▷ BINOMIAL DISTRIBUTION

PMF

$$P(r) = {}^nC_r \, p^r q^{n-r}$$

n = no. of trials

r = no. of success $\quad 0 \leq r \leq n$

p = prob of success

q = prob of failure

MGF

$$M_r(t) = E[e^{rt}] = \int_0^n e^{rt} \cdot {}^nC_r \, p^r q^{n-r} \, dr$$

$$M_x(t) = E[e^{xt}] = \sum_{x=0}^{n} {}^nC_x\, p^x q^{n-x} e^{xt}$$

$$= q^n \sum_{x=0}^{n} {}^nC_x \left(\frac{pe^t}{q}\right)^x$$

$$M_x(t) = q^n\left(1+\frac{pe^t}{q}\right)^n = (q+pe^t)^n$$

$$\boxed{M_x(t) = (q+pe^t)^n}$$

$$\mu_1' = n(q+pe^t)^{n-1} pe^t \Big|_{t=0} = n(q+p)p$$

$$= np$$

$$\boxed{E[X] = np}$$

$$\mu_2' = n\left[(n-1)(q+pe^t)^{n-2} pe^t pe^t + (q+pe^t)^{n-1} pe^t\right]_{t=0}$$

$$= n\left[(n-1)p^2 + p\right] = np + n(n-1)p^2$$

$$Var(x) = np + n(n-1)p^2 - n^2p^2$$

$$= np + \cancel{n^2p^2} - np^2 - \cancel{n^2p^2} = npq$$

$$\boxed{Var(x) = npq}$$

▷ POISSON DISTRIBUTION

when in binomial distribution

$n \to \infty \quad p \to 0 \qquad \lambda = np = \text{const}$

$$\therefore \; {}^nC_r \, p^r q^{n-r} \longrightarrow \frac{e^{-\lambda} \lambda^r}{r!}$$

\# $$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$$

$$\to \; {}^nC_r \left(\frac{\lambda}{n}\right)^r \left(1 - \frac{\lambda}{n}\right)^{n-r}$$

$$\lim_{n \to \infty} \frac{n!}{r!(n-r)!} \times \frac{\lambda^r}{n^r} \times \left(1 - \frac{\lambda}{n}\right)^n \times \left(1 - \frac{\lambda}{n}\right)^{-r}$$

$$\frac{\lambda^r}{r!} \lim_{n \to \infty} \frac{\left|\overbrace{n(n-1)\ldots(n-r+1)}^{r \text{ terms}}\right|}{n^r} \left(1 - \frac{\lambda}{n}\right)^n \cancel{\left(1 - \frac{\lambda}{n}\right)^r}^{\,1}$$

$$\frac{\lambda^r}{r!} \lim_{n \to \infty} \underbrace{\left| \frac{n}{n} \times \left(1 - \frac{1}{n}\right) \times \ldots \left(1 - \frac{r+1}{n}\right) \right|}_{\to 1} \underbrace{\left(1 - \frac{\lambda}{n}\right)^n}_{\to e^{-\lambda}}$$

$$= \frac{\lambda^r e^{-\lambda}}{r!}$$

$$M_x(t) = E[e^{xt}] = \sum_{x=0}^{\infty} e^{xt} \cdot \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} \cdot \exp(\lambda e^t) = e^{-\lambda} \cdot e^{\lambda e^t}$$

$$M_x(t) = e^{\lambda(e^t-1)}$$

$$M_1' = e^{\lambda(e^t-1)} \cdot \lambda e^t \Big|_{t=0}$$

$$= \boxed{\lambda = E(x)}$$

$$M_2' = \lambda \left[e^{\lambda(e^t-1)} \cdot \lambda e^t e^t + e^{\lambda(e^t-1)} \cdot e^t \right]_{t=0}$$

$$= \lambda [\lambda + 1] = \lambda^2 + \lambda$$

$$\boxed{Var(x) = \lambda}$$

Q3]

