csi503-s21-lecture05

Tuesday, February 16, 2021

problem: hiring assistant

$\begin{cases} c_i: \text{cost of interview} \\ c_h: \text{cost of hiring} \end{cases}$ $\quad c_h > c_i$

commited to have hired the best candidate

Q: what's the expected cost of this strategy

HIRE-ASSISTANT(n)

```
best = 0          // candidate 0 is a least-qualified dummy candidate
for i = 1 to n
    interview candidate i        ⟶ c_i
    if candidate i is better than candidate best
        best = i
        hire candidate i         ⟶ c_h
```

we interview n candidates, we hire m candidates

Cost: $O(n c_i + m c_h) \xrightarrow[c_h > c_i]{} O(m c_h)$

worst case: hire all n candidates $\longrightarrow O(n c_h)$

Q: how about on average?

worst case: max cost of any input

avg case: expected cost over all inputs

$1, \ldots, i, \ldots, n$

assign a rank to candidate i : rank(i)

$\langle \text{rank}(1), \text{rank}(2), \ldots, \text{rank}(n) \rangle$

$\Longrightarrow$ $n!$ permutations, each is equally likely

Indicator Random Variable

flip a coin $\to$ H, T

$$I\{A\} = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{if } A \text{ does not occur} \end{cases}$$

$$X_A = I\{A\} \xrightarrow{\text{expectation}} E[X_A] = Pr\{A\}$$

expected value of indicator random var = its probability

in single coin flip:

$Pr\{H\} = Pr\{T\} = 1/2$

$E(X_H) = 1/2$

flipping coin several times

X: random var for number of heads in n flips

$E[X] = \sum_{k=0}^{n} k \cdot Pr(X = k)$ ← def. of expected value

$i = 1, \dots, n$, $X_i = I\{\text{ith flip is head}\}$

$$X = \sum_{i=1}^{n} X_i$$

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right]$$

linearity of expected values

$$= \sum_{i=1}^{n} E[X_i]$$

$$= \sum_{i=1}^{n} 1/2 = n/2$$

analysis of hiring problem

X: number of times we hire (m)

$X_i = I\{\text{candidate } i \text{ is hired}\}$

$X = \sum_{i=1}^{n} X_i$

candidate i is hired $\iff$ candidate i is better than $1 \dots i-1$

assumption: candidates arrive in random $\Rightarrow$ any one of first i cand. can be best

$Pr\{\text{candidate } i \text{ be best so far}\} = 1/i \Longrightarrow E[X_i] = 1/i$

$$E[X] = E\left[\sum X_i\right]$$

$$= \sum E[X_i] = \sum_{i=1}^{n} 1/i = \ln n + O(1)$$

$\Longrightarrow$ expected hiring cost $= O(m c_h) = \boxed{O(c_h \ln n)}$

Randomized Algorithms

- don't know input distribution → probabilistic analysis
- might not be able to model it not possible

⟶ impose your distribution of choice

→ impose your distribution of choice

Original hiring problems { - deterministic algo. : time is deterministic

input A $\xrightarrow[\text{always}]{}$ output B

m hires

↗ order of candidates

worst case : ranking ⟨1, 2, 3 .. , 5⟩ → n hires

best case : first person is best → 1 hire

→ all other cases in between

Let's make it randomized :

uniformly randomize order of candidates (uniform permutation)

Random Number Generator (RNG)

property of randomized case } no deterministic best/worst case anymore

RANDOMIZED-HIRE-ASSISTANT(n)
randomly permute the list of candidates
HIRE-ASSISTANT(n)

→ expected cost $O(c_h \ln n)$

→ additional step : cost ? $O(n)$

Goal: produce uniform random permutation

(n inputs → $n!$ permutations)

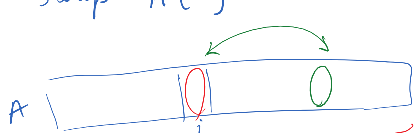
each must be equally likely

Randomize-in-place (A, n)

for $i = 1$ to n

swap $A[i]$ with $A[\text{RANDOM}(i, n)]$

RNG



swap with one of these

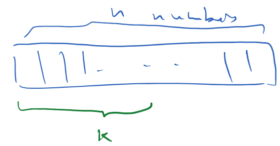
observation { $O(n)$

will not change $A[i]$ after ith iteration

Correctness criteria

n elements, k-permutation: sequence of k of the n elements

How many k-permutations? $n!/(n-k)!$



Loop invariant: just prior to ith iteration, for each $(i-1)$-permutation, $A[1 \ldots i-1]$ contains that with probability $\frac{(n-i+1)!}{n!}$

initialization: just before $i=1$: $\underbrace{\text{0-permutation}}_{\text{for each possible}}$, $A[1..0]$ contain that with prob. $\frac{n!}{n!} = 1$

maintenance: Given, assume each $(i-1)$-permutation ———— (loop invariant)

Show after ith iteration, each i-permutation appears in $A[1..i]$ with prob. $\frac{(n-i)!}{n!}$

Consider particular i-permutation: $\pi = \langle \underbrace{x_1, x_2, \ldots,} x_i \rangle$

$(i-1)$-permutation $\pi' = \langle x_1, x_2, \ldots x_{i-1} \rangle$

E_1: π' will be in $A[1..i-1]$ $\longrightarrow P\{E_1\} = \frac{(n-i+1)!}{n!}$

E_2: x_i will be in $A[i]$

to get 0-permutation π in $A[1..i]$ $\Longleftrightarrow$ E_1 and E_2

↳ Probability of producing π $\Pr\{E_1 \cap E_2\}$

conditional prob. → $\Pr\{E_2 \cap E_1\} = \Pr\{E_2 | E_1\} \Pr\{E_1\}$

$$= \frac{1}{n-(i-1)} \cdot \frac{(n-i+1)!}{n!}$$

$$= \frac{(n-i)!}{n!}$$

termination: $i = n+1 \longrightarrow \frac{(n-(n+1)+1)!}{n!} = \frac{1}{n!}$ ∎