Higher-Order Methods & Richardson Extrapolation

Group Name : Cygnus
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1 Objective

The objective of this experiment is to:

- Implement second-order $\mathcal{O}(h^2)$ and fourth-order $\mathcal{O}(h^4)$ finite difference formulas for numerical differentiation.
- Compare their convergence behavior.
- Demonstrate the accuracy improvement obtained using higher-order methods.

2 Problem Description

We approximate the first derivative of the function

$$f(x) = \sin(x)$$

at the point $x = 1.0$. The exact derivative is

$$f'(x) = \cos(x)$$

which is used to compute the numerical error.

3 Numerical Methods

3.1 $\mathcal{O}(h^2)$ Central Difference

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

3.2 $\mathcal{O}(h^4)$ Central Difference

$$f'(x) \approx \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h}$$

4 Implementation

The numerical methods were implemented in C using the `math.h` library.

```c
int main() {
    double x1 = 1.0;
    double exact = cos(x1);

    double h[] = {0.4, 0.2, 0.1, 0.05, 0.025};
    int N = 5;

    printf("\nExact derivative = %.10f\n\n", exact);
    printf("h        D2 Error        D4 Error\n");
    printf("-------------------------------------\n");

    for (int i = 0; i < N; i++) {
        double d2 = D2(x1, h[i]);
        double d4 = D4(x1, h[i]);

        printf("%-8.5f %.3e     %.3e\n",
               h[i],
               fabs(d2 - exact),
               fabs(d4 - exact));
    }
```

Figure 1: Derivative O(h^2)$andO(h^4)$

The step sizes used were:

$$h = \{0.4,\ 0.2,\ 0.1,\ 0.05,\ 0.025\}$$

5 Results

The exact derivative at $x = 1$ is:

$$f'(1) = \cos(1)$$

Table 1 shows the absolute errors for both methods.

Table 1: Error comparison for $\mathcal{O}(h^2)$ and $\mathcal{O}(h^4)$ methods

h	$\lvert D_2 - f'(x)\rvert$	$\lvert D_4 - f'(x)\rvert$
0.400	1.429e-02	4.524e-04
0.200	3.595e-03	2.868e-05
0.100	9.001e-04	1.799e-06
0.050	2.251e-04	1.125e-07
0.025	5.628e-05	7.035e-09

6 Convergence Comparison

From the numerical results, it is observed that:

- The $\mathcal{O}(h^2)$ method shows quadratic convergence, with error decreasing proportionally to h^2.
- The $\mathcal{O}(h^4)$ method converges significantly faster, with error decreasing proportional to h^4.
- For all step sizes, the $\mathcal{O}(h^4)$ method produces much smaller errors than the $\mathcal{O}(h^2)$ method.

7 Richardson Extrapolation

Richardson extrapolation can be applied to the $\mathcal{O}(h^2)$ method to eliminate the leading error term:

$$D_{\text{Rich}} = \frac{4D(h/2) - D(h)}{3}$$

This improves the accuracy to $\mathcal{O}(h^4)$, demonstrating how extrapolation can systematically increase the order of convergence.

8 Error Decay Plot

A log–log plot of error versus step size h would show:

- A slope of approximately 2 for the $\mathcal{O}(h^2)$ method.
- A slope of approximately 4 for the $\mathcal{O}(h^4)$ method.

9 Conclusion

Higher-order finite difference methods provide substantially improved accuracy with minimal additional computational cost. The fourth-order scheme clearly outperforms the second-order scheme, and Richardson extrapolation offers a systematic way to further enhance numerical accuracy.