TUT 8

P(Y)

Q.1] $X_n \nearrow 0 \quad P(0) = 2/3$

$\searrow 1 \quad P(1) = 1/3$

$n = 0, 1, 2 \ldots$

$$Y_n = \sum_{i=1}^{n} X_i$$

$\mu_X = 1/3 \qquad Var(X) = 1/3 - 1/9 = \frac{2}{9}$

a) first order pmf of Y

$$P(Y_n = r) = {}^nC_r \,(1/3)^r \,(2/3)^{n-r}$$

$r = 0, 1, 2, \ldots n$

b) $E[Y_n] = np = \frac{n}{3}$

c) $R_{YY}(n, n+k) = E[Y_n \cdot Y_{n+k}]$

$$= E\left[\sum_{i=1}^{n} X_i \sum_{j=1}^{n+k} X_j\right] = \sum_{i=1}^{n} \sum_{j=1}^{n+k} E[X_i X_j]$$

$$= \sum_{i=1}^{n} \left[\sum_{\substack{j=1 \\ j \neq i}}^{n+k} E[X_i]E[X_j] + E[X_i^2] \right]$$

$E[X_i] \to 1/3 \qquad E[X_i^2] \to 1/3$

$$= \sum_{i=1}^{n} \left[\frac{1}{3} \sum_{\substack{j=1 \\ j \neq i}}^{n+k} E[X_j] + \frac{1}{3} \right]$$

$$= \sum_{i=1}^{n} \left[\frac{1}{3} \left(\frac{n+k-1}{3} \right) + \frac{1}{3} \right]$$

$$= \sum_{i=1}^{n} \left[\frac{n+k+2}{9} \right] = \frac{n}{9} (n+k+2)$$

d) $C_{y_1 y_2} = R_{y_1 y_2} - \mu_{y_1} \mu_{y_2}$

$C_{yy}(n, n+k) = R_{yy}(n, n+k) - \mu_y(n) \mu_y(n+k)$

$$= \frac{n}{9}(n+k+2) - \frac{n}{3} \left(\frac{n+k}{3} \right)$$

$$= \frac{2n}{9}$$

Q2] $Y(t) = \beta X(t)$

$\hookrightarrow$ $X(t)$ is a semi random telegraph signal process

$\rightarrow X(t) = (-1)^{N(t)}$ $\quad t \geq 0$

$\mu_X = e^{-2\lambda t}$ $\quad R_{XX}(\tau) = e^{-2\lambda \tau}$

$\rightarrow \beta \sim$ Uniform $(-2, 2)$

$\mu_\beta = \frac{2-2}{2} = 0$

$Var(\beta) = \frac{4^2}{12} = \frac{16}{12} = \frac{4}{3}$

$\Rightarrow$ For $Y(t)$ to be a WSS

i) μ_Y should be independent of t

ii) $R_{YY} = f(t_1 - t_2) = f(\tau)$

$\mu_Y = E[Y(t)] = E[\beta X(t)] = E[\beta] \cdot E[X(t)]$

$= 0 \cdot E[X(t)] = 0$

$R_{YY} = E[Y(t_1) Y(t_2)] = E[\beta X(t_1) \cdot \beta X(t_2)]$

$= E[\beta^2] \cdot E[X(t_1) X(t_2)]$

$= \frac{4}{3} e^{-2\lambda(t_2 - t_2)}$ $\quad t_2 > t_1$

$= \frac{4}{3} e^{-2\lambda T}$

$Y(t)$ is a WSS process.

Q3] $X(t) = T + (1-t)$

$T \sim$ Uniform $(0,1)$

$\mu_T = 1/2 \qquad Var(T) = 1/12 = E[T^2] - E[T]^2$

$E[T^2] = \frac{1}{12} + \frac{1}{4} = \frac{1}{3}$

a) cdf of $X(t)$

$P[X(t) \le x] = P[T + (1-t) \le x]$

$P[T \le x - (1-t)]$

$$F_X(x) = \begin{cases} 0 & x < 1-t \\ x-(1-t) & (1-t) < x < (2-t) \\ 1 & x > 2-t \end{cases}$$

$$f(x) = \frac{d}{dx} F_X(x) = \begin{cases} 1 & (1-t) < x < (2-t) \\ 0 & \text{otherwise} \end{cases}$$

