Department of Mathematics

Probability and Random Processes — **15B11MA301**
Probability Theory and Random Processes — **10B11MA411**
Tutorial Sheet 3 — **2-D Random Variables, MGF, CF** — **B.Tech. Core**

1. Define the following: (a) two dimensional random variable (b) marginal and conditional probability distributions (c) If X denotes the number of kings and Y denotes number of aces when two cards are drawn at random without replacement from a deck of well shuffled pack of 52 cards, find
   (i) The joint probability distribution of (X, Y). (ii) The marginal distribution
   (iii) P(X=2/Y=1) (iv) P(X<2/0<Y<2) (v) $P(1 \le X \le 2 / Y = 0,2)$.
2. Let the joint pdf of a random variable (X, Y) is defined as $f(x, y) = k(xy + y^2)$, $0 \le x \le 2$, $0 \le y \le 1$. Find (i) the value of k, (ii) P(X>1) (iii) P(X+Y<1) (vi) P(X<1,Y>1/2) (v) $f_X(x)$ and $f_Y(y)$. Also test whether X and Y are independent?
3. The pdf of (X,Y) be defined as $f(x, y) = (1/4)\, e^{-|x|-|y|}$, $-\infty \le x < \infty$, $-\infty \le y < \infty$. Are X and Y independent? Find the proability that $X \le 1$ and $Y \le 0$.
4. Random variable (X, Y) have a joint probability density function $f(x, y)=(2x+y)/27$, where x and y can assume only integer values 0, 1, 2. Find the conditional distribution of Y *for* $X = x$.
5. Two ideal dice are thrown. Let X_1 be the score on the first die and X_2 the score on the other die. Let Y denote the maximum of X_1 and X_2 i.e. $\max(X_1, X_2)$.
   (a) Write down the joint distribution of Y and X_1, (b) Find E(Y) and Var(Y).
6. Let $f(x_1,x_2) = \begin{cases} 21x_1^2x_2^3 & 0 < x_1 < x_2 < 1 \\ 0 & \text{elsewhere} \end{cases}$ be the joint pdf of X_1 and X_2. Find the conditional mean and variance of X_1, given $X_2 = x_2$ and $0 < x_2 < 1$.
7. A pair of fair dice is thrown and let X be the number of 6's turned up. Find the moment generating function (MGF), mean and variance of X.
8. Find CF and MGF of X whose probability density function is given by $f(x) = k\,\frac{e^{-|x|}}{5}$ $-\infty < x < \infty$. Find first three moments of X about the origin. What is the variance of X?
9. The joint pdf of a two dimensional random variables (X, Y) is

$$f(x, y) = \begin{cases} \frac{3}{2}\left(x^2 + y^2\right) & \text{if } 0 \le x, y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

   Find C_{XY}, $E(XY)$ and ρ_{XY}.
10. Let X and Y be two independent Poisson random variables with $P_X(k) = \frac{1}{k!}e^{-2}2^k$, $P_Y(k) = \frac{1}{k!}e^{-3}3^k$. Compute the MGF of Z = 2X + 3Y.
11. Compute the characteristic function of discrete random variables X and Y if the joint pmf

$$P_{XY}(k, l) = \begin{cases} 1/3, & k = l = 0 \\ 1/6, & k = \pm 1, l = 0 \\ 1/6, & k = l = \pm 1 \\ 0, & \text{else.} \end{cases}$$

12. Find the density function of the distribution for which the characteristic function is given by $\phi(t) = e^{-\sigma^2 t^2/2}$.

Q.1] X : no. of kings   Y : no. of aces

X & Y are mutally exclusive events

no. of trials = 2

i)

$x \backslash y$	0	1	2
0	$^{44}C_2/^{52}C_2$ $\frac{44\times43}{52\times51}$	$^{44}C_1\cdot{}^{4}C_1/^{52}C_2$ $\frac{44\times4\times2}{52\times51}$	$^{4}C_2/^{52}C_2$ $\frac{4\times3}{52\times51}$
1	$^{44}C_1\cdot{}^{4}C_1/^{52}C_2$ $\frac{44\times4\times2}{52\times51}$	$^{4}C_1\cdot{}^{4}C_1/^{52}C_2$ $\frac{4\times4\times2}{52\times51}$	0
2	$^{4}C_2/^{52}C_2$ $\frac{4\times3}{52\times51}$	0	0

