MATH 316 - Spring 2008 - Dr. Bogacki

TEST 1

SOLUTION KEY

1. Given the matrices $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 2 \end{pmatrix}, B = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & -2 \end{pmatrix}$, and $C = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$, evaluate each expression below, or state that the expression cannot be evaluated.

a. $B + C$ cannot be evaluated

b. $A^T + B = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & -2 \end{pmatrix} = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 4 & 0 \end{pmatrix}$

c. $AB = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 5 & 3 & -2 \\ 0 & 6 & -4 \end{pmatrix}$

d. $BA = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 9 & -1 \end{pmatrix}$

e. BC cannot be evaluated

2. Find **all** solutions of each linear system, using either Gaussian elimination with backsubstitution or Gauss-Jordan reduction:

a.
$$\begin{array}{rcrcrcrcr} x & - & 2y & + & z & & & = & 0 \\ -2x & + & y & + & z & + & 3w & = & 0 \\ & & 2y & - & 2z & + & w & = & 3 \end{array}$$

The augmented matrix $\begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ -2 & 1 & 1 & 3 & 0 \\ 0 & 2 & -2 & 1 & 3 \end{pmatrix}$, has the reduced row echelon form:

$\begin{pmatrix} 1 & 0 & -1 & 0 & 2 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$.

Infinitely many solutions:

$x = z + 2$

$y = z + 1$

$z - arbitrary$

$w = 1$

b.
$$\begin{array}{rcrcrcr} 2x & & & + & 4z & = & 2 \\ & & y & + & 3z & = & 0 \\ 3x & - & 2y & & & = & 1 \end{array}$$

The augmented matrix $\begin{pmatrix} 2 & 0 & 4 & 2 \\ 0 & 1 & 3 & 0 \\ 3 & -2 & 0 & 1 \end{pmatrix}$ has the reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since the last row corresponds to the equation 0=1, the system is inconsistent (has no solution).

3. Find the elementary matrices that perform the following row operations on a 3×5 matrix:

a. $-4r_2 \to r_2$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

b. $r_1 - 5r_2 \to r_1$.

$$\begin{pmatrix} 1 & -5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4. Evaluate $\det\begin{pmatrix} -1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 2 & 1 & -2 & 0 \\ 1 & 0 & 3 & 1 \end{pmatrix}$.

e.g., expand along the second row:

$$0+1\begin{vmatrix} -1 & 1 & 4 \\ 2 & -2 & 0 \\ 1 & 3 & 1 \end{vmatrix} + 0 + 2\begin{vmatrix} -1 & 0 & 1 \\ 2 & 1 & -2 \\ 1 & 0 & 3 \end{vmatrix} = 32 + 2(-4) = 24$$

5. If possible, find the inverse of each matrix:

a. $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$,

The matrix $\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{pmatrix}$ has the reduced row echelon form

$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -4 & 2 & -1 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{pmatrix}$. Therefore, the inverse is $\begin{pmatrix} 1 & 0 & 0 \\ -4 & 2 & -1 \\ 2 & -1 & 1 \end{pmatrix}$.

b. $\begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$.

The matrix $\begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{pmatrix}$ has the reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & -1 \end{pmatrix}.$$ Therefore, the original matrix is singular.

6. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be a matrix whose determinant is 2. Calculate the determinant of

a. $\det((A^T)^{-1}) = \frac{1}{2}$

b. $\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} - a_{11} & a_{22} - a_{12} & a_{23} - a_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = 2$

c. $\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = 0$

d. $\det(-4A) = (-4)^3(2) = -128$

7. Let A be a 3×3 matrix such that $A^{-1} = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

a. Find $\det(A)$.

$$\det(A^{-1}) = \begin{vmatrix} 2 & 0 & 3 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 4$$

$$\det(A) = \frac{1}{\det(A^{-1})} = \frac{1}{4}$$

b. $\operatorname{adj}(A) = \det(A)A^{-1} = \begin{pmatrix} 1/2 & 0 & 3/4 \\ 0 & 1 & 0 \\ 1/4 & 0 & 1/2 \end{pmatrix}$

c. Find a 3×1 solution matrix $\vec{x}$ to the linear system $A\vec{x} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$.

$$\vec{x} = A^{-1}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}.$$

8. Let $V = R^2$ with the operations defined by $\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix}$ and

$c\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cx \end{bmatrix}$. Determine whether the following properties hold:

a. $LHS = (c+d)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (c+d)x \\ (c+d)x \end{bmatrix}$

$RHS = (c\begin{bmatrix} x \\ y \end{bmatrix}) + (d\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} cx \\ cx \end{bmatrix} + \begin{bmatrix} dx \\ dx \end{bmatrix} = \begin{bmatrix} cx \\ cx \end{bmatrix}$

$LHS \neq RHS \Rightarrow$ Property does not hold

b. $LHS = c(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x' \\ y' \end{bmatrix}) = c\begin{bmatrix} x \\ x \end{bmatrix} = \begin{bmatrix} cx \\ cx \end{bmatrix}$

$RHS = (c\begin{bmatrix} x \\ y \end{bmatrix}) + (c\begin{bmatrix} x' \\ y' \end{bmatrix}) = \begin{bmatrix} cx \\ cx \end{bmatrix} + \begin{bmatrix} cx' \\ cx' \end{bmatrix} = \begin{bmatrix} cx \\ cx \end{bmatrix}$

$LHS = RHS \Rightarrow$ Property holds

9. **a**. Is the set of all polynomials $a_2t^2 + a_1t + a_0$ such that $a_2 > 0$ a subspace of P_2?

