Prof T. Mori's Paper

From Heisenberg's equation of motion -

$$i\hbar \frac{d}{dt}\hat{A} = [\hat{A}, H]$$

So, $\boxed{\frac{d}{dt}\hat{A} = -[H, \hat{A}]}$ $\boxed{\text{Neglecting } i \text{ \& } \hbar}$

In a similar manner, the averge of operators demands the equation -

$$\boxed{\frac{d}{dt}\langle Q\rangle = -\langle [H, Q]\rangle}$$

We will use these two above equation to calculate the time evolution of the system.

Mori's equation (9) Page (4) $\boxed{H_{MF} = -\langle\sigma^z\rangle\sigma^z - h_z\sigma^z - h_x\sigma^x}$

1. $$\frac{d}{dt}\langle\sigma^x\rangle_{MF} = -\langle[H_{MF}, \sigma^x]\rangle$$

$$= -\langle(-\langle\sigma^z\rangle\sigma^z - h_z\sigma^z - h_x\sigma^x), \sigma^x\rangle$$

$$= -\langle -\langle\sigma^z\rangle\sigma^z\sigma^x + \sigma^x\langle\sigma^z\rangle\sigma^z - h_z\sigma^z\sigma^x + h_z\sigma^x\sigma^z - h_x\sigma^x\sigma^x + h_x\sigma^x\sigma^x\rangle$$

$$= -\langle -\langle\sigma^z\rangle\sigma^y - \langle\sigma^z\rangle\sigma^y - h_z\sigma^y - h_z\sigma^y\rangle$$

$$= \langle 2\langle\sigma^z\rangle\sigma^y + 2h_z\sigma^y\rangle$$

$$\boxed{\frac{d}{dt}\langle\sigma^x\rangle_{MF} = 2\langle\sigma^z\rangle_{MF}\langle\sigma^y\rangle_{MF} + 2h_z\langle\sigma^y\rangle_{MF}} \quad (9a)$$

2. $$\frac{d}{dt}\langle\sigma^y\rangle_{MF} = -\langle[H, \sigma^y]\rangle$$

$$= -\langle -\langle\sigma^z\rangle\sigma^z\sigma^y + \sigma^y\langle\sigma^z\rangle\sigma^z - h_z\sigma^z\sigma^y + h_z\sigma^y\sigma^z - h_x\sigma^x\sigma^y + h_x\sigma^y\sigma^x\rangle$$

$$= -\langle 2\langle\sigma^z\rangle\sigma^x + 2h_z\sigma^x - 2h_x\sigma^y\rangle$$

$$\frac{d}{dt}\langle\sigma^y\rangle_{MF} = -2\langle\sigma^z\rangle_{MF}\langle\sigma^x\rangle_{MF} - 2h_z\langle\sigma^x\rangle_{MF} + 2h_x\langle\sigma^y\rangle_{MF} \quad (9b)$$

$$3. \quad \frac{d}{dt}\langle\sigma^y\rangle_{MF} = -\langle -\langle\sigma^z\rangle\sigma^z - h_z\sigma^z - h_x\sigma^x, \sigma^z\rangle$$

$$= -\langle -h_x\sigma^x\sigma^z + \sigma^z h_x\sigma^x\rangle$$

$$= -\langle +h_x\sigma^y + h_x\sigma^y\rangle$$

$$\therefore \boxed{\frac{d}{dt}\langle\sigma^y\rangle_{MF} = -2h_x\langle\sigma^y\rangle_{MF}} \qquad (9c)$$

