Wiener Process as Limiting form of Random Walk

In (1), put $nT = t$, $md = x$ and $d^2 = \alpha T$ and take limits as $T \to 0$ and $n \to \infty$.

In the limit, $\{X(t)\}$ becomes a continuous process.

Now $$\frac{m}{\sqrt{n}} = \frac{x/d}{\sqrt{t/T}} = \frac{x}{\sqrt{d^2\, t/T}} = \frac{x}{\sqrt{\alpha t}} \quad (2)$$

Also $$\begin{aligned} \frac{n}{4} &= E\{X^2(nT)\} \\ &= nd^2 \\ &= n\alpha T \\ &= \alpha t \end{aligned} \quad (3)$$

If we use (2) and (3) in (1), when we proceed to limits,

$$P\{x \le X(t) \le x + dx\} = \frac{1}{\sqrt{2\pi\alpha t}} e^{-x^2/2\alpha t}\, dx, \ -\infty < x < \infty$$

i.e., the pdf of Wiener process $\{X(t)\}$ is

$$f_{x(t)}(x) = \frac{1}{\sqrt{2\pi\alpha t}}\, e^{-x^2/2\alpha t}$$

which is $N(0, \sqrt{\alpha t})$.

(ii) Obviously, the random walk $\{X(nT)\}$ is a process with independent increments.

i.e., $\{X(n_2 T) - X(n_1 T)\}$ and $\{X(n_1 T) - X(0)\}$ are independent.

Since Wiener process $\{X(t)\}$ is the limiting form of random walk, $\{X(t_2) - X(t_1)\}$ and $X(t_1)$ are independent.

Let $t_1 < t_2$

Then $$E[\{X(t_2) - X(t_1)\} \times X(t_1)]$$

$$= E\{X(t_2) - X(t_1)\} \times E\{X(t_1)\}$$

$$= 0 \qquad \text{[since } E\{X(t)\} = 0\text{]}$$

i.e., $E\{X(t_1) \times X(t_2)\} = E\{X^2(t_1)\}$

$$= \alpha t_1 \text{ [since Var } \{X(t)\} = \alpha t\text{]}$$

i.e., $R(t_1, t_2) = \alpha t_1$

Similarly, when $t_2 < t$, $R(t_1, t_2) = \alpha t_2$.

$\therefore$ $R(t_1, t_2) = \alpha \min(t_1, t_2)$

$$C(t_1, t_2) = R(t_1, t_2) - \mu(t_1) \times \mu(t_2)$$

$$= \alpha \min(t_1, t_2) \qquad \text{[since } \mu(t) = 0\text{]}$$

Example 14 *If $X(t)$ with $X(0) = 0$ and $\mu = 0$ is a W*

Properties of R(τ)

1. $R(t)$ is an even function of τ

Proof $R(\tau) = E\{X(t) \times X(t-\tau)\}$

$$\therefore \quad R(-\tau) = E\{X(t) \times X(t+\tau)\}$$
$$= E\{X(t+\tau) \times X(t)\}$$
$$= R(\tau)$$

Therefore, $R(\tau)$ is an even function of τ.

2. $R(\tau)$ is maximum at $\tau = 0$

i.e., $$|R(\tau)| \le R(0)$$

Proof Cauchy–Schwarz inequality is

$$\{E(XY)\}^2 \le E(X^2) \times E(Y^2)$$

Put $$X = X(t) \text{ and } Y = X(t-\tau)$$

Then $$[E(X(t) \times X(t-\tau)\}]^2 \le E\{X^2(t)\} \times E\{X^2(t-\tau)\}$$

i.e., $$\{R(\tau)\}^2 \le [E\{X^2(t)\}]^2$$

[since $E\{X(t)\}$ and Var $\{X(t)\}$ are constant for a stationary process]

i.e., $$\{R(\tau)\}^2 \le \{R(0)\}^2$$

Taking square-root on both sides

$$|R(\tau)| \le R(0) \qquad [\text{since } R(0) = E\{X^2(t)\} \text{ is positive}]$$

3. If the autocorrelation function $R(\tau)$ of a real stationary process $\{X(t)\}$ is continuous at $\tau = 0$, it is continuous at every other point.

Proof Consider $E[\{X(t) - X(t-\tau)\}^2]$

$$= E\{X^2(t)\} + E\{X^2(t-\tau)\} - 2E\{X(t) \times X(t-\tau)\}$$
$$= R(0 + R(0) - 2R(\tau)$$
$$= 2[R(0) - R(\tau)] \qquad (1)$$

Since $R(\tau)$ is continuous at $\tau = 0$, $\lim_{\tau \to 0} R(\tau) = R(0)$

i.e, $$\lim_{\tau \to 0} \{\text{R.S. of } (1)\} = 0$$

$\therefore \quad \lim_{\tau \to 0} \{\text{L.S. of (1)}\} = 0$

$\therefore \quad \lim_{\tau \to 0} \{X(t - \tau)\} = X(t)$

i.e. $\quad X(t)$ is continuous for all t (2)

Consider $\quad R(\tau + h) - R(\tau)$

$= E[X(t) \times X\{t - (\tau + h)\}] - E[X(t) \times X(t - \tau)]$

$= E[X(t) \{X(t - \tau - h) - X(t - \tau)\}]$ (3)

Now $\quad \lim_{h \to 0} [X\{(t - \tau) - h\} - X(t - \tau)] = 0$, by (2)

$\therefore \quad \lim_{h \to 0} \{\text{R.S. of (3)}\} = 0$

$\therefore \quad \lim_{h \to 0} \{\text{L.S. of (3)} = 0$

i.e., $\quad \lim_{h \to 0} \{R(\tau + h)\} = R(\tau)$

i.e., $\quad R(\tau)$ is continuous for all τ

4. If $R(\tau)$ is the autocorrelation function of a stationary process $\{X(t)\}$ with no periodic component, then $\lim_{\tau \to \infty} R(\tau) = \mu^2_x$, provided the limit exists.

Proof $\quad R(\tau) = E\{X(t) \times X(t - \tau)\}$

When τ is very large, $X(t)$ and $X(t - \tau)$ are two sample functions (members) of the process $\{X(t)\}$ observed at a very long interval of time.