$p(t) = 0$ is not in the set

The set is not a subspace of P_2.

b. Is the set of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $b + c = 0$ a subspace of M_{22}?

(Note that $b + c = 0$ is equivalent to $c = -b$.)

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is in the set (since $0 + 0 = 0$)

Closed under addition since in

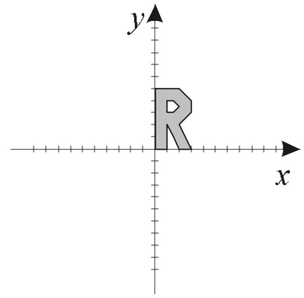
$\begin{pmatrix} a & b \\ -b & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ -b' & d' \end{pmatrix} = \begin{pmatrix} a+a' & b+b' \\ -b-b' & d+d' \end{pmatrix}$ we have $(b+b') + (-b-b') = 0$.

Closed under scalar multiplication since in

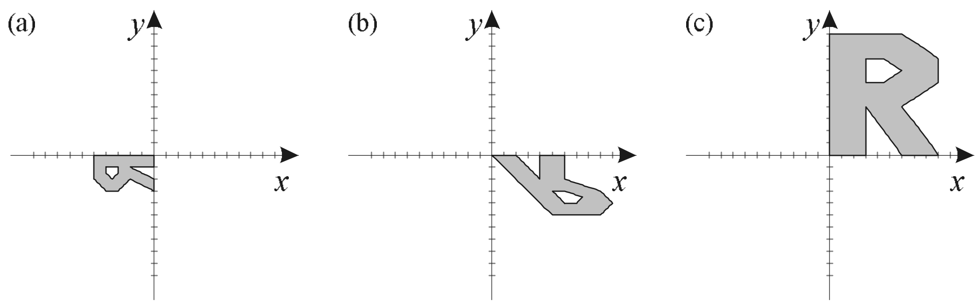
$k\begin{pmatrix} a & b \\ -b & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ -kb & kd \end{pmatrix}$ we have $kb + (-kb) = 0$.

The set is a subspace of M_{22}.

10. If the position vector $\vec{v}$ of each of the corner points of the letter "R" pictured here



undergoes the linear transformation $F(\vec{v}) = A\vec{v}$, and the corresponding points are connected in the same way, write the matrix A that results in each transformed letter using $-3, -2, -1, 0, 1, 2,$ or 3 as entries.



$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} \qquad A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

11. Show that the set $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$ is linearly dependent by expressing one of the vectors in the set as a linear combination of the remaining vectors.

$\begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ -1 & 1 & 1 & 0 & 0 \end{pmatrix}$ has the reduced row echelon form: $\begin{pmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix}$. Therefore c_4 is arbitrary. Letting $c_4 = 1$ we obtain a solution $c_1 = -2,\ c_2 = -3,\ c_3 = 1,\ c_4 = 1$ so that

$$-2\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

12. Check one box (TRUE or FALSE) for each of the following statements:

a. Every diagonal matrix is nonsingular.
FALSE (a zero square matrix is diagonal and singular)

b. For all nonsingular $n \times n$ matrices A and B, $(A^{-1}B^{-1})^T = (B^T A^T)^{-1}$.
FALSE
$(A^{-1}B^{-1})^T = (B^{-1})^T (A^{-1})^T = (B^T)^{-1}(A^T)^{-1} = (A^T B^T)^{-1}$

c. If $\det(A) = 0$ then the system $A\vec{x} = \vec{0}$ has nontrivial solutions.
TRUE (equivalent conditions)

d. If a square matrix A has a zero row then A is row equivalent to an identity matrix.
FALSE

e. The vectors $\begin{bmatrix} 2 \\ -2 \\ 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \end{bmatrix}$ are orthogonal.
TRUE

f. For all $n \times n$ matrices A and B, we have $(AB)^2 = A^2 B^2$.
FALSE

g. The set of all polynomials of degree exactly 2 (with the usual operations) is a vector space.
FALSE

h. If $\vec{u}, \vec{v}$, and $\vec{w}$ are vectors in R^5 such that $\vec{w} = 3\vec{u} + \vec{v}$ then the set $S = \{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent.
FALSE

i. A linearly independent set cannot contain a zero vector.
TRUE

j. For every square matrix A, $\det(A^2) = (\det(A))^2$.
TRUE

Score distribution:
100,98,95,94,94,92,91,91,90,89,89,89,88,87,86,86,81,80,
71,70,68,67,65,65,61,59,53,49,47,37,30,29,27,25,22,8