Department of Mathematics

Probability and Random Processes (15B11MA301)/Probability and Random Processes (10B11MA411)
Tutorial Sheet 9 **B.Tech. Core**

Special Random Processes

1. Define a Poisson process with suitable example. State and prove all the properties of Poisson process.

2. The particles are emitted from a radioactive source at the rate of 40 per hour. Find the probability that exactly 6 particles are emitted during a 25 minutes period.

3. Customers arrive at the complain department of a store at the rate of 5 per hour for male customers and 10 per hour for female customers. If arrivals in each case follow Poisson process, calculate the probabilities that (a) at most 4 male customers, (b) at most 4 female customers will arrive in a 30 minute period (c) the inter arrival time for male candidates exceeds 15 minutes.

4. If customers arrive at a service counter in accordance with a Poisson process with a mean rate of 5 per minute, find the probability that the interval between 2 successive arrivals is (i) more than 3 minute, (ii) between 4 to 7 minutes and (iii) less than 6 minutes.

5. The transition probability matrix (tpm) of a three state system is given as $\begin{array}{cccc} & 1 & 2 & 3 \\ 1 & .2 & .3 & .5 \\ 2 & .4 & .4 & .2 \\ 3 & .4 & .6 & 0 \end{array}$

   Find the probability of moving (a) from state 1 to state 3 in one step (b) from state 1 to state 2 in exactly two steps.

6. The transition probability matrix (tpm) of a three state 0, 1, 2 Markov chain is $P = \begin{bmatrix} .75 & .25 & 0 \\ .25 & .5 & .25 \\ 0 & .75 & .25 \end{bmatrix}$ and the initial state distribution of the chain is $P\{X_0 = i\} = \frac{1}{3},\ i = 0, 1, 2$. Find (i) $P\{X_2 = 2\}$ & (ii) $P\{X_0 = 2, X_1 = 1, X_2 = 2, X_3 = 1\}$.

7. A businessman sells his goods in three cities A, B and C. He never sells in the same city on successive days. If he sells in city A, then the next day he sells in city B. However, if he sells either in B or C, then next day he is twice as likely to sell in city A as in the other city. How often does he sell in each of the cities in the steady state?

8. Three persons P, Q and R are throwing a ball to each other. P always throws the ball to Q and Q always throws the ball to R , but R is twice as likely to throw the ball to Q as to P. Is the process Markovian? If yes, find the transition probability matrix and classify the states.

9. Consider a Markov Chain with two states and transition probability matrix $P = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, find the stationary distribution of the chain. Is the chain periodic? If yes find its period.

10. Let the TPM of a Markov chain be $P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$. Find the steady state distribution of the chain. Is the chain irreducible, aperiodic and non-null persistent? Justify.

TUT 9

Q2] $\lambda = 40/\text{hour}$ $t = 25 \text{ min} = \frac{25}{60} \text{ hour}$

$$\lambda t = \cancel{40}^{2} \times \frac{25}{\cancel{60}_{3}} = \frac{100}{6} = \frac{50}{3}$$

$$P[x = 6] = \frac{e^{-\lambda t}(\lambda t)^x}{x!}$$

$$= \frac{e^{-50/3}(50/3)^6}{6!}$$

Q3] $\lambda_m = 5/h$ $\lambda_f = 10/h$

a) $x \le 4$ $t = 30 \text{ min} = 1/2 \text{ hour}$

$$P[x \le 4] = \sum_{x=0}^{4} \frac{e^{-5/2}(5/2)^x}{x!}$$

b) $x \le 4$ $t = 1/2 \text{ h}$

$$P[x \le 4] = \sum_{x=0}^{4} \frac{e^{-5}(5)^x}{x!}$$

c) Inter-arrival time follow exp. dist

$$P[T \ge 15/60] = e^{-(5/4)}$$

Q4] $\lambda = 5/\text{min}$

i) $P[T>3] = e^{-5\times 3} = e^{-15}$

ii) $P[4<T<7] = \int_4^7 5e^{-5T}\,dT$

$= 5 \left| \frac{e^{-5T}}{-5} \right|_4^7 = e^{-20} - e^{-35}$

iii) $P[T<6] = 1 - e^{-30}$

Q5]

