Example 6 *If n biscuits are distributed at random among m children, what is the probability that a particular child receives r biscuits, where $r < n$?*

(MKU — Nov. 96)

Solution The first biscuit can be given to any one of the m children, i.e., in m ways.

Similarly the second biscuit can be given in m ways.

Therefore 2 biscuits can be given in m^2 ways.

Extending, n biscuits can be distributed in m^n ways. The r biscuits received by the particular child can be chosen from the n biscuits in nC_r ways. If this child has got r biscuits, the remaining $(n - r)$ biscuits can be distributed among the remaining $(m - 1)$ children in $(m - 1)^{n-r}$ ways.

$\therefore$ No. of ways of distributing in the required manner

$$= nC_r(m-1)^{n-r}$$

$$\therefore \text{ Required probability} = \frac{nC_r(m-1)^{n-r}}{m^n}$$

Example 6 *For a certain binary communication channel, the probability that a transmitted '0' is received as a '0' is 0.95 and the probability that a transmitted '1' is received as '1' is 0.90. If the probability that a '0' is transmitted is 0.4, find the probability that (i) a '1' is received and (ii) a '1' was transmitted given that a '1' was received.*

Solution Let A = the event of transmitting '1', $\overline{A}$ = the event of transmitting '0', B = the event of receiving '1' and, $\overline{B}$ = the event of receiving '0'.

Given: $P(\overline{A}) = 0.4$, $P(B/A) = 0.9$ and $P(\overline{B}/\overline{A}) = 0.95$

$\therefore$ $P(A) = 0.6$ and $P(B/\overline{A}) = 0.05$

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By the theorem of total probability

$$P(B) = P(A) \times P(B/A) + P(\bar{A}) \times P(B/\bar{A})$$
$$= 0.6 \times 0.9 + 0.4 \times 0.05$$
$$= 0.56$$

By Bayes' theorem,

$$P(A/B) = \frac{P(A) \times P(B/A)}{P(B)} = \frac{0.6 \times 0.9}{0.56} = \frac{27}{28}$$

(3) 9 9

Example 16 *If the RV k is uniformly distributed over (0, 5) what is the probability that the roots of the equation $4x^2 + 4kx + (k + 2) = 0$ are real?*

Solution The RV k is U (0, 5).

$$\therefore \quad \text{pdf of } k = \frac{1}{5}, \; 0 < k < 5$$

P(Roots of $4x^2 + 4kx + k + 2 = 0$ are real)

$$= P(\text{Discriminant of the equation} \geq 0)$$
$$= P(k^2 - k - 2 \geq 0) = P[(k-2)(k+1) \geq 0]$$
$$= P[(k \geq -1 \text{ and } k \geq 2) \text{ or } (k \leq 2 \text{ and } k \leq -1)]$$
$$= P(k \geq 2 \text{ or } k \leq -1) = P(k \geq 2) \; [\textit{since } k \textit{ takes values in } (0, 5)]$$
$$= \int_2^5 f(k)dk = \frac{1}{5}(5-2) = \frac{3}{5}.$$

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Example 18 *If the continuous RV. X represents the time of failure of a system, that has been put into operation at $t = 0$, find the conditional density function of X, given that the system has survived upto time t. Deduce the same when X follows an exponential distribution with parameter λ.*

Solution The conditional distribution function of X, subject to the given condition, is given by

$$F(x/X > t) = \frac{P[X \le x \text{ and } X > t]}{P(X > t)} \quad [\text{since unconditional } F(x) = P(X \le x)]$$

$$= \frac{P[t < X \le x]}{P[t < X < \infty]}$$

$$= \frac{F(x) - F(t)}{1 - F(t)} \text{ for } x > t$$

$$= 0 \text{ for } x < t$$

Therefore, the conditional density function $f(x/X > t)$ is given by

$$f(x/X > t) = \frac{d}{dx} F(x/X > t)$$

$$= \frac{f(x)}{1 - F(t)}, \; x > t$$

For the exponential distribution with parameter λ,

$$f(x) = \lambda e^{-\lambda x}, x > 0, \text{ and } F(x) = \int_0^X \lambda e^{-\lambda x} dx = 1 - e^{-\lambda x}.$$

$$\therefore \quad f(x/X > t) = \frac{\lambda e^{-\lambda x}}{e^{-\lambda t}} = \lambda e^{-\lambda(x-t)} = f(x - t)$$

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... independent.

Example 5 *The joint pdf of a two-dimensional RV (X, Y) is given by* $f(x, y) = xy^2 + \frac{x^2}{8}$, $0 \le x \le 2, 0 \le y \le 1$.

