Calculation for XXZ model

We consider Hamiltonian as -

$$\mathcal{H} = -J\sum \sigma^x \sigma^x - h_x \sum \sigma^x - h_z \sum \sigma^z$$

in our transverse ~~Is~~ model we will set $h_x = 0$

But for sake of generalised calculation we keep it for now.

$$\frac{d}{dt}\langle \hat{A} \rangle = -\langle [H, \hat{A}] \rangle \quad , \quad \frac{d}{dt}\hat{A} = -[H, \hat{A}]$$

neglecting i & $\hbar$.

$$H_{MF} = -\langle \sigma^x \rangle \sigma^z - h_x \sigma^x - h_z \sigma^z$$

Equation for MF

1. $$\frac{d}{dt}\langle \sigma^x \rangle_{MF} = -\langle [(-\langle \sigma^x \rangle \sigma^x - h_x \sigma^x - h_z \sigma^z), \sigma^x] \rangle$$

$$= -\langle -\langle \sigma^x \rangle \sigma^x \sigma^x - h_x \sigma^x \sigma^x - h_z \sigma^z \sigma^x + \sigma^x \langle \sigma^x \rangle \sigma^x + h_x \sigma^x \sigma^x + h_z \sigma^x \sigma^z \rangle$$

$$= -\langle -\cancel{\langle \sigma^x \rangle} - \cancel{h_x} - h_y \sigma^y + \cancel{\langle \sigma^x \rangle} + \cancel{h_x} - h_z \sigma^y \rangle$$

$$\boxed{\frac{d}{dt}\langle \sigma^x \rangle_{MF} = 2 h_y \langle \sigma^y \rangle_{MF}}$$

2. $$\frac{d}{dt}\langle\sigma^y\rangle_{MF} = -\left\langle\left[\left(-\langle\sigma^x\rangle\sigma^x - h_x\sigma^x - h_z\sigma^z\right), \sigma^y\right]\right\rangle$$

$$= +\left\langle \langle\sigma^x\rangle\sigma^x\sigma^y - \sigma^y\langle\sigma^x\rangle\sigma^x + h_x\sigma^x\sigma^y - h_x\sigma^y\sigma^x + h_z\sigma^z\sigma^y - h_z\sigma^y\sigma^z \right\rangle$$

$$= \left\langle \langle\sigma^x\rangle\sigma^z + \langle\sigma^x\rangle\sigma^z + 2h_x\sigma^z - 2h_z\sigma^x \right\rangle$$

$$\frac{d}{dt}\langle\sigma^y\rangle_{MF} = 2\langle\sigma^x\rangle\langle\sigma^z\rangle + 2h_x\langle\sigma^z\rangle - 2h_z\langle\sigma^x\rangle$$ ✓

3. $$\frac{d}{dt}\langle\sigma^z\rangle_{MF} = -\left\langle\left[\left(-\langle\sigma^x\rangle\sigma^x - h_x\sigma^x - h_z\sigma^z\right), \sigma^z\right]\right\rangle$$

$$= \left\langle \langle\sigma^x\rangle\sigma^x\sigma^z - \sigma^z\langle\sigma^x\rangle\sigma^x + h_x\sigma^x\sigma^z - h_x\sigma^z\sigma^x + h_z\sigma^z\sigma^z - h_z\sigma^z\sigma^z \right\rangle$$

$$= \left\langle -\langle\sigma^x\rangle\sigma^y - \langle\sigma^x\rangle\sigma^y - h_x\sigma^y - h_x\sigma^y \right\rangle$$

$$\frac{d}{dt}\langle\sigma^z\rangle_{MF} = -2\langle\sigma^x\rangle\langle\sigma^y\rangle - 2h_x\langle\sigma^y\rangle$$ ✓

☒ Timescale of initial stage of relaxation.

