Electrical Science-2 (15B11EC211)

Unit-3

Operational Amplifier and Filters

Lecture-3

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- Frequency response
- Transfer function with example
- Resonance circuits
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- Parallel resonance
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Frequency response

- **In order to find out the desired frequency characteristics of the electric filters, the frequency response of circuits play significant role.**
- **The frequency response of a circuit is the variation in its behavior with change in signal frequency.**
- **The frequency response of a circuit may also be considered as the variation of the gain and phase with frequency.**
- **Now consider the frequency response of simple circuits using their transfer functions.**

Ref:1

Transfer Function

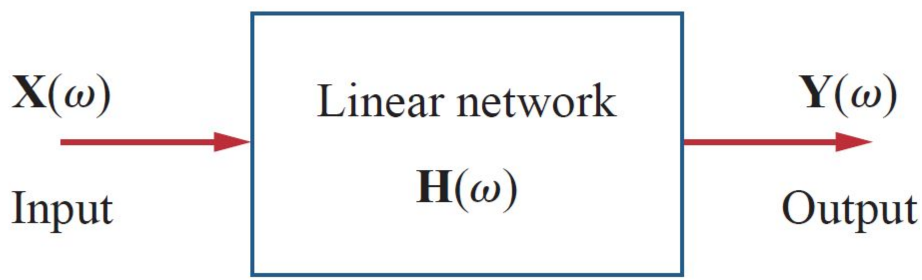
- The transfer function H(ω) (also called the network function) is a useful analytical tool for finding the frequency response of a circuit.

- In fact, the frequency response of a circuit is the plot of the circuit's transfer function H(ω) versus ω, with ω varying from ω = 0 to ω = ∞.

- A transfer function H(ω) of a circuit is the frequency-dependent ratio of a phasor output Y(ω) (an element voltage or current) to a phasor input X(ω) (source voltage or current).

- Thus

$$\mathbf{H}(\omega) = \frac{\mathbf{Y}(\omega)}{\mathbf{X}(\omega)}$$



assuming zero initial conditions

Ref: 1, 2

Transfer Function

- Since the input and output can be either voltage or current at any place in the circuit, there are four possible transfer functions:

$$\mathbf{H}(\omega) = \text{Voltage gain} = \frac{\mathbf{V}_o(\omega)}{\mathbf{V}_i(\omega)}$$

$$\mathbf{H}(\omega) = \text{Current gain} = \frac{\mathbf{I}_o(\omega)}{\mathbf{I}_i(\omega)}$$

$$\mathbf{H}(\omega) = \text{Transfer Impedance} = \frac{\mathbf{V}_o(\omega)}{\mathbf{I}_i(\omega)}$$

$$\mathbf{H}(\omega) = \text{Transfer Admittance} = \frac{\mathbf{I}_o(\omega)}{\mathbf{V}_i(\omega)}$$

- To obtain the transfer function, we first obtain the frequency-domain equivalent of the circuit by replacing resistors, inductors, and capacitors with their impedances R, jωL, and 1/jωC.

Ref: 1, 2

Transfer Function

- The transfer function can be expressed in terms of its numerator polynomial $N(\omega)$ and denominator polynomial $D(\omega)$ as

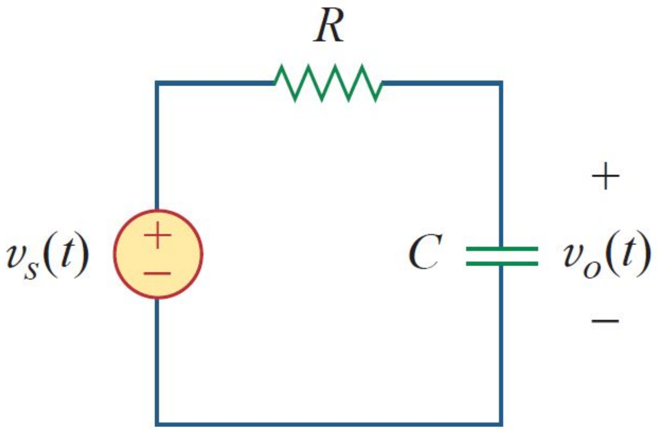
$$\mathbf{H}(\omega) = \frac{\mathbf{N}(\omega)}{\mathbf{D}(\omega)}$$

- The roots of $N(\omega) = 0$ are called the zeros of $H(\omega)$ and are usually represented as $j\omega = z_1, z_2$ ……..
- Similarly, the roots of $D(\omega) = 0$ are the poles of $H(\omega)$ and are represented as $j\omega = p_1, p_2$, ……
- To avoid complex algebra, it is expedient to replace $j\omega$ temporarily with s when working with $H(\omega)$ and replace s with $j\omega$ at the end.