Next come to "Time scale of initial stage of relaxation"

for mean field and $N \to \infty$ with time dependency

$$\langle S_i S_j\rangle = \langle S_i\rangle\langle S_j\rangle$$

Let proof equation (38) Page (11)

$$H = -J\sigma^z\sigma^z - h_z\sigma^z - h_x\sigma^x$$

$$\frac{d}{dt}\langle\sigma^x\rangle = -\langle[H, \sigma^x]\rangle$$

$$= -\langle[-J\sigma^z\sigma^z - h_z\sigma^z - h_x\sigma^x, \sigma^x]\rangle$$

$$= -\langle -J\sigma^z\sigma^z\sigma^x + J\sigma^x\sigma^z\sigma^z - h_z\sigma^z\sigma^x + h_z\sigma^x\sigma^z\rangle$$

$$= -\langle -J\sigma^z\sigma^y - J\sigma^y\sigma^z - 2h_z\sigma^y\rangle$$

$$= -\langle -J(\langle\sigma^z\rangle + \delta\sigma^z)(\langle\sigma^y\rangle + \delta\sigma^y) \cancel{-J(\langle\sigma^y\rangle + \delta\sigma^y)}$$

$$= -\langle -J(\langle\sigma^z\rangle + \delta\sigma^z)(\langle\sigma^y\rangle + \delta\sigma^y) - 2h_z\sigma^y$$
$$- J(\langle\sigma^y\rangle + \delta\sigma^y)(\langle\sigma^z\rangle + \delta\sigma^z)$$

$$= -\langle -J\langle\sigma^z\rangle\langle\sigma^y\rangle - J\langle\sigma^z\rangle\delta\sigma^y - J\delta\sigma^z\langle\sigma^y\rangle$$
$$- J\delta\sigma^z\delta\sigma^y - J\langle\sigma^y\rangle\langle\sigma^z\rangle - J\langle\sigma^y\rangle\delta\sigma^z$$
$$- J\delta\sigma^y\langle\sigma^z\rangle - J\delta\sigma^y\delta\sigma^z - 2h_z\sigma^y\rangle$$

$$= -\langle -2J\langle\sigma^z\rangle\langle\sigma^y\rangle - 2J\,\delta\sigma^y\delta\sigma^z - 2h_z\sigma^y\rangle$$

[Due to smallness of $\delta\sigma^y$ & $\delta\sigma^z$ we neglect the values]

$$= 2J\langle\sigma^z\rangle\langle\sigma^y\rangle + 2J\sum G^{yz} + 2h_z\langle\sigma^y\rangle$$

$$\text{--- } (38a)$$

$$\frac{d}{dt}\langle\sigma^y\rangle = -\langle[H,\sigma^y]\rangle$$

$$= -\langle -J\sigma^z\sigma^z\sigma^y + J\sigma^y\sigma^z\sigma^z - h_z\sigma^z\sigma^y + h_z\sigma^y\sigma^z - h_x\sigma^x\sigma^y + h_x\sigma^y\sigma^x\rangle$$

$$= -\langle + J\sigma^z\sigma^x + J\sigma^x\sigma^z + 2h_z\sigma^x + 2h_x\sigma^z\rangle$$

$$= -\langle J(\langle\sigma^z\rangle+\delta\sigma^z)(\langle\sigma^x\rangle+\delta\sigma^x) + J(\langle\sigma^x\rangle+\delta\sigma^x)(\langle\sigma^z\rangle+\delta\sigma^z) + 2h_z\sigma^x + 2h_x\sigma^z\rangle$$

$$= -\langle J\langle\sigma^z\rangle\langle\sigma^x\rangle + J\langle\sigma^z\rangle\delta\sigma^x + J\delta\sigma^z\langle\sigma^x\rangle + J\delta\sigma^z\delta\sigma^x + J\langle\sigma^x\rangle\langle\sigma^z\rangle + J\langle\sigma^x\rangle\delta\sigma^z + J\delta\sigma^x\langle\sigma^z\rangle + J\delta\sigma^x\delta\sigma^z + 2h_z\sigma^x + 2h_x\sigma^z\rangle$$

$$= -2J\langle\sigma^z\rangle\langle\sigma^x\rangle - 2h_z\langle\sigma^x\rangle - 2h_x\langle\sigma^z\rangle - 2J\,\Sigma\, G^{xz} \qquad (30b)$$

