▷ BINOMIAL DISTRIBUTION

PMF

$$P(x) = {}^nC_x \, p^x q^{n-x}$$

n = no. of trials
x = no. of success $0 \le x \le n$
p = prob of success
q = prob of failure

MGF

$$M_x(t) = E[e^{xt}] = \int_0^n e^{xt} \cdot {}^nC_x \, p^x q^{n-x} \, dx$$

$$M_x(t) = E[e^{xt}] = \sum_{x=0}^{n} {}^nC_x \, p^x q^{n-x} e^{xt}$$

$$= q^n \sum_{x=0}^{n} {}^nC_x \left(\frac{pe^t}{q}\right)^x$$

$$M_x(t) = q^n \left(1 + \frac{pe^t}{q}\right)^n = (q + pe^t)^n$$

$$\boxed{M_x(t) = (q + pe^t)^n}$$

$$\mu_1' = n(q + pe^t)^{n-1} pe^t \Big|_{t=0} = n(q+p)p$$

$$= np$$

$$\boxed{E[X] = np}$$

$$\mu_2' = n\left[(n-1)(q+pe^t)^{n-2} pe^t pe^t + (q+pe^t)^{n-1} pe^t\right]_{t=0}$$

$$= n\left[(n-1)p^2 + p\right] = np + n(n-1)p^2$$

$$Var(x) = np + n(n-1)p^2 - n^2p^2$$

$$= np + \cancel{n^2p^2} - np^2 - \cancel{n^2p^2} = npq$$

$$\boxed{Var(x) = npq}$$

▷ POISSON DISTRIBUTION

when in binomial distribution

$n \to \infty \quad p \to 0 \quad \lambda = np = \text{const}$

$$\therefore \; {}^nC_r \, p^r q^{n-r} \longrightarrow \frac{e^{-\lambda}\lambda^r}{r!}$$

$$\# \quad e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

$$\to \; {}^nC_r \left(\frac{\lambda}{n}\right)^r \left(1-\frac{\lambda}{n}\right)^{n-r}$$

$$\lim_{n\to\infty} \frac{n!}{r!(n-r)!} \times \frac{\lambda^r}{n^r} \times \left(1-\frac{\lambda}{n}\right)^n \times \left(1-\frac{\lambda}{n}\right)^{-r}$$

$$\frac{\lambda^r}{r!} \lim_{n\to\infty} \frac{\left|\overbrace{n(n-1)\ldots(n-r+1)}^{r \text{ terms}}\right|}{n^r} \left(1-\frac{\lambda}{n}\right)^n \cancel{\left(1-\frac{\lambda}{n}\right)^r} \to 1$$

$$\frac{\lambda^r}{r!} \lim_{n\to\infty} \left|\frac{n}{n} \times \left(1-\frac{1}{n}\right) \times \ldots \left(1-\frac{r+1}{n}\right)\right| \left(1-\frac{\lambda}{n}\right)^n$$

$$\qquad \to 1 \qquad\qquad \to e^{-\lambda}$$

$$= \frac{\lambda^r e^{-\lambda}}{r!}$$

$$M_x(t) = E[e^{xt}] = \sum_{x=0}^{\infty} e^{xt} \cdot \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} \cdot \exp(\lambda e^t) = e^{-\lambda} \cdot e^{\lambda e^t}$$

$$M_x(t) = e^{\lambda(e^t - 1)}$$

$$M_1' = e^{\lambda(e^t-1)} \cdot \lambda e^t \Big|_{t=0}$$

$$= \boxed{\lambda = E(x)}$$

$$M_2' = \lambda \left[e^{\lambda(e^t-1)} \cdot \lambda e^t e^t + e^{\lambda(e^t-1)} \cdot e^t \right]_{t=0}$$

$$= \lambda [\lambda + 1] = \lambda^2 + \lambda$$

$$\boxed{Var(x) = \lambda}$$

Q3] 5-comp 3-comp

$$P[x \geq 3] > P[x \geq 2]$$

$$\sum_{x=3}^{5} {}^{5}C_x\, p^x (1-p)^{5-x} > \sum_{x=2}^{3} {}^{3}C_x\, p^x (1-p)^{3-x}$$

