e^{-\lambda_i t},$$

$$R_P(t) = 1 - (1 - e^{-\lambda_1 t})\,(1 - e^{-\lambda_2 t})$$

$$= e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t}$$

and MTTF = $\int_0^\infty R_P(t)\, dt$

$$= \int_0^\infty e^{-\lambda_1 t} dt + \int_0^\infty e^{-\lambda_2 t} dt + \int_0^\infty e^{-(\lambda_1+\lambda_2)t} dt$$

$$= \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1+\lambda_2}$$

Example An electronic circuit consists of 5 silicon transistors, 3 silicon diodes, 10 composition resistors and 2 ceramic capacitors connected in series configuration. The hourly failure rate of each component is given below:

Silicon transistor: $\lambda_t = 4 \times 10^{-5}$; Silicon Diode: $\lambda_D = 3 \times 10^{-5}$

Composition Resistor: $\lambda_r = 2 \times 10^{-4}$; Ceramic Capacitor: $\lambda_C = 2 \times 10^{-4}$

Calculate the reliability of the circuit for 10 hours, when the components follow exponential distribution.

Solution: Since the components are connected in series, the system (circuit) reliability is given by

$$\boldsymbol{R_s(t)} = \boldsymbol{R_1(t)} \times \boldsymbol{R_2(t)} \times \boldsymbol{R_3(t)} \times \boldsymbol{R_4(t)}$$

$$= e^{-\lambda_1 t}\, e^{-\lambda_2 t} e^{-\lambda_3 t}\, e^{-\lambda_4 t}$$

$$= e^{-(5\lambda_t + 3\lambda_D + 10\lambda_r + 2\lambda_C)t}$$

Therefore, $\boldsymbol{R_s(10)} = e^{-(20\times10^{-5} + 9\times10^{-5} + 20\times10^{-4} + 4\times10^{-4})10}$

$$= e^{-0.0269} = 0.9735$$

Example Thermocouples of a particular design have a failure rate of 0.008 per hour. How many thermocouples must be placed in parallel if the system is to run for 100 hours with a system failure probability of no more than 0.05? Assume that all failures are independent.

Solution: If T is the time to failure of the system, it is required that

$$P(T \leq 100) \leq 0.05$$

i.e., $1 - R_P(100) \leq 0.05$

Let the number of thermocouples to be connected in parallel be n,

Then $R_P(t) = 1 - (1 - R)^n$, where R is the reliability of each couple.

Given, the failure rate of each couple =0.008 (constant)

Therefore, $R = e^{-0.008t}$

So, we have $1 - R_P(t) = (1 - R)^n = (1 - e^{-0.008t})^n$

$$1 - R_P(100) = (1 - e^{-0.8})^n \leq 0.05$$

$$= (0.55067)^n \leq 0.05$$

By trials, above equation is satisfied for n= 6.

Example A system consists of two subsystems in parallel. The reliability of each sub system is given by (Weibull failure) $R(t) = e^{-\left(\frac{t}{\theta}\right)^2}$. Determine the system MTTF.

Solution: The system reliability is given by

$$R_s(t) = -\left[1 - e^{-\left(\frac{t}{\theta}\right)^2}\right]^2$$

$$= 1 - \left[1 - 2e^{-t^2/\theta^2} + e^{-2t^2/\theta^2}\right]$$

$$= 2e^{-t^2/\theta^2} - e^{-2t^2/\theta^2}$$

$$\text{System MTTF} = \int_0^\infty R_s(t)dt = 2\int_0^\infty e^{-t^2/\theta^2}dt - \int_0^\infty e^{-2t^2/\theta^2}dt$$

$$= 2\int_0^\infty e^{-x^2} \cdot \theta dx = \int_0^\infty e^{-y^2} \cdot \frac{\theta}{\sqrt{2}}dy,$$

(on putting $t = \theta x$ in the first integral and $t = \dfrac{1}{\sqrt{2}}\theta y$ in the second integral.)

$$= 2\theta \times \frac{\sqrt{\pi}}{2} - \frac{\theta}{2} \times \frac{\sqrt{\pi}}{2}$$

$$= 1.15\ \theta.$$

Practice Questions

1. A power supply consists of three rectifiers in series. Each rectifier has a Weibull failure distribution with $\beta = 2.1$. However they have different characteristic lifetimes given by 12,000 hours, 18,500 hours and 21,500 hours. Find the MTTF and the design life of the power supply corresponding to a reliability of 0.90. (Ans. 8837.5 hours; 3417 hours)

2. Which of the following systems has the higher reliability at the end of 100 operating hours?

(i) Two constant failure rate components in parallel each having an MTTF of 1000 hours.

(ii) A Weibull component with a shape parameter of 2 and a characteristic life of 10,000 hours in series with a constant failure rate component with a failure rate of 0.00005.

(Ans. (i) 0.9909, (ii) 0.9949; system (ii) has higher reliability)

References

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4. Miller, S., Childers, D., Probability and Random Processes, Academic Press, 2012.