$$\frac{d}{dt}\langle\sigma^z\rangle = -\langle[H,\sigma^z]\rangle$$

$$= -\langle -J\sigma^z\sigma^z\sigma^z + J\sigma^z\sigma^z\sigma^z - h_z\sigma^z\sigma^z + h_z\sigma^z\sigma^z - h_x\sigma^x\sigma^z + h_x\sigma^z\sigma^x\rangle$$

$$= -2h_x\langle\sigma^y\rangle \qquad (30c)$$

Now $G^{ab} = \langle\delta\sigma^a\delta\sigma^b\rangle$

And $\sigma_i(t) = \langle\sigma\rangle_t + \delta\sigma_i(t)$

It is noted that $\langle\sigma_i\rangle$ is independent of i

and $G_K^{ab}(t) = \langle\delta\sigma_1^a\delta\sigma_K^b\rangle_t = \langle\delta\sigma_{i+1}^a\,\delta\sigma_{i+K}^b\rangle_t$

And initially $\langle\delta\sigma_i\rangle_t = \langle\delta\sigma_i(t)\rangle = 0$

$$\langle \sigma^x \rangle \, \delta\sigma^z = \langle \sigma^x \rangle (\sigma^z - \langle \sigma^z \rangle) = \langle \sigma^x \rangle \sigma^z - \langle \sigma^x \rangle \langle \sigma^z \rangle$$

$$\cancel{= \langle \sigma^x \rangle \langle \sigma^z \rangle - \langle} \quad \cancel{= \langle \sigma^x \rangle + \langle \sigma^x \rangle \langle \sigma}$$

$$\delta\sigma^z \langle \sigma^x \rangle = \sigma^z \langle \sigma^x \rangle - \langle \sigma^z \rangle \langle \sigma^x \rangle$$

So, $\underline{\langle \sigma^x \rangle \, \delta\sigma^z = - \, \delta\sigma^z \langle \sigma^x \rangle}$

$$\frac{d}{dt}\delta\sigma^x = -[H, \delta\sigma^x]$$

$$= -\left[-J\sigma^z\sigma^z - h_x\sigma^x - h_z\sigma^z, \delta\sigma^x\right]$$

$$= +J\sigma^z\sigma^z\delta\sigma^x - J\delta\sigma^x\sigma^z\sigma^z + h_x\sigma^x\delta\sigma^x - h_x\delta\sigma^x\sigma^x + h_z\sigma^z\delta\sigma^x - h_z\delta\sigma^x\sigma^z$$

$$= J(\langle\sigma^z\rangle + \delta\sigma^z)(\langle\sigma^z\rangle + \delta\sigma^z)\delta\sigma^x - J\delta\sigma^x(\langle\sigma^z\rangle + \delta\sigma^z)(\langle\sigma^z\rangle + \delta\sigma^z)$$
$$+ h_x(\langle\sigma^x\rangle + \delta\sigma^x)\delta\sigma^x - h_x\delta\sigma^x(\langle\sigma^x\rangle + \delta\sigma^x)$$
$$+ h_z(\langle\sigma^z\rangle + \delta\sigma^z)\delta\sigma^x - h_z\delta\sigma^x(\langle\sigma^z\rangle + \delta\sigma^z)$$

$$= J\cancel{\langle\sigma^z\rangle\langle\sigma^z\rangle}\delta\sigma^x + J\langle\sigma^z\rangle\delta\sigma^z\delta\sigma^x + J\delta\sigma^z\langle\sigma^z\rangle\delta\sigma^x - J\delta\sigma^x\cancel{\langle\sigma^z\rangle\langle\sigma^z\rangle}$$
$$+ J\delta\sigma^z\delta\sigma^z\delta\sigma^x - J\delta\sigma^x\langle\sigma^z\rangle\delta\sigma^z - J\delta\sigma^x\delta\sigma^z\delta\sigma^z$$
$$- J\delta\sigma^x\delta\sigma^z\langle\sigma^z\rangle$$
$$+ h_x\langle\sigma^x\rangle\delta\sigma^x - h_x\delta\sigma^x\langle\sigma^x\rangle + h_z\cancel{\langle\sigma^z\rangle}\delta\sigma^x + h_z\delta\sigma^z\delta\sigma^x$$
$$- h_z\cancel{\delta\sigma^x\langle\sigma^z\rangle} - h_z\delta\sigma^x\delta\sigma^z$$