Therefore, $X(t)$ and $X(t - \tau)$ tend to become independent [$X(t)$ and $X(t - \tau)$ may be dependent, when $X(t)$ contains a periodic component, which is not true].

$$\therefore \quad \lim_{\tau \to \infty} \{R(\tau)\} = E\{X(t)\} \times E\{X(t - \tau)\}$$

$$= \mu_x^2 \qquad \text{[since } E\{X(t)\} \text{ is a constant]}$$

i.e., $\quad \mu_x = \sqrt{\lim_{\tau \to \infty} R(\tau)}$

Example 3 *Prove that the random process $\{X(t)\}$ with constant mean is mean-ergodic, if* $\lim\limits_{T \to \infty} \left[\dfrac{1}{4T^2} \int_{-T}^{T} \int_{-T}^{T} C(t_1, t_2)\, dt_1\, dt_2 \right] = 0$

Solution As per mean-ergodic theorem, the condition for the mean-ergodicity of the process $\{X(t)\}$ is

$$\lim_{T \to \infty} \{\mathrm{Var}\left(\overline{X}_T\right)\} = 0, \text{ where}$$

$$\overline{X}_T = \frac{1}{2T} \int_{-T}^{T} X(t)\mathrm{d}t \text{ and } E\left(\overline{X}_T\right) = E\{X(t)\}$$

Now
$$\overline{X}_T^2 = \frac{1}{4T^2} \int_{-T}^{T} \int_{-T}^{T} X(t_1)\, X(t_2)\mathrm{d}t_1\mathrm{d}t_2$$

$\therefore$
$$E\left\{\overline{X}_T^2\right\} = \frac{1}{4T^2} \int_{-T}^{T} \int_{-T}^{T} R(t_1, t_2)\mathrm{d}t_1\mathrm{d}t_2$$

$$\therefore \quad \operatorname{Var}\left(\overline{X}_T\right) = E\left\{\overline{X}_T^2\right\} - E^2\left(\overline{X}_T\right)$$

$$= \frac{1}{4T^2} \int_{-T}^{T} \int_{-T}^{T} [R(t_1, t_2) - E\{X(t_1)\}\, E\{X(t_2)\}] dt_1 dt_2$$

$$= \frac{1}{4T^2} \int_{-T}^{T} \int_{-T}^{T} C(t_1, t_2) dt_1 dt_2. \tag{1}$$

Therefore the condition $\lim_{T \to \infty} \{\operatorname{Var}\left(\overline{X}_T\right)\} = 0$ is equivalent to the condition

$$\lim_{T \to \infty} \left[\frac{1}{4T^2} \int_{-T}^{T} \int_{-T}^{T} C(t_1, t_2) dt_1\, dt_2 \right] = 0$$

Hence the result.

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$$E\{X^2(t)\} = R(0) = \int_{-\infty}^{\infty} S(f)df, \text{ which is the given property.}$$

3. The spectral density function of a real random process is an even function.

Proof $S(\omega) = \int_{-\infty}^{\infty} R(\tau)\, e^{-i\omega\tau}\, d\tau$, by definition

$$\therefore \quad S(-\omega) = \int_{-\infty}^{\infty} R(\tau) e^{i\omega\tau}\, d\tau$$

Putting $\tau = -u$,

$$S(-\omega) = \int_{-\infty}^{\infty} R(-u)\, e^{-i\omega u}\, du$$

$$= \int_{-\infty}^{\infty} R(u)\, e^{-i\omega u}\, du$$

[since $R(\tau)$ is an even function of τ]

$$= S(\omega)$$

Therefore, $S(\omega)$ is an even function of ω.

4. The spectral density of a process $\{X(t)\}$, real or complex, is a real function of ω and non negative.

Proof $R(\tau) = E\{X(t)X^*(t-\tau)\}$

$$\therefore \quad R(-\tau) = E\{X(t)\, X^*(t+\tau)\}$$

$$\therefore \quad R^*(-\tau) = E\{X(t+\tau)\, X^*(t)\}$$

$$= R(\tau)$$

(or) $$R^*(\tau) = R(-\tau) \qquad (1)$$

Now $$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau}\, d\tau$$

$$\therefore \quad S^*(\omega) = \int_{-\infty}^{\infty} R^*(\tau)\, e^{i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} R(-\tau) e^{i\omega\tau}\, d\tau \text{ by (1)}$$

$$= \int_{-\infty}^{\infty} R(u)\, e^{-i\omega u}\, du, \quad \text{by putting } u = -\tau$$

$$= S(\omega)$$

Hence $S(\omega)$ is a real function of ω.

Note: It will be proved that $S(\omega) \geq 0$, in Worked Example 14).

5. The spectral density and the autocorrelation function of a real WSS process form a Fourier cosine transform pair.

Proof $S(w) = \int_{-\infty}^{\infty} R(t)\,\{\cos wt - i \sin wt\}\,dt$

$$= 2\int_{-\infty}^{\infty} R(\tau)\cos\omega\tau\, d\tau$$

[since $R(\tau)$ is even]

= Fourier cosine transform of $[2R(\tau)]$

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\,(\cos\tau\omega + i\sin\tau\omega)\,d\omega$$

$$= \frac{1}{\pi}\int_{0}^{\infty} S(\omega)\cos\tau\omega\, d\omega$$

[since $S(\omega)$ is even]

= Fourier inverse cosine transform of $\left[\frac{1}{2}S(\omega)\right]$

$$= \begin{cases} \dfrac{e^{-\lambda t_3}\, \lambda^{n_3}\, t_1^{n_1}\, (t_2 - t_1)^{n_2 - n_1}\, (t_3 - t_2)^{n_3 - n_2}}{\underline{|n_1}\ \underline{|n_2 - n_1}\ \underline{|n_3 - n_2}}, & n_3 \geq n_2 \geq n_1 \\ 0, & \text{otherwise} \end{cases}$$

Mean and Autocorrelation of the Poisson Process

The probability law of the Poisson process $\{X(t)\}$ is the same as that of a Poisson distribution with parameter λt.

$$\therefore \quad E\{X(t)\} = \text{Var}\{X(t)\} = \lambda t$$

$$\therefore \quad E\{X_2(t)\} = \lambda t + \lambda^2 t^2 \tag{1}$$

$$R_{xx}(t_1, t_2) = E\{X(t_1)\, X(t_2)\}$$

$$= E[X(t_1)\, \{X(t_2) - X(t_1) + X(t_1)\}]$$

$$= E[X(t_1)\, \{X(t_2) - X(t_1)\}] + E\{X^2(t_1)\}$$

$$= E[X(t_1)]\, E[X(t_2) - X(t_1)] + E\{X^2(t_1)\},$$

since $\{X(t)\}$ is a process of independent increments.

