Solution Tute 3, Physics-2 (15B11PH211) 2021

① Maxwell's equations:

$$\oint_S \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}, \qquad \nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\oint_S \vec{B}\cdot d\vec{a} = 0, \qquad \nabla\cdot\vec{B} = 0$$

$$\oint_P \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{a}, \qquad \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\oint_P \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} + \mu_0\varepsilon_0\frac{d}{dt}\int_S \vec{E}\cdot d\vec{a}, \qquad \nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

② Boundary conditions:

① $\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = \sigma_f$ ② $E_1^{\parallel} = E_2^{\parallel}$

③ $B_1^{\perp} = B_2^{\perp}$ ④ $\frac{1}{\mu_1}B_1^{\parallel} - \frac{1}{\mu_2}B_2^{\parallel} = \vec{K}_f \times \hat{n}$

③

$$\nabla^2 V = -\rho/\varepsilon_0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = -\rho/\varepsilon_0$$

$$r^2\frac{\partial V}{\partial r} = -\frac{\rho r^3}{3\varepsilon_0} + A$$

$$V = -\frac{\rho r^2}{6\varepsilon_0} - \frac{A}{r} + B$$

④

$$\nabla^2 V = 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 0$$

$$r^2\frac{\partial V}{\partial r} = A$$

$$V = -\frac{A}{r} + B$$

Boundary conditions: $V = -25\,V$ at $r = 0.02\,m$ and $V = 150\,V$ at $r = 0.35\,m$

$$\therefore\ -25 = -\frac{A}{0.02} + B \quad \text{and} \quad 150 = -\frac{A}{0.35} + B$$

Solving $A = 3.71\,Vm$ and $B = 160.61\,V$

$$\therefore\ V = \frac{-3.71}{r} + 160.61$$

$$\vec{E} = -\nabla V = -\frac{dV}{dr} = -\frac{d}{dr}\left(\frac{-3.71}{r} + 160.69\right)\hat{r} \;\; (V/m)$$

$$\vec{E} = -\frac{3.71}{r^2}\hat{r} \;\; V/m$$

$$\vec{D} = \varepsilon\vec{E} = \varepsilon_0 \varepsilon_r \vec{E} = 8.86\times10^{-12}\frac{C^2}{N.m^2}\times 3.12 \times\left(-\frac{3.71}{r^2}\right)\hat{r}$$

$$\vec{D} = -\frac{0.103}{r^2}\hat{r} \;\left(\frac{nC}{m^2}\right)$$

For conductor surface, $D^{\perp} = \sigma$

$$\text{at } r = 0.02m, \quad \sigma = \frac{-0.103}{r^2} = \frac{-0.103}{(0.02)^2} = -256 \; nC/m^2$$

$$\text{at } r = 0.35m, \quad \sigma = \frac{+0.103}{(0.35)^2} = +0.837 \; nC/m^2$$

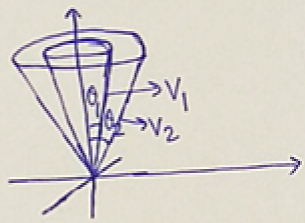
⑤ The potential is constant with r and ϕ. Laplace's eq. becomes

$$\nabla^2 V = 0$$

$$\frac{1}{r^2 \sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dV}{d\theta}\right) = 0$$

$$\sin\theta\frac{dV}{d\theta} = A$$

$$V = A\ln\left(\tan\frac{\theta}{2}\right) + B$$



Boundary conditions: $V = V_1$ at $\theta = \theta_1$ and $V = 0$ at $\theta = \theta_2$

$$\therefore V_1 = A\ln\left(\tan\frac{\theta_1}{2}\right) + B \quad \text{and} \quad 0 = A\ln\left(\tan\frac{\theta_2}{2}\right) + B$$

$$B = -A\ln\left(\tan\frac{\theta_2}{2}\right) \quad \text{and} \quad A = \frac{V_1}{\ln\left(\tan\frac{\theta_1}{2}\right) - \ln\left(\tan\frac{\theta_2}{2}\right)}$$

