Tutorial Sheet-ODD Semester 2022

15B11CI212 Theoretical Foundation of Computer Science

Tutorial 7 Solutions

Q1. Define an operation ∗ on R as x ∗ y = x + y + xy. Prove or disprove: (R, ∗) is a group

Solution:

a. [Closure] Obvious.
b. [Associativity] We have (x∗y)∗z = (x+y+xy)∗z = (x+y+xy)+z+(x+y+xy)z = x+y+z+xy+xz+ yz+xyz x∗(y∗z) = x∗(y+z+yz) = x+(y+z+yz)+x(y+z+yz) = x+y+z+xy+xz+yz+xyz, i.e., (x ∗ y) ∗ z = x ∗ (y ∗ z).
c. [Identity] It is easy to check that 0 is the identity with respect to ∗.
d. [Inverse] Let x ∈ R have the inverse y ∈ R, i.e., x ∗ y = x + y + xy = 0, i.e., y = −x 1+x , i.e., y exists if and only if x 6= −1. Since −1 does not have an inverse under ∗, (R, ∗) is not a group.

Q2. Prove or disprove that the set G = {1, 3, 7, 9} is a group under multiplication modulo 10.

Solution:The composition table of G with respect to ⨀10 is as follows:

$\odot_{10}$	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

From the composition table, we have

a. a ⨀ b ∈ G ∀ a, b ∈ G
b. (a ⨀ b) ⨀ c = a ⨀ (b ⨀ c) ∀ a, b ∈ G
c. a ⨀ 1 = a ∀ a ∈ G
d. The inverse elements of 1, 3, 7, 9 are 1, 7, 3, 9 respectively.

Q3. If G is a group such that $(ab)^2 = a^2b^2$ for all a, b ∈ G, then show that G must be abelian.

Solution: abab =a^2b^2 apply a $^{-1}$ from left and b $^{-1}$ from right. We obtain ba = ab for all a, b ∈ G. Hence G is abelian.

Q4. let (A,+,.) be a ring such that a.a =a for all a in A.
a. Show that a + a = 0 for all a, where 0 is the additive identity.
b. Show that the operation . is commutative.

Solution:

$(A, +, \cdot)$ is a ring

a) Let 0 be the additive identity.

Then $a \cdot 0 = 0 \cdot a = 0$ for all a is A

$(a + a) \cdot a = a \cdot a + a \cdot a = a + a = a \cdot (a + a)$

Hence $a + a = 0$

b) $(a + b) \cdot (a + b) = a + b$

$a \cdot a + b \cdot a + a \cdot b + b \cdot b = a + b$

$a + b \cdot a + a \cdot b + b = a + b$

Hence $b \cdot a + a \cdot b = 0$

But $a \cdot b + a \cdot b = 0$

Hence $a \cdot b$ and $b \cdot a$ are additive inverse of $a \cdot b$.

Since the additive inverse is unique $b \cdot a = a \cdot b$.

Hence $\cdot$ is commutative.

Q.5 Let (Z, *) be an algebraic structure, where Z is the set of integers and the operation * is defined by n * m = maximum of (n, m). Show that (Z, *) is a semi group. Is (Z, *) a monoid ?. Justify your answer

Solution: Let a , b and c are any three integers. Closure property: Now, a * b = maximum of (a, b) ∈ Z for all a,b ∈ Z Associativity : (a * b) * c = maximum of {a,b,c} = a * (b * c) ∴ (Z, *) is a semi group. Identity : There is no integer x such that a * x = maximum of (a, x) = a for all a ∈ Z ∴ Identity element does not exist. Hence, (Z, *) is not a monoid.

Q.6 Show that the set of all strings S is a monoid under the operation concatenation of strings. Is S a group w.r.t the above operation? Justify your answer.

Solution: Let us denote the operation concatenation of strings by + . Let s1, s2, s3 are three arbitrary strings in S. Closure property: Concatenation of two strings is again a string. i.e., s1+s2 ∈ S Associativity: Concatenation of strings is associative. (s1+ s2) + s3 = s1+ (s2 + s3) Identity: We have null string , l ∈ S such that s1 + l = S. ∴ S is a monoid. Note: S is not a group, because the inverse of a non empty string does not exist under concatenation of strings.

Q.7 If (G, *) is a group and a ∈ G such that a * a = a , then show that a = e , where e is identity element in G.

Proof: Given that, a * a = a a * a = a * e (Since, e is identity in G) a = e (By left cancellation law) Hence, the result follows. Ex. If every element of a group is its own inverse, then show that the group must be abelian . Proof: Let (G, *) be a group. Let a and b are any two elements of G. Consider the identity, (a * b)-1 = b-1 * a-1 (a * b) = b * a (Since each element of G is its own inverse) Hence, G is abelian. Note: a2 = a * a a3 = a * a * a etc.