$$\frac{d}{dt}\langle\sigma^x\rangle = -\langle[H, \sigma^x]\rangle$$

$$= -\left\langle\left[-J\sigma^x\sigma^x - h_x\sigma^x - h_z\sigma^z, \sigma^x\right]\right\rangle$$

$$= \left\langle J\sigma^x\sigma^x\sigma^x - J\sigma^x\sigma^x\sigma^x + h_x\sigma^x\sigma^x - h_x\sigma^x\sigma^x + h_z\sigma^z\sigma^x - h_z\sigma^x\sigma^z \right\rangle$$

$$\boxed{\frac{d}{dt}\langle\sigma^x\rangle = 2h_z\langle\sigma^y\rangle}$$ ✓

$$\frac{d}{dt}\langle\sigma^y\rangle = -\langle[(-J\sigma^x\sigma^x - h_x\sigma^x - h_z\sigma^z), \sigma^y]\rangle$$

$$= \langle J\sigma^x\sigma^x\sigma^y - J\sigma^y\sigma^x\sigma^x + h_x\sigma^x\sigma^y - h_x\sigma^y\sigma^x + h_z\sigma^z\sigma^y - h_z\sigma^y\sigma^z\rangle$$

$$= \langle J\sigma^x\sigma^z - J\sigma^z\sigma^x + 2h_x\sigma^z + 2h_z\sigma^x\rangle$$

$$= \langle J(\langle\sigma^x\rangle + \delta\sigma^x)(\langle\sigma^z\rangle + \delta\sigma^z) - J(\langle\sigma^z\rangle + \delta\sigma^z)(\langle\sigma^x\rangle + \delta\sigma^x) + 2h_x\sigma^z + 2h_z\sigma^x\rangle$$

$$= \langle J\langle\sigma^x\rangle\langle\sigma^z\rangle + J\cancel{\langle\sigma^x\rangle}\delta\sigma^z + J\delta\sigma^x\cancel{\langle\sigma^z\rangle} + J\delta\sigma^x\delta\sigma^z - J\langle\sigma^z\rangle\langle\sigma^x\rangle - J\cancel{\langle\sigma^z\rangle}\delta\sigma^x - J\delta\sigma^z\cancel{\langle\sigma^x\rangle} - J\delta\sigma^z\delta\sigma^x + 2h_x\sigma^z + 2h_z\sigma^x\rangle$$

$$= \langle 2J\langle\sigma^x\rangle\langle\sigma^z\rangle + 2h_x\sigma^z + 2h_z\sigma^x + \cancel{2\Sigma G}\; 2JG^{xz}\rangle$$

$$\frac{d}{dt}\langle\sigma^y\rangle = 2J\langle\sigma^x\rangle\langle\sigma^z\rangle + 2h_x\langle\sigma^z\rangle + 2h_z\langle\sigma^x\rangle + 2\sum JG^{xz}$$ ✓

$$\frac{d}{dt}\langle\sigma^z\rangle = -\langle[(-J\sigma^x\sigma^x - h_x\sigma^x - h_z\sigma^z), \sigma^z]\rangle$$

$$= \langle[J\sigma^x\sigma^x + h_x\sigma^x + h_z\sigma^z, \sigma^z]\rangle$$

$$= \langle J\sigma^x\sigma^x\sigma^z - J\sigma^z\sigma^x\sigma^x + h_x\sigma^x\sigma^z - h_x\sigma^z\sigma^x\rangle$$

$$\boxed{\cancel{= \langle -2J\sigma^x\sigma^y - 2h_x\sigma^y\rangle = -\langle 2J(\langle\sigma^x\rangle + \delta\sigma^x)(\langle\sigma^y\rangle + \delta\sigma^y) + 2h_x(\langle\sigma^y\rangle + \delta\sigma^y)\rangle = -\langle 2J(\langle\sigma^x\rangle\langle\sigma^y\rangle + \langle\sigma^x\rangle\delta\sigma^y + \delta\sigma^x\langle\sigma^y\rangle + \delta\sigma^x\delta\sigma^y) + 2h_x\delta\sigma^y\rangle}}$$

$$= +\langle -J\sigma^x\sigma^y + J\sigma^y\sigma^x + 2h_x\sigma^y\rangle$$

$$= +\langle -J(\langle\sigma^x\rangle + \delta\sigma^x)(\langle\sigma^y\rangle + \delta\sigma^y) + J(\langle\sigma^y\rangle + \delta\sigma^y)(\langle\sigma^x\rangle + \delta\sigma^x) + 2h_x\sigma^y\rangle$$

$$= +\langle -J(\langle\sigma^x\rangle\langle\sigma^y\rangle + \langle\sigma^x\rangle\delta\sigma^y + \delta\sigma^x\langle\sigma^y\rangle + \delta\sigma^x\delta\sigma^y) + J(\langle\sigma^y\rangle\langle\sigma^x\rangle + \langle\sigma^y\rangle\delta\sigma^x + \delta\sigma^y\langle\sigma^x\rangle + \delta\sigma^y\delta\sigma^x) + 2h_x\sigma^y\rangle$$

$$= -2J\langle\sigma^x\rangle\langle\sigma^y\rangle - 2\sum JG^{xy} - 2h_x\langle\sigma^y\rangle$$

