Probability and Random Processes (15B11MA301)

Lecture-36

Department of Mathematics

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Topics to Be covered

- **Power Spectral Density Function**
- **Properties**

Power Spectral Density Function(PSDF)

- The autocorrelation function(ACF) $R(\tau)$ gives the information about the random signal $X(t)$ can be expected to change as a function of time.
- If the ACF decays rapidly, it indicates that the process can be expected to change rapidly.
- If the ACF has periodic components, then the corresponding process will also have periodic components.
- Therefore, the ACF contains the information about the expected frequency content of the random processes.

- $S(\omega)$ is called the **power spectral density (PSD) function** or **spectral density**, or **power spectral of the stationary process** $\{X(t)\}$ gives the distribution of power of $\{X(t)\}$ as a function of frequency.
- The power spectral density (PSD) $S_x(\omega)$ for a signal is a measure of its power distribution as a function of frequency.
- It is a useful concept which allows us to determine the bandwidth required of a transmission system.

PSDF: Mathematical Definition

- If $\{X(t)\}$ is a stationary process (either SSS or WSS) with ACF $R_{XX}(\tau), then\ the\ power\ spectral\ density\ of\ \{X(t)\}$ is the Fourier transform of $R_{XX}(\tau)$,*i.e.*

$$S_{XX}(\omega) = S(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau$$

- If the spectral density $S_{XX}(\omega)$ is given, then the ACF is

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{i\omega\tau} d\omega$$

- The power spectral density of the random process $\{X(t)\}$ is also defined by

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{E(\left|X_T(\omega)\right|^2)}{2T}$$

Where

$$X_T(\omega) = \int_{-\infty}^{\infty} X_T(t) e^{-i\omega t} dt = \int_{-T}^{T} X_T(t) e^{-i\omega t} dt$$

$$and \qquad X_T(t) = \begin{cases} X(t), & -T < t < T \\ 0, & otherwise \end{cases}$$

- The average power of the random processes $\{X(t)\}$ is defined by
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$$P_{XX} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega$$

- Average power is also given by the time average of its second moment

$$P_{XX} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} E[X^2(t)]\, dt$$

Cross-Spectral Density Function

- The cross spectral density of the random processes {X(t)} and {Y(t)} is defined as

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-i\omega\tau} d\tau$$

- The cross- correlation is defined as
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$$R_{XY}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega) e^{i\omega\tau} d\omega$$

Wiener-Khinchine Theorem

- If $X_T(\omega)$ is the Fourier transform of the truncated random processes defined as

$$X_T(t) = \begin{cases} X(t), & |t| \leq T \\ 0, & for\ |t| > T \end{cases}$$

Where $\{X(t)\}$ is a real WSS process with power spectral density function $S(\omega), then$

$$S(\omega) = \underset{T \to \infty}{Lim} \left\{ \frac{1}{2T} E[|X_T(\omega)|^2] \right\}$$

Remark: **This theorem provides an alternative method for finding PSD function of a WSS process.**

THANK YOU