$$2J\langle\sigma^z\rangle\delta\sigma^y + 2J\delta\sigma^z\delta\sigma^y + J\delta\sigma^z\langle\sigma^z\rangle\delta\sigma^x - J\delta\sigma^x\langle\sigma^z\rangle\delta\sigma^z + 2h_z\delta\sigma^y$$

$$= 2J\langle\sigma^z\rangle\delta\sigma^y + 2J\delta\sigma^z\delta\sigma^y + 2h_z\delta\sigma^y + J\delta\sigma^z\langle\sigma^z\rangle(\sigma^x - \langle\sigma^x\rangle) - J(\sigma^x - \langle\sigma^x\rangle)\langle\sigma^z\rangle\delta\sigma^z$$

$$= 2J\langle\sigma^z\rangle\delta\sigma^y + 2J\delta\sigma^z\delta\sigma^y + 2h_z\delta\sigma^y + J\delta\sigma^z\langle\sigma^z\rangle\sigma^x - J\delta\sigma^z\langle\sigma^z\rangle\langle\sigma^x\rangle - J\sigma^x\langle\sigma^z\rangle\delta\sigma^z + J\langle\sigma^x\rangle\langle\sigma^z\rangle\delta\sigma^z$$

$$\cancel{= 2J\langle\sigma^z\rangle\delta\sigma^y + 2J\delta\sigma^z\delta\sigma^y + 2h_z\delta\sigma^y - 2J\delta\sigma^z\langle\sigma^y\rangle}$$
$$\cancel{= 2J\langle\sigma^z\rangle\delta\sigma^y + 2h_z\delta\sigma^y + 2[JG^{zy} - 2\langle\sigma^y\rangle[J\delta\sigma^z}$$
$$\cancel{= 2\langle\sigma^z\rangle\delta\sigma^y}$$

31a

$$= 2J\langle\sigma^z\rangle\delta\sigma^y + 2J\delta\sigma^z\delta\sigma^y + 2h_z\delta\sigma^y + \cancel{J\langle\sigma^x\rangle\langle\sigma^z\rangle\delta} + J\langle\sigma^x\rangle\langle\sigma^z\rangle\delta\sigma^z + J\delta\sigma^x\langle\sigma^z\rangle\delta\sigma^z$$

$$= 2\langle\sigma^z\rangle\delta\sigma^y + 2h_z\delta\sigma^y + 2[JG^{zy} + 2\langle\sigma^y\rangle[J\delta\sigma^z$$

31a

$$\frac{d}{dt}\delta\sigma^y = -\left[-J\sigma^z\sigma^z - h_x\sigma^x - h_z\sigma^z, \delta\sigma^y\right]$$

$$= J\sigma^z\sigma^z\delta\sigma^y - J\delta\sigma^y\sigma^z\sigma^z + h_x\sigma^x\delta\sigma^y - h_x\delta\sigma^y\sigma^x$$
$$+ h_z\sigma^z\delta\sigma^y - h_z\delta\sigma^y\sigma^z$$

$$= J(\langle\sigma^z\rangle + \delta\sigma^z)(\langle\sigma^z\rangle + \delta\sigma^z)\delta\sigma^y - J\delta\sigma^y(\langle\sigma^z\rangle + \delta\sigma^z)(\langle\sigma^z\rangle + \delta\sigma^z)$$
$$+ h_x(\langle\sigma^x\rangle + \delta\sigma^x)\delta\sigma^y - h_x\delta\sigma^y(\langle\sigma^x\rangle + \delta\sigma^x)$$
$$+ h_z(\langle\sigma^z\rangle + \delta\sigma^z)\delta\sigma^y - h_z\delta\sigma^y(\langle\sigma^z\rangle + \delta\sigma^z)$$