Considering $\delta\sigma^x, \delta\sigma^y, \delta\sigma^z$ are two small to account.

$$\frac{d}{dt}\delta\sigma^x = -\left[\left(-J\sigma^x\sigma^x - h_x\sigma^x - h_z\sigma^z\right), \delta\sigma^x\right]$$

$$= J\sigma^x\sigma^x\delta\sigma^x - J\delta\sigma^x\sigma^x\sigma^x + h_x\sigma^x\delta\sigma^x - h_x\delta\sigma^x\sigma^x + h_z\sigma^z\delta\sigma^x - h_z\delta\sigma^x\sigma^z$$

$$= J(\langle\sigma^x\rangle + \delta\sigma^x)(\langle\sigma^x\rangle + \delta\sigma^x)\delta\sigma^x - J\delta\sigma^x(\langle\sigma^x\rangle + \delta\sigma^x)(\langle\sigma^x\rangle + \delta\sigma^x)$$
$$+ h_x(\langle\sigma^x\rangle + \delta\sigma^x)\delta\sigma^x - h_x\delta\sigma^x(\langle\sigma^x\rangle + \delta\sigma^x)$$
$$+ h_z(\langle\sigma^z\rangle + \delta\sigma^z)\delta\sigma^x - h_z\delta\sigma^x(\langle\sigma^z\rangle + \delta\sigma^z)$$

$$= J\langle\sigma^x\rangle\langle\sigma^x\rangle\delta\sigma^x + J\langle\sigma^x\rangle\delta\sigma^x\delta\sigma^x + J\delta\sigma^x\langle\sigma^x\rangle\delta\sigma^x$$
$$+ J\delta\sigma^x\delta\sigma^x\delta\sigma^x - J\delta\sigma^x\langle\sigma^x\rangle\langle\sigma^x\rangle - J\delta\sigma^x\langle\sigma^x\rangle\delta\sigma^x$$
$$- J\delta\sigma^x\delta\sigma^x\langle\sigma^x\rangle - J\delta\sigma^x\delta\sigma^x\delta\sigma^x + h_x\langle\sigma^x\rangle\delta\sigma^x - h_x\delta\sigma^x\langle\sigma^x\rangle$$
$$+ h_z\langle\sigma^z\rangle\delta\sigma^x + \underline{h_z\delta\sigma^z\delta\sigma^x} - h_z\delta\sigma^x\langle\sigma^z\rangle - \underline{h_z\delta\sigma^x\delta\sigma^z}$$

$$= +h_z\delta\sigma^y - h_z(-\delta\sigma^y)$$

$$\boxed{\frac{d}{dt}\delta\sigma^x = 2h_z\delta\sigma^y}$$

✓

$$\frac{d}{dt}\delta\sigma^y = -\left[\left(-J\sigma^x\sigma^x - h_x\sigma^x - h_z\sigma^z\right), \delta\sigma^y\right]$$

$$= J\sigma^x\sigma^x\delta\sigma^y - J\delta\sigma^y\sigma^x\sigma^x + h_x\sigma^x\delta\sigma^y - h_x\delta\sigma^y\sigma^x + h_z\sigma^z\delta\sigma^y - h_z\delta\sigma^y\sigma^z$$

$$= J(\langle\sigma^x\rangle + \delta\sigma^x)(\langle\sigma^x\rangle + \delta\sigma^x)\delta\sigma^y - J\delta\sigma^y(\langle\sigma^x\rangle + \delta\sigma^x)(\langle\sigma^x\rangle + \delta\sigma^x)$$
$$+ h_x(\langle\sigma^x\rangle + \delta\sigma^x)\delta\sigma^y - h_x\delta\sigma^y(\langle\sigma^x\rangle + \delta\sigma^x)$$
$$+ h_z(\langle\sigma^z\rangle + \delta\sigma^z)\delta\sigma^y - h_z\delta\sigma^y(\langle\sigma^z\rangle + \delta\sigma^z)$$