Ref:1, 2

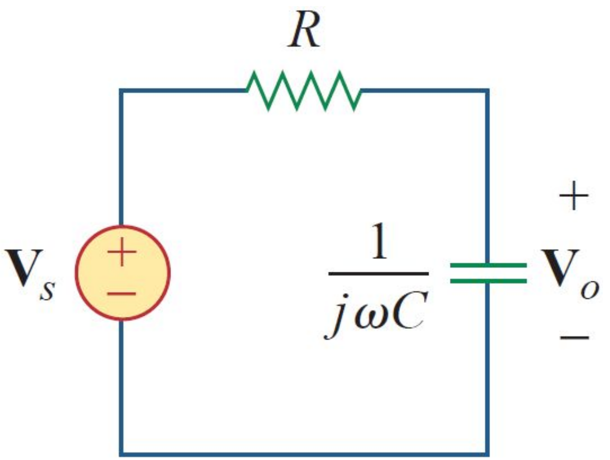
Example

Example 1: For the RC circuit shown in Figure, obtain the transfer function Vo/Vs and its frequency response. Let $v_s = V_m \cos \omega t$



Solution:

- **The frequency-domain equivalent of the circuit is shown in Figure**



Ref:1, 2

Solution Cont….

By voltage division, the transfer function is given by

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o}{\mathbf{V}_s} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1 + j\omega RC}$$

the magnitude and phase of H(ω) as

$$H = \frac{1}{\sqrt{1 + (\omega/\omega_0)^2}}, \qquad \phi = -\tan^{-1}\frac{\omega}{\omega_0}$$

Where $\omega_0 = 1/RC$

To plot H and ϕ for $0 < \omega < \infty$

At $\omega = 0$, $H = 1$ *and* $\phi = 0$

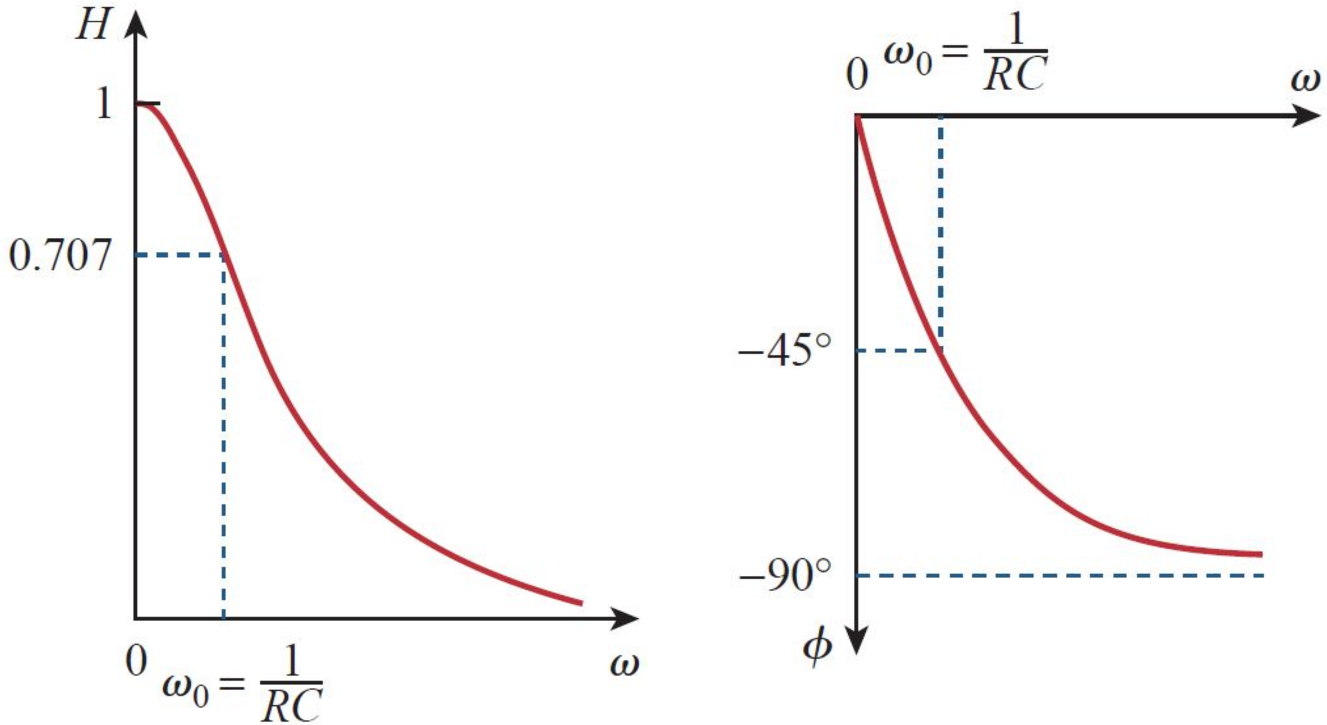
At $\omega = \omega_0$, $H = 1/\sqrt{2}$ *and* $\phi = -45^\circ$

