Probability and Random Processes (15B11MA301)

Lecture-12

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Moment Generating Function

- The moment generating function of a random variable X denoted by $M_X(\mathrm{t})$ is defined as
  $M_X(\mathrm{t}) = E[e^{tX}]$ where t is a real variable.

- If X is a discrete random variable with PMF p(x), then $M_X(\mathrm{t}) = E[e^{tX}] = \sum_x e^{tx} p(x)$

- If X is a continuous random variable with PDF f(x), then

$$M_X(\mathrm{t}) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

Properties of Moment Generating Function

- The coefficient $\frac{t^r}{r!}$ in $M_X(\mathrm{t})$ is $\mu_r^{'}$, r = 1, 2, 3…and $\mu_r^{'} = E[X^r]$ gives moments about the origin.

Proof: We know that $M_X(\mathrm{t}) = E[e^{tX}]$

$$= E\left[1 + \frac{tX}{1!} + \frac{(tX)^2}{2!} + \cdots\right]$$

$$= E(1) + \frac{t}{1!}E[X] + \frac{t^2}{2!}E[X^2] + \cdots + \frac{t^r}{r!}E[X^r] + \cdots$$

$$= 1 + \frac{t}{1!}\mu_1^{'} + \frac{t^2}{2!}\mu_2^{'} + \cdots + \frac{t^r}{r!}\mu_r^{'} + \cdots$$

Hence, $$M_X(\mathrm{t}) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r^{'} \qquad \text{..............(1)}$$

this gives the MGF in terms of the moments.

- The moments $\mu_r^{'}$ can also be obtained as

Differentiating equation (1) with respect to t, r times and putting t =0 provides moments

$$\mu_r^{'} = \left[\frac{d^r}{dt^r} M_X(\mathrm{t})\right]_{t=0}, \text{ r =1, 2, 3…} \qquad \text{.................. (2)}$$

Properties of Moment Generating Function

- $M_{aX}(t) = M_X(at)$, a being a constant.

  Proof: By definition, $M_{aX}(t) = E[e^{taX}] = E[e^{(at)X}]$

  $M_{aX}(t) = M_X(at)$

- If Y = aX +b, then $M_Y(t) = e^{bt} M_X(at)$.

  Proof: We know that,

  $$\begin{aligned} M_Y(t) &= E[e^{tY}] \\ &= E\left[e^{t(aX+b)}\right] \\ &= E\left[e^{t(aX)}\right]E[e^{tb}] \\ &= e^{bt}\, E\left[e^{(ta)X}\right] = e^{bt} M_X(at). \end{aligned}$$

Properties of Moment Generating Function

- The moment generating function of the sum of n independent random variables is equal to the product of their respective moment generating functions, i.e.

$$M_{X_1+X_2+\cdots+X_n}(t) = M_{X_1}(t)\,.\,M_{X_2}(t) \ldots\ldots M_{X_n}(t)$$

Proof: Using the definition of MGF, we have

$$M_{X_1+X_2+\cdots+X_n}(t) = E\left[e^{t(X_1+X_2+\cdots+X_n)}\right]$$

$$= E[e^{tX_1}]\, E[e^{tX_2}]\, E[e^{tX_3}] \ldots\ldots E[e^{tX_n}]$$ (since variables are independent)

Therefore,

$$\boldsymbol{M_{X_1+X_2+\cdots+X_n}(t) = M_{X_1}(t)\,.\,M_{X_2}(t) \ldots\ldots M_{X_n}(t)}$$

Properties of Moment Generating Function

- Effect of Origin and Scale on MGF

Let the random variable X be transformed to a new variable U by changing both the origin and scale in X as $U = \frac{X-a}{h}$ where a and h are constants.

Then, the MGF of U (about origin) is given by

$$M_U(\mathrm{t}) = E[e^{tU}]$$

$$= E\left[e^{t\left(\frac{X-a}{h}\right)}\right]$$

$$= e^{\left(\frac{-at}{h}\right)} E\left[e^{t\left(\frac{X}{h}\right)}\right]$$

$$= e^{\left(\frac{-at}{h}\right)} M_X(t/h)$$

Limitations of Moment Generating Function

- A random variable X may have no moments although its moment generating function exists.

  For example: $f(x) = \begin{cases} \frac{1}{x(x+1)}, x = 1,2,3 \ldots \end{cases}$ and $f(x) = 0\ otherwise$.

- A random variable X can have MGF and some or all moments, yet the MGF does not generate the moments.

