

Department of Mathematics

Odd Semester 2017

Probability and Random Processes

15B11MA301

/Probability and Random Processes

10B11MA411

Tutorial Sheet 8

B.Tech. Core

Random Walk, Telegraph Signal Process, Ergodic Process

1. Let $\{X_n, n = 1, 2, \dots\}$ be a sequence of independent random variables with sample space as $\{0, 1\}$, $P[X_n = 0] = 2/3$, $P[X_n = 1] = 1/3$. Let $Y_n = X_1 + X_2 + \dots + X_n$. Find first order pmf of Y_n , $E[Y_n]$, $R_{YY}(n, n+k)$, and $C_{YY}(n, n+k)$.
2. Let $X(t)$ is semi random telegraph signal process and $Y(t) = \beta X(t)$, where β is uniformly distributed random variable in the interval $(-2, 2)$. Is $Y(t)$ a WSS process?
3. A random process is defined by $X(t) = T + (1-t)$, where T is a uniform random variable in $(0, 1)$. (a) Find the cdf of $X(t)$, (b) Find $E[X(t)]$ and $C_{XX}(t_1, t_2)$.
4. If $\{X(t)\}$ is a WSS process with $E\{X(t)\} = 2$ and $R_{XX}(\tau) = 4 + e^{-|\tau|/10}$, find the mean and variance of $S = \int_0^1 X(t)dt$.
5. The WSS process $X(t)$ is given by $X(t) = 10 \cos(100t + \theta)$, where θ is uniformly distributed over $(-\pi, \pi)$. Check whether $\{X(t)\}$ is (i) mean ergodic random process, (ii) correlation ergodic random process, (iii) ergodic random process?
6. A random binary transmission process $\{X(t)\}$ is a WSS process with zero mean and autocorrelation function $R(\tau) = 1 - |\tau|/T$, where T is a constant. Find the variance of the time average of $\{X(t)\}$ and also the mean over $(0, T)$. Is $\{X(t)\}$ mean ergodic?
7. If $\{X(t)\}$ is the random telegraph signal process with $E\{X(t)\}=0$ and $R(\tau) = \exp(-2\lambda|\tau|)$, find the mean and variance of the time average of $\{X(t)\}$ over $(-T, T)$. Is it mean ergodic?
8. Define power spectral density function (PSDF) of a stationary process. State and prove all the properties of PSDF.
9. Find the autocorrelation function of the process $\{X(t)\}$, for which the power spectral density function is given by $S(\omega) = \begin{cases} 1 + \omega^2, & \text{for } |\omega| \leq 1 \\ 0, & \text{for } |\omega| > 1 \end{cases}$
10. Find the PSDF of a process whose autocorrelation function is given by $R(\tau) = \begin{cases} 1 - \frac{|\tau|}{T}, & \text{for } |\tau| \leq T \\ 0, & \text{for } |\tau| > T \end{cases}$

$$Y_n = \sum_{i=1}^n X_i$$

$$\mu_x = \frac{1}{3} \quad \text{Var}(x) = \frac{1}{3} - \frac{1}{9} = \frac{2}{9}$$

a) first order pmf of y

$$P(Y_n = x) = {}^n C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{n-x}$$

$$b) E[Y_n] = np = \frac{n}{M}$$

$$c) R_{YY}(n, n+k) = E[Y_n \cdot Y_{n+k}]$$

$$= E \left[\sum_{i=1}^n x_i \sum_{j=1}^{n+k} x_j \right] = \sum_{i=1}^n \sum_{j=1}^{n+k} E [x_i x_j]$$

$$= \sum_{i=1}^n \left[\sum_{\substack{j=1 \\ j \neq i}}^{n+k} E[x_i]E[x_j] + E[x_i^2] \right]$$

$$= \sum_{i=1}^n \left[\frac{1}{3} \sum_{\substack{j=1 \\ j \neq i}}^{n+k} E[X_j] + \frac{1}{3} \right]$$

$$= \sum_{i=1}^n \left[\frac{1}{3} \left(\frac{n+k-1}{3} \right) + \frac{1}{3} \right]$$