At $\omega = \infty$, $H = 0$ *and* $\phi = -90^\circ$

Ref:1, 2

Solution Cont….

Figure Frequency response of the *RC circuit:*

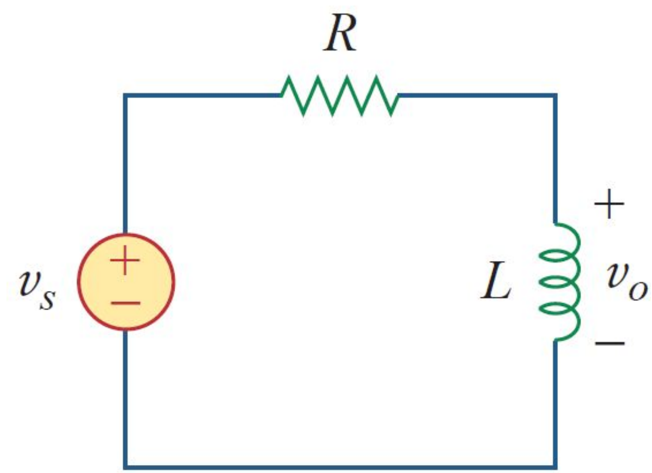


(a) Amplitude response, **(b) Phase response**

Ref:1, 2

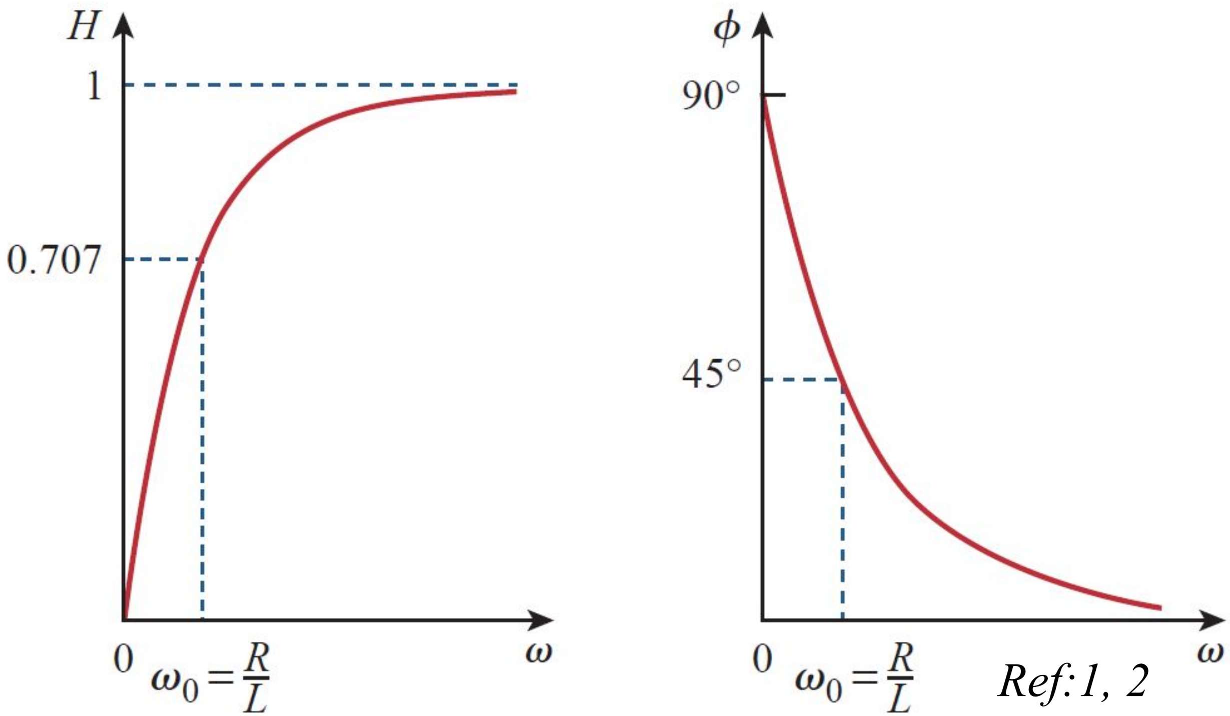
Practice Example

Obtain the transfer function Vo/Vs of the RL circuit shown in Figure, and its frequency response.
Let $v_s = V_m \cos \omega t$



Answer:

jωL/(R + jωL)



Ref:1, 2

Resonance Circuits

- **Resonant circuits (series or parallel) are useful for constructing filters as their transfer functions can be highly frequency selective.**

- They are used in many applications such as selecting the desired stations in radio and TV receivers.
- Resonance occurs in any system that has a complex conjugate pair of poles; it is the cause of oscillations of stored energy from one form to another.

- It is the phenomenon that allows frequency discrimination in communications networks.
- Resonance occurs in any circuit that has at least one inductor and one capacitor.
- **Resonance is a condition in an RLC circuit in which the capacitive and inductive reactances are equal in magnitude, thereby resulting in a purely resistive impedance.**