$$= \lambda t_1, \lambda\, (t_2 - t_1) + \lambda t_1 + \lambda t^2{}_1, \text{ if } t_2 \geq t_1 \qquad \text{[by (1)]}$$

$$= \lambda^2\, t_1 t_2 + \lambda t_1, \text{ if } t_2 \geq t_1$$

or $\quad R_{xx}(t_1, t_2) = \lambda^2 t_1 t_2 + \lambda \min\,(t_1, t_2)$

$$C_{xx}(t_1, t_2) = R_{xx}\,(t_1, t_2) - E\{X(t_1)\}\, E\{X(t_2)\}$$

$$= \lambda^2\, t_1\, t_2 + \lambda t_1 - \lambda^2\, t_1\, t_2$$

$$= \lambda t_1, \text{ if } t_2 \geq t_1$$

or $\quad = \min\,(t_1, t_2)$

$$r_{xx}(t_1, t_2) = \frac{C_{xx}\,(t_1, t_2)}{\sqrt{\text{var}\,\{X(t_1)\}\ \text{var}\,\{X(t_2)\}}}$$

$$= \frac{\lambda t_1}{\sqrt{\lambda t_1\, \lambda t_2}} = \sqrt{\frac{t_1}{t_2}}\,, \text{ if } t_2 \geq t_1$$

Note *Poisson process is not a stationary process.*

Properties of Poisson Process

1. The Poisson process is a Markov process.

Proof Consider $P[X(t_3) = n_3 / X(t_2) = n_2,\, X(t_1) = n_1]$

$$= \frac{P[X(t_1) = n_1,\, X(t_2) = n_2,\, X(t_3) = n_3]}{P[X(t_1) = n_1,\, X(t_2) = n_2]}$$

$$= \frac{e^{-\lambda(t_3 - t_2)} \lambda^{n_3 - n_2} (t_3 - t_2)^{n_3 - n_2}}{\lfloor n_3 - n_2}$$

[refer to the second-and third-order probability functions of the Poisson process]

$$= P[X(t_3) = n_3/X(t_2) = n_2]$$

This means that the conditional probability distribution of $X(t_3)$ given all the past values $X(t_1) = n_1$, $X(t_2) = n_2$ depends only on the most recent value $X(t_2) = n_2$.

That is, the Poisson process possesses the Markov property. Hence the result.

2. **Additive property:** Sum of two independent Poisson processes is a Poisson process.

Proof We have already derived in Chapter IV the characteristic function of a Poisson distribution with parameter λ as $e^{-\lambda(1 - e^{i\omega})}$

Therefore, the characteristic functions of $X_1(t)$ and $X_2(t)$ are given by

$$\phi_{x_1(t)}(\omega) = e^{-\lambda_1 t(1 - e^{i\omega})} \text{ and } \phi_{X_2(t)}(\omega) = e^{-\lambda_2 t(1 - e^{i\omega})}$$

Since $X_1(t)$ and $X_2(t)$ are independent,

$$\phi_{x_1(t) + x_2(t)}(\omega) = \phi_{x_1(t)}(\omega)\, \phi_{x_2(t)}(\omega)$$

$$= e^{-(\lambda_1 + \lambda_2)t(1 - e^{i\omega})}$$

which is the characteristic function of Poisson distribution with parameter $(\lambda_1 + \lambda_2)t$.

Therefore, $\{X_1(t) + X_2(t)\}$ is a Poisson process.

Alternative proof

Let $\quad X(t) = X_1(t) + X_2(t).$

$$P\{X(t) = n\} = \sum_{r=0}^{n} P\{X_1(t) = r\}\, P\{X_2(t) = n - r\}$$

$$= \sum_{r=0}^{n} \frac{e^{-\lambda_1 t} (\lambda_1 t)^r}{\lfloor r} \frac{e^{-\lambda_2 t} (\lambda_2 t)^{n-r}}{\lfloor n - r}$$

$$= e^{-(\lambda_1 + \lambda_2)t} \frac{1}{\lfloor n} \sum_{r=0}^{n} nC_r (\lambda_1 t)^r (\lambda_2 t)^{n-r}$$

$$= e^{-(\lambda_1 + \lambda_2)t} \left(\lambda_1 + \lambda_2\right) n / \lfloor n$$

Therefore, $X_1(t) + X_2(t)\}$ is a Poisson process with parameter $(\lambda_1 + \lambda_2)t$.

Note *The additive property holds good for any number of independent Poisson processes.*

3. Difference of two independent Poisson processes is not a Poisson process.

Proof Let $X(t) = X_1(t) - X_2(t)$

$$E\{X(t)\} = E\{X_1(t)\} - E\{X_2(t)\}$$
$$= (\lambda_1 - \lambda_2)t$$
$$E\{X^2(t)\} = E\{X_1^2(t)\} + E\{X_2^2(t)\} - 2E\{X_1(t)\}\ E(X_2(t)\}$$

(by independence)

$$= (\lambda_1^2 t^2 + \lambda_1 t) + (\lambda_2^2 t^2 + \lambda_2 t) - 2(\lambda_1 t)(\lambda_2 t)$$
$$= (\lambda_1 + \lambda_2)t + (\lambda_1 - \lambda_2)^2 t^2$$
$$\neq (\lambda_1 - \lambda_2)t + (\lambda_1 - \lambda_2)^2 t^2$$

Recall that $E\{X^2(t)\}$ for a Poisson process $\{X(t)\}$ with parameter λ is given by $E\{+ X^2_{(t)}\} = \lambda t + \lambda^2 t^2$.

Therefore, $\{X_1(t) - X_2(t)\}$ is not a Poisson process.

4. The interarrival time of a Poisson process, i.e., the interval between two successive occurrences of a Poisson process with parameter λ has an exponential distribution with mean $1/\lambda$.

Proof Let two consecutive occurrences of the event be E_i and E_{i+1}.

Let E_i take place at time instant t_i and T be the interval between the occurrences of E_i and E_{i+1}. T is a continuous RV.

$$P(T > t) = P\{E_{i+1} \text{ did not occur in } (t_i, t_i + t)\}$$
$$= P\{\text{No event occurs in an interval of length } t\}$$
$$= P\{X(t) = 0\}$$
$$= e^{-\lambda t}$$

Therefore, the cdf of T is given by

$$F(t) = P\{T \le t\} = 1 - e^{-\lambda t}$$

Therefore, the pdf of T is given by

$$f(t) = \lambda \quad e^{-\lambda t} \quad (t \ge 0)$$

which is an exponential distribution with mean $1/\lambda$.