$$= \sum_{i=1}^n \left[\frac{n+k+2}{9} \right] = \frac{n}{9} (n+k+2)$$

d) $C_{yy_2} = R_{yy_2} - M_y M_{y_2}$

$$\cdot C_{yy}(n, n+k) = R_{yy}(n, n+k) - M_y(n) M_{y_2}(n+k)$$

$$= \frac{n}{9} (n+k+2) - \frac{n}{3} \left(\frac{n+k}{3} \right)$$

$$= \frac{2n}{9}$$

$$Q_2] \quad Y(t) = \beta X(t)$$

$\hookrightarrow X(t)$ is a semi random telegraph signal process

$$\rightarrow X(t) = (-1)^{N(t)} \quad t \geq 0$$

$$M_x = e^{-2\lambda t} \quad R_{xx}(\tau) = e^{-2\lambda \tau}$$

$$\rightarrow \beta \sim \text{Uniform } (-2, 2)$$

$$M_\beta = \frac{2-2}{2} = 0$$

$$\text{Var}(\beta) = \frac{4^2}{12} = \frac{16}{12} = \frac{4}{3}$$

\Rightarrow For $Y(t)$ to be a WSS

- i) M_y should be independent of t
- ii) $R_{yy} = f(t_1 - t_2) = f(\tau)$

$$\begin{aligned} M_y &= E[Y(t)] = E[\beta X(t)] = E[\beta] \cdot E[X(t)] \\ &= 0 \cdot E[X(t)] = 0 \end{aligned}$$

$$R_{yy} = E[Y(t_1)Y(t_2)] = E[\beta X(t_1), \beta X(t_2)]$$

$$= E[\beta^2] \cdot E[X(t_1), X(t_2)]$$

$$= \frac{4}{3} e^{-2\lambda(t_2 - t_1)} \quad t_2 > t_1$$

$$-\frac{4}{3} e^{-2\lambda T}$$

$y(t)$ is a WSS process.

$$g_3] x(t) = T + (1-t)$$

$T \sim \text{Uniform}(0, 1)$

$$M_T = \frac{1}{2} \quad \text{Var}(T) = \frac{1}{12} = E[T^2] - t[T]^2$$

$$E[T^2] = \frac{1}{12} + \frac{1}{4} = \frac{1}{3}$$

a) cdf of $X(t)$

$$P[X(t) \leq n] = P[T + (1-t) \leq n]$$

$$P[T \leq n - (1-t)]$$

$$F_X(n) = \begin{cases} 0 & n < 1-t \\ n - (1-t) & (1-t) \leq n \leq (2-t) \\ 1 & n > 2-t \end{cases}$$

$$f(n) = \frac{d}{dn} F_X(n) = \begin{cases} 1 & (1-t) < n < (2-t) \\ 0 & \text{otherwise} \end{cases}$$

$$b) M_X = \int_{1-t}^{2-t} 1 \, dn = \frac{3}{2} - t$$

$$c) \overbrace{C_{xx}(t_1, t_2)} = R_{xx}(t_1, t_2) - M_x(t_1) M_x(t_2)$$

$$\rightarrow R_{xx}(t_1, t_2) = E[x(t_1)x(t_2)]$$

$$= E[(T + (1-t_1))(T + (1-t_2))]$$

$$= E[T^2 + T[(1-t_1) + (1-t_2)] + (1-t_1)(1-t_2)]$$

$$= \frac{1}{3} + \frac{1}{2}(2-t_1-t_2) + (1-t_1)(1-t_2)$$

$$C_{xx} = \frac{1}{3} + 1 - \frac{t_1}{2} - \frac{t_2}{2} + 1 - t_1 - t_2 + t_1 t_2$$

$$- \left(\frac{3}{2} - t_1 \right) \left(\frac{3}{2} - t_2 \right)$$

$$\frac{1}{12}$$

Q4] $x(t) \sim \text{WSS}$

$$N_x = 2, R_{xx}(t) = e^{-|t|/10} + 4$$

$$S(t) = \int_0^t x(\tau) d\tau$$

$$\begin{aligned} M_s &= E \left[\int_0^t x(\tau) d\tau \right] = \int_0^t E[x(\tau)] d\tau \\ &= \int_0^t 2 d\tau = 2t \end{aligned}$$