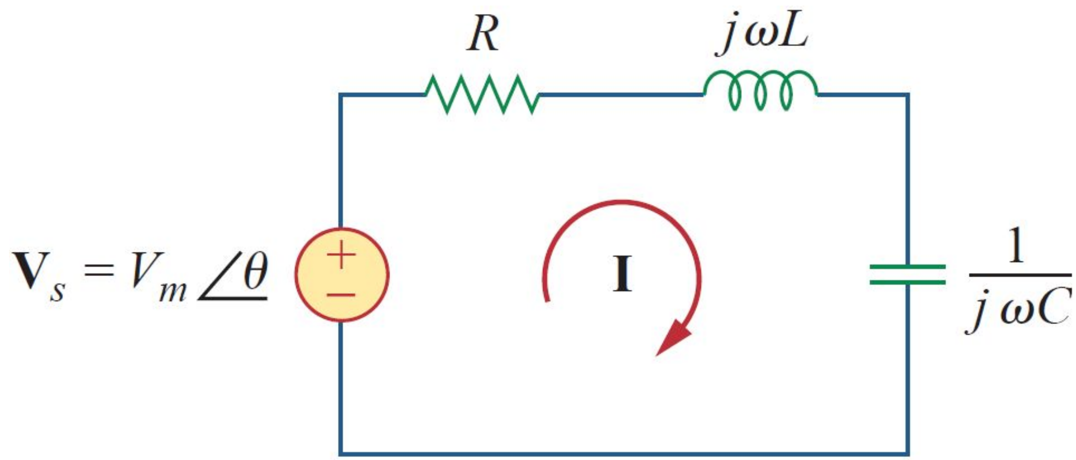
Ref: 1, 2

Series Resonance

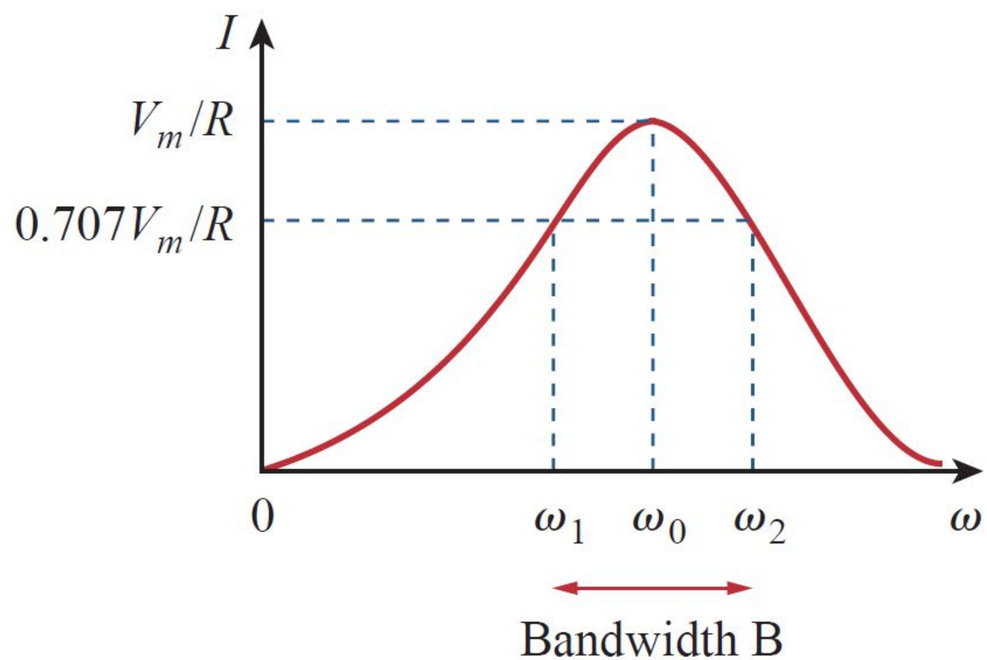
- **The value of ω that satisfies the resonance condition is called the resonant frequency ω_0 .**

$$\omega_0 L = \frac{1}{\omega_0 C}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}$$



Series resonant circuit



Series Resonance

- The half-power frequencies are obtained by setting $Z = (\sqrt{2})R$

$$\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} = \sqrt{2}R$$

Solving for ω , we obtain

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

- Resonant frequency is the geometric mean of the half-power frequencies

