Department of Mathematics

15B11MA211 **Mathematics-II**

Tutorial Sheet 14 **B.Tech. Core**

Topic: Evaluation of Real Integrals Using Residues

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1. Evaluate the real integral $\int_0^{2\pi} \frac{1}{3-2\cos\theta} d\theta$ using suitable contour.
2. Evaluate $\int_0^{\pi} \frac{4\cos 2\theta}{4-4\cos\theta+1} d\theta$, using suitable contour.
3. Evaluate the integral $\int_{-\infty}^{\infty} \frac{x^2+1}{1+x^4} dx$, using suitable contour.
4. Evaluate $\int_0^{\infty} \frac{x^2}{(1+x^4)(x^2+1)} dx$, using suitable contour.

Ans:

1. $2\pi/\sqrt{5}$
2. $2\pi/3$
3. $\sqrt{2}\,\pi$
4. $\pi/6$

1.

[redacted] . $\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta}$

let $z = e^{i\theta}$, $|z|=1 \Rightarrow dz = iz\,d\theta \Rightarrow d\theta = \frac{dz}{iz}$

$\cos\theta = \frac{z+z^{-1}}{2} = \frac{z^2+1}{2z}$ [redacted]

$$\int_{|z|=1} \frac{dz}{iz\left(3-2\left(\frac{z^2+1}{2z}\right)\right)} = \frac{1}{i}\int_{|z|=1} \frac{dz}{(3z-z^2-1)}$$

$$= \quad = i\int_{|z|=1} \frac{dz}{(z-a_1)(z-a_2)}$$

where $a_1 = \frac{3+\sqrt{5}}{2}$

$a_2 = \frac{3-\sqrt{5}}{2}$, only it lies inside.

$z^2-3z+1=0$

$z = \frac{3\pm\sqrt{9-4}}{2}$

$= \frac{3\pm\sqrt{5}}{2}$

[redacted]

$$= i \cdot 2\pi i \cdot \left(\text{Res}\left(\text{at } z = \frac{3-\sqrt{5}}{2}\right)\right)$$

$$= -2\pi \cdot \frac{1}{\left(\frac{3-\sqrt{5}}{2} - \frac{3+\sqrt{5}}{2}\right)} = -4\pi \cdot \frac{1}{\cancel{3}-\sqrt{5}-\cancel{3}-\sqrt{5}}$$

$$= \frac{2\pi}{\sqrt{5}} \quad \text{or} \quad \frac{2\sqrt{5}}{5}\pi$$ [redacted]

2.

$$\int_0^{\pi} \frac{4\cos 2\theta}{5-4\cos\theta}\, d\theta$$

$$\int_0^{2\pi} \frac{4\cos 2\theta}{5-4\cos\theta}\, d\theta = \int_0^{\pi} \text{''}\, d\theta + \int_{\pi}^{2\pi} \text{''}\, d\theta$$

$2\pi - \phi = \theta$

$d\theta = -d\phi$

$$\int_0^{\pi} \frac{4\cos 2\theta\, d\theta}{5-4\cos\theta} - \int_{\pi}^{0} \frac{4\cos 2\phi}{5-4\cos\phi}\, d\phi$$

$$= 2\int_0^{\pi} \frac{4\cos 2\theta}{5-4\cos\theta}\, d\theta$$

Put $z = e^{i\theta}$

$dz = i e^{i\theta}\, d\theta$

$d\theta = \dfrac{dz}{iz}$

$\cos\theta = \dfrac{z + 1/z}{2}$

$\cos 2\theta = \dfrac{e^{2i\theta} + e^{-2i\theta}}{2} = \dfrac{z^2 + 1/z^2}{2}$

$C: |z| = 1$

$0 \le \theta \le 2\pi$

$$I = \int_C \frac{2(z^2 + 1/z^2)}{5 - 2(z + 1/z)} \frac{dz}{iz}$$

$$= \frac{2}{i} \int_C \frac{(z^4+1)/z^2}{5 - 2\left(\frac{z^2+1}{z}\right)} \cdot \frac{dz}{z}$$

$$= \frac{2}{i} \int_C \frac{(z^4+1)/z^{\cancel{2}}}{(5z - 2z^2 - 2)/\cancel{z}} \frac{dz}{z} = -\frac{2}{i} \int_C \frac{z^4+1}{z^2(2z^2 - 5z + 2)}\, dz$$

$$= -\frac{2}{i}\int_C \frac{z^4+1}{z^2(2z-1)(z-2)}\,dz$$

Res (at $z = 0$) $= \frac{d}{dz}\left(\frac{z^4+1}{2z^2-5z+2}\right)$ at $z=0$

$$= \frac{(2z^2-5z+2)(4z^3)-(z^4+1)(4z-5)}{(2z^2-5z+2)^2}$$

$$= \frac{-(+1)(-5)}{2^2} = \frac{5}{4}$$

Res (at $z = \frac{1}{2}$) $= \lim_{z\to \frac{1}{2}} \cancel{(z-\tfrac{1}{2})}\,\frac{z^4+1}{2z^2\cancel{(z-\frac{1}{2})}(z-2)}$

$$= \frac{\frac{1}{16}+1}{2\times\frac{1}{4}\left(\frac{1}{2}-2\right)} = \frac{17/16}{-3/4}$$

$$= -\frac{17}{12}$$

~~Res~~ and $z = 2$ doesnot lie inside $|z| = 1$.

$$\therefore I = -\frac{2}{i}\int_C \frac{z^4+1}{z^2(2z-1)(z-2)}\,dz$$

$$= -\frac{2}{i}(2\pi i)\left(\frac{5}{4}-\frac{17}{12}\right)$$

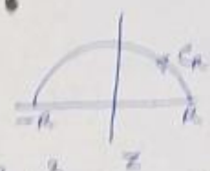
$$= -4\pi\left(\frac{15-17}{\cancel{12}\,3}\right)$$

$$= \frac{2\pi}{3}$$

3.

