MATH 316U

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SAMPLE TEST 1

SOLUTION KEY

Spring 2008

1. (a) $\begin{bmatrix} 1 & 2 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 2 & 3 \\ 0 & 1 \end{bmatrix}$ cannot multiply –number of columns of the first matrix (4) does not match the number of rows of the second one (3)

   (b) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & -2 \end{bmatrix}$

2. Since $F(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ (every horizontal vector, including $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, is reversed)

   and $F(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ (every vertical vector, including $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, is doubled)

   then the matrix of F is $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$. (see the discussion on p.36 and Example 1.24)

3. $x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 3x_3 \begin{bmatrix} 0 \\ 4 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -12 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

   The matrix of T is $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & -12 \end{bmatrix}$

4. (a) Augmented matrix:

   $$\begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ -2 & -4 & 1 & -3 & -3 \\ 3 & 6 & -1 & 4 & 5 \end{bmatrix} \quad r_2 + 2r_1 \longrightarrow r_2 \quad \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 3 & 6 & -1 & 4 & 5 \end{bmatrix} \quad r_3 - 3r_1 \longrightarrow r_3$$

   $$\begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & 1 & -1 \end{bmatrix} \quad r_3 + r_2 \longrightarrow r_3 \quad \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   This r.r.e.f. matrix corresponds to the system

   $$\begin{array}{ccccccccc} x & + & 2y & & & + & w & = & 2 \\ & & & & z & - & w & = & 1 \\ & & & & & & 0 & = & 0 \end{array}$$

   Solution: $x = 2 - 2y - w$; $y - arbitrary$; $z = 1 + w$; $w - arbitrary$

   (b) The augmented matrix:

   $$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 5 \end{bmatrix} \quad r_3 - 2r_1 \longrightarrow r_3 \quad \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

   $$r_3 - r_2 \longrightarrow r_3 \quad \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   The last equation is $0 = 1 \Rightarrow$ No solution.

(c) Augmented matrix:

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 7 & 5 \\ 1 & 4 & 4 \end{bmatrix} \quad r_2 - 2r_1 \longrightarrow r_2 \quad \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 1 & 4 & 4 \end{bmatrix} \quad r_3 - r_1 \longrightarrow r_3$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix} \quad r_3 - r_2 \longrightarrow r_3 \quad \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad r_1 - 3r_2 \longrightarrow r_1$$

$$\begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

This r.r.e.f. matrix corresponds to the system

$$\begin{aligned} x \quad &= -8 \\ y &= 3 \\ 0 &= 0 \end{aligned}$$

Solution: $x = -8$; $y = 3$.

5. (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}$; b. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

6. (a) $A = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & -2 & 2 \end{bmatrix}$; Form the matrix $(A|I_3)$: $\begin{bmatrix} -1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ -1 & -2 & 2 & 0 & 0 & 1 \end{bmatrix}$

$$-1r_1 \longrightarrow r_1 \quad \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ -1 & -2 & 2 & 0 & 0 & 1 \end{bmatrix} \quad r_2 - r_1 \longrightarrow r_2 \quad \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ -1 & -2 & 2 & 0 & 0 & 1 \end{bmatrix}$$

$$r_3 + r_1 \longrightarrow r_3 \quad \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & -2 & 1 & -1 & 0 & 1 \end{bmatrix} \quad r_3 + 2r_2 \longrightarrow r_3 \quad \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 \end{bmatrix}$$

$$r_1 + r_3 \longrightarrow r_1 \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 \end{bmatrix}; \quad \text{Answer } A^{-1} = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

(b) $B = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$; Form the matrix $(B|I_4)$: $\begin{bmatrix} 1 & 2 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$

$$r_3 - r_1 \longrightarrow r_3 \quad \begin{bmatrix} 1 & 2 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \quad r_3 - r_2 \longrightarrow r_3 \quad \begin{bmatrix} 1 & 2 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The third row in the left hand side part of the matrix is 0; B is singular.

7. (a) $(B^{-1}A)^{-1} = A^{-1}B = \begin{bmatrix} 2 & 0 & 0 \\ 3 & -3 & 0 \\ 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 2 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 4 \\ 9 & -15 & 6 \\ 3 & 20 & 3 \end{bmatrix}$

(b) $A\overrightarrow{x} = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} \Rightarrow A^{-1}A\overrightarrow{x} = A^{-1} \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}$

$$\Rightarrow \overrightarrow{x} = A^{-1} \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 3 & -3 & 0 \\ 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 17 \end{bmatrix}$$

(c) The matrices A and B row equivalent since they are both nonsingular, which makes them both row equivalent to I .