5. If the number of occurrences of an event E is an interval of length t is a Poisson process $\{X(t)\}$ with parameter λ and if each occurrence of E has a constant probability p of being recorded and the recordings are independent of each other, then the number $N(t)$ of the recorded occurrences in t is also a Poisson process with parameter λp.

Proof $P\{N(t) = n\} = \sum_{r=0}^{\infty} P\{E$ occurs $(n + r)$ times in t and n of them are recorded$\}$

$$= \sum_{r=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^{n+r}}{\underline{|n+r}} (n+r)C_n p^n q^r, \quad q = 1 - p$$

$$= \sum_{r=0}^{\infty} \frac{e^{-\lambda t} (\lambda t)^{n+r}}{\lfloor n+r} \frac{\lfloor n+r}{\lfloor n \lfloor r} p^n q^r$$

$$= \frac{e^{-\lambda t} (\lambda pt)^n}{\lfloor n} \sum_{r=0}^{\infty} \frac{(\lambda qt)^r}{\lfloor r}$$

$$= \frac{e^{-\lambda t} (\lambda pt)^n}{\lfloor n} e^{\lambda qt}$$

$$= \frac{e^{-\lambda pt} (\lambda pt)^n}{\lfloor n}$$

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Example 3 *If $\{N_1(t)\}$ and $\{N_2(t)\}$ are 2 independent Poisson processes with parameters λ_1 and λ_2 respectively, show that*

$$P[N_1(t) = k/\{N_1(t) + N_2(t) = n\}] = nC_k\ p^k q^{n-k}, \text{ where}$$

$$P = \frac{\lambda_1}{\lambda_1 + \lambda_2} \text{ and } q = \frac{\lambda_2}{\lambda_1 + \lambda_2}.$$

Solution Required conditional probability

$$= \frac{P[\{N_1(t) = k\} \cap \{N_1(t) + N_2(t) = n\}]}{P\{N_1(t) + N_2(t) = n\}}$$

$$= \frac{P[\{N_1(t) = k\} \cap \{N_2(t) = n - k\}]}{P\{N_1(t) + N_2(t) = n\}}$$

$$= \frac{\dfrac{e^{-\lambda_1 t}(\lambda_1 t)^k}{\lfloor\underline{k}} \times \dfrac{e^{-\lambda_2 t}(\lambda_2 t)^{n-k}}{\lfloor\underline{n-k}}}{\dfrac{e^{-(\lambda_1+\lambda_2)t}\{(\lambda_1+\lambda_2)t\}^n}{\lfloor\underline{n}}}$$

(by indepence and additive property)

$$= \frac{\lfloor\underline{n}}{\lfloor\underline{k}\ \lfloor\underline{n-k}} \cdot \frac{(\lambda_1 t)^k (\lambda_2 t)^{n-k}}{\{(\lambda_1 + \lambda_2)t\}^n}$$

$$= nC_k \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^{n-k}$$

$$= nC_k\ p^k q^{n-k}$$

Example 8 *If {X(t)} is a Poisson process, prove that*

$$P\{X(s) = r/X(t) = n\} = nC_r \left(\frac{s}{t}\right)^r \left(1 - \frac{s}{t}\right)^{n-r} \text{ where } s < t$$

Solution $P\{X(s) = r/X(t) = n\} = \dfrac{P[\{X(s) = r\} \cap \{X(t) = n\}]}{P\{X(t) = n\}}$

$$= \frac{P\{X(s) = r \cap X(t-s) = n-r\}}{P\{X(t) = n\}}$$

$$= \frac{P\{X(s) = r\}\, P\{X(t-s) = n-r\}}{P\{X(t) = n\}} \quad \text{(by independence)}$$

$$= \frac{e^{-\lambda s}(\lambda s)^r / \lfloor r \; e^{-\lambda(t-s)}[\lambda(t-s)]^{n-r} / \lfloor n-r}{e^{-\lambda t}(\lambda t)^n / \lfloor n}$$

$$= \frac{\lfloor n}{\lfloor r \, \lfloor n-r} \; \frac{s^r (t-s)^{n-r}}{t^n}$$

$$= nC_r \left(\frac{s}{t}\right)^r \left(1 - \frac{s}{t}\right)^{n-r}$$

Exercise 7(B)

$\ldots = 0, 1, 2, \ldots, n.$

Mean and Variance of the Binomial Distribution

We have already found out $E(X)$ and $\text{Var}(X)$ for the binomial distribution $B(n, p)$ using the moment generating function in Example 3 in Worked Example 4 (d). Here we shall find them directly using the definitions of $E(X)$ and $\text{Var}(X)$.

$$E(X) = \sum_r x_r p_r$$

$$= \sum_{r=0}^{n} r \cdot nC_r\, p^r q^{n-r}$$

$$= \sum_{r=0}^{n} r \cdot \frac{n!}{r!(n-r)!} p^r q^{n-r} \tag{1}$$

$$= np \cdot \sum_{r=1}^{n} \frac{(n-1)!}{(r-1)!\{(n-1)-(r-1)\}!} p^{r-1} q^{(n-1)-(r-1)}$$

$$= np \sum_{r=1}^{n} (n-1)\, C_{r-1} \cdot p^{r-1} \cdot q^{(n-1)-(r-1)}$$

$$= np\,(q+p)^{n-1}$$

$$= np \tag{2}$$

$$E(X^2) = \sum_r x_r^2 p_r = \sum_0^n r^2 p_r$$

$$= \sum_{r=0}^{n} \{r(r-1)+r\} \frac{n!}{r!(n-r)!} p^r q^{n-r}$$

$$= n\,(n-1)p^2 \sum_{r=2}^{n} (n-2)C_{r-2}\, p^{r-2} q^{n-r} + np, \quad \text{[by (1) and(2)]}$$