$$\omega_0 = \sqrt{\omega_1 \omega_2}$$

- Half-power Bandwidth B $\quad B = \omega_2 - \omega_1$

Ref: 1, 2

Series Resonance

- Quality factor Q:
  The "sharpness" of the resonance in a resonant circuit is measured quantitatively by the quality factor Q.

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 C R}$$

- The relationship between the bandwidth *B and the quality factor Q*

$$B = \frac{R}{L} = \frac{\omega_0}{Q}$$

- Hence, the quality factor of a resonant circuit is the ratio of its resonant frequency to its bandwidth

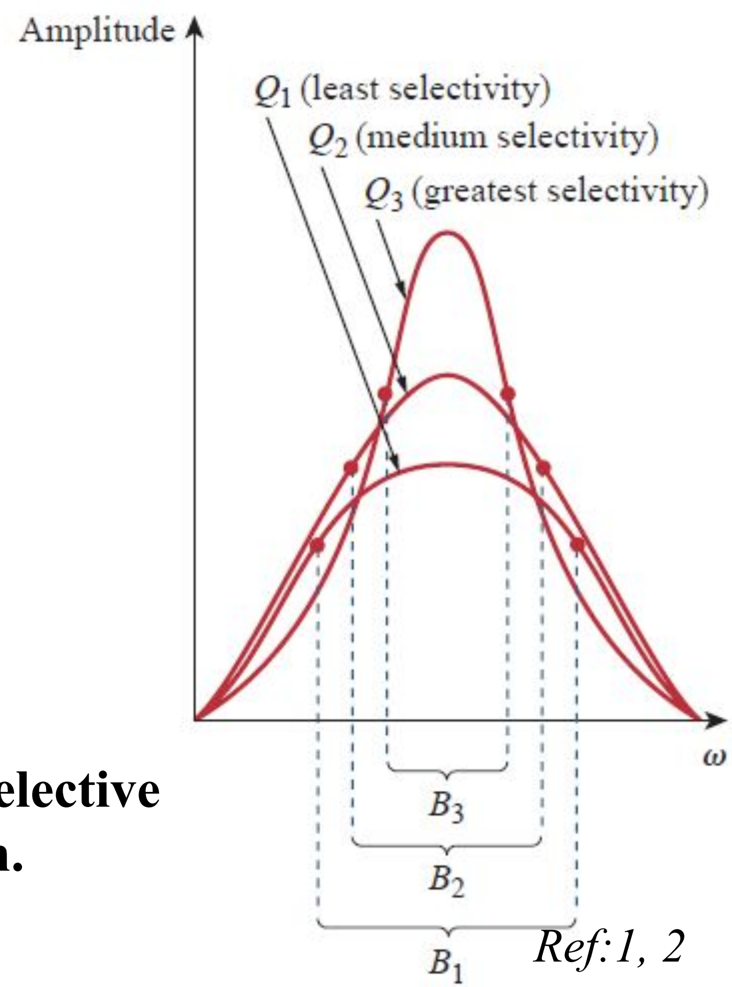


Figure higher the value of Q, the more selective the circuit is but the smaller the bandwidth.

