Probability and Random Processes (15B11MA301)

Lecture-28

(Content Covered: Random Process and its Classification)

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Contents

Order and Density of the Distribution

The first order distribution function is defined as

$$F(x;t) = P[X(t) \leq x]$$

The first order density function is defined by

$$f(x;t) = \frac{d}{dx} F(x;t)$$

These definition are generalize to the n^{th}-order case.

For any given positive integer n, let $x_1, x_2, \cdots x_n$ denote n realization variables and let $t_1, t_2, t_3, \cdots t_n$ denote n time variable. Then, define the n^{th}-order distribution function as

$$F(x_1, x_2, x_3, \cdots x_n, t_1, t_2, \cdots, t_n) = P[X(t_1) \leq x_1, X(t_2) \leq x_2, \cdots, X(t_n) \leq x_n]$$

Similarly, define the n^{th}-order density function as

$$f(x_1, x_2, \cdots, x_n: t_1, t_2, \cdots, t_n) = \frac{\partial^n F(x_1, x_2, \cdots, x_n: t_1, t_2, \cdots, t_n)}{\partial x_1 \partial x_2 \partial x_3 \cdots \partial x_n}$$

MARKOV PROCESS

A random process $\{X(t)\}$ is called a Markov process if

$$P[X(t_n) = a_n |\, X(t_{n-1}) = a_{n-1}, X(t_{n-2}) = a_{n-2}, \cdots, X(t_1) = a_1] = P[X(t_n) = a_n |\, X(t_{n-1}) = a_{n-1}]$$

for all $t_1 < t_2 < \cdots < t_n$.
In other words, we can say that if the future behaviour of a process depends upon the present but not on the past, then the process is called a Markov Process.

There are four types of the Markov processes depending upon the values taken by the time variables and the values taken by the State space $\{X(t)\}$.

1. A **continuous random process** satisfying the **Markov property** is known as **Continuous parameter Markov process.**

2. A **continuous random sequence** satisfying the **Markov property** is known as **Discrete parameter Markov process.**

3. A **discrete random sequence** satisfying the **Markov property** is known as **Discrete parameter Markov Chain.**

4. A **discrete random process** satisfying the **Markov property** is called as **Continuous parameter Markov Chain.**

CHARACTERIZATION OF RANDOM PROCESSES

The distinguish feature of a random process is the relationship of the random variables a various times.

- ❖ A random process at a given time is a random variable, it can thus be described with a probability distribution.
- ❖ In general, the form of the distribution of a random process is different for different instants of time.

In most cases, it is not possible to determine the distribution of a random processes at a certain time or the bivariate distribution at any two different times or joint distribution at many different times, because all sample functions are hardly ever known.

As a mater of fact, from practical consideration standpoint, more often that not, only one sample function of finite duration is all that is ever available.

JOINT DISTRIBUTION OF TIME SAMPLES

Let $X(t_1), X(t_2), \cdots, X(t_k)$ be the k-random variables obtained by sampling the random process $X(t)$ at time instants $t_1, t_2, \cdots, t_k$, where k is a positive integer. The k^{th}-order joint CDF of a random process is then defined as follows:

$$F_{X(t_1),\cdots,X(t_n)}(x_1, x_2, \cdots, x_n) = P[X(t_1) \leq x_1, \cdots, X(t_k) \leq x_k]$$

The join probability density functions (pdf's) or probability mass functions (pmf's) depending on whether the random process $X(t)$ is continuous-valued or discrete-valued can be obtained using the joint cumulative distribution function (cdf). The complete characterization of the random process $X(t)$ requires the massive knowledge of all the distribution as $k \to \infty$.

INDEPENDENT IDENTICALLY DISTRIBUTED RANDOM PROCESS

A discrete-time random process consisting of a sequence of independent identically-distributed random variables with common cdf $F_X(x)$ is called the independent identically distributed random process. The joint cdf of such a random process is then equal to the product of the individual cdf's, i.e., we have

$$F_{X(t_1),\cdots,X(t_k)}(x_1, x_2, \cdots, x_k) = F_X(x_1) \times F_X(x_2) \times \cdots \times F_X(x_k)$$

Thus, a first-order distribution is sufficient to characterize an independent identically distributed random process $X(t)$. The above equation clearly implies that if the sequence is discrete-valued, the joint pmf is then the product of individual pmf's, and if it is continuous-valued, the joint is then the product of individual pdf's.

Multiple Random Processes

In many practical applications, we often need to deal with more than one random process at a time as we may need to assess the inter-relatedness (i.e. positive and negative correlation) between the processes, say between two processes $X(t)$ and $Y(t)$.

Two random processes $X(t)$ and $Y(t)$ are specified by their joint CDF for all possible choices of time samples of the processes. Let $X(t_1), \cdots, X(t_k)$ be the k-random variables obtained by sampling the random process $X(t)$ at time $t_1, \cdots, t_k$ and $Y(t_1'), \cdots, Y(t_k')$ be the m-random variables obtained by sampling the random process $Y(t)$ at times $t_1', \cdots, t_m'$ where both k and m are positive integers. The joint CDF of two random processes is then defined as follows:

$$F_{X(t_1),\cdots,X(t_k),Y(t_1'),\cdots,Y(t_m')}(x_1, x_2, \cdots, x_k, y_1, \cdots, y_m)$$
$$= P[X(t_1) \leq x_1, \cdots, X(t_k) \leq x_k, Y(t_1') \leq y_1, \cdots, Y(t_m') \leq y_m]$$

Note that the above definition can be easily extended to multiple random processes, and from the joint CDF of multiple random processes, the CDF for each random process can be obtained.

INDEPENDENT RANDOM PROCESSES

The random processes $X(t)$ and $Y(t)$ are independent random processes if the random vectors $\{X(t_1), \cdots, X(t_k)\}$ and $\{Y(t'_1), \cdots, Y(t'_m)\}$ are independent for positive integers k and m, for all $t_1, \cdots, t_k$ and $t'_1, \cdots, t'_m$, i.e., the joint cdf's is equal to the product of the individual cdf's. In other words, we have

$$F_{X(t_1),\cdots,X(t_k),Y(t'_1),\cdots,Y(t'_m)}(x_1, x_2, \cdots, x_k, y_1, \cdots, y_m)$$
$$= F_{X(t_1),\cdots,X(t_k)}(x_1, \cdots, x_k, y_1, \cdots, y_m) \times F_{Y(t'_1),\cdots,Y(t'_m)}(y_1, \cdots, y_m)$$

In many applications, it is rather easy to see whether the two random processes are independent. Note that the concept of independent random processes can be extended further to more than two random processes.

INDEPENDENT INCREMENTS

A continuous random process $X(t)$, where $t \geq 0$, is said to have independent increments if the set of n random variables $X(t_1) - X(0), X(t_2) - X(t_1), \cdots, X(t_n) - X(t_{n-1})$ are jointly independent for any integer $n \geq 2$ and all time instants $0 \leq t_1 < t_2 < \cdots < t_n$. In other words, the increments are statistically independent when intervals are non-overlapping. The independent increment property simplifies analysis, as it makes easy to get the higher-order distributions.

For instance, with the independent increments in the context of counting processes, the probability of having m events in the interval $(t_{i-1}, t_i]$ and k events in the interval $\left(t_{j-1}, t_j\right]$, when the two intervals are disjoint, is equal to the product of the probability of having m events in the interval $(t_{i-1}, t_i]$ and the probability of having k-events in the interval $\left(t_{j-1}, t_j\right]$.

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