MATH 316 | SAMPLE TEST 2 | Spring 2008

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SAMPLE TEST 2

SOLUTION KEY

1. The equation $c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ corresponds to the linear system with the augmented matrix $\begin{bmatrix} 1 & 1 & 2 & x \\ 0 & 1 & -1 & y \\ 2 & 1 & 5 & z \end{bmatrix}$, after row operation $r_3 - 2r_1 \to r_3$ becomes $\begin{bmatrix} 1 & 1 & 2 & x \\ 0 & 1 & -1 & y \\ 0 & -1 & 1 & z-2x \end{bmatrix}$, then after $r_3 + r_2 \to r_3$ $\begin{bmatrix} 1 & 1 & 2 & x \\ 0 & 1 & -1 & y \\ 0 & 0 & 0 & z-2x+y \end{bmatrix}$

   The first two columns of the left hand side contain leading entries, and the third one does not - therefore, the first two of the original vectors, $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ form a basis for span S - its dimension is 2. Span S is a plane in R^3.

   A vector outside the span of S can be obtained by taking $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $z - 2x + y \neq 0$, e.g. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

2. Follow the procedure of Example 4.32 on p.175

   Append the columns of I_3 to the given vectors and set up the homogeneous system

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_5 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

   The augmented matrix $\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \end{bmatrix}$ has the r.r.e.f. $\begin{bmatrix} \boxed{1} & 0 & 0 & 1 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \boxed{1} & -1 & -\frac{1}{2} & 0 \end{bmatrix}$. The leading entries point to the columns of the original matrix that form a basis: $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

3. $A = \begin{bmatrix} 1 & 1 & 0 & 2 & 1 \\ 2 & 1 & 1 & 3 & 2 \\ 0 & 0 & 2 & 2 & 0 \\ -1 & 1 & -1 & 1 & -1 \end{bmatrix}$ .has r.r.e.f. $\begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 2 & 0 \\ 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(a) Find a basis for the null space of A.

The last two columns contain no leading entries, therefore, the corresponding unknowns, x_4 and x_5 are arbitrary.

We solve for x_1, x_2, and x_3 :

$x_1 + x_5 = 0$

$x_2 + 2x_4 = 0$

$x_3 + x_4 = 0$

In the vector form,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_5 \\ -2x_4 \\ -x_4 \\ x_4 \\ x_5 \end{bmatrix} = x_4 \begin{bmatrix} 0 \\ -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\{ \begin{bmatrix} 0 \\ -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \}$ is a basis for the null space of A.

(b) Find the nullity of A.

nullity of A is the dimension of the solution space of $A\overrightarrow{x} = \overrightarrow{0}$;

From part (a), nullity of A is 2.

(c) Find a basis for the row space of A.

The nonzero rows of the r.r.e.f. of A can be used as a basis for the row space of A :

$$\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \}$$

(d) Find a basis for the column space of A.

Leading entries of the r.r.e.f. of A point to the columns of A that can be used as a basis for the column space of A :

$$\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix} \}$$

(e) Find the rank of A.

The r.r.e.f. of A has three leading entries $\Rightarrow$ rank $A = 3$.

(f) Are the rows of A linearly independent?

If the rows of A were L.I., they would have spanned a subspace of R^5 with dimension 4. Since the dimension of the row space is 3 instead, the rows of A are linearly dependent.

Do they span R^5?

No set with fewer than 5 vectors can span a space of dimension 5.

4. Consider the set of vectors $S = \{\overrightarrow{u_1}, \overrightarrow{u_2}, \overrightarrow{u_3}\}$ where $\overrightarrow{u_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\overrightarrow{u_2} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\overrightarrow{u_3} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

(a) Show that the set S is a basis for R^3.

S is L.I. if and only if the homogeneous system $A\overrightarrow{x} = \overrightarrow{0}$ with $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ has nontrivial solutions.

r.r.e.f. of A is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ therefore $A\overrightarrow{x} = \overrightarrow{0}$ has the unique (trivial) solution $\Rightarrow S$ is L.I.

(b) Find $[\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}]_S$

We need to find the vector $[\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}]_S = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ such that

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Therefore, $[\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}]_S$ is the solution of the linear system with augmented matrix: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 \end{array}\right]$.

The r.r.e.f. is: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$, therefore $[\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}]_S = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

(Check: $(1) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (0) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ $\surd$)

5. Consider the basis $T = \left\{ \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \right\}$ for the vector space M_{22}.