$$= n\,(n-1)\,p^2\,(q+p)^{n-2} + np$$

$$= n\,(n-1)\,p^2 + np$$

$$\text{Var}(X) = E(X^2) - \{E(X)\}^2$$

$$= n\,(n-1)\,p^2 + np - n^2p^2$$

$$= np\,(1-p)$$

$$= npq$$

Mean and Variance of Geometric Distribution

$$E(X) = \sum_{r} x_r p_r$$

$$= \sum_{r=1}^{\infty} r q^{r-1} p$$

$$= p[1 + 2q + 3q^2 + \ldots + \infty]$$

$$= p\,(1-q)^{-2} = \frac{1}{p}$$

$$E(X^2) = \sum_{r} x_r^2\, pr$$

$$= \sum_{r=1}^{\infty} r^2 q^{r-1} p$$

$$= p \sum_{r=1}^{\infty} \{r(r+1) - r\}\, q^{r-1}$$

$$= p[\{1 \times 2 + 2 \times 3q + 3 \times 4\, q^2 + \ldots + \infty\} - \{1 + 2q + 3q^2 + \ldots + \infty\}]$$

$$= p[2(1-q)^{-3} - (1-q)^{-2}]$$

$$= p\left(\frac{2}{p^3} - \frac{1}{p^2}\right) = \frac{1}{p^2}\,(2-p) = \frac{1}{p^2}\,(1+q)$$

$$\text{Var}(X) = E(X^2) - \{E(X)\}^2$$

$$= \frac{1}{p^2}\,(1+q) - \frac{1}{p^2} = \frac{q}{p^2}.$$

Note *Sometimes the probability mass function of a geometric RV X is taken as*

$$P(X = r) = q^r p;\ r = 0, 1, 2, \ldots, \infty \text{ where } p + q = 1$$

It is this definition that was given in chapter II. If this definition is assumed, then

$$E(X) = \frac{q}{p} \text{ and } Var(X) = \frac{q}{p^2} \quad \text{[see example (5) in section 4(d)]}$$

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Moments of the Uniform Distribution $U(a, b)$

Raw moments μ_r of the uniform distribution $U(a, b)$ about the origin are given by

$$\mu'_r = E\{X^r\}, \text{ where } X \text{ follows } U(a, b)$$

$$= \int_a^b x^r \frac{1}{b-a}\, dx$$

$$= \frac{b^{r+1} - a^{r+1}}{(r+1)(b-a)} \qquad (1)$$

$$\therefore \qquad E(X) = \text{Mean of } U(a, b) = \mu_1' = \frac{1}{2}(b+a) \qquad (2)$$

Central moments μ_r of the uniform distribution $U(a, b)$ are given by

$$\mu_r = E[\{X - E(X)\}^r]$$

$$= E\{(X - \frac{1}{2}(b+a)\}^r]$$

$$= \int_a^b \frac{\left\{x - \frac{1}{2}(b+a)\right\}^r}{b-a}\, dx$$

$$= \frac{1}{b-a}\int_{-c}^{c} t^r\, dt, \text{ on putting } t = x - \frac{1}{2}(b+a) \text{ and } c = \frac{1}{2}(b-a)$$

$$= \begin{cases} 0 & \text{if } r \text{ is odd} \\ \dfrac{1}{r+1}\cdot\left(\dfrac{b-a}{2}\right)^r & \text{if } r \text{ is even} \end{cases}$$

Thus $\mu_{2n-1} = 0$ and $\mu_{2n} = \dfrac{1}{2n+1}\cdot\left(\dfrac{b-a}{2}\right)^{2n}$ for $n = 1, 2, 3, \ldots$ (3)

In particular, $\mu_2 = \text{variance of } U(a, b) = \dfrac{1}{12}(b-a)^2$ (4)

$$\mu_3 = 0 \text{ and } \mu_4 = \frac{1}{80}(b-a)^4$$

The absolute central moments ν_r of the uniform distribution $U(a, b)$ are given by

$$\nu_r = E\{|X - E(X)|^r\}$$

$$= \int_a^b \frac{\left|x - \frac{1}{2}(b+a)\right|^r}{b-a}\, dx$$

$$= \frac{1}{b-a}\int_{-c}^{c} |t|^r\, dt, \text{ on putting } t = x - \frac{1}{2}(b+a) \text{ and } c = \frac{1}{2}(b-a)$$

$$= \frac{2}{b-a} \int_0^c t^r \, dt, \quad \text{(since the integrand is an even function of } t\text{)}$$

$$= \frac{1}{r+1} \cdot \left(\frac{b-a}{2} \right)^r \tag{5}$$

Definition: $E\{|X - E(X)|\}$ is called the *mean deviation* (MD) about the mean of the RV X or of the corresponding distribution.

Thus the MD about the mean of the distribution $U(a, b)$ is given by

$$v_1 = \frac{1}{4}(b-a)$$

2. Exponential distribution

Definitions: A continuous RV X is said to follow an *exponential distribution* or *negative exponential distribution* with parameter $\lambda > 0$, if its probability density function is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

We note that $\int_0^\infty f(x)\, dx = \int_0^\infty \lambda e^{-\lambda x}\, dx = 1$ and hence $f(x)$ is a legitimate density function.

Mean and Variance of the Exponential Distribution

Raw moments μ_r' about the origin of the exponential distribution are given by

$$\mu_r' = E(X^r)$$

$$= \int_0^\infty x^r \cdot \lambda e^{-\lambda x}\, dx$$

$$= \frac{1}{\lambda^r} \int_0^\infty y^r e^{-y}\, dy, \text{ (on putting } y = \lambda x\text{)}$$

$$= \frac{1}{\lambda^r} \Gamma(r+1)$$

$$= \frac{r!}{\lambda^r} \tag{1}$$

$\therefore$ $E(X)$ = Mean of the exponential distribution

$$= \mu_1' = \frac{1}{\lambda}, \text{ [from (1)]}$$

Putting $r = 2$ in (1), we get

Memoryless Property of the Exponential Distribution

If X is exponentially distributed, then

$$P(X > s + t/X > s) = P(X > t), \text{ for any } s, t > 0$$

$$P(X > k) = \int_k^{\infty} \lambda\, e^{-\lambda x}\, dx$$

$$= \left(-e^{-\lambda x}\right)_k^{\infty} = e^{-\lambda k} \qquad (1)$$

Now $P(X > s + t/X > s) = \dfrac{P\{X > s + t \text{ and } X > s\}}{P\{X > s\}}$

$$= \frac{P\{X > s + t\}}{P\{X > s\}}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}, \text{ [by (1)]}$$

$$= e^{-\lambda t} = P(X > t).$$

Note *The converse of this result is also true. That is, if $P(X > s + t/X > s) = P(X > t)$, then X follows an exponential distribution. See Example (8) in Worked Example 5(b).]*

3. Erlang distribution or General Gamma distribution

Definition: A continuous RV X is said to follow an *Erlang distribution* or *General Gamma distribution* with parameters $\lambda > 0$ and $k > 0$, if its probability density function is given by

$$f(x) = \begin{cases} \dfrac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma(k)}, & \text{for } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

We note that $\displaystyle\int_0^{\infty} f(x)\, dx = \frac{\lambda^k}{\Gamma(k)} \int_0^{\infty} x^{k-1} e^{-\lambda x}\, dx$

$$= \frac{1}{\Gamma(k)} \int_0^{\infty} t^{k-1} e^{-t}\, dt, \text{ [on putting } \lambda x = t]$$

$$= 1$$

Hence $f(x)$ is a legitimate density function.