8. (a) $c(d\overrightarrow{u}) = (cd)\overrightarrow{u}$

$LHS = c(d\begin{bmatrix} x \\ y \end{bmatrix}) = c\begin{bmatrix} dy \\ dx \end{bmatrix} = \begin{bmatrix} cdx \\ cdy \end{bmatrix}$

$RHS = (cd)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cdy \\ cdx \end{bmatrix}$

Generally, $LHS \neq RHS \Rightarrow$ The property does not hold.

(b) $c(\overrightarrow{u} + \overrightarrow{v}) = (c\overrightarrow{u}) + (c\overrightarrow{v})$

$LHS = c(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x' \\ y' \end{bmatrix}) = c\begin{bmatrix} y+y' \\ x+x' \end{bmatrix} = \begin{bmatrix} c(x+x') \\ c(y+y') \end{bmatrix}$

$RHS = (c\begin{bmatrix} x \\ y \end{bmatrix}) + (c\begin{bmatrix} x' \\ y' \end{bmatrix}) = \begin{bmatrix} cy \\ cx \end{bmatrix} + \begin{bmatrix} cy' \\ cx' \end{bmatrix} = \begin{bmatrix} cx+cx' \\ cy+cy' \end{bmatrix}$

$LHS = RHS \Rightarrow$ The property holds.

9. (a) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x = 2y = 3z \right\}$,

- $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is in W (since $0 = (2)(0) = (3)(0)$)
- Closed under addition: let $\overrightarrow{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\overrightarrow{v} = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$ with $x = 2y = 3z$ and $x' = 2y' = 3z'$.

  Then $\overrightarrow{u}+\overrightarrow{v} = \begin{bmatrix} x+x' \\ y+y' \\ z+z' \end{bmatrix}$ satisfies $x+x' = 2y+2y' = 2(y+y')$ and $x+x' = 3z+3z' = 3(z+z')$ therefore $\overrightarrow{u} + \overrightarrow{v}$ is in W.
- Closed under scalar multiplication: let $\overrightarrow{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ with $x = 2y = 3z$. Then $k\overrightarrow{u} = \begin{bmatrix} kx \\ ky \\ kz \end{bmatrix}$ satisfies $kx = k(2y) = 2(ky)$ and $kx = k(3z) = 3(kz)$ therefore $k\overrightarrow{u}$ is in W.
- Answer: W is a subspace of R^3

(b) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x \geq y \right\}$,

- Not closed under scalar multiplication, e.g. $(-2)\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -6 \\ -4 \\ 0 \end{bmatrix}$ has $-6 \not\geq -4$.
- Answer: W is not a subspace of R^3

(c) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : z = 4 \right\}$.

- Zero vector is not in W $(0 \neq 4)$
- Answer: W is not a subspace of R^3

10. (a) Is the set of all polynomials $a_2t^2 + a_1t + a_0$ such that $a_0 + 3a_2 = 0$ a subspace of P_2?

- Zero polynomial $p(t) \equiv 0$ is in the set ($0 + (3)(0) \stackrel{\checkmark}{=} 0$).
- Take two arbitrary polynomials in the set: $p(t) = a_2t^2 + a_1t - 3a_2$ and $q(t) = b_2t^2 + b_1 - 3b_2$. The sum $p(t)+q(t) = (a_2+b_2)t^2+(a_1+b_1)t+(-3a_2-3b_2)$ satisfies $(-3a_2-3b_2)+3(a_2+b_2) = 0$ therefore it is in the set.

  The set is closed under vector addition.

- Take an arbitrary scalar c and an arbitrary polynomial in the set, $p(t) = a_2t^2 + a_1t - 3a_2$. The scalar multiple $cp(t) = ca_2t^2 + ca_1t + c(-3a_2)$ satisfies $c(-3a_2) + 3(ca_2) = 0$ therefore it is in the set.
  The set is closed under scalar multiplication.
- Answer: the set is a subspace of P_2.

(b) Is the set of all 2×2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ that are **symmetric** a subspace of M_{22}?

- Zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is symmetric, therefore it is in the set.
- Take two arbitrary matrices in the set: $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ and $B = \begin{bmatrix} a' & b' \\ b' & d' \end{bmatrix}$. The sum $A+B = \begin{bmatrix} a+a' & b+b' \\ b+b' & d+d' \end{bmatrix}$ is symmetric therefore it is in the set.
  The set is closed under vector addition.
- Take an arbitrary scalar k and an arbitrary matrix in the set $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$. The scalar multiple $kA = \begin{bmatrix} ka & kb \\ kb & kd \end{bmatrix}$ is symmetric, therefore it is in the set.
  The set is closed under scalar multiplication.
- Answer: the set is a subspace of M_{22}..

11. The equation $c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ corresponds to the linear system with the augmented matrix $\begin{bmatrix} 1 & 1 & 2 & x \\ 0 & 1 & -1 & y \\ 2 & 1 & 5 & z \end{bmatrix}$, after row operation $r_3 - 2r_1 \to r_3$ becomes $\begin{bmatrix} 1 & 1 & 2 & x \\ 0 & 1 & -1 & y \\ 0 & -1 & 1 & z-2x \end{bmatrix}$, then after $r_3 + r_2 \to r_3$ $\begin{bmatrix} 1 & 1 & 2 & x \\ 0 & 1 & -1 & y \\ 0 & 0 & 0 & z-2x+y \end{bmatrix}$

    The first two columns of the left hand side contain leading entries, and the third one does not - therefore, the first two of the original vectors, $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ form a basis for span S - its dimension is 2. Span S is a plane in R^3.

    A vector outside the span of S can be obtained by taking $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $z - 2x + y \neq 0$, e.g. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

12. (a) The set S is **linearly independent** if and only if the equation

$$c_1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

can **only** be solved by $c_1 = c_2 = c_3 = 0$. Otherwise (i.e. if other solutions exist) then S is linearly dependent.

We can rewrite the equation as a linear system with the augmented matrix:

$$\begin{bmatrix} 2 & 2 & 0 & 0 \\ 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad r_1 \leftrightarrow r_4 \quad \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 2 & 2 & 0 & 0 \end{bmatrix} \quad r_2 - r_1 \to r_2 \quad \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 2 & 2 & 0 & 0 \end{bmatrix}$$

$$r_4 - 2r_1 \to r_4 \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 0 \end{bmatrix} \quad r_2 \leftrightarrow r_3 \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & -2 & 0 \end{bmatrix}$$

$$r_3 + r_2 \to r_3 \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 \end{bmatrix} \quad r_4 - 2r_2 \to r_4 \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The third column of the r.r.e.f. does not contain a leading entry, therefore c_3 is arbitrary (does not have to be zero). Consequently, S is linearly dependent.

We can solve this system to express one vector as a linear combination of the rest:

$$\begin{aligned} c_1 &= -c_3 \\ c_2 &= c_3 \end{aligned}$$

Let c_3 be a nonzero number, e.g. $c_3 = 1$. Then

$$\begin{aligned} c_1 &= -1 \\ c_2 &= 1 \\ c_3 &= 1 \end{aligned}$$

leading to

$$(-1) \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} + (1) \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + (1) \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We can express the third vector as a linear combination of the other two:

$$\begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix} = (1) \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

(Check!)

(b) $c_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ can be rewritten as

$$\begin{bmatrix} c_1 + c_2 + c_3 & c_2 \\ c_3 & c_1 + c_2 + c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Two matrices are equal only if their corresponding elements are equal, which leads to the linear system of four equations:

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_2 \quad\;\; &= 0 \\ c_3 &= 0 \\ c_1 + c_2 + c_3 &= 0 \end{aligned}$$

The augmented matrix of this system is $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ $r_4 - r_1 \to r_4$ $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This is a r.e.f., in which every left-hand-side column contains a leading entry. Therefore, the system has only one solution (the trivial solution) $c_1 = c_2 = c_3 = 0$. Thus, S is linearly independent. (We cannot express one of its vectors as a linear combination of the rest.)

(c) $c_1\left(t^3+t+2\right)+c_2\left(t^3+t^2+t+3\right)+c_3\left(t^3-2t^2+t\right)=0$ can be rewritten as

$$(c_1+c_2+c_3)\,t^3+(c_2-2c_3)t^2+(c_1+c_2+c_3)t+(2c_1+3c_2)=0$$

Two polynomials are equal only if the coefficients associated with the corresponding powers on each side are equal, i.e.

$$\begin{aligned} c_1+c_2+c_3 &= 0 \\ c_2-2c_3 &= 0 \\ c_1+c_2+c_3 &= 0 \\ 2c_1+3c_2 \quad &= 0 \end{aligned}$$