$$E[S^2(t)] = E \left[\int_0^t x_1(\tau_1) d\tau_1 \int_0^t x_2(\tau_2) d\tau_2 \right]$$

$$= \int_0^t \int_0^t E[x_1(\tau_1) x_2(\tau_2)] d\tau_1 d\tau_2$$

$$= \int_0^t \int_0^t (4 + e^{-|t_1 - t_2|/10}) d\tau_1 d\tau_2$$

$$= 4 + \int_{-10}^t e^{-|\tau|/10} (1 - |\tau|) d\tau$$

$$= 4 - 10 \left[e^{-\tau/10} \right]_{-10}^0 + \int_{-10}^0 \tau e^{-\tau/10} d\tau - \int_0^{\tau} \tau e^{-\tau/10} d\tau$$

$$= 4 - 10 \left[e^{-1/10} - e^{1/10} \right] - \left[10 \tau e^{-\tau/10} + 100 e^{-\tau/10} \right]_{-10}^0$$

$$+ \left[10 \tau e^{-\tau/10} + 100 e^{-\tau/10} \right]_0^{\tau}$$

$$\begin{aligned}
 &= 4 - 10e^{-1/10} + 10e^{1/10} - 100 + 10(e^{1/10} \\
 &\quad - 100e^{-1/10} + 10e^{-1/10}) + 100e^{-1/10} - 100 \\
 &= 4 - 200 + e^{1/10} [10 + 10 - 100] \\
 &\quad + e^{-1/10} [-10 + 10 + 100] \\
 &= 100e^{-1/10} - 80e^{1/10} - 196
 \end{aligned}$$

$$\begin{aligned}
 \text{Var}(S) &= E[S^2] - (E[S])^2 \\
 &= 100e^{-1/10} - 80e^{1/10} - 196 - 4 \\
 &= \underline{\underline{100e^{-1/10} - 80e^{1/10} - 200}}
 \end{aligned}$$

$$\text{Q.S.] } x(t) = 10 \cos(100t + \varphi)$$

$\varphi \sim \text{Uniform}(-\pi, \pi)$

$$M_\varphi = 0 \quad \text{Var } (\varphi) = \frac{\pi^2}{3}$$

$$M_{\cos(\varphi)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\varphi) d\varphi = 0$$

$$M_{\sin(\varphi)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin(\varphi) d\varphi = 0$$

iii) We cannot prove ergodicity of a process

i) Mean Ergodicity

$$\begin{aligned} \bar{x}_T &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T 10 \cos(100t + \varphi) dt \\ &= \lim_{T \rightarrow \infty} \frac{5}{T} \left[\text{finite } (-1, 1) \right] = \frac{1}{\infty} = 0 \end{aligned}$$

$$E[x] = E[10 \cos(100t + \varphi)] = 10 E[\cos(100t + \varphi)]$$

$$= 10 E[\cos(100t) \cos \varphi - \sin(100t) \sin \varphi]$$

$$= 10 \left\{ E[\cos 100t] \cdot E[\cos \varphi] - E[\sin 100t] E[\sin \varphi] \right\}$$

$$= 0$$

$$\boxed{\bar{x}_T = E[x]} \Rightarrow \text{Mean Ergodic}$$

ii) Correlation Ergodicity

first we need ACF of $X(t)$

$$R_{xx}(t_1, t_2) = E[X(t_1)X(t_2)]$$

$$= E[10 \cos(100t_1 + \phi) \cdot 10 \cos(100t_2 + \phi)]$$

$$= \frac{100}{2} \left\{ E[\cos(100(t_1 + t_2) + \phi)] + E[\cos 100(t_1 - t_2)] \right\}$$

$$= 50 \times 0 + 50 \cos(100(t_1 - t_2))$$

$$R_{xx}(\tau) = 50 \cos(100\tau)$$

$$\overline{X^2}_{T, T+\tau} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T X(t) \cdot X(t+\tau) dt$$

$$= \lim_{T \rightarrow \infty} \frac{100}{2T} \int_{-T}^T \cos(100t + \phi) \cos(100(t+\tau) + \phi) dt$$

$$= \lim_{T \rightarrow \infty} \frac{50}{T} \int_{-T}^T \cancel{\cos(100(2t+\tau) + \phi)} dt$$