$$= J\langle\sigma^z\rangle\cancel{\langle\sigma^z\rangle}\delta\sigma^y + J\delta\sigma^z\langle\sigma^z\rangle\delta\sigma^y + \boxed{\delta\sigma^z} - J\delta\sigma^y\cancel{\langle\sigma^z\rangle}\langle\sigma^z\rangle$$
$$+ J\langle\sigma^z\rangle\delta\sigma^z\delta\sigma^y + J\delta\sigma^z\delta\sigma^z\delta\sigma^y - J\delta\sigma^y\delta\sigma^z\langle\sigma^z\rangle - J\delta\sigma^y\delta\sigma^z\delta\sigma^z$$
$$- J\delta\sigma^y\langle\sigma^z\rangle\delta\sigma^z - J\delta\sigma^y\delta\sigma^z\langle\sigma^z\rangle - h_x\cancel{\delta\sigma^y}\langle\sigma^x\rangle - h_x\delta\sigma^y\delta\sigma^x$$
$$+ h_x\cancel{\langle\sigma^x\rangle}\delta\sigma^y + h_x\delta\sigma^x\delta\sigma^y - h_z\cancel{\delta\sigma^y}\langle\sigma^z\rangle - h_z\delta\sigma^y\delta\sigma^z$$
$$+ h_z\cancel{\langle\sigma^z\rangle}\delta\sigma^y + h_z\delta\sigma^z\delta\sigma^y$$

$$= J\delta\sigma^z\langle\sigma^z\rangle\delta\sigma^y - J\delta\sigma^y\langle\sigma^z\rangle\delta\sigma^z + 2J\langle\sigma^z\rangle\delta\sigma^x - 2J\delta\sigma^z\delta\sigma^x$$
$$+ 2h_x\delta\sigma^z - 2h_z\delta\sigma^x$$

$$= -2J\langle\sigma^z\rangle\delta\sigma^x - 2J\delta\sigma^z\delta\sigma^x - 2h_z\delta\sigma^x + 2h_x\delta\sigma^z - J(\sigma^y - \langle\sigma^y\rangle)\langle\sigma^z\rangle\delta\sigma^z$$
$$J\delta\sigma^z\langle\sigma^z\rangle(\sigma^y - \langle\sigma^y\rangle)$$

$$= -2J\langle\sigma^z\rangle\delta\sigma^x - 2J\delta\sigma^z\delta\sigma^x - 2h_z\delta\sigma^x + 2h_x\delta\sigma^z$$
$$+ 2J\delta\sigma^z\langle\sigma^x\rangle$$

$$= -2\langle\sigma^z\rangle\delta\sigma^x - 2h_z\delta\sigma^x + 2h_x\delta\sigma^z$$
$$J\delta\sigma^z - 2\left[J G^{zx}\right.$$
$$+2\langle\sigma^x\rangle\left[\right.$$

$$\frac{d}{dt}\delta\sigma^z = -[H, \delta\sigma^z]$$

$$= -[-J\sigma^z\sigma^z - h_x\sigma^x - h_z\sigma^z, \delta\sigma^z]$$

$$= J\sigma^z\sigma^z\delta\sigma^z - J\delta\sigma^z\sigma^z\sigma^z + h_x\sigma^x\delta\sigma^z - h_x\delta\sigma^z\sigma^x$$
$$+ h_z\sigma^z\delta\sigma^z - h_z\delta\sigma^z\sigma^z$$

$$= -2h_x\delta\sigma^y \quad \text{—— } (3/c)$$

In the form $\to$ $\boxed{\frac{d}{dt}\delta\sigma = W\delta\sigma + \Delta_i + \xi_i}$