$$\overset{\text{5-comp}}{P[x \geq 3]} > \overset{\text{3-comp}}{P[x \geq 2]}$$

$$\sum_{r=3}^{5} {}^5C_r\, p^r (1-p)^{5-r} > \sum_{r=2}^{3} {}^3C_r\, p^r (1-p)^{3-r}$$

$$\frac{\cancel{5!}^{\,5\times\cancel{4}^{2}}}{\cancel{3!}\,\cancel{2!}} p^3(1-p)^2 + 5p^4(1-p) + p^5$$

$$> 3p^2(1-p) + p^3$$

$$10p^3(1+p^2-2p) + 5p^4(1-p) + p^5$$

$$> 3(p^2-p^3) + p^3$$

$$10p^3 + 10p^5 - 20p^4 + 5p^4 - 5p^5 + p^5$$

$$> 3p^2 - 3p^3 + p^3$$

$$10p^3 + 6p^5 - 15p^4 > 3p^2 - 2p^3$$

$$10p + 6p^3 - 15p^2 > 3 - 2p$$

$$6p^3 - 15p^2 + 12p - 3 > 0$$

$$3(p-1)^2(2p-1) > 0$$

$$\rightarrow p < 1 \;\&\; p > \frac{1}{2}$$

$$\boxed{1/2 < p < 1}$$

Q4] Poisson Dist.

$n = 100{,}000 \quad p = 2 \times 10^{-5}$

$\lambda = np = 2 \times 10^{-5} \times 10^{5} = 2 = \text{const}$

$$P(x) = \frac{e^{-2}\, 2^{x}}{x!}$$

$$P[x \geq 5] = 1 - P[x < 5]$$

$$= 1 - e^{-2}\left[\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!}\right]$$

$$= 1 - e^{-2}\left[1 + 2 + 2 + \frac{4}{3} + \frac{\cancel{4} \times \cancel{4} \times 2}{\cancel{4} \times 3 \times \cancel{2}}\right]$$

$$= 1 - e^{-2}\left[1 + 2 + 2 + 2\right] = \underline{\underline{1 - 7e^{-2}}}$$

Q5] $p = \frac{25}{100} = \frac{1}{4} \quad q = \frac{3}{4}$

$$P(2) = {}^{10}C_2 \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^8 = \frac{\overset{5}{\cancel{10}} \times \cancel{9}\, 3 \times 3}{\cancel{2} \times 4 \times 4} \times \left(\frac{3}{4}\right)^8$$

$$= \underline{\underline{5 \times \left(\frac{3}{4}\right)^{10}}}$$

Q6] Ratio $= \frac{390}{520} = \frac{x}{5}$

$x = \frac{39\cancel{0}}{52\cancel{0}} \times 5 = 3.75$ error /page

$\lambda = 3.75$

$P(0) = \frac{e^{-3.75}(3.75)^0}{0!} = \underline{\underline{e^{-3.75}}}$

Q10] $\lambda = 2$

$P\left[\frac{X = 1 \text{ or } 2}{X \geq 1}\right] = \frac{P(X = 1 \text{ or } 2 \cap X \geq 1)}{1 - P(X=0)}$

$= \frac{P(X=1) + P(X=2)}{1 - P(X=0)}$ # events are independent

$= \frac{\frac{e^{-2}.2^1}{1!} + \frac{e^{-2}2^2}{2!}}{1 - \frac{e^{-2}2^0}{0!}} = \frac{4e^{-2}}{1-e^{-2}} = \underline{\underline{0.62607}}$

$E\left[\frac{X=1 \text{ or } X=2}{X \geq 1}\right] = \frac{\lambda - 0 \times P(X=0)}{1 - P(X=0)} = \frac{\lambda}{1-e^{-2}} = \frac{2}{1-e^{-2}}$

$= \underline{\underline{2.313}}$