$$+ \frac{50}{T} \int_{-T}^T \cos 100\tau dt$$

$$= 50 \cos 100\tau$$

$$\boxed{\overline{X^2}_{T, T+\tau} = R_{xx}(\tau)} \Rightarrow \text{correlation ergodic}$$

$$Q_6] \quad x(t) \sim WSS$$

$$M_x = 0 \quad R_{xx}(\tau) = 1 - \frac{|\tau|}{T}$$

$$\text{Var}(\bar{x}_T) = \frac{1}{(T/2)} \int_0^T C_{xx} \left[1 - \frac{|\tau|}{T} \right] d\tau$$

$$C_{xx} = R_{xx}(\tau) - \underbrace{M_x(t_1) M_x(t_2)}_{= 0}$$

$$\text{Var}(\bar{x}_T) = \frac{2}{T} \int_0^T \left(1 - \frac{\tau}{T} \right)^2 d\tau$$

$$= -\frac{2}{T} \left[\frac{\left(1 - \frac{\tau}{T} \right)^3}{3} \right]_0^T$$

$$= -\frac{2 \times T}{T} \left[\frac{0^3}{3} - \frac{1^3}{3} \right] = \frac{2T}{3T} = \frac{2}{3}$$

$$\lim_{T \rightarrow \infty} \text{Var}(\bar{x}_T) = 0$$

for Mean Ergodic Process

Hence, it is not ergodic process

$$87] \quad x(t) \sim RTP \quad x(t) = \alpha (-1)^{N(t)}$$

$$M_x = 0 \quad R_{xx}(\tau) = e^{-2\lambda|\tau|}$$

$$\begin{aligned} E[\bar{x}_T] &= E\left[\frac{1}{T} \int_{-T}^T x(t) dt\right] = \lim_{T \rightarrow \infty} \int_{-T}^T E[x(t)] dt \\ &= \int_{-T}^T 0 dt = 0 \end{aligned}$$

$$\text{Var}(\bar{x}_T) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T}^T C_{xx} \left[1 - \frac{\tau}{T}\right] d\tau$$

$$= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T}^T e^{-2\lambda|\tau|} \left[1 - \frac{|\tau|}{T}\right] d\tau$$

$$= \lim_{T \rightarrow \infty} \frac{1}{T} \left\{ \int_{-T}^0 e^{2\lambda\tau} \left[1 + \frac{\tau}{T}\right] d\tau + \int_0^T e^{-2\lambda\tau} \left[1 - \frac{\tau}{T}\right] d\tau \right\}$$

$$= \lim_{T \rightarrow \infty} \frac{1}{T} \left\{ \left[\frac{e^{2\lambda T}}{2\lambda} \right]_{-T}^0 + \frac{1}{T} \left[\tau \frac{e^{2\lambda\tau}}{2\lambda} - \frac{e^{2\lambda\tau}}{(2\lambda)^2} \right]_{-T}^0 \right.$$

$$\left. + \left[\frac{e^{-2\lambda T}}{2\lambda} \right]_0^T - \frac{1}{T} \left[\tau \left(\frac{e^{-2\lambda\tau}}{-2\lambda} \right) - \frac{e^{-2\lambda\tau}}{(2\lambda)^2} \right]_0^T \right\}$$

$$= \frac{1}{T} \left\{ \cancel{\frac{1}{2\lambda} - \frac{e^{-2\lambda T}}{2\lambda}} + \frac{1}{T} \left[\frac{-1}{(2\lambda)^2} + T \frac{e^{-2\lambda T}}{2\lambda} - \frac{e^{-2\lambda T}}{(2\lambda)^2} \right] \right\}$$

$$+ \left[\cancel{\frac{e^{-2\lambda T}}{2\lambda}} - \cancel{\frac{1}{2\lambda}} \right] - \frac{1}{T} \left[\frac{T e^{-2\lambda T}}{(-2\lambda)} - \frac{e^{-2\lambda T}}{(2\lambda)^2} + \frac{1}{(2\lambda)^2} \right]$$