$$W = \begin{pmatrix} 0 & 2(\langle\sigma^z\rangle + h_z) & 0 \\ -2(\langle\sigma^z\rangle + h_z) & 0 & 2h_x \\ 0 & -2h_x & 0 \end{pmatrix}$$

$$\Delta_i = 2\sum J\,\delta\sigma^z \begin{pmatrix} +\langle\sigma^y\rangle \\ -\langle\sigma^x\rangle \\ 0 \end{pmatrix}$$

where $\Delta_i = \sum J\,\delta\sigma^z\,\vec{v}$

$$\frac{d}{dt} G^{ab}_k(t) = \left\langle \frac{d\delta\sigma^a_i(t)}{dt}\, \delta\sigma^b_j(t) \right\rangle + \left\langle \delta\sigma^a_i(t)\, \frac{d\delta\sigma^b_j(t)}{dt} \right\rangle$$

$$\frac{d}{dt} G^{xz} = \left\langle \left(2\langle\sigma^z\rangle\delta\sigma^y + 2h_z\delta\sigma^y - 2\langle\sigma^y\rangle\sum J\,\delta\sigma^z + 2\sum J\,\delta\sigma^y\delta\sigma^z\right)\delta\sigma^z\right\rangle + \left\langle \delta\sigma^x\left(-2h_x\delta\sigma^y\right)\right\rangle$$

$$= \Big\langle \underbrace{2\langle\sigma^z\rangle\delta\sigma^y\delta\sigma^z + 2h_z\delta\sigma^y\delta\sigma^z}_{(\mathrm{I})} - \underbrace{2\langle\sigma^y\rangle\sum J\,\delta\sigma^z\delta\sigma^z}_{(\mathrm{III})} + 2\sum J\,\delta\sigma^y\delta\sigma^z\delta\sigma^z\Big\rangle - \underbrace{\langle 2h_x\,\delta\sigma^x\delta\sigma^y\rangle}_{(\mathrm{II})}$$

$$\frac{d}{dt} G^{xz}_k = \cancelto{0}{W_{xx}} G^{xz} + W_{xy}G^{yz} + \cancelto{0}{W_{xz}} G^{zz} + \cancelto{0}{W_{zx}} G^{xx} + W_{zy}G^{xy} + W_{zz}G^{xz} + 2\langle\sigma^y\rangle\sum J\,G^{zz} + J_{1k}\, 2\langle\sigma^y\rangle f_z$$

$$= \underbrace{2(\langle\sigma^z\rangle + h_z)\langle\delta\sigma^y\delta\sigma^z\rangle}_{(\mathrm{I})} + \underbrace{2h_x\langle\delta\sigma^x\delta\sigma^y\rangle}_{(\mathrm{II})} - \underbrace{2\langle\sigma^y\rangle\sum J\,G^{zz}}_{(\mathrm{III})} - J\,2\langle\sigma^y\rangle f_z$$

$$W = \begin{pmatrix} xx & xy & xz \\ yx & yy & yz \\ zx & zy & zz \end{pmatrix}$$

So,

$$2\sum J\,\delta\sigma^y\delta\sigma^z\delta\sigma^z = -2J\langle\sigma^y\rangle f_z$$

$$(\sigma^y - \langle\sigma^y\rangle)(\sigma^z - \langle\sigma^z\rangle)(\sigma^z - \langle\sigma^z\rangle) = -\langle\sigma^y\rangle f_z$$

$$(\sigma^y - \langle\sigma^y\rangle)(1 - \langle\sigma^z\rangle^2) = -f_z\langle\sigma^y\rangle$$

$$\boxed{f_z = 1 - \langle\sigma^z\rangle^2}$$

$\therefore$

$$\frac{d}{dt} G^{xy} = \left\langle \frac{d}{dt}(\delta\sigma^x)\,\delta\sigma^y\right\rangle + \left\langle \delta\sigma^x \frac{d}{dt}\delta\sigma^y\right\rangle$$