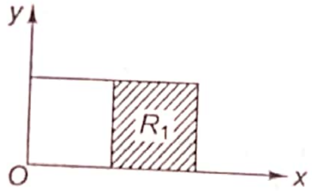
Compute $P(X > 1)$, $P\left(Y < \frac{1}{2}\right)$, $P(X > 1/Y < 1/2)$

$P\left(Y < \frac{1}{2}/X > 1\right)$, $P(X < Y)$ and $P(X + Y \le 1)$.

Solution Here the rectangle defined by $0 \le x \le 2, 0 \le y \le 1$ is the range space R. $R_1, R_2, \ldots$ are event spaces.

(i) $$P(X > 1) = \underset{\substack{R_1 \\ (x>1)}}{\int\int} f(x, y)\, dx\, dy$$

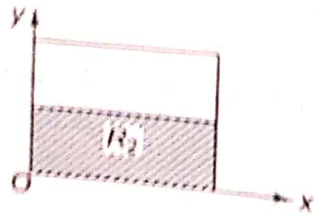
$$= \int_0^1 \int_1^2 \left(xy^2 + \frac{x^2}{8}\right) dx dy = \frac{19}{24}$$



(ii) $$P(Y < 1/2) = \underset{\substack{R_2 \\ \left(y < \frac{1}{2}\right)}}{\int} \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$$

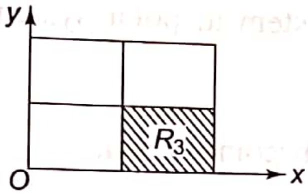
$$= \int_0^{1/2} \int_0^2 \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$$

$$= \frac{1}{4}$$



(iii) $P(X > 1, Y < 1/2) = \underset{\substack{R_3 \\ \left(x>1\,\&\,y<\frac{1}{2}\right)}}{\int\int} \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$

$$= \int_0^{1/2}\int_1^2 \left(xy^2 + \frac{x^2}{8}\right) dx\, dy = \frac{5}{24}$$

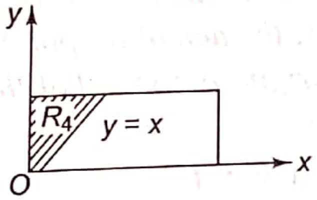


(iv) $P(X > 1/Y < \frac{1}{2}) = \dfrac{P\left(X>1, Y<\frac{1}{2}\right)}{P\left(Y<\frac{1}{2}\right)} = \dfrac{5/24}{1/4} = \dfrac{5}{6}$

(v) $P(Y < \frac{1}{2}/X > 1) = \dfrac{P\left(X>1, Y<\frac{1}{2}\right)}{P(X>1)} = \dfrac{5/24}{19/24} = \dfrac{5}{19}$

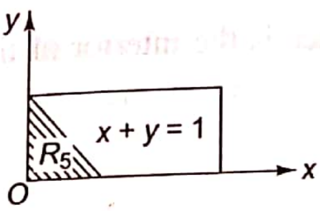
(vi) $P(X < Y) = \underset{\substack{R_4 \\ (x<y)}}{\int\int} \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$

$$= \int_0^1\int_0^y \left(xy^2 + \frac{x^2}{8}\right) dx\, dy = \frac{53}{480}$$



(vii) $P(X + Y \le 1) = \underset{\substack{R_5 \\ (x+y\le 1)}}{\int\int} \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$

$$= \int_0^1\int_0^{1-y} \left(xy^2 + \frac{x^2}{8}\right) dx\, dy = \frac{13}{480}$$



Example 15 *A Box contains 2^n tickets of which nC_r tickets bear the number r ($r = 0, 1, 2\ldots, n$). Two tickets are drawn from the box. Find the expectation of the sum of their numbers.*

Solution Total number of tickets in the box.

$$\sum_{r=0}^{n} nC_r = nC_0 + nC_1 + \ldots + nC_n$$

$$= (1 + 1)^n = 2^n, \text{ as given.}$$

Let the RVs X and Y represent the numbers on the first and second tickets respectively.

Then $$E(X + Y) = E(X) + E(Y)$$

X can take the values $0, 1, 2, \ldots, n$ with probabilities $\frac{nC_0}{2^n}, \frac{nC_1}{2^n}, \ldots \frac{nC_n}{2^n}$ respectively.

$$\therefore \quad E(X) = 1 \times \frac{nC_1}{2^n} + 2 \times \frac{nC_2}{2^n} + \ldots + n \times \frac{nC_n}{2^n}$$

$$= \frac{n}{2^n}\{(n-1)C_0 + (n-1)C_1 + \ldots + (n-1)C_{n-1}\}$$

$$= \frac{n}{2^n}(1+1)^{n-1} = \frac{n}{2}$$

Similarly, $$E(Y) = \frac{n}{2}$$

$$\therefore \quad E(X + Y) = n$$

Example 14 *If a discrete random variable X takes the values 0, 1, 2, … n, with probabilities* $\frac{1}{2^n}, \frac{nc_1}{2^n}, \frac{nc_2}{2^n}, \ldots, \frac{nc_n}{2^n}$ *respectively, show that the coefficient of variation of the distribution of X is* $\frac{100}{\sqrt{n}}$.