$$= J\langle\sigma^x\rangle\cancel{\langle\sigma^x\rangle}\delta\sigma^y + J\langle\sigma^x\rangle\delta\sigma^x\delta\sigma^y + J\delta\sigma^x\langle\sigma^x\rangle\delta\sigma^y$$
$$+ J\delta\sigma^x\delta\sigma^x\delta\sigma^y - J\delta\sigma^y\cancel{\langle\sigma^x\rangle}\langle\sigma^x\rangle - J\delta\sigma^y\langle\sigma^x\rangle\delta\sigma^x$$
$$- J\delta\sigma^y\delta\sigma^x\langle\sigma^x\rangle - J\delta\sigma^y\delta\sigma^x\delta\sigma^x + h_x\langle\sigma^x\rangle\delta\sigma^y$$
$$+ h_x\delta\sigma^x\delta\sigma^y - h_x\delta\sigma^y\langle\sigma^x\rangle - h_x\delta\sigma^y\delta\sigma^x + h_z\langle\sigma^z\rangle\delta\sigma^y$$
$$+ h_z\delta\sigma^z\delta\sigma^y - h_z\delta\sigma^y\langle\sigma^z\rangle - h_z\delta\sigma^y\delta\sigma^z$$

$$= 2J\langle\sigma^x\rangle\delta\sigma^z + 2J\delta\sigma^x\delta\sigma^z + J\delta\sigma^x\langle\sigma^x\rangle\delta\sigma^y - J\delta\sigma^y\langle\sigma^x\rangle\delta\sigma^x$$
$$+ 2h_x\delta\sigma^z - 2h_z\delta\sigma^x$$

$$= 2J\langle\sigma^x\rangle\delta\sigma^z + 2J\delta\sigma^x\delta\sigma^z + 2h_x\delta\sigma^z - 2h_z\delta\sigma^x$$
$$+ J\delta\sigma^x\langle\sigma^x\rangle(\sigma^y - \langle\sigma^y\rangle) - J(\sigma^y - \langle\sigma^y\rangle)\langle\sigma^x\rangle\delta\sigma^x$$

$$= 2J\langle\sigma^x\rangle\delta\sigma^z + 2J[\delta\sigma^x\delta\sigma^z + 2\langle\sigma^z\rangle][J\delta\sigma^x$$

$$\frac{d}{dt}\delta\sigma^z = -[H, \delta\sigma^z]$$

$$= J\sigma^x\sigma^x\delta\sigma^z - J\delta\sigma^z\sigma^x\sigma^x + h_x\sigma^x\delta\sigma^z - h_x\delta\sigma^z\sigma^x + h_z\sigma^z\delta\sigma^z - h_z\delta\sigma^z\sigma^z$$

$$= J(\langle\sigma^x\rangle + \delta\sigma^x)(\langle\sigma^x\rangle + \delta\sigma^x)\delta\sigma^z - J\delta\sigma^z(\langle\sigma^x\rangle + \delta\sigma^x)(\langle\sigma^x\rangle + \delta\sigma^x)$$
$$+ h_x(\langle\sigma^x\rangle + \delta\sigma^x)\delta\sigma^z - h_x\delta\sigma^z(\langle\sigma^x\rangle + \delta\sigma^x)$$
$$+ h_z(\langle\sigma^z\rangle + \delta\sigma^z)\delta\sigma^z - h_z\delta\sigma^z(\langle\sigma^z\rangle + \delta\sigma^z)$$

$$= J\langle\sigma^x\rangle\cancel{\langle\sigma^x\rangle}\delta\sigma^z + J\langle\sigma^x\rangle\delta\sigma^x\delta\sigma^z + J\delta\sigma^x\langle\sigma^x\rangle\delta\sigma^z$$
$$+ J\delta\sigma^x\delta\sigma^x\delta\sigma^z - J\delta\sigma^z\cancel{\langle\sigma^x\rangle}\langle\sigma^x\rangle - J\delta\sigma^z\langle\sigma^x\rangle\delta\sigma^x$$
$$- J\delta\sigma^z\delta\sigma^x\langle\sigma^x\rangle - J\delta\sigma^z\delta\sigma^x\delta\sigma^x \cancel{2h_x\langle\sigma^x\rangle\delta\sigma^y}$$
$$- 2h_x\delta\sigma^y$$

