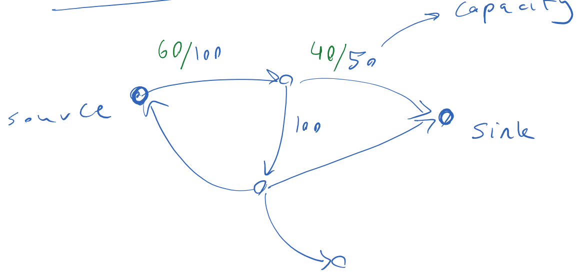
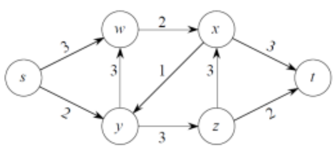
Maximum Flow



$G = (V, E)$

- edge (u,v) has capacity $c(u,v) \geq 0$
- if $(u,v) \in E$ then its reverse edge $(v,u) \notin E$
- source vertex : s
- sink vertex : t

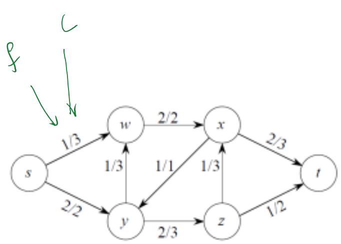


Flow function : $f : V \times V \longrightarrow \mathbb{R}$ that satisfies the following properties:

- Capacity Constraint : $\forall\, u, v \in V,\ 0 \leq f(u,v) \leq c(u,v)$
- Flow conservation : $\forall\ u \in V - \{s,t\},\ \underbrace{\sum_{v \in V} f(v,u)}_{\text{inflow of } u} = \underbrace{\sum_{v \in V} f(u,v)}_{\text{out-flow of } u}$

$\Downarrow$

$$\sum_{v \in V} f(u,v) - \sum_{v \in V} f(v,u) = 0$$



$\longrightarrow |f| = 3$

Value of flow f $= |f| = \underbrace{\sum_{v \in V} f(s,v)}_{\text{flow out of } s} - \underbrace{\sum_{v \in V} f(v,s)}_{\text{flow into } s}$

Maximum Flow Problem:

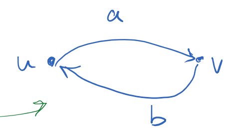
$G = (V, E)$

Source s, sink t

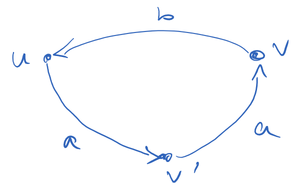
capacity c,

find a flow that has maximum value.

Workaround to antiparallel edges:



$\Downarrow$ work-around by adding new vertex v'



cut (S, T)

partition V into S, $T = V - S$ such that $s \in S$, $t \in T$

net flow of cut (S,T):

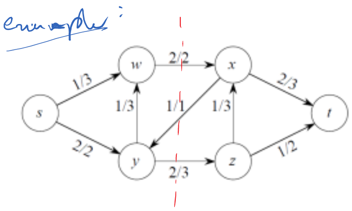
$$f(S,T) = \underbrace{\sum_{u \in S} \sum_{v \in T} f(u,v)}_{\text{flow from } S \text{ to } T} - \underbrace{\sum_{u \in S} \sum_{v \in T} f(v,u)}_{\text{flow from } T \text{ to } S}$$

Capacity of cut (S,T):

$$c(S,T) = \sum_{u \in S} \sum_{v \in T} c(u,v)$$

minimum cut: cut that its capacity is minimum in G.

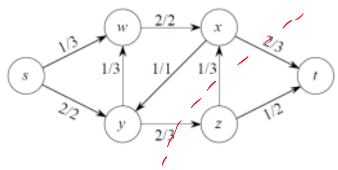
example:



$S = \{s, w, y\}$, $T = \{x, t, z\}$

$f(S,T) = 2 + 2 - 1 = 3$

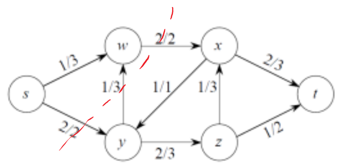
$c(S,T) = 2 + 3 = 5$



$S = \{s, w, x, y\}$, $T = \{t, z\}$

$f(S,T) = 2 + 2 - 1 = 3$

$c(S,T) = 3 + 3 = 6$



<u>Lemma:</u> For any cut (S,T), $f(S,T) = |f|$.

<u>Corollary:</u> value of any flow $\leq$ capacity of any cut.