$$P = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & .2 & .3 & .5 \\ 2 & .4 & .4 & .4 \\ 3 & .4 & .6 & 0 \end{array} \qquad P^2 = \begin{bmatrix} & \otimes_{a_{12}} & \\ & & \\ & & \end{bmatrix}$$

a) $a_{13}^{(1)} = \underline{0.5}$

b) $a_{12}^{(2)} = .3\times.2 + .4\times.3 + .6\times.5$

$= 0.06 + 0.12 + 0.30$

$= \underline{0.48}$

Q 6]

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} .75 & .25 & 0 \\ .25 & .50 & .25 \\ 0 & .75 & .25 \end{bmatrix} \end{array}$$

$$P_0 = \begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix}$$

i) $P[X_2 = 2]$

$$⓪ \xrightarrow{p^2} ②$$
$$① \xrightarrow{p^2} ②$$
$$② \xrightarrow{p^2} ②$$

$$P^2 = \begin{bmatrix} 0.625 & 0.3125 & 0.0625 \\ 0.3125 & 0.5 & 0.1875 \\ 0.1875 & 0.5625 & 0.25 \end{bmatrix}$$

$$P[X_2 = 2] = \underline{0.16666}$$

ii) $P[X_0 = 2, X_1 = 1, X_2 = 2, X_3 = 1]$

$$\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array}$$

$$\overset{1/3}{②} \xrightarrow{0.75} ① \xrightarrow{.25} ② \xrightarrow{0.75} ① = (1/3)(0.75) \times (0.25)(0.75)$$

$$= \underline{0.046875}$$

Q7]

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{bmatrix} \end{array}$$

Let stead state vector $\pi = [\pi_1 \ \pi_2 \ \pi_3]$

$$\pi P = \pi$$

$$[\pi_1 \ \pi_2 \ \pi_3] \begin{bmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{bmatrix} = [\pi_1 \ \pi_2 \ \pi_3]$$

$\pi_1 + \pi_2 + \pi_3 = 1$ —— (1)

$2/3\,\pi_2 + 2/3\,\pi_3 = \pi_1$ —— (2)

$2\pi_2 + 2\pi_3 = 3\pi_1$

$\pi_1 + 1/3\,\pi_3 = \pi_2$

$1/3\,\pi_2 = \pi_3$ —— (3)

$\rightarrow$ $2\pi_1 + \cancel{2\pi_2} + \cancel{2\pi_3} = 2$

$-\cancel{2\pi_2} - \cancel{2\pi_3} = -3\pi_1$

$5\pi_1 = 2$

$$\boxed{\pi_1 = \frac{2}{5}}$$

$$\frac{2}{5} + \pi_2 + \frac{1}{3}\pi_2 = 1$$

$$\frac{2}{5} + \frac{4}{3}\pi_2 = 1$$

$$\frac{4}{3}\pi_2 = \frac{3}{5}$$

$$\boxed{\pi_2 = \frac{9}{20}}$$

$$\boxed{\pi_3 = \frac{3}{20}}$$

Q8]

$$P = \begin{array}{c} \\ P \\ Q \\ R \end{array} \begin{array}{c} \begin{array}{ccc} P & Q & R \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 2/3 & 0 \end{bmatrix} \end{array} \quad n(\text{zero}) = 5$$

The Prob of $P[X_n]$ depends only on $P[X_{n-1}]$ hence it is <u>markov chain</u>.

$$P^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1/3 & 2/3 & 0 \\ 0 & 1/3 & 2/3 \end{bmatrix} \quad n(\text{zero}) = 4$$

$$P^3 = \begin{bmatrix} 1/3 & 2/3 & 0 \\ 0 & 1/3 & 2/3 \\ 2/9 & 4/9 & 3/9 \end{bmatrix} \quad n(\text{zero}) = 2$$

