Probability and Random Processes (15B11MA301)

Lecture-3

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Contents of the Lecture:

❑   Axiomatic approach of Probability

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Event Space

The sample space of an experiment contains all possible outcomes, the event space contains all *sets of outcomes*; all subsets of the sample space.

Example: Coin Flip

For a simple coin flip, the two possible outcomes are either heads or tails, so the **sample space** is given by

S = {H, T}

The **event space** is a little different. The possible events are:

- {H}— getting heads,
- {T}— getting tails,
- {H,T}— getting either heads *or* tails.

Because each of these are different subsets of the sample space, they count as different events, even though {H} (heads) would imply {H, T} (either H or T). The event space contains all three of those events:

$$A=\{\{H\}, \{T\}, \{H, T\}\}$$

Axiomatic Definition of Probability:

Let S be the sample space and A be an event associated with a random experiment. Then the probability of the event A, denoted by $P(A)$, is designed as a real number satisfying the following axioms.

(i) $0 \le P(A) \le 1$

(ii) $P(S) = 1$

(iii) If A and B are mutually exclusive events, $P(A \cup B) = P(A) + P(B)$ and

(iv) If $A_1, A_2, \ldots, A_n, \ldots$ are a set of mutually exclusive events,

$$P(A_1 \cup A_2 \cup \ldots \cup A_n \ldots) = P(A_1) + P(A_2) + \ldots + P(A_n) + \ldots$$

Revision

Definition: Mutually Exclusive Events

A set of events is said to be mutually exclusive if the occurrence of any one of them excludes the occurrence of the others.

Two events A and B are mutually exclusive if A occurs and B does not occur and vice versa.

In other words, A and B cannot occur simultaneously, i.e. $P(A \cap B) = 0$.

Some results derived using axioms of probability:

Result 1: The probability of the impossible event is zero, i.e., if ϕ is the subset (event) containing no sample point, $P(\phi)=0$.

Proof: The certain event **S** and the impossible event ϕ are mutually exclusive.

Hence $\qquad P(S \cup \phi) = P(S) + P(\phi)$ $\qquad$ [Axiom (iii)]

But $\qquad S \cup \phi = S$.

$\therefore \qquad P(S) = P(S) + P(\phi)$

$\therefore \qquad P(\phi) = 0$

Result 2: If $\overline{A}$ is the complementary event of A, $P(\overline{A})=1-P(A)\le 1$.

Proof: A and $\overline{A}$ are mutually exclusive events, such that $A\cup\overline{A}=S$.

$$\therefore \qquad P(A\cup\overline{A}) = P(S)$$

$$P(A\cup\overline{A}) = P(S)=1 \qquad \textbf{[Axiom (ii)]}$$

i.e., $\qquad P(A)+P(\overline{A})=1 \qquad \textbf{[Axiom (iii)]}$

$$\therefore \qquad P(\overline{A}) = 1-P(A)$$

Since $P(A)\ge 0$, it follows that $P(\overline{A})\le 1$.

ADDITION LAW OF PROBABILITY

Result 3: If A and B are any two events, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Proof: A is the union of the mutually exclusive events $A \cap \bar{B}$ and $A \cap B$

and B is the union of the mutually exclusive events $\bar{A} \cap B$ and $A \cap B$.

$\therefore \qquad P(A) = P(A \cap \bar{B}) + P(A \cap B) \qquad$ **[Axiom(iii)]**

and $\qquad P(B) = P(\bar{A} \cap B) + P(A \cap B) \qquad$ **[Axiom(iii)]**

$\therefore \quad P(A) + P(B) = [P(A \cap \bar{B}) + P(A \cap B) + P(\bar{A} \cap B) + P(A \cap B)]$

$$= P(A \cup B) + P(A \cap B)$$

$As, \; A \cup B = (A \cap \bar{B}) \cup (A \cap B) \cup (\bar{A} \cap B)$

By Axiom (iii)

$\Rightarrow P(A \cup B) = P(A \cap \bar{B}) + P(A \cap B) + P(\bar{A} \cap B)$

This implies $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Observations:

- If A and B are any two events, $P(A \cup B) \le P(A) + P(B)$.

  It follows from **Addition Law of Probability**: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

  as $P(A \cap B) \ge 0$.

  This implies $P(A \cup B) \le P(A) + P(B)$.

- If A, B and C are any three events, then

  $P(A \cup B \cup C) = P$ (at least one of A, B and C occurs)

  $\qquad = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

Result 4: If $B \subseteq A,\ P(B) \le P(A)$.

Proof: B and $A \cap \bar{B}$ are mutually exclusive events

such that $B \cup (A \cap \bar{B}) = A$.

$\therefore \quad P(B \cup (A \cap \bar{B})) = P(A)$

i.e. $P(B) + P(A \cap \bar{B}) = P(A)$ **[Axiom (iii)]**

$\therefore \quad P(B) \le P(A)$

Example: Determine the range for $P(A \cup B)$ and $P(A \cap B)$ when $P(A) = \dfrac{3}{4}$ and $P(B) = \dfrac{5}{8}$.

Example: Determine the range for $P(A \cup B)$ and $P(A \cap B)$ when $P(A) = \dfrac{3}{4}$ and $P(B) = \dfrac{5}{8}$.

Sol: Since

$$\Rightarrow \max[P(A),\ P(B)] \le P(A \cup B)$$

Here, $P(A \cup B) \ge \dfrac{3}{4}$

$$\Rightarrow \dfrac{3}{4} \le P(A \cup B) \le 1$$

By addition law: $P(A \cup B) = P(A) + P(B) - P(A \cap B) \le 1$

$$\Rightarrow \dfrac{3}{8} \le P(A \cap B) \le \dfrac{5}{8}$$

Practice Questions

1. If A and B are any two events, prove that $P(\bar{A} \cap B) = P(B) - P(A \cap B)$.

2. If $A \cap B = \phi$, then show that $P(A) \leq P(\bar{B})$.

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