SOUTHEAST UNIVERSITY

BANGLADESH

GROUP ASSIGNMENT REPORT

CSE261: Numerical Methods

Assignment Topic:

Higher-Order Methods and Richardson Extrapolation for Numerical Differentiation

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Abstract

This report presents a comprehensive study of higher-order numerical differentiation methods and Richardson extrapolation. Second-order and fourth-order finite difference schemes are implemented to approximate first derivatives of smooth functions. Richardson extrapolation is applied to further improve accuracy by eliminating leading truncation error terms. Numerical experiments, error analysis, and convergence studies demonstrate that higher-order methods achieve significantly faster convergence and improved accuracy compared to low-order schemes. The results validate theoretical expectations and highlight the effectiveness of Richardson extrapolation in numerical differentiation.

1 Introduction

Numerical differentiation is a fundamental topic in numerical methods and plays a crucial role in scientific computing, engineering simulations, and data analysis. In many real-world problems, analytical expressions for derivatives are either unavailable or computationally expensive to evaluate. In such cases, numerical differentiation provides an efficient alternative by approximating derivatives using discrete function values.

However, numerical differentiation is sensitive to truncation and rounding errors, particularly when low-order methods are used. To address these limitations, higher-order finite difference methods and extrapolation techniques have been developed. The objective of this work is to implement second-order and fourth-order numerical differentiation schemes, apply Richardson extrapolation, and analyze their accuracy, stability, and convergence behavior.

2 Theoretical Background

Finite difference methods approximate derivatives by replacing limits with finite step sizes. The first derivative of a function $f(x)$ is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

In numerical computation, the step size h cannot be taken arbitrarily small, leading to truncation errors. The accuracy of a finite difference method is characterized by its order of accuracy, denoted by $O(h^p)$, where p indicates how fast the error decreases as h decreases.

3 Methodology

This study employs the following numerical differentiation techniques:

- Second-order central difference method $O(h^2)$
- Fourth-order central difference method $O(h^4)$
- Richardson extrapolation for error reduction

The second-order central difference formula is given by:

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}.$$

A fourth-order accurate central difference approximation is:

$$f'(x) \approx \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h}.$$

Richardson extrapolation is applied using:

$$D_R = \frac{4D(h/2) - D(h)}{3},$$

which improves the accuracy from $O(h^2)$ to $O(h^4)$.

4 Implementation

The numerical algorithms were implemented using Python. Separate functions were written to compute numerical derivatives using different step sizes and orders of accuracy. Error analysis routines were implemented to compare numerical results with exact derivatives. All source codes, figures, and documentation were maintained in a structured GitHub repository, ensuring reproducibility and version control.

The full implementation is available on GitHub:

`https://github.com/rahibraihan/cygnus.git`

5 Numerical Results

Numerical experiments were conducted using the test function

$$f(x) = \sin x,$$

with exact derivative

$$f'(x) = \cos x.$$

The derivative was evaluated at $x = 1$ for step sizes $h = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}$.

The numerical results clearly show that higher-order methods produce smaller errors for the same step size. Richardson extrapolation further improves the accuracy by reducing truncation error.

6 Error, Stability and Convergence Analysis

The numerical error is defined as:

$$\text{Error} = |f'\text{numerical} - f'\text{exact}| .$$

Error analysis confirms that the second-order method exhibits error proportional to h^2, while the fourth-order method shows error proportional to h^4. Log-log plots of error versus step size demonstrate convergence slopes close to the theoretical values.

Stability analysis indicates that excessively small step sizes may introduce rounding errors due to finite machine precision. Therefore, an optimal step size range is necessary to balance truncation and rounding errors.

7 Plots, Tables and Interpretation

Error decay curves were generated by plotting the absolute error against step size on a log-log scale. The plots reveal that higher-order methods converge significantly faster than lower-order methods. Richardson extrapolation produces the steepest error decay curve, confirming its effectiveness.

These visual results strongly support the theoretical analysis and demonstrate the superiority of higher-order numerical differentiation methods.

8 Discussion

The results obtained in this study demonstrate that higher-order finite difference methods significantly outperform low-order schemes in terms of accuracy and convergence. Richardson extrapolation effectively enhances accuracy without increasing computational cost substantially. However, careful selection of step size is essential to avoid numerical instability caused by rounding errors.

9 Conclusion

- Higher-order numerical differentiation methods reduce truncation error.

- Fourth-order methods converge faster than second-order methods.
- Richardson extrapolation further improves accuracy.
- Numerical results agree well with theoretical predictions.
- Step size selection plays a crucial role in numerical stability.

Future work may include adaptive step-size control and applications to noisy experimental data.

10 References

- R. L. Burden and J. D. Faires, *Numerical Analysis*, Cengage Learning.
- S. C. Chapra and R. P. Canale, *Numerical Methods for Engineers*, McGraw-Hill.
- K. Atkinson, *An Introduction to Numerical Analysis*, Wiley.

A Appendix

Complete source codes, Richardson tables, and additional plots are included in the GitHub repository associated with this project.