Find the vector $\overrightarrow{v}$ such that $[\overrightarrow{v}]_T = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}$.

$$\overrightarrow{v} = (1) \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} + (-1) \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} + (0) \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + (2) \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -4 & 1 \end{bmatrix}$$

6. Decide if each of the following is a linear transformation:

(a) $F : R^2 \to R^3$ defined by $F(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} 0 \\ 1 \\ x_2 - x_1 \end{bmatrix}$.

Not a linear transformation, e.g. $F(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}) = F(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ does not equal

$$F(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) + F(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

(b) $G : M_{22} \to R$ defined by $G(A) = \det A$.

Not a linear transformation, e.g., $G(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}) = G(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = 1$ does not equal $G(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}) + G(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}) = 0 + 0 = 0$

(c) $H : R^2 \to P_1$ defined by $H(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = x_1 t$.

A linear transformation - satisfies both properties:

- $H(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}) = H(\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}) = (x_1 + y_1)t$

  $H(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) + H(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}) = x_1 t + y_1 t$ - equal to the expression above
- $H(k \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = H(\begin{bmatrix} kx_1 \\ kx_2 \end{bmatrix}) = kx_1 t$

  $kH(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = kx_1 t$

7. (a) Kernel of F is the set of all 2-vectors $\overrightarrow{u} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that $F(\overrightarrow{u}) = \overrightarrow{0}$, i.e., $\ker F$ is the solution space of the homogeneous system

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ 2x_1 + x_2 &= 0 \end{aligned}$$

Since the augmented matrix of the system $\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 1 & 0 \end{array}\right]$ has the r.r.e.f. $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$, the system has only the trivial solution $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Consequently, $\ker F$ has no basis.

The range of F is the space comprised of all images of F, i.e., the set of all vectors

$$\begin{bmatrix} x_1 + 2x_2 \\ 2x_1 + x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Since the vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ span range F and they are L.I. (see above), they form a basis for range F.

(b) Since $\ker F = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$, F is one-to-one.

Since $\dim$ range $F = 2 = \dim R^2$, F is onto.

(c) The matrix C such that $[F(\overrightarrow{x})]_S = C\,[\overrightarrow{x}]_S$ can be obtained from $[[F(\overrightarrow{v_1})]_S \,|\, [F(\overrightarrow{v_2})]_S]$

$F(\overrightarrow{v_1}) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$; $F(\overrightarrow{v_2}) = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$.

$\begin{bmatrix} 1 & 1 & -1 \\ -1 & -2 & 1 \end{bmatrix}$ has reduced row echelon form: $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow [F(\overrightarrow{v_1})]_S = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & -3 \\ -1 & -2 & 0 \end{bmatrix}$ has reduced row echelon form: $\begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 3 \end{bmatrix} \Rightarrow [F(\overrightarrow{v_2})]_S = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$

Consequently, $C = \begin{bmatrix} -1 & -6 \\ 0 & 3 \end{bmatrix}$.

(d) $\begin{bmatrix} 1 & 1 & -3 \\ -1 & -2 & 4 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix}$ therefore $\begin{bmatrix} -3 \\ 4 \end{bmatrix}_S = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$.

$[F(\overrightarrow{x})]_S = C\,[\overrightarrow{x}]_S = \begin{bmatrix} -1 & -6 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \end{bmatrix}$.

$$F(\overrightarrow{x}) = 8\begin{bmatrix} 1 \\ -1 \end{bmatrix} - 3\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}.$$

(e) $\begin{bmatrix} 5 \\ -2 \end{bmatrix}$

8. Consider the following orthogonal basis for R^5: $T = \{\underbrace{\begin{bmatrix} 3 \\ 0 \\ -4 \\ 0 \\ 0 \end{bmatrix}}_{\overrightarrow{v_1}}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}}_{\overrightarrow{v_2}}, \underbrace{\begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \\ -1 \end{bmatrix}}_{\overrightarrow{v_3}}, \underbrace{\begin{bmatrix} 0 \\ -2 \\ 0 \\ 0 \\ 2 \end{bmatrix}}_{\overrightarrow{v_4}}, \underbrace{\begin{bmatrix} 4 \\ 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}}_{\overrightarrow{v_5}}\}$.