ii) Marginals of X & Y

$$P(X=0) = P(Y=0) = \frac{44(51)+12}{52\times51}$$

$$P(X=1) = P(Y=1) = \frac{8\times48}{52\times51}$$

$$P(X=2) = P(Y=2) = \frac{4\times3}{52\times51}$$

iii) $P(X=2/Y=1) = 0$

iv)

$$P(X<2/0<Y<2) = \frac{P(X<2,\, Y=1)}{P(Y=1)} = \frac{^{44}C_1\cdot{}^{4}C_1/^{52}C_2 + (^{4}C_1\cdot{}^{4}C_1/^{52}C_2)}{^{44}C_1\cdot{}^{4}C_1/^{52}C_2 + (^{4}C_1\cdot{}^{4}C_1/^{52}C_2)}$$

$$= 1$$

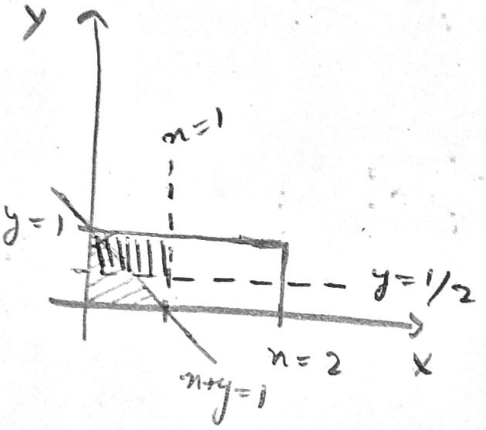
v)

$$P(1\le X\le 2 / Y=0,2) = \frac{P(X=1,2,\, Y=0,2)}{P(Y=0,2)} = \frac{\frac{44\times8+12}{\cancel{52\times51}}}{\frac{44\times51+12+12}{\cancel{52\times51}}}$$

$$= \frac{364}{2268}$$

Q2] $f(x,y) = k(xy + y^2)$ $\quad 0 \le x \le 2$, $0 \le y \le 1$

support domain



i) To find k

$$1 = k\int_0^1\int_0^2 xy + y^2 \, dx\, dy$$

$$= k\int_0^1 \left| \frac{x^2 y}{2} + xy^2 \right|_0^2 dy$$

$$= 2k \left| \frac{y^2}{2} + \frac{y^3}{3} \right|_0^1 = 2k \times \frac{5}{6}$$

$$\boxed{k = 3/5}$$

ii) $$P(X>1) = \frac{3}{5}\int_0^1\int_1^2 xy + y^2 \, dx\, dy = \frac{3}{5}\int_0^1 \left| \frac{x^2 y}{2} + xy^2 \right|_1^2 dy$$

$$= \frac{3}{5}\int_0^1 \frac{3}{2}y + y^2 \, dy = \frac{3}{5}\left| \frac{3y^2}{4} + \frac{y^3}{3} \right|_0^1$$

$$= \frac{3}{5}\left[\frac{9+4}{12} \right] = \frac{13}{20}$$

iii) $$P(X+Y<1) = \frac{3}{5}\int_0^1\int_0^{1-y} xy + y^2 \, dx\, dy = \frac{3}{5}\int_0^1 \frac{(1-y)^2 y}{2} + (1-y)y^2 \, dy$$

$$= \frac{3}{10}\int_0^1 y - y^3 \, dy = \frac{3}{10} \times \frac{1}{4} = \frac{3}{40}$$

iv) $$P(X<1, Y>1/2) = \frac{3}{5}\int_{1/2}^1\int_0^1 xy + y^2 \, dx\, dy = \frac{3}{5}\int_{1/2}^1 \left[y/2 + y^2 \right] dy$$

$$= \frac{3}{5}\left| \frac{y^2}{4} + \frac{y^3}{3} \right|_{1/2}^1 = \frac{3}{5}\left[\frac{1}{4} + \frac{1}{3} - \frac{1}{16} - \frac{1}{24} \right]$$

$$= \frac{3}{5} \times \frac{23}{48} = \frac{23}{80}$$

2) ii) $f_x(x) = \frac{3}{5}\int_0^1 xy + y^2\,dy = \frac{3}{5}\left[\frac{x}{2} + \frac{1}{3}\right]$