Ref:1, 2

Series Resonance

- A resonant circuit is designed to operate at or near its resonant frequency.
- It is said to be a high-Q circuit when Q ≥ 10
- For high-Q circuits (Q ≥ 10) the half-power frequencies can be approximated as

$$\omega_1 \simeq \omega_0 - \frac{B}{2}, \qquad \omega_2 \simeq \omega_0 + \frac{B}{2}$$

Ref:1, 2

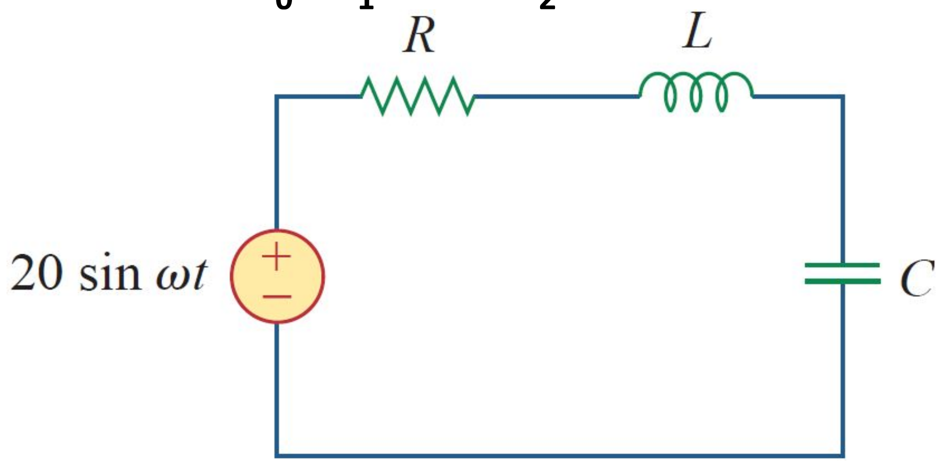
Example

In the circuit of Figure, R = 2 Ω, L = 1 mH, and C = 0.4 μF.

(a) Find the resonant frequency and the half-power frequencies.

(b) Calculate the quality factor and bandwidth.

(c) Determine the amplitude of the current at ω_0, ω_1, and ω_2.



Solution:

The resonant frequency is

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10^{-3} \times 0.4 \times 10^{-6}}} = 50 \text{ krad/s}$$

$$Q = \frac{\omega_0 L}{R} = \frac{50 \times 10^3 \times 10^{-3}}{2} = 25$$

Ref:1, 2

Solution Cont….

From *Q, we find*

$$B = \frac{\omega_0}{Q} = \frac{50 \times 10^3}{25} = 2 \text{ krad/s}$$

Since Q > 10, this is a high-*Q circuit and we can obtain the half-power* frequencies as