Solution $E(X) = \sum_{r=0}^{n} p_r x_r$

$$= \sum_{r=0}^{n} r \cdot nC_r / 2^n \tag{1}$$

$$= \frac{1}{2^n} \sum_{r=0}^{n} r \frac{n!}{r!(n-r)!}$$

$$= \frac{n}{2^n} \sum_{r=1}^{n} r \frac{(n-1)!}{(r-1)!(n-r)!}$$

$$= \frac{n}{2^n} \sum_{r=1}^{n} (n-1)C_{(r-1)}$$

$$= \frac{n}{2^n} (1+1)^{n-1} [\because (1-1)^{n-1} = (n-1)\, C_0 + (n-1)\, C_1 + \ldots$$
$$+ (n-1)\, C_{n-1}]$$

$$= \frac{n}{2} \quad (2)$$

$$E(X^2) = \sum_{r=0}^{n} p_r x_r^2$$

$$= \frac{1}{2^n} \sum_{r=0}^{n} r^2 \cdot nC_r$$

$$= \frac{1}{2^n} \sum_{r=0}^{n} \{r(r-1)+r\} \frac{n!}{r!(n-r)!}$$

$$= \frac{n(n-1)}{2^n} \sum_{r=2}^{n} \frac{(n-2)!}{(r-2)!(n-r)!} + \frac{1}{2^n} \sum_{r=0}^{n} r \cdot nC_r$$

$$= \frac{n(n-1)}{2^n} \sum_{r=2}^{n} (n-2)C_{r-2} + E(X), \text{ from (1)}$$

$$= \frac{n(n-1)}{2^n} (1+1)^{n-2} + \frac{n}{2}, \text{ using (2)}$$

$$= \frac{n(n-1)}{4} + \frac{n}{2} = \frac{n(n-1)}{4}$$

$$\therefore \quad \text{Var}(X) = E(X^2) - \{E(X\}^2$$

$$= \frac{n(n+1)}{4} - \frac{n^2}{4}$$

$$= \frac{n}{4}$$

standard deviation

$$(\text{C.V.})_X = \frac{\sigma_X}{E(X)} \times 100$$

$$= \frac{\sqrt{n}}{2} \times \frac{2}{n} \times 100$$

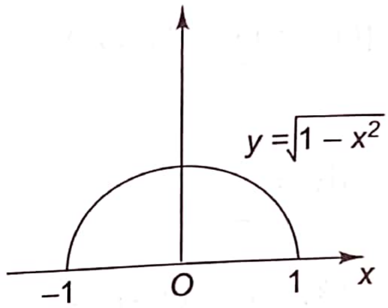
$$= \left(\frac{100}{\sqrt{n}}\right)\%$$

$$= \frac{1}{2}(1-x)^2 - \frac{4}{9}(1-x)^2$$

$$= \frac{1}{18}(1-x)^2$$

Example 19 *If (X, Y) is uniformly distributed over the semicircle bounded by $y = \sqrt{1-x^2}$ and $y = 0$, find E(X/Y) and E(Y/X). Also verify the E{E(X/Y)} = E(X) and E{E(Y/X)} = E(Y).*

Solution



$$f(x, y) = k$$

$$\iint f(x, y)\, dy\, dx = 1$$

i.e., $$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} k\, dy\, dx = 1$$

i.e., $$2k\int_{0}^{1}\sqrt{1-x^2}\, dx = 1$$

$\therefore$ $$k = \frac{2}{\pi}$$

$$f_X(x) = \int_{0}^{\sqrt{1-x^2}} \frac{2}{\pi}\, dy = \frac{2}{\pi}\sqrt{1-x^2}\,, \quad -1 \le x \le 1$$

$$f_Y(y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{2}{\pi}\, dx = \frac{4}{\pi}\sqrt{1-y^2}\,,\ 0 \le y \le 1$$

$$f(x/y) = \frac{f(x, y)}{f_Y(y)} = \frac{1}{2\sqrt{1-y^2}}\,,\ -\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$$

$$f(y/x) = \frac{1}{\sqrt{1-x^2}}\,,\ 0 \le y \le \sqrt{1-x^2}$$

$$E(X) = \int_{-1}^{1} x f_X(x)\, dx = \frac{2}{\pi}\int_{-1}^{1} x\sqrt{1-x^2}\, dx = 0$$