$f_y(y) = \frac{3}{5}\int_0^2 xy + y^2\,dx = \frac{3}{5}\left[2y + 2y^2\right]$

$f_x \cdot f_y = \frac{3}{5}\left[\frac{x}{2} + \frac{1}{3}\right] \times \frac{3}{5}\left[2y + 2y^2\right] = \cancel{2\times}\frac{3}{5} \times \frac{3}{5}\left(\frac{3x+2}{\cancel{6}}\right)(y+y^2)$

$= \frac{3}{25}(3x+2)(y+y^2) = \frac{3}{25}(3xy + 3xy^2 + 2y + 2y^2)$

$\Rightarrow$ X & Y are not Independent
$f_X f_Y \neq f_{XY}$

Q3] $f(x,y) = \frac{1}{4} e^{-|x| - |y|}$

$P(X \leq 1, Y \leq 0) = \int_{-\infty}^{0}\int_{-\infty}^{1} \frac{1}{4} e^{-|x| - |y|}\,dx\,dy$

$= \frac{1}{4}\int_{-\infty}^{0} e^{-|y|}\,dy \int_{-\infty}^{1} e^{-|x|}\,dx = \frac{1}{4}\int_{-\infty}^{0} e^{+y}\,dy \left[\int_{-\infty}^{0} e^{x}\,dx + \int_{0}^{1} e^{-x}\,dx\right]$

$= \frac{1}{4}(1)\left[(1) + (-e^{-1} + 1)\right] = \frac{1}{4}\left(2 - \frac{1}{e}\right) = \frac{2e-1}{4e}$

Q4] $f(x,y) = (2x+y)/27$ $x = 0,1,2$
$y = 0,1,2$ 5

$x \backslash y$	0	1	2	
0	0	1/27	2/27	3/27
1	2/27	3/27	4/27	9/27
2	4/27	5/27	6/27	15/27
	6/27	9/27	12/27	

conditional distribution of Y for given X = x

$$f(y/x) = \frac{f_{xy}}{f_x}$$

$x \backslash y$	0	1	2
0	0	1/3	2/3
1	2/9	3/9	4/9
2	4/15	5/15	6/15

Q5] X_1 = no. of first dice (1, 2, 3, 4, 5, 6)

X_2 = no of second dice (1, 2, 3, 4, 5, 6)

$Y = \max(X_1, X_2)$

a)

X_1 \ Y	1	2	3	4	5	6
1	1/36	1/36	1/36	1/36	1/36	1/36
2	0	2/36	1/36	1/36	1/36	1/36
3	0	0	3/36	1/36	1/36	1/36
4	0	0	0	4/36	1/36	1/36
5	0	0	0	0	5/36	1/36
6	0	0	0	0	0	6/36
	1/36	3/36	5/36	7/36	9/36	11/36

b) $E[Y] = \sum_{j=1}^{6} Y_j \times P_{*j}$

$= 1 \times \frac{1}{36} + 2 \times \frac{3}{36} + 3 \times \frac{5}{36} + 4 \times \frac{7}{36} + 5 \times \frac{9}{36} + 6 \times \frac{11}{36}$

$= \frac{1 + 6 + 15 + 28 + 45 + 66}{36} = \frac{161}{36}$

c) $Var(Y) = E[Y^2] - E[Y]^2$

$E[Y^2] = \frac{1 + 12 + 45 + 112 + 225 + 396}{36} = \frac{791}{36}$

$Var(Y) = \frac{791}{36} - \left(\frac{161}{36}\right)^2 = \frac{28,476 - 25921}{1296}$

$= \frac{2555}{1296}$

Q6] $f(x,y) = \begin{cases} 21x^2y^3 & 0 < x < y < 1 \\ 0 & \text{else} \end{cases}$

Q7]

a) $E[X/Y] = \int x \, f_{X/Y} \, dx$

$$f_{X/Y} = \frac{f_{XY}}{f_Y} = \frac{21x^2y^3}{\int_0^y 21x^2y^3 \, dx} = \frac{x^2 \cancel{y^3}}{\cancel{y^3} \frac{y^3}{3}} = \frac{3x^2}{y^3}$$

$$E[X/Y] = \int_0^y \frac{3x^3}{y^3} dx = \frac{3}{\cancel{y^3}} \times \frac{y^4}{4} = \frac{3}{4} y$$

b) $Var(X/Y) = E[X^2/Y] - E[X/Y]^2$

$$E[X^2/Y] = \int_0^y x^2 \left(\frac{3x^2}{y^3}\right) = \frac{3}{y^3} \times \frac{y^5}{5} = \frac{3}{5} y^2$$

$$Var(X/Y) = \frac{3}{5}y^2 - \frac{9}{16}y^2 = \frac{3}{80}y^2$$

Q7] X : no of 6 when 2 dice are thrown

X =	0	1	2
$P[x]$ =	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

a) $E[x] = \sum x\, P[x] = \frac{0\times 25 + 10 + 2\times 1}{36} = \frac{12}{36} = \frac{1}{3}$