$$\omega_1 = \omega_0 - \frac{B}{2} = 50 - 1 = 49 \text{ krad/s}$$

$$\omega_2 = \omega_0 + \frac{B}{2} = 50 + 1 = 51 \text{ krad/s}$$

At $\omega = \omega_0$

$$I = \frac{V_m}{R} = \frac{20}{2} = 10 \text{ A}$$

At $\omega = \omega_1, \omega_1$

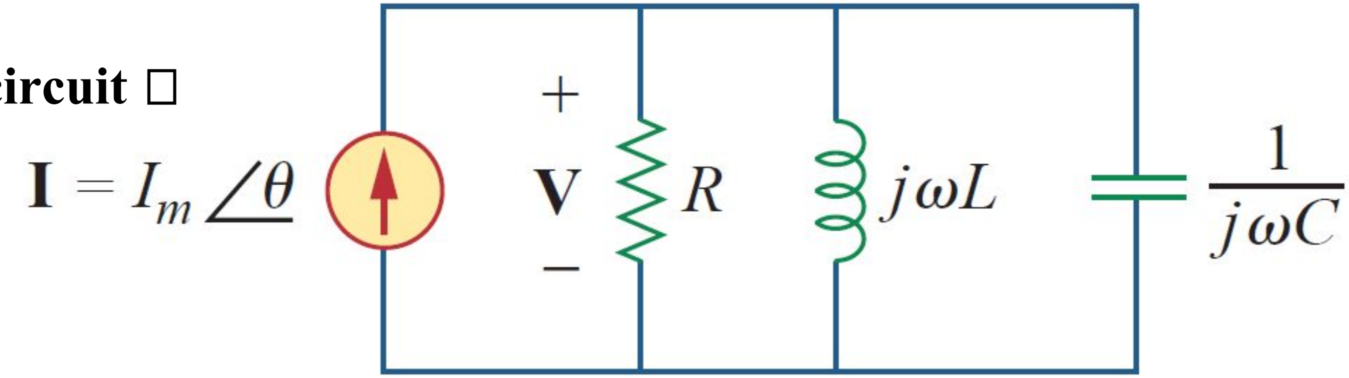
$$I = \frac{V_m}{\sqrt{2}R} = \frac{10}{\sqrt{2}} = 7.071 \text{ A}$$

Ref:1, 2

Parallel Resonance

Note The parallel RLC circuit is the dual of the series RLC circuit.

Parallel resonant circuit



Admittance Y

$$\mathbf{Y} = H(\omega) = \frac{\mathbf{I}}{\mathbf{V}} = \frac{1}{R} + j\omega C + \frac{1}{j\omega L}$$

$$\mathbf{Y} = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$$

Resonance occurs when the imaginary part of Y is zero

Ref:1, 2

Parallel Resonance

$$\omega C - \frac{1}{\omega L} = 0$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}$$

Note □ Resonant frequency is the same as for the series resonant circuit.

Notice that at resonance, the parallel LC combination acts like an open circuit, so that the entire current flows through R.

Note: Because of duality, the parameters (the two half-power frequencies ω_1 and ω_2 the resonant frequency ω_0 , the bandwidth B, and the quality factor Q) for parallel resonant circuit can be obtained by replacing R, L, and C in the expression for the series resonant circuit with 1/R, C, and L respectively

Ref: 1, 2

Parallel Resonance

$$\omega_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

$$\omega_2 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

$$B = \omega_2 - \omega_1 = \frac{1}{RC}$$

$$Q = \frac{\omega_0}{B} = \omega_0 \, RC = \frac{R}{\omega_0 L}$$

Again, for high-Q circuits ($Q \geq 10$)

$$\omega_1 \simeq \omega_0 - \frac{B}{2}, \qquad \omega_2 \simeq \omega_0 + \frac{B}{2}$$

Ref: 1, 2

Summary of the characteristics of resonant RLC circuits

Characteristic	Series circuit	Parallel circuit
Resonant frequency, ω_0	$\dfrac{1}{\sqrt{LC}}$	$\dfrac{1}{\sqrt{LC}}$
Quality factor, Q	$\dfrac{\omega_0 L}{R}$ or $\dfrac{1}{\omega_0 RC}$	$\dfrac{R}{\omega_0 L}$ or $\omega_0 RC$
Bandwidth, B	$\dfrac{\omega_0}{Q}$	$\dfrac{\omega_0}{Q}$
Half-power frequencies, ω_1, ω_2	$\omega_0\sqrt{1+\left(\dfrac{1}{2Q}\right)^2} \pm \dfrac{\omega_0}{2Q}$	$\omega_0\sqrt{1+\left(\dfrac{1}{2Q}\right)^2} \pm \dfrac{\omega_0}{2Q}$
For $Q \geq 10$, ω_1, ω_2	$\omega_0 \pm \dfrac{B}{2}$	$\omega_0 \pm \dfrac{B}{2}$

Ref:1, 2

References

1. R.C. Dorf and James A. Svoboda, "Introduction to Electric Circuits" , Chapter 17, 9th ed, John Wiley & Sons, 2013.

2. Charles K. Alexander and Matthew N. O. Sadiku, "Fundamentals of Electric Circuits" , Chapter 19, 4th ed, Mcgraw Hill, 2009.