Department of Mathematics

15B11MA211 **Mathematics-II**

Tutorial Sheet 3 **B.Tech. Core**

Series Solutions, Bessel and Legendre functions [C106.3]

1. Find the singular points of the following differential equations and classify them.
   (a) $x^2 y''-5y'+3x^2 y=0$ (b) $x^2 y''+(\sin x)y'+(\cos x)y=0$
   (c) $(x^2+x-2)^2 y''+3(x+2)y'+(x-1)y=0$ (d) $x^4 y''+4x^3 y'+y=0$.
2. Solve the following differential equations in series:
   (a) $(1-x^2)y''+2xy'+y=0$ (b) $xy''+y'+xy=0$ (c) $x^2 y''+xy'+(x^2-n^2)y=0$
   (d) $(x-x^2)y''+(1-5x)y'-4y=0$ (e) $8x^2 y''+10xy'-(1+x)y=0$
   (f) $x(1+x)y''+(x+5)y'-4y=0$.
3. Express the following polynomials in terms of Legendre polynomials
   (a) $f(x)=x^2$ (b) $f(x)=4x^3-2x^2-3x+8$ (c) $f(x)=x^4+2x^3-6x^2+5x-3$.
4. State and prove Rodrigue's formula and hence obtain $P_0(x), P_1(x), P_2(x) \,\&\, P_3(x)$.
5. Prove that $P_n(0)=0$, for n odd and $P_n(0)=\dfrac{(-1)^{n/2}}{2^n}\dfrac{n!}{\{(n/2)!\}^2}$, for n even.
6. Prove the following recurrence relations:
   (a) $nP_n = xP_n' - P_{n-1}'$ (b) $(1+2n)P_n = P_{n+1}' - P_{n-1}'$ (c) $(1-x^2)P_n' = n(P_{n-1}-xP_n)$.
7. Show that $J_2'(x)=(1-\dfrac{4}{x^2})J_1(x)+\dfrac{2}{x}J_0(x)$.
8. Prove that $J_n J_{-n}' - J_{-n} J_n' = -\dfrac{2Sin(n\pi)}{\pi x.}$. Hence deduce that $\dfrac{d}{dx}\left(\dfrac{J_{-n}}{Jn}\right) = -\dfrac{2Sin(n\pi)}{\pi x J_n^2}$.
9. Express $J_2(x), J_3(x) \,\&\, J_4(x)$ in terms of $J_0(x) \,\&\, J_1(x)$.
10. Prove that $J_1^{//}(x)=\dfrac{J_2(x)}{x}-J_1(x)$.
11. Prove that $\int J_0(x)\cos x dx = xJ_0(x)\cos x + xJ_1(x)\sin x + c$.

Answers:

1(a) $x=0$, irregular singular point (b) $x=0$,regular singular point

(c) $x=1$, irregular singular point, $x=-2$ regular singular point

(d) $x=0$, irregular singular point.

2(a) $y(x)=c_0(1-\frac{1}{2}x^2+\frac{1}{8}x^4+...)+c_1(x-\frac{1}{2}x^3+\frac{1}{80}x^5+...)$

(b) $y=a(1-\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}-\frac{x^6}{2^2.4^2.6^2}+...)+b(u\log x+(\frac{x^2}{2^2}-\frac{x^2}{2^2.4^2}(1+\frac{1}{2})+...))$

(c) $y=ax^n(1-\frac{x^2}{4(n+1)}+\frac{x^4}{4.8.(n+1).(n+2)}-...)+bx^{-n}(1-\frac{x^2}{4(1-n)}+\frac{x^4}{4.8.(1-n).(2-n)}-...)$

(d) $y=a(1+2^2.x+3^2.x^2+4^2.x^3+..)+b(u\log x-2(1.2x+2.3x^2+...))$

(e) $y=ax^{\frac{1}{4}}(1+\frac{x}{14}+\frac{x^2}{14.44}+...)+bx^{\frac{-1}{2}}(1+\frac{x}{2}+\frac{x^2}{220}+...)$

(f) $y=a(1+\frac{4}{5}x+\frac{1}{5}x^2+...)+bx^{-4}(1+4x+\frac{5}{4}x^2+...)$

3(a) $f(x)=\frac{2}{3}p_2(x)+\frac{p_0(x)}{3}$ (b) $\frac{8}{5}p_3(x)-\frac{4}{3}p_2(x)-\frac{3}{5}p_1+\frac{22}{3}$

(c) $f(x)=\frac{8}{35}p_4+\frac{4}{5}p_3-\frac{24}{7}p_2+\frac{31}{5}p_1-\frac{44}{7}$

9.

$J_2(x)=\frac{2}{x}J_1(x)-J_0(x),\ J_3(x)=(\frac{8}{x^2}-1)J_1(x)-\frac{4}{x}J_0(x)$

$J_3(x)=(\frac{6}{x^2}-1)\frac{8}{x}J_1(x)-(\frac{24}{x^2}-1)J_0(x)$