

# Probability and Random Processes (15B11MA301)

## Lecture-24

**(Content Covered: Exponential and Weibull Failure distributions)**



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# Special Failure Distributions

## (i) Exponential Distribution

If the time to failure  $T$  follows an exponential distribution with parameter  $\lambda$ , then its probability density function is given by

$$f(t) = \lambda e^{-\lambda t}, t \geq 0 \quad \dots\dots\dots(1)$$

The reliability function is given by  $R(t) = \int_t^{\infty} f(t)dt \dots\dots(2)$

Using (1) in (2), we get  $R(t) = \int_t^{\infty} f(t)dt = \int_t^{\infty} \lambda e^{-\lambda t} dt = e^{-\lambda t} \dots\dots(3)$

**Hazard Function ( $\lambda(t)$ )** : We know,  $\lambda(t) = \frac{f(t)}{R(t)}$

Using (1) and (3) in hazard function, we get  $\lambda(t) = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$

It means that, when the failure distribution is an exponential distribution with parameter  $\lambda$ , then the failure rate at any time is a constant, equal to  $\lambda$ .

**Conversely**, when hazard rate =  $\lambda$ , then using the definition of  $f(t)$ , we have

$$f(t) = \lambda(t) e^{-\int_0^t f(t) dt}$$

$$f(t) = \lambda e^{-\int_0^t \lambda dt} = \lambda e^{-\lambda t}, t \geq 0$$

Due to this property, the exponential distribution is also referred as **Constant Failure rate distribution**.

- Mean Time to failure (MTTF)  $= E(T) = \frac{1}{\lambda}$

It means that the time to failure of a component is not dependent on how long the component has been functioning. This property is known as the memoryless property of the constant failure rate distribution.

- Variance  $= \text{Var}(T) = \frac{1}{\lambda^2}$

- Conditional Reliability  $R(t/T_o) = \frac{R(T_o+t)}{R(T_o)} = \frac{e^{-\lambda(T_o+t)}}{e^{-\lambda T_o}}$

**Example:** A manufacturer determines that, on the average, a television set is used 1.8 hours per day. A one year warranty is offered on the picture tube having a MTTF of 2000 hours. If the distribution is exponential, what percentage of the tubes will fail during the warranty period ?

Solution: Since the distribution of the time to failure of the picture tube is exponential,

$$R(t) = e^{-\lambda t}, \lambda \text{ being the failure rate.}$$

Given, MTTF = 2000 hours

$$\text{i.e. } \int_0^{\infty} e^{-\lambda t} dt = 2000$$

$$\text{i.e. } \frac{1}{\lambda} = 2000 \text{ or } \lambda = 0.0005 / \text{hour}$$

$$\begin{aligned} P(T \leq 1 \text{ year}) &= P(T \leq 365 \times 1.8 \text{ hours}) \text{ (since, TV is operated for 1.8 hours/day)} \\ &= 1 - P(T > 657) = 1 - R(657) \\ &= 1 - e^{-0.0005 \times 657} = 0.28 \end{aligned}$$

i.e. 28% of the tubes will fail during the warranty period.

**Example:** Given that  $R(t) = e^{-\sqrt{0.001t}}$ ,  $t \geq 0$

- (a) Compute the reliability for a 50 hours mission.
- (b) Show that the hazard rate is decreasing.
- (c) Given a 10- hour wear-in period, compute the reliability for a 50-hour mission.
- (d) What is the design life for a reliability of 0.95, given a 10-hour wear-in period?