  For example: $P(X = \pm 2^x)=\frac{e^{-1}}{x!}$, x=0,1,2,...

- A random variable X can have all or some moments, but MGF does not exist, except perhaps at one point.

  For example: $P(X = \pm 2^x)=\frac{e^{-1}}{2x!}$, x=0,1,2,... and $P(X = \pm 2^x) = 0, otherwise$

Example 1: If a random variable X has the MGF $M_X(t) = \frac{3}{3-t}$, obtain the standard deviation of X.

Solution: $M_X(t) = \frac{3}{3-t} = 1 + \frac{t}{3} + \frac{t^2}{9} + \ldots\ldots$

$$E(X) = \text{coefficient of } \frac{t}{1!} = 1/3$$

$$E(X^2) = \text{coefficient of } \frac{t^2}{2!} = 2/9$$

$$\text{Var}(X) = E(X^2) - (E(X))^2$$

$$= \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$$

Standard Deviation = $\boldsymbol{\sigma_X}$ = 1/3

Example 2: Find the MGF of the random variable X whose probability function $(X = x) = \frac{1}{2^x}$, x = 1, 2,3...

hence, find its mean.

Solution: $M_X(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx}\, P(X = x)$

$$= \sum_{x=1}^{\infty} e^{tx}\, \frac{1}{2^x} = \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x$$

On expanding the above summation, we get

$$\boldsymbol{M_X(t) = \frac{e^t}{2}\left(\frac{2}{2-e^t}\right) = \frac{e^t}{2-e^t}}$$

Mean = $\mu_1' = \frac{d}{dt} M_X(t)$ at t =0.

$= \frac{d}{dt}\left(\frac{e^t}{2-e^t}\right)$ **at t =0**

Mean = 2

Example 3: A random variable X has the PDF given by $f(x) = \{\, 2e^{-2x}, x \geq 0 \; and \; 0 \; if \; x < 0.$

Find (i) MGF and

(ii) the first four moments of X about the origin.

Solution: $M_X(\mathrm{t}) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x) dx$

$$M_X(\mathrm{t}) = \int_0^{\infty} 2e^{tx} \, e^{-2x} \, dx = \left[\frac{e^{-(2-t)x}}{-(2-t)}\right] \text{ from } 0 \; to \; \infty$$

Therefore, $M_X(\mathrm{t}) = \frac{2}{2-\mathrm{t}}$

We know that, $M_X(\mathrm{t}) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r^{'} = \frac{2}{2-\mathrm{t}} = \frac{2}{2(1-\frac{\mathrm{t}}{2})}$

$$1 + \frac{t}{1!}\mu_1^{'} + \frac{t^2}{2!}\mu_2^{'} + \cdots + \frac{t^r}{r!}\mu_r^{'} + \cdots = \left(1 - \frac{\mathrm{t}}{2}\right)^{-1}$$

$$= 1 + \frac{t}{2} + \frac{t^2}{2^2} + \frac{t^3}{2^3} + \frac{t^4}{2^4} + \cdots$$

$$= 1 + \frac{1}{2}\frac{t}{1!} + \frac{2!}{4}\frac{t^2}{2!} + \cdots.$$

On equating the coefficients of $\frac{\mathrm{t}}{1!}, \frac{\mathrm{t}^2}{2!}$, and so on, we have..

$$\mu_1^{'} = 1/2, \;\; \mu_2^{'} = \frac{1}{2}, \; \mu_3^{'} = 3/4, \; \mu_2^{'} = 3/2$$

Practice Questions

1. A random variable X has the density function given by $f(x) = \begin{cases} \frac{1}{k}, 0 < x < k \\ 0, otherwise \end{cases}$.

   Find (i) MGF , (ii) r^{th} moment, (iii) Mean and (iv) Variance

   [Ans. (i) $\frac{(e^{tk}-1)}{kt}$ (ii) $\frac{k^r}{(r+1)!}$ (iii) $\frac{k}{2}$ (iv) $\frac{k^2}{12}$]

2. Let X be a random variable with PDF $f(x) = \begin{cases} \frac{e^{\frac{-x}{3}}}{3}, 0 < x \\ 0, otherwise \end{cases}$.

   Find (i) P(X>3) (ii) MGF of X , (iii) E(X) and Var(X).

   [Ans. (i) 1/e (ii) $(1-3t)^{-1}$ (iii) E(X) =3, Var(X)= 9]

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