$$= -2J\langle\sigma^x\rangle\delta\sigma^y - 2J\sum\delta\sigma^x\delta\sigma^y - 2h_x\delta\sigma^y$$
$$- J(\sigma^z - \langle\sigma^z\rangle)\langle\sigma^x\rangle\delta\sigma^x + J\delta\sigma^x\langle\sigma^x\rangle(\sigma^z - \langle\sigma^z\rangle)$$

$$= -2J\langle\sigma^x\rangle\delta\sigma^y - 2J\sum_i\delta\sigma^x\delta\sigma^y - 2h_x\delta\sigma^y + 2J\langle\sigma^y\rangle\sum\delta\sigma^x$$

$$W = \begin{pmatrix} 0 & 2h_z & 0 \\ -2h_z & 0 & 2(\langle\sigma^x\rangle + h_x) \\ 0 & -2(\langle\sigma^x\rangle + h_x) & 0 \end{pmatrix}\begin{pmatrix}\delta\sigma^x \\ \delta\sigma^y \\ \delta\sigma^z\end{pmatrix}$$

$$\delta_i = 2\sum J\delta\sigma^x\begin{pmatrix} 0 \\ +\langle\sigma^z\rangle \\ -\langle\sigma^y\rangle\end{pmatrix}$$

$$\frac{d}{dt} G^{xy} = \left\langle \left(\frac{d}{dt}\delta\sigma^x\right)\delta\sigma^y \right\rangle + \left\langle \delta\sigma^x \frac{d}{dt}\delta\sigma^y \right\rangle$$

$$= \left\langle 2h_z\,\delta\sigma^y\delta\sigma^y \right\rangle + \Big\langle (\delta\sigma^x)\Big(2J\langle\sigma^x\rangle\delta\sigma^z + 2J\,\delta\sigma^x\delta\sigma^z$$
$$+ 2h_x\,\delta\sigma^z - 2h_z\,\delta\sigma^x + 2\langle\sigma^z\rangle\textstyle\sum J\,\delta\sigma^x\Big)\Big\rangle$$

$$= \left\langle 2h_z\,\delta\sigma^y\delta\sigma^y \right\rangle + \Big\langle 2J\langle\sigma^x\rangle\delta\sigma^x\delta\sigma^z + 2J\,\delta\sigma^x\delta\sigma^x\delta\sigma^z$$
$$+ 2h_x\,\delta\sigma^x\delta\sigma^z - 2h_z\,\delta\sigma^x\delta\sigma^x + 2\langle\sigma^z\rangle\textstyle\sum J\,\delta\sigma^x\delta\sigma^x\Big\rangle$$

$$= W_{xy}G^{yy} + W_{yz}G^{xz} + 2J\left\langle \delta\sigma^x\delta\sigma^x\delta\sigma^z\right\rangle$$
$$+ W_{yx}G^{xx} + 2\langle\sigma^z\rangle\textstyle\sum J\,\delta\sigma^x\delta\sigma^x$$

$$= W_{xy}G^{yy} + W_{yz}G^{xz} + 2J\left\langle \delta\sigma^x\delta\sigma^x\delta\sigma^z\right\rangle$$
$$+ W_{yx}G^{xx} + v_y\textstyle\sum J G^{xx}$$

Now) $2J\langle\delta\sigma^x\delta\sigma^x\delta\sigma^z\rangle = 2J\langle\sigma^z\rangle f_x$

$$= 2J\,(\sigma^x - \langle\sigma^x\rangle)(\sigma^x - \langle\sigma^x\rangle)(\sigma^z - \langle\sigma^z\rangle)$$

$$= 2J\,(1 - \langle\sigma^x\rangle^2)(\sigma^z - \langle\sigma^z\rangle)$$

$$\therefore\ \boxed{f_x = 1 - \langle\sigma^x\rangle^2}$$

$$= W_{xy}G^{yy} + W_{yz}G^{xz} + v_y f_x + W_{yx}G^{xx} + v_y\textstyle\sum J G^{xx}$$

$$\frac{d}{dt}G^{yz} = \left\langle \frac{d\,\delta\sigma^y}{dt}\delta\sigma^z\right\rangle + \left\langle \delta\sigma^y\frac{d\,\delta\sigma^z}{dt}\right\rangle$$