Solution: (a) Given  $R(t) = e^{-\sqrt{0.001t}}$ ,  $t \geq 0$

$$R(50) = e^{-\sqrt{0.001 \times 50}} = 0.9512$$

(b) We know,  $\lambda(t) = \frac{-R'(t)}{R(t)}$

$$= \frac{-e^{-\sqrt{0.001t}} \times -\sqrt{0.001}}{e^{-\sqrt{0.001t}}} \times t^{-\frac{1}{2}}$$

$$= \sqrt{0.001} \times t^{-\frac{1}{2}}, \text{ which is a decreasing function of } t.$$

$$(c) R(t/T_o) = \frac{R(T_o+t)}{R(T_o)}$$

$$\begin{aligned} \text{Therefore, } R(50/10) &= \frac{R(60)}{R(10)} \\ &= \frac{e^{-\sqrt{0.001 \times 60}}}{e^{-\sqrt{0.001 \times 10}}} = 0.8651 \end{aligned}$$

$$(d) \text{ Given, } R(t_D/10) = 0.95$$

$$\frac{R(t_D+10)}{R(10)} = 0.95$$

$$\text{Therefore, } e^{-\sqrt{0.001 \times (t_D+10)}} = 0.95 \times e^{-\sqrt{0.001 \times 10}}$$

$$\sqrt{0.001 \times (t_D + 10)} = 0.15129$$

$$t_D = 12.89 \text{ hours}$$



## (ii) Weibull Distribution

We know that, the pdf of the Weibull Distribution is defined as

$$f(t) = \alpha \beta t^{\beta-1} e^{-\alpha t^{\beta}}, \quad t \geq 0 \quad \text{.....(4)}$$

On using,  $\alpha = \frac{1}{\theta^{\beta}}$ , in above pdf, an alternative form of Weibull's pdf can be written as:

$$f(t) = \frac{\beta}{\theta} \left(\frac{t}{\theta}\right)^{\beta-1} e^{-\left(\frac{t}{\theta}\right)^{\beta}}, \quad t \geq 0, \theta > 0, \beta > 0 \quad \text{.....(5)}$$

Where,  $\beta$  : Shape parameter

$\theta$ : Characteristic life or scale parameter of Weibull's distribution given by (5).

- If the time to failure  $T$  follows Weibull's distribution (5) , then the **reliability function** is

$$R(t) = \int_t^{\infty} f(t)dt$$

$$\begin{aligned} R(t) &= \int_t^{\infty} \frac{\beta}{\theta} \left(\frac{t}{\theta}\right)^{\beta-1} e^{-\left(\frac{t}{\theta}\right)^{\beta}} dt \\ &= \int_t^{\infty} e^{-x} dx, \text{ on putting } \left(\frac{t}{\theta}\right)^{\beta} = x \\ &= e^{-x} \\ &= e^{-\left(\frac{t}{\theta}\right)^{\beta}} \quad \dots\dots(6) \end{aligned}$$

- **Hazard Function:**  $\lambda(t) = \frac{f(t)}{R(t)}$

$$\begin{aligned} &= \frac{\frac{\beta}{\theta} \left(\frac{t}{\theta}\right)^{\beta-1} e^{-\left(\frac{t}{\theta}\right)^{\beta}}}{e^{-\left(\frac{t}{\theta}\right)^{\beta}}} \\ &= \frac{\beta}{\theta} \left(\frac{t}{\theta}\right)^{\beta-1} \quad \dots\dots(7) \end{aligned}$$

- **Mean Time to failure (MTTF)**  $= E(T) = \int_0^{\infty} t \left( \frac{\beta}{\theta} \left( \frac{t}{\theta} \right)^{\beta-1} e^{-\left( \frac{t}{\theta} \right)^{\beta}} \right) dt$

- **Variance**  $= \text{Var}(T) = \theta^2 \left\{ \left| \left( \frac{2}{\beta} + 1 \right) \right| - \left( \left| \left( \frac{1}{\beta} + 1 \right) \right| \right)^2 \right\}$   

$$= \theta^2 \left| \left( \frac{1}{\beta} + 1 \right) \right|$$

- **Conditional Reliability**  $R(t/T_o) = \frac{R(T_o+t)}{R(T_o)} = \frac{e^{-\left( \frac{T_o+t}{\theta} \right)^{\beta}}}{e^{-\left( \frac{T_o}{\theta} \right)^{\beta}}}$

$$= \exp \left[ - \left( \frac{T_o+t}{\theta} \right)^{\beta} + \left( \frac{T_o}{\theta} \right)^{\beta} \right]$$