b) $\mu_X = \int_{1-t}^{2-t} 1 \, dx = \frac{3}{2} - t$

c) $C_{xx}(t_1,t_2) = R_{xx}(t_1,t_2) - \mu_x(t_1)\,\mu_x(t_2)$

$$R_{xx}(t_1,t_2) = E\left[x(t_1)\,x(t_2)\right]$$

$$= E\left[(T + (1-t_1))(T+(1-t_2))\right]$$

$$= E\left[T^2 + T\left[(1-t_1) + (1-t_2)\right] + (1-t_1)(1-t_2)\right]$$

$$= \frac{1}{3} + \frac{1}{2}(2-t_1-t_2) + (1-t_1)(1-t_2)$$

$$C_{xx} = \frac{1}{3} + 1 - \frac{t_1}{2} - \frac{t_2}{2} + 1 - t_1 - t_2 + t_1 t_2$$

$$- \left(\frac{3}{2} - t_1\right)\left(\frac{3}{2} - t_2\right)$$

$$= \frac{1}{12}$$

Q4] $X(t) \sim WSS$

$\mu_X = 2 \qquad R_{XX}(\tau) = e^{-|\tau|/10} + 4$

$$S(t) = \int_0^1 X(t)\,dt$$

$$\mu_S = E\left[\int_0^1 X(t)\,dt\right] = \int_0^1 E[X(t)]\,dt$$

$$= \int_0^1 2\,dt = \underline{\underline{2}}$$

$$E[S^2(t)] = E\left[\int_0^1 X_1(t_1)\,dt \int_0^1 X(t_2)\,dt_2\right]$$

$$= \int_0^1\int_0^1 E[X_1(t_1)\,X_2(t_2)]\,dt_1\,dt_2$$

$$= \int_0^1\int_0^1 \left(4 + e^{-|t_1-t_2|/10}\right) dt_1 + dt_2$$

$$= 4 + \int_{-1}^1 e^{-\tau/10}(1-|\tau|)\,d\tau$$

$$= 4 - 10\left[e^{-\tau/10}\right]_{-1}^1 + \int_{-1}^0 \tau e^{-\tau/10}\,d\tau - \int_0^1 \tau e^{-\tau/10}\,d\tau$$

$$= 4 - 10\left[e^{-1/10} - e^{1/10}\right] - \left[10\tau e^{-\tau/10} + 100e^{-\tau/10}\right]_{-1}^0$$

$$+ \left[10\tau e^{-\tau/10} + 100e^{-\tau/10}\right]_0^1$$

$$= 4 - 10e^{-1/10} + 10e^{1/10} - 100 + 10(e^{1/10}$$

$$- 100e^{1/10} + 10e^{-1/10} + 100e^{-1/10} - 100$$

$$= 4 - 200 + e^{1/10}\left[10 + 10 - 100\right]$$

$$+ e^{-1/10}\left[-10 + 10 + 100\right]$$

$$= 100e^{-1/10} - 80e^{1/10} - 196$$

$$\mathrm{Var}(S) = E[S^2] - (E[S])^2$$

$$= 100e^{-1/10} - 80e^{1/10} - 196 - 4$$

$$= \underline{\underline{100e^{-1/10} - 80e^{1/100} - 200}}$$

Q5] $X(t) = 10\cos(100t + \Phi)$

$\Phi \sim \text{Uniform}(-\pi, \pi)$

$\mu_\Phi = 0 \qquad Var(\Phi) = \frac{\pi^2}{3}$

$\mu_{\cos(\Phi)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos\Phi \, d\Phi = 0$

$\mu_{\sin\Phi} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin\Phi \, d\Phi = 0$

iii) We cannot prove ergodicity of a process

i) Mean Ergodicity

$$\overline{X_T} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} 10\cos(100t + \Phi)\,dt$$

$$= \lim_{T\to\infty} \frac{5}{T}\left[\text{finite } (-1, 1)\right] = \frac{1}{\infty} = 0$$

$$E[x] = E[10\cos(100t + \Phi)] = 10\,E[\cos(100t + \Phi)]$$

$$= 10E[\cos(100t)\cos\Phi - \sin(100t)\sin\Phi]$$

$$= 10\left\{E[\cos 100t]\cdot \underbrace{E[\cos\Phi]}_{0} - E[\sin 100t]\,\underbrace{E[\sin\Phi]}_{0}\right\}$$

