Department of Mathematics

15B11MA211 **Mathematics-II**
B.Tech. Core

Solution of Tutorial Sheet 1 (Differential Equations With Constant Coefficient)

1. Find the complementary function of the following equations

   i. $(D^2 - 2D + 2)y = 0$
   ii. $(D^4 - 81)y = 0$
   iii. $(D^3 - 1)^2 y = 0.$

1) i) $m^2 - 2m + 2 = 0 \Rightarrow m = 1 \pm i$

$y(x) = e^x(c_1 \cos x + c_2 \sin x)$

ii) $m^4 - 81 = 0 \Rightarrow m = 3, -3, \pm 3i$

$y(x) = Ae^{3x} + Be^{-3x} + C\cos 3x + D\sin 3x$

iii) $(m^3 - 1)^2 = 0$

$m = 1, 1, \frac{-1 \pm \sqrt{3}i}{2}, \frac{-1 \pm \sqrt{3}i}{2}$

$y(x) = (A + Bx)e^x + e^{-x/2}\left((C + Dx)\cos\left(\frac{\sqrt{3}}{2}x\right) + (E + Fx)\sin\left(\frac{\sqrt{3}}{2}x\right)\right)$

2. Solve the following differential equations.

i. $(D^2 - 4D + 4)y = e^{2x} + x^3 + \cos 2x$

) $m^2 - 4m + 4 = 0$, $m = 2, 2$

$y_c(x) = (c_1 + c_2 x)\, e^{2x}$

$$y_p(x) = \frac{1}{(D-2)^2} e^{2x} + \frac{1}{(D-2)^2} x^3 + \frac{1}{(D-2)^2} \cos 2x$$

$$= \frac{x^2}{2} e^{2x} + \frac{1}{4}\left(1 - \frac{D}{2}\right)^{-2} x^3 - \frac{1}{4} \frac{1}{D} \cos 2x$$

$$= \frac{x^2}{2} e^{2x} + \frac{1}{4}\left(1 + 2\frac{D}{2} + 3\frac{D^2}{4} + 4\frac{D^3}{8}\right)(x^3) - \frac{1}{4} \cdot \frac{\sin 2x}{2}$$

$$= \frac{x^2}{2} e^{x} + \frac{x^3}{4} + \frac{3x^2}{4} + \frac{9x}{8} + \frac{3}{4} - \frac{\sin 2x}{8}$$

ii. $(D^2 - 6D + 13)y = 16\, e^{3x} \sin 4x + 3^x$

, $m^2 - 6m + 13 = 0$

$m = 3 \pm 2i$

$y_c(x) = e^{3x}(c_1 \cos 2x + c_2 \sin 2x)$

$$y_p(x) = \frac{1}{D^2 - 6D + 13} 16\, e^{3x} \sin 4x + \frac{1}{D^2 - 6D + 13} 3^x$$

$$= 16\, e^{3x} \frac{1}{(D+3)^2 - 6(D+3) + 13} \sin 4x + \frac{1}{D^2 - 6D + 13} e^{(\log_e 3)\, x}$$

$$= 16\, e^{3x} \frac{1}{D^2 + 4} \sin 4x + \frac{e^{(\log 3) x}}{(\log 3)^2 - 6 \log 3 + 13}$$

$D^2 \to -16$

$$= \cancel{16}^{\,4}\, e^{3x}\left(\frac{-1}{\cancel{12}_{\,3}} \sin 4x\right) + \frac{3^x}{(\log 3)^2 - 6 \log 3 + 13}$$

$$= -\frac{4}{3} e^{3x} \sin 4x + \frac{3^x}{(\log 3)^2 - 6(\log 3) + 13}$$

iii. $(D^2 + 1)y = cosec\ x$

(iii) $m^2+1=0$

$y_c(x) = c_1 \cos x + c_2 \sin x$

$y_p(x) = \frac{1}{(D+i)(D-i)} \operatorname{cosec} x$

$= \frac{1}{2i}\left[\frac{1}{D-i} \operatorname{cosec} x - \frac{1}{D+i} \operatorname{cosec} x\right]$