$E[x^2] = \sum x^2\, P[x] = \frac{0\times 25 + 1\times 10 + 4\times 1}{36} = \frac{14}{36} = \frac{7}{18}$

b) $Var(x) = E[x^2] - E[x]^2 = \frac{7}{18} - \frac{1}{9} = \frac{5}{18}$

c) $M_x(t) = E[e^{xt}] = \sum_{x=0}^{2} e^{xt}\, P[x]$

$= \frac{25}{36} + e^t \cdot \frac{10}{36} + e^{2t} \cdot \frac{1}{36}$

Q 8] $f(x) = \frac{k e^{-|x|}}{5}$

$$1 = \int_{-\infty}^{0} k \frac{e^{+x}}{5} + \int_{0}^{\infty} \frac{k e^{-x}}{5} = \frac{k}{5}\left[(1-0) + (0+1)\right]$$

$$\boxed{k = 5/2}$$

$$\Rightarrow f(x) = \frac{1}{2} e^{-|x|}$$

$$M_x(t) = E[e^{xt}] = \int_{-\infty}^{\infty} e^{xt}\left(\frac{1}{2}e^{-|x|}\right)dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{0} e^{x(t+1)} + \int_{0}^{\infty} e^{-x(1-t)}\right]dx = \frac{1}{2}\left[\frac{1}{t+1} - \frac{1}{t-1}\right]$$

$$= \frac{1}{1-t^2} = [1-t^2]^{-1}$$

$$= 1 + t^2 + t^4 + t^6 \ldots$$

$E[x] = 0 = \mu_1'$
$E[x^2] = 2! \times 1 = 2 = \mu_2'$
$E[x^3] = 3! \times 0 = 0 = \mu_3'$

$$Var(x) = \mu_2' - \mu_1'^2 = 2 - 0^2 = \underline{\underline{2}}$$

$\phi_x(i\omega) = \frac{1}{1-(i\omega)^2} = \frac{1}{1+\omega^2}$

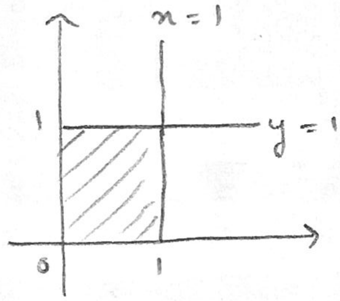
Q9] $f(x,y) = \begin{cases} \frac{3}{2}(x^2+y^2) & 0 \le x, y \le 1 \\ 0 & \text{else} \end{cases}$

$$1 = \int_0^{c_x}\int_0^1 \frac{3}{2}(x^2+y^2)\,dy\,dx = \frac{3}{2}\int_0^{c_x} \left| x^2y + \frac{y^3}{3} \right|_0^1 dx$$

$$1 = \frac{3}{2}\int_0^{c_x}\left(x^2+\frac{1}{3}\right)dx = \frac{3}{2}\left| \frac{x^3}{3} + \frac{x}{3} \right|_0^{c_x}$$

$$2 = c_x^3 + c_x = c_x(c_x^2+1)$$

$$\boxed{c_x = 1}$$



$$f(x,x) = \frac{3}{2} \times 2x^2 = 3x^2$$
$$f(y,y) = \frac{3}{2} \times 2y^2 = 3y^2$$
$$f(x,x)\big|_{x=y} = f(y,y)$$

funcn is symtric & limits are same

$$f_X = \frac{3}{2}\int_0^1 x^2+y^2\,dy = \frac{3}{2}\left| x^2y + \frac{y^3}{3} \right|_0^1 = \frac{3}{2}\left| x^2 + \frac{1}{3} \right|$$

$$f_Y = \frac{3}{2}\int_0^1 x^2+y^2\,dx = \frac{3}{2}\left| \frac{x^3}{3} + xy^2 \right|_0^1 = \frac{3}{2}\left| y^2 + \frac{1}{3} \right|$$