(a) Transform T into an orthonormal basis S for R^5.

$$\overrightarrow{w_1} = \frac{1}{\|\overrightarrow{v_1}\|}\overrightarrow{v_1} = \frac{1}{\sqrt{9+16}}\begin{bmatrix} 3 \\ 0 \\ -4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 0 \\ -4/5 \\ 0 \\ 0 \end{bmatrix}$$

$$\overrightarrow{w_2} = \frac{1}{\|\overrightarrow{v_2}\|}\overrightarrow{v_2} = \frac{1}{\sqrt{1+1+1}}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$\overrightarrow{w_3} = \frac{1}{\|\overrightarrow{v_3}\|}\overrightarrow{v_3} = \frac{1}{\sqrt{1+4+1}}\begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/\sqrt{6} \\ 0 \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$$

$$\overrightarrow{w_4} = \frac{1}{\|\overrightarrow{v_4}\|}\overrightarrow{v_4} = \frac{1}{\sqrt{4+4}}\begin{bmatrix} 0 \\ -2 \\ 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$\overrightarrow{w_5} = \frac{1}{\|\overrightarrow{v_5}\|}\overrightarrow{v_5} = \frac{1}{\sqrt{16+9}}\begin{bmatrix} 4 \\ 0 \\ 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 0 \\ 3/5 \\ 0 \\ 0 \end{bmatrix}$$

(b) Given the vector $\overrightarrow{u} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, find $[\overrightarrow{u}]_S$.

Using Theorem 6.2,

$$[\overrightarrow{u}]_S = \begin{bmatrix} \overrightarrow{u}\cdot\overrightarrow{w_1} \\ \overrightarrow{u}\cdot\overrightarrow{w_2} \\ \overrightarrow{u}\cdot\overrightarrow{w_3} \\ \overrightarrow{u}\cdot\overrightarrow{w_4} \\ \overrightarrow{u}\cdot\overrightarrow{w_5} \end{bmatrix} = \begin{bmatrix} \frac{6}{5} \\ \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} \\ 0 \\ \frac{8}{5} \end{bmatrix}$$

9. Use the Gram-Schmidt process to find an orthogonal basis for the subspace of R^4 with a basis: $\{\begin{bmatrix}1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\2\\3\end{bmatrix}, \begin{bmatrix}0\\2\\1\\4\end{bmatrix}\}$.

$$\overrightarrow{u_1} = \begin{bmatrix}1\\1\\0\\1\end{bmatrix}, \overrightarrow{u_2} = \begin{bmatrix}1\\2\\2\\3\end{bmatrix}, \overrightarrow{u_3} = \begin{bmatrix}0\\2\\1\\4\end{bmatrix}$$

$$\overrightarrow{v_1} = \overrightarrow{u_1} = \begin{bmatrix}1\\1\\0\\1\end{bmatrix}$$

$$\overrightarrow{v_2} = \overrightarrow{u_2} - \frac{\overrightarrow{u_2} \cdot \overrightarrow{v_1}}{\overrightarrow{v_1} \cdot \overrightarrow{v_1}}\overrightarrow{v_1} = \begin{bmatrix}1\\2\\2\\3\end{bmatrix} - \frac{\begin{bmatrix}1\\2\\2\\3\end{bmatrix} \cdot \begin{bmatrix}1\\1\\0\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\0\\1\end{bmatrix} \cdot \begin{bmatrix}1\\1\\0\\1\end{bmatrix}} \begin{bmatrix}1\\1\\0\\1\end{bmatrix}$$

$$= \begin{bmatrix}1\\2\\2\\3\end{bmatrix} - \tfrac{6}{3}\begin{bmatrix}1\\1\\0\\1\end{bmatrix} = \begin{bmatrix}1\\2\\2\\3\end{bmatrix} - \begin{bmatrix}2\\2\\0\\2\end{bmatrix} = \begin{bmatrix}-1\\0\\2\\1\end{bmatrix}$$

$$\overrightarrow{v_3} = \overrightarrow{u_3} - \frac{\overrightarrow{u_3} \cdot \overrightarrow{v_1}}{\overrightarrow{v_1} \cdot \overrightarrow{v_1}}\overrightarrow{v_1} - \frac{\overrightarrow{u_3} \cdot \overrightarrow{v_2}}{\overrightarrow{v_2} \cdot \overrightarrow{v_2}}\overrightarrow{v_2} =$$

$$= \begin{bmatrix}0\\2\\1\\4\end{bmatrix} - \frac{\begin{bmatrix}0\\2\\1\\4\end{bmatrix} \cdot \begin{bmatrix}1\\1\\0\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\0\\1\end{bmatrix} \cdot \begin{bmatrix}1\\1\\0\\1\end{bmatrix}} \begin{bmatrix}1\\1\\0\\1\end{bmatrix} - \frac{\begin{bmatrix}0\\2\\1\\4\end{bmatrix} \cdot \begin{bmatrix}-1\\0\\2\\1\end{bmatrix}}{\begin{bmatrix}-1\\0\\2\\1\end{bmatrix} \cdot \begin{bmatrix}-1\\0\\2\\1\end{bmatrix}} \begin{bmatrix}-1\\0\\2\\1\end{bmatrix}$$