$$= \left\langle\left(2\langle\sigma^z\rangle\delta\sigma^y + 2h_x\delta\sigma^y - 2\langle\sigma^y\rangle\sum J\,\delta\sigma^z + 2\sum J\,\delta\sigma^y\delta\sigma^z\right)\delta\sigma^y\right\rangle$$

$$+ \left\langle \delta\sigma^x\left(-2\langle\sigma^z\rangle\delta\sigma^x - 2h_z\delta\sigma^x + 2h_x\delta\sigma^z + 2\langle\sigma^x\rangle\sum J\,\delta\sigma^z - 2\sum J\,\delta\sigma^x\delta\sigma^z\right)\right\rangle$$

$$= \Big\langle \underbrace{2\langle\sigma^z\rangle\delta\sigma^y\delta\sigma^y + 2h_x\delta\sigma^y\delta\sigma^y}_{(\mathrm{I})} - \underbrace{2\langle\sigma^y\rangle\sum J\,\delta\sigma^z\delta\sigma^y}_{(\mathrm{IV})} + 2\sum J\,\delta\sigma^y\delta\sigma^z\delta\sigma^y\Big\rangle$$

$$+ \Big\langle \underbrace{-2\langle\sigma^z\rangle\delta\sigma^x\delta\sigma^x - 2h_z\delta\sigma^x\delta\sigma^x}_{(\mathrm{II})} + \underbrace{2h_x\delta\sigma^x\delta\sigma^z}_{(\mathrm{III})} + \underbrace{2\langle\sigma^x\rangle\sum J\,\delta\sigma^x\delta\sigma^z}_{(\mathrm{V})} - 2\sum J\,\delta\sigma^x\delta\sigma^x\delta\sigma^z\Big\rangle$$

$$\frac{d}{dt} G^{xy} = \cancelto{0}{W_{xx}} G^{xy} + W_{xy}G^{yy} + \cancelto{0}{W_{xz}} G^{zy} + W_{yx}G^{xx} + \cancelto{0}{W_{yy}} G^{xy}$$

$$+ W_{yz}G^{xz} + 2\langle\sigma^y\rangle\sum J\,G^{zy} + 2\langle\sigma^x\rangle\sum J\,G^{xz}$$

$$+ J_{1k}\left(-2\langle\sigma^y\rangle f_y + 2\langle\sigma^x\rangle f_x\right)$$

$$= \underbrace{2(\langle\sigma^z\rangle + h_z)\langle\delta\sigma^y\delta\sigma^y\rangle}_{(\mathrm{I})} + \Big(\underbrace{-2(\langle\sigma^z\rangle + h_z)\langle\delta\sigma^x\delta\sigma^x\rangle}_{(\mathrm{II})} + \underbrace{2h_x\langle\delta\sigma^x\delta\sigma^z\rangle}_{(\mathrm{III})}$$

$$- \underbrace{2\langle\sigma^y\rangle\sum J\,\langle\delta\sigma^z\delta\sigma^y\rangle}_{(\mathrm{IV})} + \underbrace{2\langle\sigma^x\rangle\sum J\,\langle\delta\sigma^x\delta\sigma^z\rangle}_{(\mathrm{V})}$$

$$- 2J\left(\langle\sigma^y\rangle f_y + \langle\sigma^x\rangle f_x\right)$$

Now, $\langle \delta\sigma^y \delta\sigma^z \delta\sigma^y \rangle$

$$= \langle (\sigma^y - \langle\sigma^y\rangle)(\sigma^z - \langle\sigma^z\rangle)(\sigma^y - \langle\sigma^y\rangle)\rangle$$

$$= \langle \sigma^y\sigma^z\sigma^y - \sigma^y\sigma^z\langle\sigma^y\rangle - \sigma^y\langle\sigma^z\rangle\sigma^y + \sigma^y\langle\sigma^z\rangle\langle\sigma^y\rangle$$
$$- \langle\sigma^y\rangle\sigma^z\sigma^y + \langle\sigma^y\rangle\sigma^z\langle\sigma^y\rangle + \langle\sigma^y\rangle\langle\sigma^z\rangle\sigma^y - \langle\sigma^y\rangle\langle\sigma^z\rangle\langle\sigma^y\rangle\rangle$$