$$= \frac{1}{T^2} \left\{ -\frac{1}{(2\lambda)^2} + \frac{T e^{-2\lambda T}}{2\lambda} - \frac{\cancel{e^{-2\lambda T}}}{(2\lambda)^2} \right. \\ \left. + \frac{T e^{-2\lambda T}}{2\lambda} + \frac{\cancel{e^{-2\lambda T}}}{(2\lambda)^2} - \frac{1}{(2\lambda)^2} \right]$$

$$= \frac{1}{T^2} \left\{ \frac{T e^{-2\lambda T}}{\lambda} - \frac{1}{2\lambda^2} \right\}$$

$$\lim_{T \rightarrow \infty} \text{Var}(\bar{x}_T) = \frac{1}{\infty} = 0$$

Hence Process is mean ergodic

$$89) \quad s(\omega) = \begin{cases} 1 + \omega^2 & |\omega| \leq 1 \\ 0 & |\omega| > 1 \end{cases}$$

$$s(t) \Rightarrow 1$$

$$\mathcal{F}^{-1}\{w^2\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} w^2 e^{j\omega t} dw$$

$$= \frac{1}{2\pi} \left[w^2 \left(\frac{e^{j\omega t}}{jt} \right) - w \left(\frac{e^{j\omega t}}{(jt)^2} \right) + \frac{e^{j\omega t}}{(jt)^3} \right]_0^\infty$$

$$= \frac{1}{2\pi} \left[\frac{1}{jt} e^{jt} - \frac{e^{jt}}{(jt)^2} + \frac{e^{jt}}{(jt)^3} - \left(\frac{e^{-jt}}{jt} + \frac{e^{-jt}}{(jt)^2} + \frac{e^{-jt}}{(jt)^3} \right) \right]$$

$$= \frac{1}{2\pi} \left[\frac{e^{jt} - e^{-jt}}{jt} - \left(\frac{e^{jt} + e^{-jt}}{(jt)^2} \right) + \frac{e^{jt} - e^{-jt}}{(jt)^3} \right]$$

$$= \frac{1}{\pi} \left[\frac{\sin t}{t} - \frac{\cos t}{-t^2} + \frac{\sin t}{-t^3} \right]$$

$$R_{xx}(\tau) = \mathcal{F}^{-1}\{S(\omega)\}$$

$$= S(\tau) + \frac{1}{\pi} \left\{ \frac{\sin t}{t} + \frac{\cos t}{t^2} + \frac{\sin t}{t^3} \right\}$$

Q10] $R(\tau) = \begin{cases} 1 - \frac{|\tau|}{T} & |\tau| \leq T \\ 0 & |\tau| > T \end{cases}$

$$S(\omega) = \mathcal{F}\{R(\tau)\}$$

$$1 \iff 2\pi\delta(\omega)$$

$$\mathcal{F}\{|\tau|\} = \int_{-T}^T |\tau| e^{-j\omega\tau} d\tau$$

$$= \int_{-T}^0 -\tau e^{-j\omega\tau} d\tau + \int_0^T \tau e^{-j\omega\tau} d\tau$$

$$= - \left[(\tau) \left(e^{-j\omega\tau} \right) + \frac{e^{-j\omega\tau}}{(j\omega)^2} \right]_0^T$$

$$+ \left[\tau \left(\frac{e^{-j\omega\tau}}{-j\omega} \right) + \frac{e^{-j\omega\tau}}{(j\omega)^2} \right]_0^T$$

$$= - \left[\frac{1}{\omega^2} - \left(\frac{-Te^{-j\omega T}}{-j\omega} + \frac{e^{-j\omega T}}{-\omega^2} \right) \right]$$

$$+ \left[\frac{Te^{-j\omega T}}{-j\omega} + \frac{e^{-j\omega T}}{-\omega^2} - \left(\frac{1}{-\omega^2} \right) \right]$$

$$= \frac{1}{\omega^2} + \frac{Te^{-j\omega T}}{j\omega} + \frac{e^{-j\omega T}}{\omega^2}$$

$$- \frac{Te^{-j\omega T}}{j\omega} - \frac{e^{-j\omega T}}{\omega^2} + \frac{1}{\omega^2}$$

$$= \frac{2}{\omega^2}$$

So, PSDF = $S(\omega) = 2\pi\delta(\omega) + \frac{2}{\omega^2}$