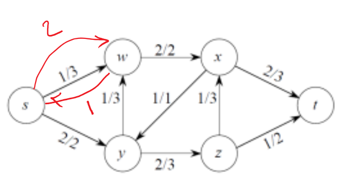
$\Rightarrow$ maximum flow $\leq$ capacity of the minimum cut

Ford-Fulkerson Method

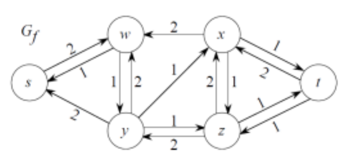
Residual Network: for any edge (u,v), how much more flow can we push through the edge? $\longrightarrow$ residual capacity

$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E \\ f(v,u) & \text{if } (v,u) \in E \\ 0 & \text{otherwise} \end{cases}$$

Residual Network $G_f = (V, E_f)$ $\quad E_f = \{ (u,v) \in V \times V : c_f(u,v) > 0 \}$



$\longrightarrow$ residual network



A flow in G_f (that satisfies the flow definition) can augment the original flow.

Given flow f in G and flow f' in G_f, $(f \uparrow f')$ is augmentation of f by f':

[illegible]

$$(f \uparrow f')(u,v) = \begin{cases} \underbrace{f(u,v)}_{\text{original flow}} + \underbrace{f'(u,v)}_{\text{residual flow that increases it}} - \underbrace{f'(v,u)}_{\text{cancelling original / additional residual flow}} & \text{if } (u,v) \in E \\ 0 & \text{otherwise} \end{cases}$$

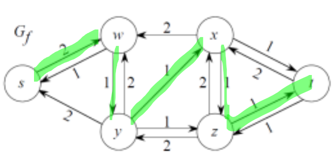
Lemma: G, flow f; G_f, flow f' $\Longrightarrow$ flow $f \uparrow f'$ is a flow in G with value $|f \uparrow f'| = |f| + |f'|$

Augmenting path:

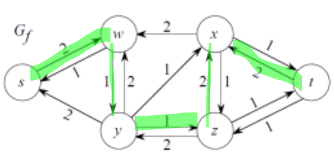
given simple path from s to t : $s \rightsquigarrow t$ in G_f

residual capacity of path:

$$c_f(p) = \min\{ c_f(u,v) : (u,v) \text{ is on path } p \}$$

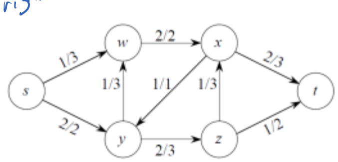


$c_f(p) = 1$

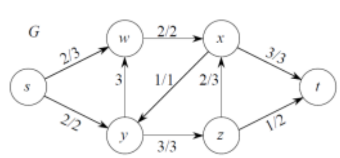


$c_f(p) = 1$

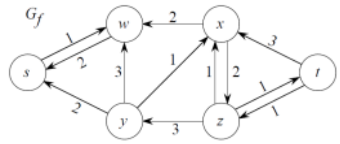
original G



augment based on $c_f(p)$



residual network



There is no more path in G_f to use for augmentation

Max-Flow Min-Cut Theorem

The followings are equivalents

- f is a maximum flow
- G_f has no augmenting path
- $|f| = c(S,T)$ for some cut (S,T)

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FORD-FULKERSON(G, s, t)
  for all (u, v) ∈ G.E
      (u, v).f = 0
  while there is an augmenting path p in G_f
      augment f by c_f(p)
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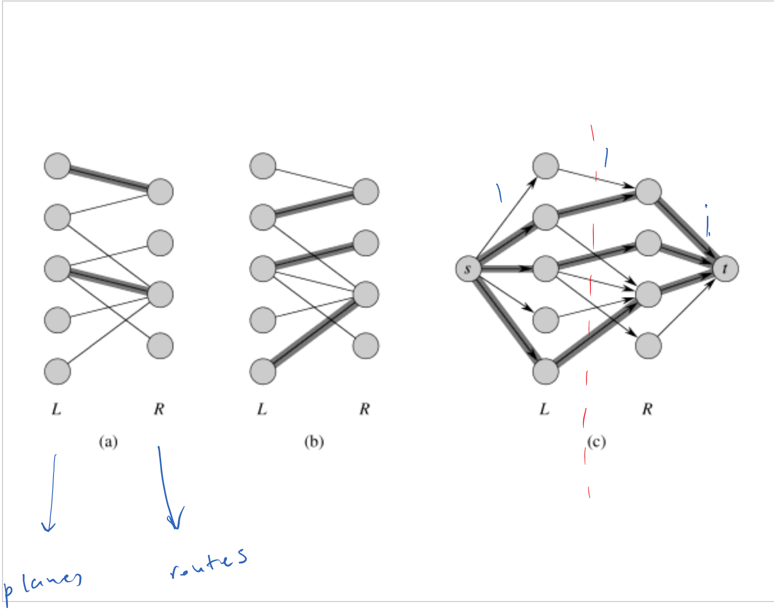
running time ≤ $O(E\,|f^*|)$

f^* is max flow

Maximum Bipartite matching



Original Bipartite Graphs G

flow Network: $G' = (V', E')$

$$\begin{cases} V' = V \cup \{s, t\} \\ E' = \{(s,u) : u \in L\} \cup E \cup \{(u,t) : u \in R\} \\ \forall u,v \in V', \quad c(u,v) = 1 \end{cases}$$