Department of Mathematics

15B11MA211 **Mathematics-II**
B.Tech. Core

Solution Tutorial Sheet 4 (Alternating Series and Power Series)

1. Test the series $\sum (-1)^{n-1} \frac{1}{n^p}$ for (a) convergence (b) absolute convergence.

$\sum (-1)^{n-1} \frac{1}{n^p}$ by Leibnitz test.

$= \frac{1}{1^p} - \frac{1}{2^p} + \frac{1}{3^p} - \cdots$

[this is an alternating series]

a) for convergence $\because u_n < u_{n+1}$ [we have $n < n+1 \Rightarrow n^p < (n+1)^p \Rightarrow \frac{1}{n^p} > \frac{1}{(n+1)^p}$, $u_n > u_{n+1}$]

b) $\lim_{n\to\infty} u_n = 0$ [$u_n = \frac{1}{n^p}$ i.e. $\lim_{n\to\infty} u_n = 0$ for $p \geq 0$]

$\therefore$ By Leibnitz test $\sum (-1)^{n-1} \frac{1}{n^p}$ cgs. Now for absolute convergence, we check $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for convergence

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ cgs for $p > 1$
dgs for $p \leq 1$ by P-test

2. Show that the series $\frac{2}{1^2}-\frac{3}{2^2}+\frac{4}{3^2}-\frac{5}{4^2}+\ldots$ converges conditionally.

(7) $\frac{2}{1^2}-\frac{3}{2^2}+\frac{4}{3^2}-\frac{5}{4^2}+\cdots$

$= \sum (-1)^{n-1}\left|\frac{n+1}{n^2}\right| = (-1)^{n-1} u_n$

$$u_{n+1}-u_n = \frac{n+2}{(n+1)^2}-\frac{n+1}{n^2}$$

$$= \frac{n^3+2n^2-(n^3+3n^2+3n+1)}{n^2(n+1)^2}$$

$$= \frac{-(n^2+3n+1)}{n^2(n+1)^2} < 0$$

$u_{n+1} < u_n$

Also $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{n+1}{n^2} = \lim_{n\to\infty}\frac{1+\frac{1}{n}}{n} = 0$

By Leibnitz rule, $\sum u_n$ convergent.

Now consider $\frac{2}{1^2}+\frac{3}{2^2}+\frac{4}{3^2}+\cdots$

$= \sum \frac{n+1}{n^2}$

$= \sum \frac{1}{n}+\frac{1}{n^2} > \sum \frac{1}{n}$ (by Comparison)

which is divergent series.

$\therefore$ By comparison test

$\sum \frac{n+1}{n^2}$ is divergent

$\therefore$ $\sum u_n$ is conditionally convergent [by Leibnitz rule]

3. Discuss the convergence including absolute convergence of the series
$1-2x+3x^2-4x^3+\ldots$

(3) $1-2x+3x^2-4x^3+\ldots$

$= \sum (-1)^{n-1} n \cdot x^{n-1}$

$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(-1)^n (n+1) x^n}{(-1)^{n-1} n x^{n-1}}\right| = \left|-(1+\frac{1}{n})x\right|$

$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = |-x| = |x| \begin{cases} <1 & \text{cgs} \\ >1 & \text{dgs} \end{cases}, =1 \text{ fail}$

Now This series is absolutly convergent & convergent for $|x|<1 \Rightarrow -1<x<1$

for $x=1$:- i.e. $\sum (-1)^{n-1} \cdot n = 1-2+3-4 - - -$ divergent

for $x=-1$ i.e. $\sum n$ which is also divergent.

4. Show that the series $x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots$ converges if and only if $-1\le x\le 1$.

$x-\frac{x^3}{3}+\frac{x^5}{5} - \ldots$ converges iff $-1\le x\le 1$.

$= \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{x^{2n-1}}{2n-1}$

by D' alembert test

$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n x^{2n+1}}{2n+1} \times \frac{2n-1}{(-1)^{n-1} x^{2n-1}}\right|$

$= \lim_{n\to\infty}\left|\left(\frac{2-\frac{1}{n}}{2+\frac{1}{n}}\right) x^2\right|$

$= x^2$ cgs if $x^2<1$
$\Rightarrow |x|<1$

Hence it is Abs cgs $\Rightarrow -1<x<1$

(5)

if $x = 1$:-

$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

$3 < 5$

$\frac{1}{3} > \frac{1}{5}$

And $u_n = \frac{1}{2n-1}$

$\lim_{n\to\infty} u_n = 0$

$\therefore$ by. Leibnitz it cgs for $x = 1$

At $x = -1$:-

$-1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \cdots$

$3 < 5 \Rightarrow \frac{1}{3} > \frac{1}{5}$

$\lim_{n\to\infty} u_n = 0$

$\therefore$ by Leibnitz it cgs for $x = -1$

5. Test for the uniform convergence for the series

$$1 + a\cos x + a^2\cos 2x + a^3\cos 3x + \cdots a^n\cos nx + \cdots \ldots$$

$\therefore$ by Leibnitz it cgs ...

(6) $1 + a\cos x + a^2\cos 2x + \cdots + a^n\cos nx + \cdots$

$= \Sigma\, a^n \cos nx$

by wirstress M-test $\xrightarrow{}$ is b/w -1 to 1

$|a^{n-1}\cos(n-1)x| \leq |a^{n-1}|$

If $\Sigma\,|a^{n-1}|$ cgs then $\Sigma\,|a^{n-1}\cos(n-1)x|$ is uniformly cgt.

Note:- $1 + |a| + |a^2| + |a^3| + \cdots$

$1 + |a| + \underbrace{|a|^2 + |a|^3}_{GP} + \cdots$ $r = \frac{1}{a}$ cgs iff $|a| < 1$ GP.

$-1 < a < 1$.

6. Find the radius of convergence and region of convergence for the following series:

$$\sum n(x+2)^n / 3^{n+1}$$

$$\sum \frac{n\,(x+2)^n}{3^{n+1}}$$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)\,(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n\,(x+2)^n}$$

$$= \left(\frac{n+1}{n}\right) \cdot \frac{1}{3}\,(x+2)$$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left(1+\frac{1}{n}\right) \left|\frac{(x+2)}{3}\right|$$

$$= \left|\frac{x+2}{3}\right|$$

$\therefore \left|\frac{x+2}{3}\right| < 1$ (By Ratio test)

$|x+2| < 3 \Rightarrow -3-2 < x < 3-2$

$\therefore -5 < x < 1$

Radius $= 3$

Region is $(-5,1)$