$$= \cancel{\langle\sigma^z\rangle} - \cancel{\langle\sigma^y\sigma^z\rangle}\langle\sigma^y\rangle - \cancel{\langle\sigma^z\rangle} + \langle\sigma^y\rangle\langle\sigma^z\rangle\langle\sigma^y\rangle - \cancel{\langle\sigma^y\rangle\langle\sigma^z\sigma^y\rangle}$$
$$+ \langle\sigma^y\rangle\langle\sigma^z\rangle\langle\sigma^y\rangle + \langle\sigma^y\rangle\cancel{\langle\sigma^z\rangle}\langle\sigma^y\rangle - \cancel{\langle\sigma^y\sigma^z\rangle}\langle\sigma^y\rangle$$

$\boxed{\text{use } \langle a\rangle\langle bc\rangle = -\langle bc\rangle\langle a\rangle}$

$$= 2\langle\sigma^y\rangle\langle\sigma^z\rangle\langle\sigma^y\rangle$$

And, $-\langle\delta\sigma^x\delta\sigma^x\delta\sigma^z\rangle$

$$= -\langle(\sigma^x-\langle\sigma^x\rangle)(\sigma^x-\langle\sigma^x\rangle)(\sigma^z-\langle\sigma^z\rangle)\rangle$$

$$= -\langle \sigma^x\sigma^x\sigma^z - \sigma^x\sigma^x\langle\sigma^z\rangle - \sigma^x\langle\sigma^x\rangle\sigma^z + \sigma^x\langle\sigma^x\rangle\langle\sigma^z\rangle - \langle\sigma^x\rangle\sigma^x\sigma^z$$
$$+ \langle\sigma^x\rangle\sigma^x\langle\sigma^z\rangle + \langle\sigma^x\rangle\langle\sigma^x\rangle\sigma^z - \langle\sigma^x\rangle\langle\sigma^x\rangle\langle\sigma^z\rangle\rangle$$

$$= -[\cancel{\langle\sigma^z\rangle} - \cancel{\langle\sigma^z\rangle} - \cancel{\langle\sigma^x\sigma^z\rangle}\langle\sigma^x\rangle + \langle\sigma^x\rangle\langle\sigma^x\rangle\langle\sigma^z\rangle - \langle\sigma^x\rangle\cancel{\langle\sigma^x\sigma^z\rangle}$$
$$+ \langle\sigma^x\rangle\langle\sigma^x\rangle\langle\sigma^z\rangle + \langle\sigma^x\rangle\cancel{\langle\sigma^x\rangle}\langle\sigma^z\rangle - \langle\sigma^x\rangle\cancel{\langle\sigma^x\rangle}\langle\sigma^z\rangle]$$

$$= -2\langle\sigma^x\rangle\langle\sigma^x\rangle\langle\sigma^z\rangle$$

Okay, we get now, considering $J=1$

$$-2\langle\sigma^y\rangle f_y + 2\langle\sigma^x\rangle f_x = 2\langle\sigma^y\rangle\langle\sigma^z\rangle\langle\sigma^y\rangle - 2\langle\sigma^x\rangle\langle\sigma^x\rangle\langle\sigma^z\rangle$$

So,
$$f_x = -\langle\sigma^x\rangle\langle\sigma^z\rangle$$
$$f_y = -\langle\sigma^y\rangle\langle\sigma^z\rangle.$$

Thus,
$$f = \begin{pmatrix} -\langle\sigma^x\rangle\langle\sigma^z\rangle \\ -\langle\sigma^y\rangle\langle\sigma^z\rangle \\ 1-\langle\sigma^z\rangle^2 \end{pmatrix}$$

This is for T. Mori's zxx model.