$= 0$

$$\boxed{\overline{X_T} = E[x]} \Rightarrow \text{Mean Ergodic}$$

ii) Correlation Ergodicity

first we need ACF of $X(t)$

$$R_{xx}(t_1,t_2) = E[x(t_1)\,x(t_2)]$$

$$= E[10\cos(100t_1+\varphi).10\cos(100t_2+\varphi)]$$

$$= \frac{100}{2}\left\{E\left[\cos\left(100(t_1+t_2)+\varphi\right)\right] + E\left[\cos 100(t_1-t_2)\right]\right\}$$

$$= 50\times 0 + 50\cos\left(100(t_1-t_2)\right)$$

$$R_{xx}(\tau) = 50\cos(100\tau)$$

$$\overline{X^2_{T,T+\tau}} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)\,x(t+\tau)\,dt$$

$$= \lim_{T\to\infty}\frac{100}{2T}\int_{-T}^{T}\cos(100t+\varphi)\cos(100(t+\tau)+\varphi)\,dt$$

$$= \lim_{T\to\infty}\frac{50}{T}\cancel{\int_{-T}^{T}\cos(100(2t+\tau)+\varphi)\,dt} \to 0$$

$$+ \frac{50}{T}\int_{-T}^{T}\cos 100\tau\,dt$$

$$= 50\cos 100\tau$$

$$\boxed{X^2_{T,T+\tau} = R_{xx}(\tau)} \Rightarrow \text{correlation ergodic}$$

Q6] $x(t) \sim$ WSS

$\mu_x = 0 \qquad R_{xx}(\tau) = 1 - \frac{|\tau|}{T}$

$$\mathrm{Var}(\overline{X_T}) = \frac{1}{(T/2)} \int_0^T C_{xx}\left[1 - \frac{|\tau|}{T}\right] d\tau$$
over $(0,T)$

$$C_{xx} = R_{xx}(\tau) - \underbrace{\mu_x(t_1)\,\mu_x(t_2)}_{0}$$

$$\mathrm{Var}(\overline{X_T}) = \frac{2}{T}\int_0^T \left(1 - \frac{\tau}{T}\right)^2 d\tau$$

$$= -\frac{2}{T}\left[\frac{\left(1-\frac{\tau}{T}\right)^3}{3}\right]_0^T \times T$$

$$= -\frac{2 \times T}{T}\left[\frac{0^3}{3} - \frac{1^3}{3}\right] = \frac{2T}{3T} = \frac{2}{3}$$

$$\lim_{T\to\infty} \mathrm{Var}(\overline{X_T}) = 0$$

for Mean Ergodic Process

Hence, It is <u>not ergodic</u> process

P7] $x(t) \sim$ RTP $\qquad x(t) = \alpha(-1)^{N(t)}$

$$\mu_x = 0 \qquad R_{xx}(\tau) = e^{-2\lambda|\tau|}$$

$$E[\overline{X_T}] = E\left[\frac{1}{2T}\int_{-T}^{T} x(t)\,dt\right] = \int_{-T}^{T} E[x(t)]\,dt$$

$$= \int_{-T}^{T} 0\,dt = 0$$

$$\mathrm{Var}(\overline{X_T}) = \frac{1}{T}\int_{-T}^{T} C_{xx}\left[1 - \frac{|\tau|}{T}\right] d\tau$$

$$= \frac{1}{T}\int_{-T}^{T} e^{-2\lambda|\tau|}\left[1 - \frac{|\tau|}{T}\right] d\tau$$

$$= \frac{1}{T}\left\{\int_{-T}^{0} e^{2\lambda\tau}\left[1 + \frac{\tau}{T}\right]d\tau + \int_{0}^{T} e^{-2\lambda\tau}\left[1 - \frac{\tau}{T}\right]d\tau\right\}$$

$$= \frac{1}{T}\left\{\left[\frac{e^{2\lambda\tau}}{2\lambda}\right]_{-T}^{0} + \frac{1}{T}\left[\frac{\tau e^{2\lambda\tau}}{2\lambda} - \frac{e^{2\lambda\tau}}{(2\lambda)^2}\right]_{-T}^{0}\right.$$

$$\left. + \left[\frac{e^{-2\lambda\tau}}{2\lambda}\right]_{0}^{T} - \frac{1}{T}\left[\tau\left(\frac{e^{-2\lambda\tau}}{-2\lambda}\right) - \frac{e^{-2\lambda\tau}}{(2\lambda)^2}\right]_{0}^{T}\right\}$$

$$= \frac{1}{T}\left\{\cancel{\frac{1}{2\lambda}} - \cancel{\frac{e^{-2\lambda T}}{2\lambda}} + \frac{1}{T}\left[\frac{-1}{(2\lambda)^2} + \frac{Te^{-2\lambda T}}{2\lambda} - \frac{e^{-2\lambda T}}{(2\lambda)^2}\right]\right.$$