$$= \Big\langle 2J\langle\sigma^x\rangle\delta\sigma^z\delta\sigma^z + 2J\textstyle\sum\delta\sigma^x\delta\sigma^z\delta\sigma^z + 2h_x\,\delta\sigma^z\delta\sigma^z + 2J\langle\sigma^x\rangle\delta\sigma^x\delta\sigma^z \Big\rangle$$
$$+ 2\langle\sigma^y\rangle\textstyle\sum J\,\delta\sigma^x\delta\sigma^z$$
$$- 2h_z\,\delta\sigma^x\delta\sigma^z - 2J\,\delta\sigma^y\delta\sigma^x\delta\sigma^y - 2h_x\,\delta\sigma^y\delta\sigma^y$$
$$+ \Big\langle -2J\,\delta\sigma^y\langle\sigma^x\rangle\delta\sigma^y - 2J\,\delta\sigma^y\delta\sigma^x\delta\sigma^y$$
$$+ 2J\langle\sigma^y\rangle\textstyle\sum\delta\sigma^y\delta\sigma^x\Big\rangle$$

$$= 2J\langle\sigma^x\rangle G^{zz} + 2J\left\langle\delta\sigma^x\delta\sigma^z\delta\sigma^z\right\rangle + 2h_x G^{zz} - 2h_z G^{xz} - 2J\langle\sigma^x\rangle G^{yy} - 2J\left\langle\delta\sigma^y\delta\sigma^x\delta\sigma^y\right\rangle$$
$$- 2\langle\sigma^z\rangle\textstyle\sum J G^{xz}$$
$$- 2h_x G^{yy} + 2J\langle\sigma^y\rangle\textstyle\sum G^{yx}$$

$$= W_{yz}G^{zz} + W_{yx}G^{xz} + W_{zx}G^{yy} + v_y\textstyle\sum J G^{xz} + v_z\sum J G^{yx}$$
$$+ 2J\left\langle\delta\sigma^x\delta\sigma^z\delta\sigma^z\right\rangle - 2J\left\langle\delta\sigma^y\delta\sigma^x\delta\sigma^y\right\rangle$$

$\langle \delta\sigma^x \delta\sigma^z \delta\sigma^z \rangle$

$= \langle (\sigma^x - \langle\sigma^x\rangle)(\sigma^z - \langle\sigma^z\rangle)(\sigma^z - \langle\sigma^z\rangle) \rangle$

$= \langle \sigma^x\sigma^z\sigma^z - \sigma^x\sigma^z\langle\sigma^z\rangle - \sigma^x\langle\sigma^z\rangle\sigma^z + \sigma^x\langle\sigma^z\rangle\langle\sigma^z\rangle$
$- \langle\sigma^x\rangle\sigma^z\sigma^z + \langle\sigma^x\rangle\sigma^z\langle\sigma^z\rangle + \langle\sigma^x\rangle\langle\sigma^z\rangle\sigma^z - \langle\sigma^x\rangle\langle\sigma^z\rangle\langle\sigma^z\rangle \rangle$

$= \cancel{\langle\sigma^x\rangle} - \langle\sigma^x\sigma^z\rangle\langle\sigma^z\rangle - \langle\sigma^z\rangle\langle\sigma^x\sigma^z\rangle + \langle\sigma^x\rangle\langle\sigma^z\rangle\langle\sigma^z\rangle$
$- \cancel{\langle\sigma^x\rangle} + \langle\sigma^x\rangle\langle\sigma^z\rangle\langle\sigma^z\rangle + \langle\sigma^x\rangle\cancel{\langle\sigma^z\rangle}\langle\sigma^z\rangle - \langle\sigma^x\rangle\cancel{\langle\sigma^z\rangle}\langle\sigma^z\rangle$

$= + \langle\sigma^y\rangle\langle\sigma^z\rangle + \langle\sigma^z\rangle\langle\sigma^y\rangle + 2\langle\sigma^x\rangle\langle\sigma^z\rangle\langle\sigma^z\rangle$

$= \langle\sigma^x\rangle - \langle\sigma^x\rangle + 2\langle\sigma^x\rangle\langle\sigma^z\rangle\langle\sigma^z\rangle$

$= 2\langle\sigma^x\rangle\langle\sigma^z\rangle\langle\sigma^z\rangle$

And, $-\langle \delta\sigma^y \delta\sigma^x \delta\sigma^y \rangle$

$= -\langle (\sigma^y - \langle\sigma^y\rangle)(\sigma^x - \langle\sigma^x\rangle)(\sigma^y - \langle\sigma^y\rangle) \rangle$