This is a linear system with the augmented matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 3 & 0 & 0 \end{bmatrix} \quad r_3-r_1\to r_3 \quad \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \end{bmatrix} \quad r_4-2r_1\to r_4$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix} \quad r_4-r_2\to r_4 \quad \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is a r.e.f. in which the third column contains no leading entry, therefore c_3 can be arbitrary, e.g. $c_3=1$ leads to $c_2=2$ and $c_1=-2-1=-3$ (by backsubstitution). We can write

$$-3\left(t^3+t+2\right)+2\left(t^3+t^2+t+3\right)+1\left(t^3-2t^2+t\right)=0$$

so that we can express the third vector (polynomial) as a linear combination of the rest:

$$t^3-2t^2+t=3\left(t^3+t+2\right)-2\left(t^3+t^2+t+3\right)$$

(Check!)

For additional information on problems of this type, refer to the Linear Algebra Toolkit http://www.math.odu.edu/~bogacki/lat/ (module: Linear independence and dependence.)

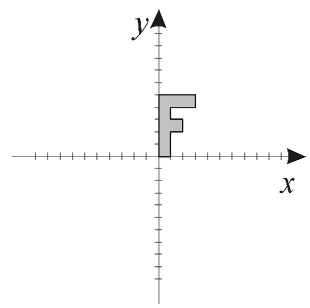
13. Follow the procedure of Example 4.32 on p.175

Append the columns of I_3 to the given vectors and set up the homogeneous system

$$c_1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}+c_2\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}+c_3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}+c_4\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}+c_5\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

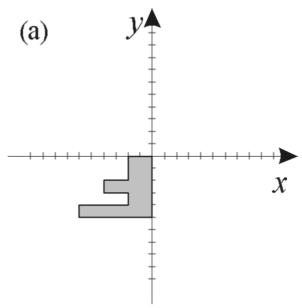
The augmented matrix $\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \end{bmatrix}$ has the r.r.e.f. $\begin{bmatrix} \boxed{1} & 0 & 0 & 1 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \boxed{1} & -1 & -\frac{1}{2} & 0 \end{bmatrix}$. The leading entries point to the columns of the original matrix that form a basis: $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

14. If the position vector $\overrightarrow{v}$ of each of the corner points of the letter "F" pictured here

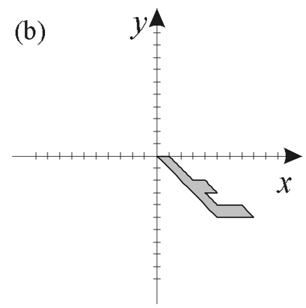


undergoes the linear transformation $F(\overrightarrow{v}) = A\overrightarrow{v}$, and the corresponding points are connected in the same way, write the matrix A that results in each transformed letter using $-2, -1, 0, 1$, or 2 as entries.

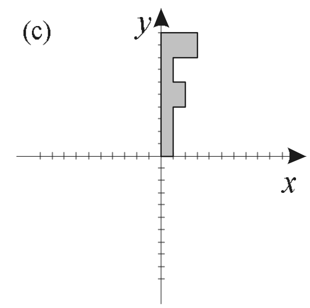
(a)



(b)



(c)



$\begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$ $\quad$ $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ $\quad$ $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

15. (a) $\det \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = 5$

$\det(A^T) = \det(A)$

(b) $\det \begin{bmatrix} a_{13} & a_{12} & a_{11} \\ a_{23} & a_{22} & a_{21} \\ a_{33} & a_{32} & a_{31} \end{bmatrix} = -5$

Interchanging two columns, the sign of the determinant changes.

(c) $\det(A^{-1})^2 = (\det(A^{-1}))^2 = (1/\det(A))^2 = (1/5)^2 = 1/25$

(d) $\det((-3)A) = (-3)^3 \det(A) = -27(5) = -135$

Calculating the scalar multiple $-3A$. is equivalent to multiplying each of its three rows by -3. Every time a row is multiplied by -3, the determinant is also multiplied by -3.

16. Expand along the second column:

$$-(-3)\det \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & -2 \\ -1 & 1 & 1 \end{bmatrix} + 2\det \begin{bmatrix} 2 & -1 & 1 \\ 3 & 1 & 1 \\ 1 & 2 & -2 \end{bmatrix}$$

$$= 3(6+2+1+2+6-1) + 2(-4-1+6-1-4-6)$$

$$= (3)(16) + (2)(-10) = 48 - 20 = 28$$