$$\frac{\cancel{5!}^{\,5\times\cancel{4}^{2}}}{\cancel{3!}\,\cancel{2!}} p^3(1-p)^2 + 5p^4(1-p) + p^5$$

$$> 3p^2(1-p) + p^3$$

$$10p^3(1+p^2-2p) + 5p^4(1-p) + p^5$$

$$> 3(p^2-p^3) + p^3$$

$$10p^3 + 10p^5 - 20p^4 + 5p^4 - 5p^5 + p^5$$

$$> 3p^2 - 3p^3 + p^3$$

$$10p^3 + 6p^5 - 15p^4 > 3p^2 - 2p^3$$

$$10p + 6p^3 - 15p^2 > 3 - 2p$$

$$6p^3 - 15p^2 + 12p - 3 > 0$$

$$3(p-1)^2(2p-1) > 0$$

$$\to p < 1 \;\&\; p > \frac{1}{2}$$

$$\boxed{1/2 < p < 1}$$

Q4] Poisson Dist.

$n = 100{,}000 \qquad p = 2 \times 10^{-5}$

$\lambda = np = 2 \times 10^{-5} \times 10^{5} = 2 = \text{const}$

$$P(x) = \frac{e^{-2}\, 2^{x}}{x!}$$

$P[X \geq 5] = 1 - P[X < 5]$

$$= 1 - e^{-2}\left[\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!}\right]$$

$$= 1 - e^{-2}\left[1 + 2 + 2 + \frac{4}{3} + \frac{\cancel{4} \times \cancel{4} \times 2}{\cancel{4} \times 3 \times \cancel{2}}\right]$$

$$= 1 - e^{-2}\left[1 + 2 + 2 + 2\right] = \underline{\underline{1 - 7e^{-2}}}$$

Q5] $p = \frac{25}{100} = \frac{1}{4} \qquad q = \frac{3}{4}$

$$P(2) = {}^{10}C_2 \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^8 = \frac{\overset{5}{\cancel{10}} \times \cancel{9}\, 3 \times 3}{\cancel{2} \times 4 \times 4} \times \left(\frac{3}{4}\right)^8$$

$$= \underline{\underline{5 \times \left(\frac{3}{4}\right)^{10}}}$$

Q6] Ratio $= \frac{390}{520} = \frac{n}{5}$

$n = \frac{39\cancel{0}}{52\cancel{0}} \times 5 = 3.75$ error /page

$\lambda = 3.75$

$P(0) = \frac{e^{-3.75}(3.75)^0}{0!} = \underline{\underline{e^{-3.75}}}$

Q10] $\lambda = 2$

$P\left[\frac{x = 1 \text{ or } 2}{x \geq 1}\right] = \frac{P(x = 1 \text{ or } 2 \cap x \geq 1)}{1 - P(x=0)}$

$= \frac{P(x=1) + P(x=2)}{1 - P(x=0)}$ # events are independent

$= \frac{\frac{e^{-2}.2^1}{1!} + \frac{e^{-2}2^2}{2!}}{1 - \frac{e^{-2}2^0}{0!}} = \frac{4e^{-2}}{1-e^{-2}} = \underline{\underline{0.62607}}$

$E\left[\frac{x=1 \text{ or } x=2}{x \geq 1}\right] = \frac{\lambda - 0 \times P(x=0)}{1 - P(x=0)} = \frac{\lambda}{1-e^{-2}} = \frac{2}{1-e^{-2}}$

$= \underline{\underline{2.313}}$

Q7] $X \sim$ Negative Binomial Dist.