$\frac{1}{D-i} \operatorname{cosec} x = e^{ix} \int e^{-ix} \operatorname{cosec} x \, dx$

$= e^{ix} \int (\operatorname{cosec} x \cos x - i)\, dx$

$= e^{ix} (\log|\sin x| - ix)$

$\frac{1}{D+i} \operatorname{cosec} x = e^{-ix} (\log|\sin x| + ix)$

$y_p(x) = \frac{1}{2i}\left[\log \sin x (e^{ix} - e^{-ix}) - ix(e^{ix} + e^{-ix})\right]$

$= \sin x \log|\sin x| - x \cos x$

$y(x) = (c_1 - x) \cos x + (c_2 + \log|\sin x|) \sin x$

3. Find the solution of the following differential equations:

i. $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

ii. $x^2y'' + 4xy' + 2y = 0$

iii. $x^2y'' - 5xy' + 9y = 0$

iv. $x^3y''' + 3x^2y'' + xy' + y = \sin(logx) + x$

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③ (i) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

putting $x = e^t \Rightarrow t = \log x$

A.E. $m(m-1) + m + 1 = 0$

$m^2 + 1 = 0 \Rightarrow m = \pm i$

$y(t) = CF = A\cos t + B\sin t$

$\Rightarrow y(x) = A\cos(\log x) + B\sin(\log x)$

(ii) $x^2y'' + 4xy' + 2y = 0$

AE $m(m-1) + 4m + 2 = 0$

$m^2 + 3m + 2 = 0 \Rightarrow m = -1, -2$

$y(t) = Ae^{-t} + Be^{-2t}$

$\Rightarrow y(x) = A/x + B/x^2$

(iii) $x^2y'' - 5xy' + 9y = 0$

AE $m(m-1) - 5m + 9 = 0$

$m^2 - 6m + 9 = 0 \Rightarrow m = 3, 3$

$y(t) = (A + Bt)e^{3t}$

$\Rightarrow y(x) = (A + B\log x)x^3$

(iv) $x^3y''' + 3x^2y'' + xy' + y = \sin(\log x) + x$

putting $x = e^t$ we get

$(D(D-1)(D-2) + 3D(D-1) + D + 1)y = \sin t + e^t$

where $D \equiv \frac{d}{dt}$

$\Rightarrow (D^3 + 1)y = \sin t + e^t$

AE $m^3 + 1 = 0 \Rightarrow m = -1, \frac{1 \pm \sqrt{3}i}{2}$

$CF = Ae^{-t} + e^{t/2}\left[B\cos\frac{\sqrt{3}}{2}t + C\sin\frac{\sqrt{3}}{2}t\right]$

$PI = \frac{1}{D^3+1}\sin t + \frac{1}{D^3+1}e^t = \frac{1}{-D+1}\sin t + \frac{e^t}{2}$

$= \frac{1+D}{1-D^2}\sin t + \frac{e^t}{2} = \frac{1+D}{2}\sin t + \frac{e^t}{2} = \frac{\sin t + \cos t + e^t}{2}$

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$\Rightarrow y(t) = CF + PI$

$= Ae^{-t} + e^{t/2}\left[B\cos\left(\frac{\sqrt{3}}{2}t\right) + C\sin\left(\frac{\sqrt{3}}{2}t\right)\right]$

$+ \frac{\sin t + \cos t + e^t}{2}$

$\Rightarrow y(x) = \frac{A}{x} + x^{1/2}\left[B\cos\left(\frac{\sqrt{3}}{2}\log x\right) + C\sin\left(\frac{\sqrt{3}}{2}\log x\right)\right]$

$+ \frac{\sin(\log x) + \cos(\log x) + x}{2}$