$$\int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1}\, dx$$

$$\int_C \frac{z^2+1}{z^4+1}\, dz = \int_C f(z)\, dz$$



$$z^4+1=0 \Rightarrow z^4=-1$$
$$= e^{(2n+1)\pi i}$$
$$\Rightarrow z = e^{(2n+1)i\pi/4}$$

$n=0,1,2,3$

$z = e^{i\pi/4}, e^{3i\pi/4}$ only lie inside C

Res (at $z=e^{i\pi/4}$)

$$= \frac{e^{i\pi/2}+1}{4e^{i3\pi/4}} = \frac{i+1}{4\left(\frac{-1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)}$$

$$e^{i3\pi/4} = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$$
$$= -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

Res (at $z=e^{3i\pi/4}$)

$$= \frac{e^{3i\pi/2}+1}{4e^{i9\pi/4}} = \frac{1-i}{4\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)}$$

~~Res (at $z=e^{5i\pi/4}$)~~

$$e^{i3\pi/2} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$$
$$= 0 + i(-1) = -i$$

$$e^{i9\pi/4} = \cos\frac{9\pi}{4} + i\sin\frac{9\pi}{4}$$

$$= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$$

Sum of residues $= \dfrac{i+1}{4\left(\frac{-1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)} + \dfrac{1-i}{4\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)}$

$$= \frac{1}{\frac{4}{\sqrt{2}}}\left[\frac{i+1}{(-1+i)} + \frac{1-i}{1+i}\right]$$

$$= \frac{\sqrt{2}}{4}\left[\frac{\cancel{1}+i^2+2i + (-\cancel{1}+2i+\cancel{1})}{i^2-1}\right]$$

$$= \frac{\sqrt{2}}{4}\left[\frac{4i}{-2}\right] = -\frac{i}{\sqrt{2}}.$$

$$\int_C f(z)\,dz = 2\pi i\left(-\frac{i}{\sqrt{2}}\right) = \sqrt{2}\,\pi$$

Now

$$\int_C f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{-R}^{R}\frac{x^2+1}{x^4+1}\,dx.$$

when $R\to\infty$

$|z|\to\infty$.

$$|f(z)| = \left|\frac{z^2+1}{z^4+1}\right| = \frac{z^2\,|1+1/z^2|}{|z^4_{z^2}|\,|1+1/z^4|}$$

$$= \frac{1}{|z^2|}\,\frac{|1+1/z^2|}{|1+1/z^4|} = \frac{1}{R^2}\left(\frac{1+1/R^2}{1+1/R^4}\right)$$

$$\to 0.$$

$$\therefore \int_{C_R} f(z)\,dz = 0.$$

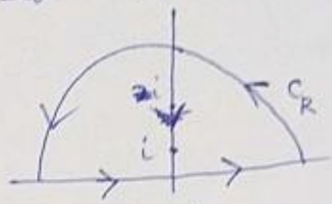
$$\therefore \int_C f(z)\,dz = \int_{-\infty}^{\infty}\frac{x^2+1}{x^4+1}\,dx = \sqrt{2}\,\pi.$$

4.

$$\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx$$

ider $\displaystyle\int_C \frac{z^2}{(z^2+1)(z^2+4)}\,dz$

$$= \int_C f(z)\,dz$$



where C is the contour consisting of ~~quarter~~ semi circle C_R of radius R together with the part of real axis from 0 to R ~~and part of imaginary axis from R to 0~~

The integrand has simple poles at $z = i, -i, 2i, -2i$ of which only $z = i, 2i$ lie inside C.

$$\text{Res (at } z=i) = \lim_{z\to i} (z-i)\, f(z).$$

$$= \lim_{z\to i} \cancel{(z-i)} \frac{z^2}{(z+i)\cancel{(z-i)}(z^2+4)}$$

$$= \frac{i^2}{2i(i^2+4)} = \frac{-1}{2i(3)} = -\frac{1}{6i}$$

$$\text{Res (at } z=2i) = \lim_{z\to 2i} (z-2i)\, f(z)$$

$$= \lim_{z\to 2i} \frac{z^2}{(z^2+1)(z+2i)}$$

$$= \frac{4i^2}{(4i^2+1)(4i)} = \frac{-4}{(-3)(4i)}$$

$$= \frac{1}{3i}$$

$\therefore$ By Residue thm

$$\int_C f(z)\, dz = 2\pi i\left(\frac{1}{3i} - \frac{1}{6i}\right)$$

$$= 2\pi\left(\frac{1}{3} - \frac{1}{6}\right) = \cancel{2}\pi\left(\frac{1}{\cancel{6}_{3}}\right) = \frac{\pi}{3}.$$

$$\int_C f(z)\, dz = \int_{C_R} f(z)\, dz + \int_{-R}^{R} f(x)\, dx + \cancel{\int_R^0 f(z)}$$

Now as $R \to \infty$

$$\frac{z^2}{(z^2+1)(z^2+4)} = \frac{z^2}{\cancel{z^4}_{z^2}\left(1+\frac{1}{z^2}\right)\left(1+\frac{4}{z^2}\right)}$$

$$= \frac{1}{z^2(1+z^{-2})(1+4z^{-2})}$$

$\frac{1}{z^2}$ decreases for any point on C_R and tends to zero as $|z| \to \infty$

$$\int_{C_R} f(z)\, dz = 0.$$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\, dx = \frac{\pi}{3}.$$

$$\int_0^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\, dx = \frac{\pi}{6}.$$