$= -\langle \sigma^y\sigma^x\sigma^y - \sigma^y\sigma^x\langle\sigma^y\rangle - \sigma^y\langle\sigma^x\rangle\sigma^y + \sigma^y\langle\sigma^x\rangle\langle\sigma^y\rangle$
$- \langle\sigma^y\rangle\sigma^x\sigma^y + \langle\sigma^y\rangle\sigma^x\langle\sigma^y\rangle + \langle\sigma^y\rangle\langle\sigma^x\rangle\sigma^y$
$- \langle\sigma^y\rangle\langle\sigma^x\rangle\langle\sigma^y\rangle \rangle$

$= -\langle\sigma^z\rangle\langle\sigma^y\rangle + \langle\sigma^x\rangle + 2\langle\sigma^y\rangle\langle\sigma^x\rangle\langle\sigma^y\rangle$

$= -2\langle\sigma^y\rangle\langle\sigma^x\rangle\langle\sigma^y\rangle$

Comparing we get –

$-2\langle\sigma^z\rangle f_z = 2\langle\sigma^x\rangle\langle\sigma^z\rangle\langle\sigma^z\rangle$

$f_z = -\underline{\langle\sigma^x\rangle\langle\sigma^z\rangle}$

and, $2\langle\sigma^y\rangle f_y = -2\langle\sigma^y\rangle\langle\sigma^x\rangle\langle\sigma^y\rangle$

$f_y = -\langle\sigma^x\rangle\langle\sigma^y\rangle$

XXZ model

$$\frac{d}{dt} G^{xy} = W_{xy} G^{yy} + W_{yz} G^{xz} + W_{yx} G^{xx} + v_y f_x + v_y \sum J G^{xx}$$

$$\frac{d}{dt} G^{yz} = W_{yz} G^{zz} + W_{yx} G^{xz} + W_{zx} G^{yy} + v_y \sum J G^{xz} + v_z \sum J G^{yx} + v_y f_z + v_z f_y$$

$$\frac{d}{dt} G^{ab} = W_{ac} G^{cb} + W_{bc} G^{ac} + v_b f_a + v_b \sum J G^{ax} + v_a \sum J G^{xb} + v_a f_b$$

So,

$$\boxed{\frac{d}{dt} G^{ab} = \sum_{c=x,y,z} \left(W_{ac} G^{cb} + W_{bc} G^{ac} \right) + v_a \sum_{j \neq 1,k} J\, G^{xb}_{r_{ij+1}} + v_b \sum_{j \neq 1,k} J\, G^{ax} + J_{1k} \left(v_a f_b + v_b f_a \right)}$$

So,

$$f = \begin{pmatrix} 1 - \langle \sigma^x \rangle^2 \\ - \langle \sigma^x \rangle \langle \sigma^y \rangle \\ - \langle \sigma^x \rangle \langle \sigma^z \rangle \end{pmatrix}$$

XXZ

$$\frac{d}{dt} G_K^{ab}(t) = \left\langle \frac{d}{dt}(\delta\sigma_i^a)\, \delta\sigma_j^b \right\rangle + \left\langle \delta\sigma_i^a \frac{d}{dt}(\delta\sigma_j^b) \right\rangle$$

$$= \sum_{c=x,y,z} \left(W_{ac} G^{cb} + W_{bc} G^{ac} \right) + v_a \sum_{j(\neq 1,k)} J_{1j}\, G^{zb}_{r_{jk}+1} + v_b \sum_{j(\neq 1,k)} J_{kj}\, G^{az}$$

$$+ J_{1k} \left(v_a f_b + v_b f_a \right)$$

$$\vec{f} = \begin{pmatrix} -\langle\sigma^x\rangle\langle\sigma^z\rangle \\ -\langle\sigma^y\rangle\langle\sigma^z\rangle \\ 1 - \langle\sigma^z\rangle^2 \end{pmatrix}$$

$$\vec{v} = 2 \begin{pmatrix} \langle\sigma^y\rangle \\ -\langle\sigma^x\rangle \\ 0 \end{pmatrix}$$

$$W = \begin{pmatrix} 0 & 2\left(\langle\sigma^z\rangle + h_z\right) & 0 \\ -2\left(\langle\sigma^z\rangle + h_z\right) & 0 & 2h_x \\ 0 & -2h_x & 0 \end{pmatrix}$$