Note

1. *When $\lambda = 1$, the Erlang distribution is called Gamma distribution or simple Gamma distribution with parameter k whose density function is* $f(x) = \dfrac{1}{\Gamma(k)} x^{k-1} e^{-x}, x \geq 0; k > 0.$
2. *When $k = 1$, the Erlang distribution reduces to the exponential distribution with parameter $\lambda > 0$.*
3. *Sometimes, the Erlang distribution itself is called Gamma distribution.*

Mean and Variance of Erlang Distribution

The raw moments μ_r' about the origin of the Erlang distribution are given by

$$\mu_r' = E(X^r)$$

$$= \int_0^\infty \frac{\lambda^k}{\Gamma(k)} x^{k+r-1} e^{-\lambda x} \, dx$$

$$= \frac{\lambda^k}{\Gamma(k)} \cdot \frac{1}{\lambda^{k+r}} \int_0^\infty t^{k+r-1} e^{-t} \, dt, \text{ (on putting } \lambda x = t)$$

$$= \frac{1}{\lambda^r} \frac{\Gamma(k+r)}{\Gamma(k)}$$

$$\therefore \quad \text{Mean} = E(X) = \frac{1}{\lambda} \cdot \frac{\Gamma(k+1)}{\Gamma(k)} = \frac{k}{\lambda}$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= \frac{1}{\lambda^2} \cdot \frac{\Gamma(k+2)}{\Gamma(k)} - \left(\frac{k}{\lambda}\right)^2$$

$$= \frac{1}{\lambda^2} \{k(k+1) - k^2\} = \frac{k}{\lambda^2}$$

Reproductive Property of Gamma Distribution

The sum of a finite number of independent Erlang variables is also an Erlang variable. That is, if $X_1, X_2, \ldots, X_n$ are independent Erlang variables with parameters $(\lambda, k_1), (\lambda, k_2), \ldots, (\lambda, k_n)$, then $X_1 + X_2 + \ldots + X_n$ is also an Erlang variable with parameter $(\lambda, k_1 + k_2 + \ldots + k_n)$.

Let us first find the moment generating function of the Erlang variable X with parameters λ and k and use it to prove this property. MGF of X is given by

$$M_X(t) = E\{e^{tx}\}$$

$$= \int_0^\infty \frac{\lambda^k}{\Gamma(k)} x^{k-1} e^{-\lambda x} e^{tx} \, dx$$

$$= \frac{\lambda^k}{\Gamma(k)} \int_0^\infty x^{k-1} e^{-(\lambda - t)x} \, dx$$

$$= \frac{\lambda^k}{\Gamma(k)} \cdot \frac{1}{(\lambda - t)^k} \int_0^\infty y^{k-1} e^{-y}\, dy, \quad \text{(on putting } \lambda - t = y\text{)}.$$

$$= \left(\frac{\lambda}{\lambda - t}\right)^k \quad [\because \text{ the integral} = \Gamma(k)\,]$$

$$= \left(1 - \frac{t}{\lambda}\right)^{-k}$$

Now $M_{(X_1 + X_2 + \ldots + X_n)}(t) = M_{X_1}(t)\, M_{X_2}(t) \ldots M_{X_n}(t)$ (since $X_1, X_2, \ldots, X_n$ are independent)

[Refer to property (4) of MGF given in section 4(d) of chapter (4)]

$$= \left(1 - \frac{t}{\lambda}\right)^{-k_1} \left(1 - \frac{t}{\lambda}\right)^{-k_2} \cdots \left(1 - \frac{t}{\lambda}\right)^{-k_n}$$

$$= \left(1 - \frac{t}{\lambda}\right)^{-(k_1 + k_2 + \ldots + k_n)}$$

which is the MGF of an Erlang variable with parameters $(\lambda, k_1 + k_2 + \ldots + k_n)$. Hence the reproductive property.

Relation Between the Distribution Functions (cdf) of the Erlang Distribution With $\lambda = 1$ (or Simple Gamma Distribution) and (Poisson Distribution)

If X is a Poisson random variable with mean λ,

then $$P(X \le K) = \sum_{r=0}^{k} \frac{e^{-\lambda} \lambda^r}{r!} \tag{1}$$

Differentiating both sides with respect to λ, we get

$$\frac{d}{d\lambda} P(X \le k) = \sum_{r=0}^{k} \frac{1}{r!} \{e^{-\lambda} \cdot r\lambda^{r-1} - e^{-\lambda} \times \lambda^r\}$$

$$= e^{-\lambda} \cdot \sum_{r=0}^{k} \left[\frac{\lambda^{r-1}}{(r-1)!} - \frac{\lambda^r}{r!}\right]$$

$$= e^{-\lambda}\left[-1 + \left(1 - \frac{\lambda}{1!}\right) + \left(\frac{\lambda}{1!} - \frac{\lambda^2}{2!}\right) + \cdots + \left\{\frac{\lambda^{k-1}}{(k-1)!} - \frac{\lambda^k}{k!}\right\}\right]$$

$$= -\frac{e^{-\lambda}\lambda^k}{k!} \tag{2}$$

Integrating both sides of (2) with respect to λ from λ to ∞, we get

$$\left[\sum_{r=0}^{k} \frac{e^{-\lambda}\lambda^r}{r!}\right]_\lambda^\infty = -\int_\lambda^\infty \frac{1}{k!} e^{-\lambda} \lambda^k \, d\lambda$$

i.e., $$\sum_{r=0}^{K} \frac{e^{-\lambda}\lambda^r}{r!} = \int_{\lambda}^{\infty} \frac{1}{k!} e^{-y} y^k \, dy$$

i.e., $$P(X \leq k) = P(Y \geq \lambda),$$

[where Y is the Erlang variable with parameters 1 and $(k + 1)$

or $$P(X \leq k) = 1 - P(Y \leq \lambda)$$

Note *The above relationship is valid only when the parameter k is a positive integer.*

4. Weibull Distribution

Definition: A continuous RV X is said to follow a *Weibull distribution* with parameters $\alpha, \beta > 0$, if the RV $Y = \alpha X^{\beta}$ follows the exponential distribution with density function $f_y(y) = e^{-y}, y > 0$.