$$E[X] = E[Y] = \frac{3}{2}\int_0^1 x\left(x^2+\frac{1}{3}\right)dx = \frac{3}{2}\left| \frac{x^4}{4} + \frac{x^2}{6} \right|_0^1$$

$$= \frac{3}{2}\left| \frac{1}{4} + \frac{1}{6} \right| = \frac{5}{8}$$

$$E[x^2] = E[y^2] = \frac{3}{2}\int_0^1 x^2\left(x^2+\frac{1}{3}\right)dx = \frac{3}{2}\left|\frac{x^5}{5}+\frac{x^3}{9}\right|_0^{1,0}$$

$$= \frac{3}{2}\left|\frac{1}{5}+\frac{1}{9}\right| = \frac{\cancel{3}}{2}\times\frac{14}{\cancel{45}\ 15} = \frac{14}{30} = \frac{7}{15}$$

$$Var(x) = Var(y) = \frac{7}{15} - \left(\frac{5}{8}\right)^2 = \frac{7}{15} - \frac{25}{64} = \frac{448-375}{960}$$

$$= \frac{73}{960}$$

$$E[xy] = \frac{3}{2}\int_0^1\int_0^1 (xy)(x^2+y^2)\,dx\,dy = \frac{3}{2}\int_0^1 \left|\frac{x^4}{4}y + \frac{x^2}{2}y^3\right|_0^1$$

$$= \frac{3}{2}\int_0^1 \frac{y}{4} + \frac{y^3}{2} = \frac{3}{2}\left|\frac{y^2}{8} + \frac{y^4}{8}\right|_0^1 = \frac{3}{8}$$

$$Cov(x,y) = E[xy] - E[x]E[y] = \frac{3}{8} - \frac{5}{8}\times\frac{5}{8}$$

$$= \frac{-1}{64}$$

$$\delta_{xy} = \frac{Cov(x,y)}{\sqrt{Var(x)}\sqrt{Var(y)}} = \frac{-1/64}{(73/960)} = -\frac{960}{73\times 64} = \frac{-15}{73}$$

3.10] Poisson Distribution

$$P_x(r) = \frac{e^{-\lambda} \lambda^r}{r!} \qquad r = 0, 1 \ldots \infty$$

$$\lambda = \text{const}$$

$$M_x(t) = E[e^{rt}] = \sum_{r=0}^{\infty} \frac{e^{tr} \cdot e^{-\lambda} \lambda^r}{r!}$$

$$= e^{-\lambda} \sum_{r=0}^{\infty} \frac{(e^t \cdot \lambda)^r}{r!}$$

$$= e^{-\lambda} e^{(e^t \cdot \lambda)} = e^{\lambda(e^t - 1)}$$

Now, given $P_x(k) = \frac{1}{k!} e^{-2} 2^k \leftarrow \lambda = 2$

$$P_y(k) = \frac{1}{k!} e^{-3} 3^k \leftarrow \lambda = 3$$

$$M_x(t) = e^{2(e^t - 1)} \quad \& \quad M_y(t) = e^{3(e^t - 1)}$$

$$Z = 2X + 3Y$$

$$M_z(t) = M_{2x+3y}(t) = M_{2x}(t) \cdot M_{3y}(t)$$

$$= M_x(2t) \cdot M_y(3t) = e^{2(e^{2t} - 1)} \cdot e^{3(e^{3t} - 1)}$$

$$= e^{2e^{2t} - 2 + 3e^{3t} - 3}$$

$$= \underline{\underline{e^{2e^{2t} + 3e^{3t} - 5}}}$$

Q.11]

$$P_{XY}(\cancel{k}_x, \cancel{l}_y) = \begin{cases} 1/3 & k = l = 0 \\ 1/6 & k = \pm 1, \ l = 0 \\ 1/6 & k = l = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\phi(\omega) = \sum_k \sum_l e^{i\omega_1 x_0} e^{i\omega_2 y_0} p_{ij}$$

$$= \frac{1}{3} + \frac{1}{6}\left[e^{i\omega_1} + e^{-i\omega_1} + e^{i\omega_1} e^{i\omega_2} + e^{-i\omega_1} e^{-i\omega_2}\right]$$

$$= \frac{1}{3} + \frac{1}{6}\left[e^{i\omega_1}\left(1 + e^{i\omega_2}\right) + e^{-i\omega_1}\left(1 + e^{-i\omega_2}\right)\right]$$