$$= \begin{bmatrix}0\\2\\1\\4\end{bmatrix} - \tfrac{6}{3}\begin{bmatrix}1\\1\\0\\1\end{bmatrix} - \tfrac{6}{6}\begin{bmatrix}-1\\0\\2\\1\end{bmatrix} = \begin{bmatrix}0\\2\\1\\4\end{bmatrix} - \begin{bmatrix}2\\2\\0\\2\end{bmatrix} - \begin{bmatrix}-1\\0\\2\\1\end{bmatrix} = \begin{bmatrix}-1\\0\\-1\\1\end{bmatrix}$$

(Check that $\overrightarrow{v_1} \cdot \overrightarrow{v_2} = \overrightarrow{v_1} \cdot \overrightarrow{v_3} = \overrightarrow{v_2} \cdot \overrightarrow{v_3} = 0$!)

10. Refer to Example 6.18.

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 2 & -3 \end{bmatrix} \text{ has the reduced row echelon form: } \begin{bmatrix} \boxed{1} & 0 & 2 & -3 \\ 0 & \boxed{1} & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Null space contains vectors of the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 + 3x_4 \\ x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$

A basis for the orthogonal complement is formed by the vectors $\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$

11. (a) Exercises 33-38, p.183

    33. FALSE
        Any vector space (except for $\{\vec{0}\}$) has infinitely many different bases.
    34. TRUE
        See Theorem 4.9.
    35. TRUE
        Dimension 1 means any basis for the space has one vector in it, which spans the space.
    36. TRUE
        A set containing a zero vector is linearly dependent, and cannot be a basis.
    37. TRUE
        If the set S is L.I. then it is a basis for span S, which is a subspace of V.
    38. FALSE
        For a set to span a vector space V, it must contain at least $\dim V$ vectors.

    (b) Exercises 6-14, p. 206.

    6. FALSE
       $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has rank 0
    7. TRUE
    8. TRUE
       If 5 columns span R^5, they must form a basis for R^5, and be L.I.
       By equivalent statements, the 5 rows must also be L.I., therefore they must be a basis for R^5, consequently, they must span R^5 as well.
    9. TRUE
       There are 7 columns in R^4 - no more than 4 vectors can be L.I. in R^4.
    10. TRUE
        Since rank$A$+nullity$A = 9$ then rank $A = 9 - 4 = 5$.
        By the equivalent statements, the columns are L.I., therefore they are a basis for R^5 so that they span R^5 as well.
    11. e.g., $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
    12. Such matrix cannot exist -
        rank+nullity=3, therefore, both must be no bigger than 3
    13. Such matrix cannot exist -
        rank+nullity=3, therefore, both must be no bigger than 3
    14. Such matrix cannot exist -
        Not enough rows (the rank cannot exceed 3)

(c) Exercises 21-24, p. 238.

21. FALSE
    If F is one-to-one then $\ker F = \{\overrightarrow{0}\}$ so that $\dim \ker F = 0$.
    Consequently, rank $F = 3 - 0 = 3 \neq 4$.
22. TRUE
    If F is onto R^3 then range $F = R^3$, making rank $F = \dim \operatorname{range} F = 3$.
23. TRUE
    nullity $F = 0 \Rightarrow F$ is one-to-one
    rank $F = \dim R^4 -$nullity $F = 4 - 0 = 4 \Rightarrow F$ is onto.
    Therefore, F is invertible.
24. FALSE
    e.g., $F(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} x \\ x \\ y \\ y \end{bmatrix}$ is one-to-one (ker $F = \{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\}$)

(d) Exercises 11-15, p. 248.

11. TRUE
    Zero vector is orthogonal to any vector in the same space.
12. FALSE
    One of the vectors in S could be $\overrightarrow{0}$.
13. FALSE
    If T has fewer than 7 vectors then it cannot be a basis for R^7.
14. TRUE
    If $A^{-1} = A^T$ then $(A^T)^{-1} = \left(A^{-1}\right)^T = \left(A^T\right)^T$ which means A^T is orthogonal.
15. TRUE
    (see the solution of the previous exercise).