(since the integrand is odd)

$$E(X/Y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x f(x/y)\, dx$$

$$= \frac{1}{2\sqrt{1-y^2}} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} |x|\, dx = 0$$

(since the integrand is odd)

$$\therefore \quad E\{E(X/Y)\} = E\{0\} = 0 = E(X)$$

$$E(Y) = \int_{0}^{1} y f_Y(y)\, dy = \frac{4}{\pi}\int_{0}^{1} y\sqrt{1-y^2}\, dy = \frac{4}{3\pi}$$

$$E(Y/X) = \int_{0}^{\sqrt{1-x^2}} y f(y/x)\, dy = \frac{1}{\sqrt{1-x^2}} \cdot \left(\frac{y^2}{2}\right)_0^{\sqrt{1-x^2}} = \frac{1}{2}\sqrt{1-x^2}$$

$$\therefore \quad E\{E(Y/X)\} = E\left\{\frac{1}{2}\sqrt{1-X^2}\right\}$$

$$= \int_{-1}^{1} \frac{1}{2}\sqrt{1-x^2}\, f_X(x)\, dx$$

$$= \frac{2}{\pi}\int_{0}^{1} (1-x^2)\, dx = \frac{4}{3\pi}$$

$$\therefore \quad E\{E(Y/X)\} = E(Y)$$

Example 6 *If the independent random variables X and Y have the variances 36 and 16 respectively, find the correlation coefficient between (X + Y) and (X − Y).*

Solution Let $U = X + Y$ and $V = X - Y$

$$E(U) = E(X) + E(Y);\ E(V) = E(X) - E(Y)$$

$$E(UV) = E(X^2 - Y^2) = E(X^2) - E(Y^2)$$

$$E(U^2) = E\{(X + Y)^2\} = E(X^2) + E(Y^2) + 2E(XY)$$

$$E(V^2) = E(X^2) + E(Y^2) - 2E(XY)$$

$$\begin{aligned} C_{UV} &= E(UV) - E(U)\cdot E(V) \\ &= E(X^2) - E(Y^2) - \{E^2(X) - E^2(Y)\} \\ &= [E(X^2) - E^2(X)] - [E(Y^2) - E^2(Y)] \\ &= \sigma^2_X - \sigma^2_Y = 36 - 16 = 20 \end{aligned}$$

$$\begin{aligned} \sigma^2_U &= E(U^2) - E^2(U) \\ &= \{E(X^2) + E(Y^2) + 2E(XY)\} - \{E^2(X) + E^2(Y) + 2E(X)\cdot E(Y)\} \\ &= [E(X^2) - E^2(X)] + [E(Y^2) - E^2(Y)] + 2[E(XY) - E(X)\cdot E(Y)] \\ &= 36 + 16 + 2\times 0 \end{aligned}$$

[$\because$ X and Y are independent and hence uncorrelated]

$$= 52$$

Similarly, $\sigma^2_V = 52$

Now $$r_{UV} = \frac{C_{UV}}{\sigma_U \cdot \sigma_V} = \frac{20}{52} = \frac{5}{13}$$

Example 7 *If X, Y and Z are uncorrelated RV's with zero means and standard deviations 5, 12 and 9 respectively and if $U = X + Y$ and $V = Y + Z$, find the correlation coefficient between U and V.*

Solution $E(X) = E(Y) = E(Z) = 0$

$$\text{Var}(X) = E(X^2) - E^2(X) = 25 \quad \therefore \quad E(X^2) = 25$$

Similarly $E(Y^2) = 144$ and $E(Z^2) = 81$

X and Y are uncorrelated

$\therefore$ $r_{XY} = 0$. i.e., $E(XY) - E(X) \cdot E(Y) = 0$

$\therefore$ $E(XY) = 0$. Similarly $E(YZ) = 0$; $E(ZX) = 0$

Now $E(U) = E(X + Y) = 0$ and $E(V) = 0$

$$E(U^2) = E(X^2 + Y^2 + 2XY)$$
$$= 25 + 144 + 2 \times 0 = 169$$
$$E(V^2) = E(Y^2 + Z^2 + 2YZ)$$
$$= 144 + 81 + 2 \times 0 = 225$$

$\therefore$ $\sigma^2_U = E(U^2) - E^2(U) = 169$

and $\sigma^2_V = E(V^2) - E^2(V) = 225$

$$E(UV) = E\{(X + Y)(Y + Z)\}$$
$$= E(XY) + E(XZ) + E(YZ) + E(Y^2)$$
$$= 0 + 0 + 0 + 144 = 144$$

$$r_{UV} = \frac{E(UV) - E(U) \cdot E(V)}{\sigma_U \cdot \sigma_V} = \frac{144}{13 \times 15} = \frac{48}{65}$$

Example 8 *If X and Y are two RV's with varian...*