Probability and Random Processes (15B11MA301)

Lecture-4

Department of Mathematics

Jaypee Institute of Information Technology, Noida

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Conditional Probability

The conditional probability of an event B, assuming that the event A has happened, is denoted by P(B/A) and defined as

$$P(B/A) = \frac{P(A \cap B)}{P(A)}, \text{ provided } P(A) \neq 0$$

For example: when a fair dice is tossed, the conditional probability of getting '1', given that an odd number has been obtained, is equal to 1/3

as $S = \{1, 2, 3, 4, 5, 6\}$; $A = \{1, 3, 5\}$; $B = \{1\}$

$$\therefore \quad P(B/A) = \frac{n(A \cap B)}{n(A)} = \frac{1}{3}$$

As per the definition given above, $P(B/A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{1/6}{1/2} = \dfrac{1}{3}$.

Product theorem of Probability:

For any two events A and B, $P(A \cap B) = P(A) \times P(B / A)$.

Proof: Let n_A, n_{AB} be the number of cases favourable to the events A and $A \cap B$, out of the total number n of the cases.

$$\therefore \quad P(A \cap B) = \frac{n_{AB}}{n} = \frac{n_A}{n} \times \frac{n_{AB}}{n_A} = P(A) \times P(B / A)$$

- The product theorem can be extended to 3 events A, B and C as follows:

$$P(A \cap B \cap C) = P(A) \times P(B / A) \times P(C / A \, and \, B)$$

Results: These following properties are easily deduced from the definition of conditional probability:

1. If $A \subset B, P(B/A)=1,$ since $A \cap B = A$

2. If $B \subset A, P(B/A) \geq P(B),$ since $A \cap B = B,$ and $\frac{P(B)}{P(A)} \geq P(B),$ as $P(A) \leq P(S) = 1$

3. If A and B are mutually exclusive events, P(B/A)=0, since $P(A \cap B) = 0$

4. If $P(A) > P(B), P(A/B) > P(B/A)$

5. If $A_1 \subseteq A_2, P(A_1/B) \leq P(A_2/B)$

Independent Events:

A set of events is said to be independent if the occurrence of any one of them does not depend on the occurrence of non-occurrence of the others.

When 2 events A and B are independent,

then from the definition that P(B/A) = P(B).

If the events A and B are independent, the product theorem takes the form $P(A \cap B) = P(A) \times P(B)$.

Conversely, if $P(A \cap B) = P(A) \times P(B)$, the events A and B are said to be independent (Pairwise independent).

The product theorem can be extended to any number of independent events: If $A_1, A_2, \ldots\ldots, A_n$ are ***n*** independent events.

$$P(A_1 \cap A_2 \cap \ldots\ldots \cap A_n) = P(A_1) \times P(A_2) \times \ldots\ldots \times P(A_n)$$

When this condition is satisfied, the events $A_1, A_2, \ldots\ldots, A_n$ are also said to be totally independent.

Result 1: If the events A and B are independent, the events $\bar{A}$ and B (and similarly A and $\bar{B}$) are also independent.

Proof. The events $A \cap B$ and $\bar{A} \cap B$ are mutually exclusive such that $(A \cap B) \cup (\bar{A} \cap B) = B$

$\therefore \quad P(A \cap B) + P(\bar{A} \cap B) = P(B)$ **(by addition theorem)**

$\therefore \quad P(\bar{A} \cap B) = P(B) - P(A \cap B)$

$= P(B) - P(A)P(B)$ **(by product theorem)**

$= P(B)[1 - P(A)]$

$= P(B)P(\bar{A})$

Result 2: If the events A and B are independent, then so are $\bar{A}$ and $\bar{B}$.