Density Function of the Weibull Distribution

Since $Y = \alpha \cdot X^{\beta}$, we have $y = \alpha \cdot x^{\beta}$.

By the transformation rule, derived in chapter 3, we have $f_X(x) = f_Y(y) \left| \frac{dy}{dx} \right|$, where $f_X(x)$ and $f_Y(y)$ are the density functions of X and Y respectively.

$$\therefore \quad f_X(x) = e^{-y} \alpha \beta x^{\beta - 1}$$

$$= \alpha \beta x^{\beta - 1} e^{-\alpha x^{\beta}}; \; x > 0 \qquad [\because y > 0]$$

Note *When $\beta = 1$, Weibull distribution reduces to the exponential distribution with parameter α.*

Mean and Variance of the Weibull Distribution

The raw moments μ_r' about the origin of the Weibull distribution are given by

$$\mu_r' = E(X^r)$$

$$= \alpha\beta \int_0^{\infty} x^{r + \beta - 1} e^{-\alpha x^{\beta}} \, dx$$

$$= \int_0^{\infty} \left(\frac{y}{\alpha}\right)^{\frac{r}{\beta} + 1 - \frac{1}{\beta}} e^{-y} \left(\frac{y}{\alpha}\right)^{\frac{1}{\beta} - 1} dy,$$

on putting $y = \alpha x^{\beta}$ or $x = \left(\frac{y}{\alpha}\right)^{\frac{1}{\beta}}$

$$= \alpha^{-r/\beta} \int_0^{\infty} y^{r/\beta} e^{-y} \, dy$$

$$= \alpha^{-r/b}\,\Gamma\left(\frac{r}{\beta}+1\right)$$

$$\therefore \qquad \text{Mean} = E(X) = \mu_1' = \alpha^{-\frac{1}{\beta}}\,\Gamma\left(\frac{1}{\beta}+1\right)$$

$$\text{Var}(X) = E(X^2) - \{E(X)\}^2$$

$$= \alpha^{-2/\beta}\left[\Gamma\left(\frac{2}{\beta}+1\right) - \left\{\Gamma\left(\frac{1}{\beta}+1\right)\right\}^2\right]$$

Note *Weibull distribution finds frequent applications in Reliability Theory. It is assumed as the probability distribution of the time to failure (or length of life) of a component in a system. Other distributions used to describe the failure law are the exponential and normal distributions. See Example (19) in Worked Example 5(b).*

7. Additive property of normal distribution

If $X_i (i = 1, 2, \ldots, n)$ be n independent normal RVs with mean μ_i and variance σ_i^2, then $\sum_{i=1}^{n} a_i X_i$ is also a normal RV with mean $\sum_{i=1}^{n} a_i \mu_i$ and variance $\sum_{i=1}^{n} a_i^2 \sigma_i^2$.

$$M_{\left(\sum_{i=1}^{n} a_i X_i\right)}(t) = M_{a_1 X_1}(t) . M_{a_2 X_2}(t) \ldots M_{a_n X_n}(t), \quad \text{(by independence)}$$

$$= e^{a_1 \mu_1 t + a_1^2 \sigma_1^2 t^2/2} \times e^{a_2 \mu_2 t + a_2^2 \sigma_2^2 t^2/2} \times e^{a_n \mu_n t + a_n^2 \sigma_n^2 t^2/2}$$

$$= e^{(\Sigma a_i \mu_i) t + \Sigma a_i^2 \sigma_i^2 t^2/2}$$

which is the MGF of a normal RV with mean $\sum a_i \mu_i$ and variance $\sum a_i^2 \sigma_i^2$.

Hence the property.

$.. + \infty$

8. *Normal distribution as limiting form of binomial distribution*

When n is very large and neither p nor q is very small, the standard normal distribution can be regarded as the limiting form of the standardised binomial distribution.

When X follows the binomial distribution $B(n, p)$, the standardised binomial variable Z is given by $Z = \dfrac{X - np}{\sqrt{npq}}$. As X varies from 0 to n with step size 1, Z varies from $\dfrac{-np}{\sqrt{npq}}$ to $\dfrac{np}{\sqrt{npq}}$ with step size $\dfrac{1}{\sqrt{npq}}$. When neither p nor q is very small and n is very large, Z varies from $-\infty$ to ∞ with infinitesimally small step size. Hence, in the limit, the distribution of Z may be expected to be a continuous distribution extending from $-\infty$ to ∞ and having mean 0 and standard deviation 1. In fact the limiting form of the distribution of Z is standard normal distribution as seen below:

If X follows $B(n, p)$, then the MGF of X is given by $M_X(t) = (q + p\,e^t)^n$.

If $$Z = \frac{X - np}{\sqrt{npq}}, \text{ then}$$

$$M_Z(t) = M_{\frac{1}{\sqrt{npq}}X - \frac{np}{\sqrt{npq}}}(t) = e^{\frac{-npt}{\sqrt{npq}}} \{q + p\,e^{t/\sqrt{npq}}\}^n$$

$$\therefore \quad \log M_Z(t) = -\frac{npt}{\sqrt{npq}} + n \log \{q + p\,e^{t/\sqrt{npq}}$$

$$= -\frac{npt}{\sqrt{npq}} + n \log\left[q + p\left\{1 + \frac{t}{\sqrt{npq}} + \frac{t^2}{2\,npq} + \frac{t^3}{6\,(npq)^{3/2}} + \cdots\right\}\right]$$

$$= -\frac{npt}{\sqrt{npq}} + n \log\left[1 + \left\{\frac{pt}{\sqrt{npq}} + \frac{pt^2}{2\,npq} + \frac{pt^3}{6\,(npq)^{3/2}} + \cdots\right\}\right]$$