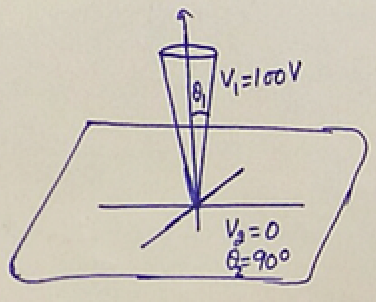
$$B = -\frac{V_1 \ln\left(\tan\frac{\theta_2}{2}\right)}{\ln\left(\tan\frac{\theta_1}{2}\right) - \ln\left(\tan\frac{\theta_2}{2}\right)}$$

$$\therefore \boxed{V = V_1\frac{\ln\left(\tan\frac{\theta}{2}\right) - \ln\left(\tan\frac{\theta_2}{2}\right)}{\ln\left(\tan\frac{\theta_1}{2}\right) - \ln\left(\tan\frac{\theta_2}{2}\right)}}$$

Now $V_1 = 100\,V$ at $\theta_1 = 10^\circ$ and $\theta_2 = 90^\circ$

$$\therefore V = 100\frac{\ln\left(\tan\frac{\theta}{2}\right)}{\ln(\tan 5^\circ)}$$

$$\therefore \vec{E} = -\nabla V = -\frac{1}{r}\frac{dV}{d\theta}\hat{\theta}$$



$$\vec{E} = -\frac{100\left(\frac{1}{2}\right)}{r\left(\tan\frac{\theta}{2}\right)\left(\cos^2\frac{\theta}{2}\right)\ln(\tan 5^\circ)}\,\hat{\theta} = -\frac{100}{(r\sin\theta)\ln(\tan 5^\circ)}\,\hat{\theta}$$

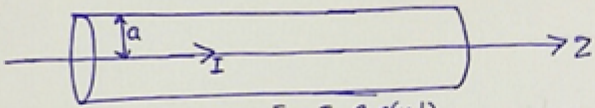
$$\vec{E} = \frac{41.05}{r\sin\theta}\,\hat{\theta}$$

$$\vec{D} = \varepsilon_0\vec{E} = \frac{3.63\times10^{-10}}{r\sin\theta}\,\hat{\theta} \quad (C/m^2)$$

For $\theta = 90^\circ$ plane, $\sin\theta = 1$ and direction of $\vec{D}$ points in $-\hat{z}$. (as $D^\perp = \sigma$)

$$\therefore \quad \sigma = -\frac{3.63\times10^{-10}}{r} \quad (C/m^2)$$

⑥



$I = I_0\cos(\omega t)$

$$\vec{E}(r,t) = \frac{\mu_0 I_0\omega}{2\pi}\sin(\omega t)\ln\left(\frac{a}{r}\right)\hat{z}$$

ⓐ $\vec{J}_d = \varepsilon_0\frac{\partial\vec{E}}{\partial t} = \frac{\mu_0\varepsilon_0\omega^2}{2\pi} I \ln\left(\frac{a}{r}\right)\hat{z}$

ⓑ $I_d = \int\vec{J}_d\cdot d\vec{a} = \frac{\mu_0\varepsilon_0\omega^2}{\cancel{2\pi}}\int_0^a I\ln\left(\frac{a}{r}\right)(\cancel{2\pi}\, r\,dr)$

$$= \mu_0\varepsilon_0\omega^2 I\int_0^a (r\ln a - r\ln r)\,dr$$

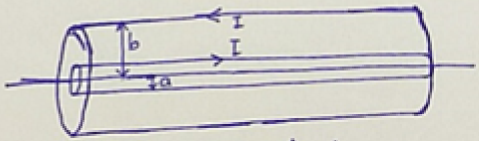
$$= \mu_0\varepsilon_0\omega^2 I\left[(\ln a)\frac{r^2}{2} - \frac{r^2}{2}\ln r + \frac{r^2}{4}\right]_0^a$$