Proof $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$

$$= 1 - [P(A) + P(B) - P(A \cap B)] \quad \text{(by addition theorem)}$$

$$= 1 - P(A) - P(B) + P(A) \times P(B) \text{ (since } A \text{ and } B \text{ are independent)}$$

$$= [1 - P(A)] - P(B)[1 - P(A)]$$

$$= P(\bar{A}) \times P(\bar{B})$$

Note: It follows that when the events A and B are independent,

$$P(A \cup B) = 1 - P(\bar{A}) \times P(\bar{B}).$$

Example: For any three events A, B and C

$$P(A \cup B / C) = P(A / C) + P(B / C) - P(A \cap B / C)$$

Proof: For any three events A, B and C

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C),$$

$$P[(A \cup B) \cap C] = P[(A \cap C) \cup (B \cap C)]$$

$$P[(A \cup B) \cap C] = P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$$

Divide both side by $P(C) > 0$, we get

$$\frac{P[(A \cup B) \cap C]}{P(C)} = \frac{P(A \cap C)}{P(C)} + \frac{P(B \cap C)}{P(C)} - \frac{P(A \cap B \cap C)}{P(C)}$$

By definition of probability:

$$P(A \cup B / C) = P(A / C) + P(B / C) - P(A \cap B / C)$$

Example: A box contains 4 bad and 6 good tubes. Two are drawn out from the box at a time. One of them is tested and found to be good. What is the probability that the other one is also good?

Solution:

Event A: **First tube is good.**

Event B: **Second tube is good.**

$$P(B / A) = \frac{P(A \cap B)}{P(A)}$$

$$P(A \cap B) = P(\text{both tubes are good})$$

$$P(A \cap B) = \frac{{}^{6}C_2}{{}^{10}C_2} = \frac{1}{3}$$

$$P(A) = P(\text{first tube is good})$$

$$= \frac{{}^{6}C_1}{{}^{10}C_1} = \frac{3}{5}$$

$$P(B / A) = \frac{P(A \cap B)}{P(A)} = \frac{1/3}{3/5} = \frac{5}{9}.$$

Example: If $P(A) = 0.5, P(B) = 0.3$ and $P(A \cap B) = 0.15$, find $P(A / \overline{B})$.

Solution: Here,

$$P(A \cap B) = 0.15 = 0.5 \times 0.3 = P(A) \times P(B)$$

Hint :- here P(A) and P(B) are independent as if we see carefully

This implies, A and B are independent events.

Hence, A and $\overline{B}$ are independent events.

Therefore,

$$P(A \cap \overline{B}) = P(A) \times P(\overline{B}) = 0.5(1 - 0.3) = 0.35$$

$$\Rightarrow P(A / \overline{B}) = \frac{P(A \cap \overline{B})}{P(\overline{B})} = \frac{0.35}{(1 - 0.3)} = 0.5$$

$$\Rightarrow P(A / \overline{B}) = 0.5$$

Example: A manufacturer of airplane parts knows that the probability is 0.8 that an order will be ready for shipment on time and it is 0.7 that an order will be ready for shipment and will be delivered on time. What is the probability that such an order will be delivered on time given that it was also ready for shipment on time?

Solution: Let A be the event that an order will be ready for shipment on time and B be an event that an order will be delivered on time.

Given $P(A) = 0.8$ and $P(A \cap B) = 0.7$

Therefore,

$$P(B / A) = \frac{P(A \cap B)}{P(A)} = \frac{0.7}{0.8} = 7/8$$

$$\Rightarrow P(B / A) = 7/8.$$

Practice Questions

1. For any three events A, B and C, prove that

$$P(A \cap B / C) = P(A / C) - P(A \cap \bar{B} / C).$$

2. A bag contains 5 white and 3 black balls. Two balls are drawn at random one after the other without replacement. Find the probability that both balls drawn are black. [**Ans:** 3/28]

3. A pair of dice is thrown simultaneously. If A denotes the number on the first die is 1 and B be the event that the number on the second die is 6 and C be the event that the sum of the two numbers on the dice is 7. Find whether A, B and C are mutually independent. [**Ans:** No]

References

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