Matrix is <u>Irreducible</u> because

$a_{11} > 0 \rightarrow P^3$
$a_{12} > 0 \rightarrow P, P^3$
$a_{13} > 0 \rightarrow P^2$

$a_{21} > 0 \rightarrow P^2$
$a_{22} > 0 \rightarrow P^2, P^3$
$a_{23} > 0 \rightarrow P, P^3$

$a_{31} > 0 \rightarrow P, P^3$
$a_{32} > 0 \rightarrow P, P^2, P^3$
$a_{33} > 0 \rightarrow P^2, P^3$

$$P^4 = \begin{bmatrix} 0 & 1/3 & 2/3 \\ 2/9 & 4/9 & 3/9 \\ 1/9 & 4/9 & 4/9 \end{bmatrix} \qquad n(\text{zero}) = 1$$

$$P^5 = \begin{bmatrix} 2/9 & 4/9 & 3/9 \\ 1/9 & 4/9 & 4/9 \\ 4/27 & 11/27 & 12/27 \end{bmatrix} \qquad n(\text{zero}) = 0$$

Thus matrix is <u>Regular</u>

Matrix is finite & Regular, so state of $a_{11} = a_{22} = a_{33}$

State of $a_{22} = \gcd(2, 3, 4, 5) = 1$

thus all states are <u>Aperiodic</u>

Matrix is finite, regular, aperiodic hence <u>Ergodic</u> in nature.

Q9]

$$P = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 11/16 & 5/16 \\ 5/8 & 3/8 \end{bmatrix}$$

a) To find stationary dist.

$$\pi P = \pi$$

$$[\pi_1 \quad \pi_2] \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix} = [\pi_1 \quad \pi_2]$$

$$\pi_1 + \pi_2 = 1 \quad \text{--- } (1)$$

$$3/4\,\pi_1 + 1/2\,\pi_2 = \pi_1 \quad \text{--- } (2)$$

$$1/2\,\pi_2 = 1/4\,\pi_1$$

$$2\pi_2 = \pi_1 \longrightarrow \boxed{\begin{aligned} \pi_2 &= 1/3 \\ \pi_1 &= 2/3 \end{aligned}}$$

$$\pi = [2/3 \quad 1/3]$$

b)

If chain is stationary, it's period is infinite

9.10]

$$P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

a) $\pi P = \pi$

$$\begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix} \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix}$$

$\pi_1 + \pi_2 + \pi_3 = 1$ —— ①

$\pi_2 + \pi_3 = 2\pi_1$ —— ②

$4\pi_1 + 3\pi_3 = 6\pi_2$

$\cancel{4\pi_1} + 3\pi_3 = 6\pi_2$

$2\pi_2 + 2\pi_3 = \cancel{4\pi_1}$

$5\pi_3 = 4\pi_2$

$1 - \pi_1 = 2\pi_1 \Rightarrow \boxed{\pi_1 = 1/3}$

$\pi_2 + \frac{4}{5}\pi_2 = \frac{2}{3} \Rightarrow \boxed{\pi_2 = \frac{10}{27}}$

$\pi_3 = \frac{4}{5} \times \frac{10}{27} \Rightarrow \boxed{\pi_3 = \frac{8}{27}}$

$$\pi = \begin{bmatrix} \frac{9}{27} & \frac{10}{27} & \frac{8}{27} \end{bmatrix}$$

This markov chain has a stationary distribution, hence all states are <u>non null persistant</u>.

$$p^2 = \begin{bmatrix} 3/6 & 1/6 & 2/6 \\ 3/12 & 7/12 & 2/12 \\ 3/12 & 4/12 & 5/12 \end{bmatrix}$$

↳ Regular

↳ Irreducible

period = ∞ (as stationary)

or Non - periodic