$n = 10 \qquad p = 0.05 = 1/20$

$m = 5 \qquad q = 0.95 = 19/20$

$$P(x) = {}^{n-1}C_{m-1}\, p^m q^{n-m}$$

$$P(10) = {}^9C_4 \left(\frac{1}{20}\right)^5 \left(\frac{19}{20}\right)^5 = \frac{9 \times \overset{2}{\cancel{8}} \times 7 \times \cancel{6}}{\cancel{4} \times \cancel{3 \times 2}} \frac{(19)^5}{(20)^{10}}$$

$$= 126 \times \frac{19^5}{20^{10}}$$

Q8] $X \sim$ Geometric Dist.

X = no. of failures for 1^{st} Success

$P(x = r) = q^r p \qquad p = 0.4$ (given)

$$P(r < 5) = \sum_{r=0}^{4} (0.6)^r (0.4) = (0.4) \sum_{r=0}^{4} (0.6)^r$$

$$= (0.4) \left[\frac{1 - (0.6)^5}{1 - 0.6} \right] = \frac{\cancel{(0.4)}\,(1 - (0.6)^5)}{\cancel{0.4}}$$

$$= \underline{0.9223}$$

$X \sim$ Geometric Dist

$X =$ no of failures

$P(X=r) = q^r p$

$P(r = \text{odd}) = \sum_{k=0}^{\infty} q^{2k} \cdot p$ # substituting $r = 2k$ for even values

$= p \sum_{k=0}^{\infty} (q^2)^k$ # $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$

given $\rightarrow$ $0.6 =$

$\frac{p}{1-q^2} = \frac{p}{1-(1-p)^2} = \frac{p}{1-1-p^2+2p}$

$\frac{3}{5}\cancel{\frac{6}{10}} = \frac{p}{2p-p^2} \Rightarrow 6p - 3p^2 = 5p$

$\Rightarrow 3p^2 - p = 0 \Rightarrow p(3p-1) = 0$

$p = 0$ or $p = 1/3$

× ✓

We chose even values for $X = r$ because then total no. of trials will be odd

$\underbrace{\text{failures}}_{r\ =\ \text{even}} + \underbrace{\text{success}}_{1^{st}} = \text{Odd}$

Q11] $X \sim$ Negative Binomial Dist

$X = n =$ no. of trials of 2nd success

$m = 2$

$p = \frac{5}{100} = \frac{1}{20}$

$m \leq n \leq \infty$

$P(X = n \geq 4) = 1 - P(X < 4)$

$= 1 - \sum_{n=2}^{3} {}^{n-1}C_1 \left(\frac{1}{20}\right)^2 \left(\frac{19}{20}\right)^{n-2}$

$= 1 - {}^1C_1 \left(\frac{1}{20}\right)^2 - {}^2C_1 \left(\frac{1}{20}\right)^2 \left(\frac{19}{20}\right)$

$= 1 - \left(\frac{1}{20}\right)^2 - \frac{2 \times 19}{(20)^3} = 1 - 0.0025 - 0.00475$

$= \underline{\underline{0.99275}}$

Q12] $X \sim$ negative Binomial Dist.

$X =$ no. of shots of 10 baskets

$10 \leq n \leq \infty$

$m = 10$

$p = \frac{8}{10} = \frac{4}{5}$

a) $pmf = {}^{n-1}C_9 \left(\frac{4}{5}\right)^{10} \left(\frac{1}{5}\right)^{n-10}$ $n \geq 10$

b) $E[X] = \frac{m}{p} = \frac{10}{4/5} = \underline{\underline{12.5}}$

$Var(X) = \frac{mq}{p^2} = \frac{10 \times 1/5}{(4/5)^2} = \underline{\underline{3.125}}$

$$(x=12) = {}^{11}C_9 \left(\frac{4}{5}\right)^{10}\left(\frac{1}{5}\right)^2$$

$$= \frac{11\times 10}{2} \times \frac{4^{10}}{5^{12}}$$

$$= \frac{11\times \cancel{10}}{\cancel{2\times 5}} \times \frac{4^{10}}{5^{11}} = \frac{11}{5} \times \left(\frac{4}{5}\right)^{10}$$

$$= \underline{\underline{0.2362}}$$

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17 sep 18