**Example** For a system having a Weibull failure distribution with a shape parameter of 1.4 and a scale parameter of 550 days, find

- (a)  $R(100 \text{ days})$
- (b) the B1 life
- (c) MTTTF
- (d) The standard deviation
- (e) The design life for a reliability of 0.90

Solution: We know, the pdf of Weibull distribution is given by

$$f(t) = \frac{\beta}{\theta} \left(\frac{t}{\theta}\right)^{\beta-1} e^{-\left(\frac{t}{\theta}\right)^{\beta}}, \quad t \geq 0, \theta > 0, \beta > 0$$

Given  $\beta = 1.4$  and  $\theta = 550 \text{ days}$

(a) We know,  $R(t) = e^{-\left(\frac{t}{\theta}\right)^\beta}$

$$\begin{aligned} R(100) &= e^{-\left(\frac{100}{550}\right)^{1.4}} \\ &= 0.9122 \end{aligned}$$

(b) B1 Life: The lifetime corresponding to a **reliability of 0.99** is called the B1 Life.

Let  $t_R$  be the B1 life of the system, Then

$$R(t_R) = 0.99$$

$$R(t_R) = e^{-\left(\frac{t_R}{550}\right)^{1.4}} = 0.99$$

$$\left(\frac{t_R}{550}\right)^{1.4} = 0.01005$$

$$\text{Therefore, } t_R = 550 (0.01005)^{(1/1.4)}$$

$$= 20.6 \text{ days}$$

$$\begin{aligned}
\text{(c) MTTF} &= \theta \sqrt{\left(\frac{1}{\beta} + 1\right)} \\
&= 550 \sqrt{\left(\frac{1}{1.4} + 1\right)} \\
&= 550 \sqrt{(1.71)} = 550 (0.91057) \text{ ( using the gamma tables)} \\
&= 500.8 \text{ days}
\end{aligned}$$

$$\begin{aligned}
\text{(d) (S.D.)}^2 &= \theta^2 \left\{ \sqrt{\left(\frac{2}{\beta} + 1\right)} - \left( \sqrt{\left(\frac{1}{\beta} + 1\right)} \right)^2 \right\} \\
&= (550)^2 \left\{ \sqrt{\left(\frac{2}{1.4} + 1\right)} - \left( \sqrt{\left(\frac{1}{1.4} + 1\right)} \right)^2 \right\} \\
&= (550)^2 \left\{ \sqrt{2.43} - (\sqrt{1.71})^2 \right\} \\
&= 363.96 \text{ days}
\end{aligned}$$

(e) Let  $t_D$  be the required design life for  $R = 0.90$ , Then

$$R(t_D) = 0.90$$

$$R(t_D) = e^{-\left(\frac{t_D}{550}\right)^{1.4}} = 0.90$$

$$\left(\frac{t_D}{550}\right)^{1.4} = 0.010536$$

$$\begin{aligned}\text{Therefore, } t_D &= 550 (0.010536)^{(1/1.4)} \\ &= 110.2 \text{ days}\end{aligned}$$

## Practice Question

1. A turbine blade has demonstrated a Weibull failure pattern with a decreasing failure rate characterised by a shape parameter of 0.6 and a scale parameter of 800 hours.

(a) Compute the reliability for a 100-hour mission.

(b) If there is a 200-hour burn-in of the blades, what is the reliability for a 100-hour mission.

(Ans. 0.75, 0.89)

2. The one-month reliability on an indicator lamp is 0.95 with the failure rate specified as constant. What is the probability that more than two spare bulbs will be needed during the first year of operation (ignore replacement time).

(Ans. 0.0246)



## References

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