$$= \mu_0\varepsilon_0\omega^2 I\left[\frac{a^2}{2}\cancel{(\ln a)} - \frac{a^2}{2}\cancel{(\ln a)} + \frac{a^2}{4}\right]$$

$$I_d = \frac{\mu_0\varepsilon_0\omega^2 I a^2}{4}$$

ⓒ $\frac{I_d}{I} = \frac{\mu_0\varepsilon_0\omega^2 a^2}{4} = \left(\frac{\omega a}{2c}\right)^2$

⑦



Magnetic field between the cylinders is

$$\oint\vec{B}\cdot d\vec{l} = \vec{B}(2\pi r) = \mu_0 I\,\hat{\phi}$$

$$\Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi r}\,\hat{\phi}$$

If λ is the charge per unit length then electric field between the cylinders is

$$\oint\vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0} \Rightarrow \vec{E}(2\pi r l) = \frac{\lambda l}{\varepsilon_0} \Rightarrow \vec{E} = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{r}\,\hat{r}$$

$$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = \frac{\lambda I}{4\pi^2\varepsilon_0 r^2}\hat{z}$$

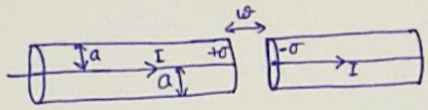
$$P = \int \vec{S}\cdot d\vec{a} = \int_0^{2\pi}\int_a^b \left(\frac{\lambda I}{4\pi^2\varepsilon_0 r^2}\hat{z}\right).\left(r\,dr\,d\phi\,\hat{z}\right) = \frac{\lambda I}{4\pi^2\varepsilon_0}(2\pi)\int_a^b \frac{dr}{r}$$

$$P = \frac{\lambda I}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right)$$

or $\quad V = \int_a^b \vec{E}.d\vec{l} = \frac{\lambda}{2\pi\varepsilon_0}\int_a^b \frac{1}{r}dr = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right)$

and $P = IV = \frac{\lambda I}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right)$

(8)



$w \ll a$

(a) $\vec{E}(t) = \frac{\sigma(t)}{\varepsilon_0}\hat{z}$, $\sigma(t) = \frac{Q(t)}{\pi a^2}$, $Q(t) = It$

$$\Rightarrow \vec{E}(t) = \frac{It}{\pi\varepsilon_0 a^2}\hat{z}$$

$$\oint \vec{B}.d\vec{l} = \vec{B}(2\pi r) = \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}(\pi r^2)$$

$$\vec{B}(2\pi r) = \frac{I\pi r^2}{\pi\varepsilon_0 a^2} \Rightarrow \vec{B}(r,t) = \frac{\mu_0 I r}{2\pi a^2}\hat{\phi}$$

(b) $\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = \frac{1}{\mu_0}\left(\frac{It}{\pi\varepsilon_0 a^2}\right)\left(\frac{\mu_0 I r}{2\pi a^2}\right)(-\hat{r})$

$$\boxed{\vec{S} = -\frac{I^2 t}{2\pi^2\varepsilon_0 a^2} r\,\hat{r}}$$

(9) $\vec{k} = -\frac{\omega}{c}\hat{x}$, $\hat{n} = \hat{z}$, $\vec{k}.\vec{r} = \left(-\frac{\omega}{c}\hat{x}\right).(x\hat{x}+y\hat{y}+z\hat{z}) = -\frac{\omega}{c}x$

$$\hat{k}\times\hat{n} = -\hat{x}\times\hat{z} = \hat{y}$$

$\vec{E}(x,t) = E_0\cos\left(\frac{\omega}{c}x + \omega t\right)\hat{z}$ and $\vec{B}(x,t) = \cos\left(\frac{\omega}{c}x + \omega t\right)\hat{y}$