$$= -\frac{npt}{\sqrt{npq}} + n\left[\frac{pt}{\sqrt{npq}}\left\{1 + \frac{t}{2\sqrt{npq}} + \frac{t^2}{6n^2p^2q^2} + \cdots\right\}\right.$$

$$\left. - \frac{1}{2}\cdot\frac{p^2t^2}{npq}\left\{1 + \frac{t}{2\sqrt{npq}} + \frac{t^2}{6n^2p^2q^2} + \cdots\right\}^2 + \cdots\right]$$

$$= \frac{t^2}{2} + \text{terms containing } \frac{1}{\sqrt{n}} \text{ and lower powers of } n$$

$$\therefore \lim_{n\to\infty} \log M_Z(t) = \frac{t^2}{2}$$

i.e., $$\log_e\left[\lim_{n\to\infty} M_z(t)\right] = \frac{t^2}{2}$$

$$\therefore \qquad \lim_{n\to\infty} M_Z(t) = e^{t^2/2},$$

which is the MGF of the standard normal distribution. Hence the limit of the standardised binomial distribution, as n tends to ∞, is the standard normal distribution.

Note *We recall De Moivre–Laplace approximation for the sum of a large number of terms of the form $nC_r\, p^r q^{n-r}$ in terms of the integral of standard normal density function, which was discussed in section 1(c). It was stated that*

$$\sum_{r=r}^{r_2} nC_r\, p^r q^{n-r} = \int_{z_1}^{z_2} \phi(z)\, dz$$

where $z_1 = \dfrac{r_1 - np - \frac{1}{2}}{\sqrt{npq}}$ *and* $z_2 = \dfrac{r_2 - np + \frac{1}{2}}{\sqrt{npq}}$ *and $\phi(z)$ is the density function of the standard normal distribution.*

$$= -\frac{npt}{\sqrt{npq}} + n\log\left[q + p\left\{1 + \frac{t}{\sqrt{npq}} + \frac{t^2}{2npq} + \frac{t^3}{6(npq)^{3/2}} + \cdots\right\}\right]$$

$$= -\frac{npt}{\sqrt{npq}} + n\log\left[1 + \left\{\frac{pt}{\sqrt{npq}} + \frac{pt^2}{2npq} + \frac{pt^3}{6(npq)^{3/2}} + \cdots\right\}\right]$$

$$= -\frac{npt}{\sqrt{npq}} + n\left[\frac{pt}{\sqrt{npq}}\left\{1 + \frac{t}{2\sqrt{npq}} + \frac{t^2}{6n^2p^2q^2} + \cdots\right\}\right.$$

$$\left. - \frac{1}{2}\cdot\frac{p^2t^2}{npq}\left\{1 + \frac{t}{2\sqrt{npq}} + \frac{t^2}{6n^2p^2q^2} + \cdots\right\}^2 + \cdots\right]$$

$$= \frac{t^2}{2} + \text{terms containing } \frac{1}{\sqrt{n}} \text{ and lower powers of } n$$

$$\therefore \lim_{n\to\infty} \log M_Z(t) = \frac{t^2}{2}$$

i.e., $$\log_e\left[\lim_{n\to\infty} M_z(t)\right] = \frac{t^2}{2}$$

$$\therefore \quad \lim_{n\to\infty} M_Z(t) = e^{t^2/2},$$

which is the MGF of the standard normal distribution. Hence the limit of the standardised binomial distribution, as n tends to ∞, is the standard normal distribution.

Note *We recall De Moivre–Laplace approximation for the sum of a large number of terms of the form $nC_r\, p^r q^{n-r}$ in terms of the integral of standard normal density function, which was discussed in section 1(c). It was stated that*

$$\sum_{r=r}^{r_2} nC_r\, p^r q^{n-r} = \int_{z_1}^{z_2} \phi(z)\, dz$$

where $z_1 = \dfrac{r_1 - np - \frac{1}{2}}{\sqrt{npq}}$ *and* $z_2 = \dfrac{r_2 - np + \frac{1}{2}}{\sqrt{npq}}$ *and $\phi(z)$ is the density function of the standard normal distribution.*

Importance of Normal Di...

Example 16 *If the conditional distribution of Y, given X = x, is an exponential distribution with parameter x and if the unconditional distribution of X is an Erlang distribution with parameters $\lambda(> 0)$ and $k(> 2)$, prove that the conditional distribution of X, given Y = y, is an Erlang distribution with parameters $\lambda + y$ and $k + 1$.*

Solution **Given:** $F_{Y/X}(y) = x\, e^{-xy}$, $y > 0$ and $x > 0$

and $$f_X(x) = \frac{\lambda^k}{\Gamma(k)} x^{k-1} e^{-\lambda x},\ x > 0$$

If $f(x, y)$ denotes the joint density function of (X, Y), then $f_{Y/X}(y) = \dfrac{f(x, y)}{f_X(x)}$

$$\therefore \quad f(x, y) = \frac{\lambda^k}{\Gamma(k)} x^k e^{-(\lambda + y)x},\ x > 0,\ y > 0.$$

Now $f_Y(y)$ = the marginal density function of Y

$$= \int_0^\infty f(x, y)\, dx$$

$$= \frac{\lambda^k}{\Gamma(k)} \cdot \int_0^\infty x^k e^{-(\lambda + y)x}\, dx$$

$$= \frac{\lambda^k}{\Gamma(k)} \frac{1}{(\lambda + y)^{k+1}} \cdot \int_0^\infty t^k e^{-t}\, dt \quad \text{[on putting } (\lambda + y)x = t]$$

$$= \frac{\lambda^k}{\lceil(k)} \frac{1}{(\lambda + y)^{k+1}} \cdot \lceil(k+1) = \frac{k\,\lambda^k}{(\lambda + y)^{k+1}}, \quad y > 0$$

Now $\quad f_{X/Y}(x) = \dfrac{f(x, y)}{f_Y(y)}$

$$= \frac{\dfrac{\lambda^k}{\lceil(k)} x^k e^{-(\lambda + y)x}}{\dfrac{k\,\lambda^k}{(\lambda + y)^{k+1}}} \quad x > 0 \text{ and } y > 0$$

$$= \frac{(\lambda + y)^{k+1}}{\lceil(k+1)} \cdot x^k e^{-(\lambda + y)x} \quad x > 0$$

This is the density function of an Erlang distribution with parameters $\lambda + y$ and $k + 1$.