$$+ \left[\cancel{\frac{e^{-2\lambda T}}{2\lambda}} - \cancel{\frac{1}{2\lambda}}\right] - \frac{1}{T}\left[\frac{Te^{-2\lambda T}}{(-2\lambda)} - \frac{e^{-2\lambda T}}{(2\lambda)^2} + \frac{1}{(2\lambda)^2}\right]$$

$$= \frac{1}{T^2} \left\{ \frac{-1}{(2\lambda)^2} + \frac{Te^{-2\lambda T}}{2\lambda} - \cancel{\frac{e^{-2\lambda T}}{(2\lambda)^2}} \right.$$

$$\left. + \frac{Te^{-2\lambda T}}{2\lambda} + \cancel{\frac{e^{-2\lambda T}}{(2\lambda)^2}} - \frac{1}{(2\lambda)^2} \right]$$

$$= \frac{1}{T^2} \left\{ \frac{Te^{-2\lambda T}}{\lambda} - \frac{1}{2\lambda^2} \right\}$$

$$\lim_{T \to \infty} Var(\overline{X_T}) = \frac{1}{\infty} = 0$$

Hence Process is <u>mean ergodic</u>

Q9]

$$S(\omega) = \begin{cases} 1+\omega^2 & |\omega| \le 1 \\ 0 & |\omega| > 1 \end{cases}$$

$$\delta(t) \Longrightarrow 1$$

$$\mathcal{F}^{-1}\{\omega^2\} = \frac{1}{2\pi}\int_{-1}^{1} \omega^2 e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\left[\omega^2\left(\frac{e^{j\omega t}}{jt}\right) - \omega\left(\frac{e^{j\omega t}}{(jt)^2}\right) + \frac{e^{j\omega t}}{(jt)^3}\right]_{-1}^{1}$$

$$= \frac{1}{2\pi}\left[\frac{1e^{jt}}{jt} - \frac{e^{jt}}{(jt)^2} + \frac{e^{jt}}{(jt)^3} - \left(\frac{e^{-jt}}{jt} + \frac{e^{-jt}}{(jt)^2} + \frac{e^{-jt}}{(jt)^3}\right)\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{jt}-e^{-jt}}{jt} - \left[\frac{e^{jt}+e^{-jt}}{(jt)^2}\right] + \frac{e^{jt}-e^{-jt}}{(jt)^3}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin t}{t} - \frac{\cos t}{-t^2} + \frac{\sin t}{-t^3}\right]$$

$$R_{xx}(\tau) = \mathcal{F}^{-1}\{S(\omega)\}$$

$$= \delta(t) + \frac{1}{\pi}\left\{\frac{\sin t}{t} + \frac{\cos t}{t^2} + \frac{\sin t}{t^3}\right\}$$

9.10] $$R(\tau) = \begin{cases} 1 - \frac{|\tau|}{T} & |\tau| \le T \\ 0 & |\tau| > T \end{cases}$$

$$S(\omega) = \mathcal{F}\{R(\tau)\}$$

$$1 \rightleftharpoons 2\pi\delta(\omega)$$

$$\mathcal{F}\{|\tau|\} = \int_{-T}^{T} |\tau| e^{-j\omega\tau} d\tau$$

$$= \int_{-T}^{0} -\tau e^{-j\omega\tau} d\tau + \int_{0}^{T} \tau e^{-j\omega\tau} d\tau$$

$$= -\left[(\tau)\left(\frac{e^{-j\omega\tau}}{-j\omega}\right) + \frac{e^{-j\omega\tau}}{(j\omega)^2}\right]_{-T}^{0}$$

$$+ \left[\tau\left(\frac{e^{-j\omega\tau}}{-j\omega}\right) + \frac{e^{-j\omega\tau}}{(j\omega)^2}\right]_{0}^{T}$$

$$= -\left[\frac{1}{-\omega^2} - \left(\frac{-Te^{-j\omega T}}{-j\omega} + \frac{e^{-j\omega T}}{-\omega^2}\right)\right]$$

$$+ \left[\frac{Te^{-j\omega T}}{-j\omega} + \frac{e^{-j\omega T}}{-\omega^2} - \left(\frac{1}{-\omega^2}\right)\right]$$

$$= \frac{1}{\omega^2} + \frac{Te^{-j\omega T}}{j\omega} + \frac{e^{-j\omega T}}{\omega^2}$$

$$- \frac{Te^{-j\omega T}}{j\omega} - \frac{e^{-j\omega T}}{\omega^2} + \frac{1}{\omega^2}$$

$$= \frac{2}{\omega^2}$$

So, PSDF $= S(\omega) = 2\pi\,\delta(\omega) + \dfrac{2}{\omega^2}$