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المدرسة التونسية للتقنيات

Financial Engineering with Design Project Report

Evaluation of an American Barrier Option

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1 Introduction to American Barrier Options

American barrier options are a sophisticated type of financial derivative that combines the flexibility of American options with the conditional features of barrier options. These instruments provide unique opportunities for investors to tailor their risk-reward profiles while reducing costs compared to standard options.
An American option allows the holder to exercise the option at any time before its expiration, offering greater flexibility to respond to market changes. A barrier option, on the other hand, incorporates a predefined price level, known as the barrier, which activates or deactivates the option depending on the underlying asset's price movements.
When these two features are combined, the resulting American barrier option can suit various market conditions and investor strategies. For instance, a trader might use an Up-and-Out American Call Option to bet on moderate price increases while mitigating costs, as the option is deactivated if the price rises above a certain level.
The key appeal of American barrier options lies in their ability to reduce the premium (cost of the option) by incorporating conditions tied to the barrier. However, this comes at the cost of increased complexity and the risk of the option becoming worthless if the barrier condition is triggered.

2 Binomial Model for Option Pricing

The binomial model, introduced by Cox, Ross, and Rubinstein (1979), provides a simple yet powerful numerical method for pricing a wide variety of options that do not have closed-form analytical solutions. This technique models the price evolution of a financial instrument by assuming it can move in only two directions at each time step: up or down. By reducing the time interval dt between these movements to an infinitesimally small value, the binomial model closely approximates the stochastic process governing the price dynamics of the underlying asset.

3 Construction of the Binomial Tree

3.1 Define Input Parameters

S_0: Current price of the underlying asset.
σ: Volatility of the underlying asset.
K: Strike price.
T: Time to maturity.
n: Number of time steps in the tree.
B: Barrier price (e.g., Up-and-Out or Down-and-Out).

r: Risk-free interest rate (annualized).
$\Delta t = \frac{T}{n}$: Length of each time step.

3.2 Calculate Tree Parameters

Upward price factor (u):

$$u = e^{\sigma\sqrt{\Delta t}}$$

Downward price factor (d):

$$d = \frac{1}{u}$$

Risk-neutral probability (p):

$$p = \frac{e^{r\Delta t} - d}{u - d}$$

Discount Factor (disc):

$$disc = e^{-r\Delta t}$$

3.3 Construct the Asset Price Tree

At each node in the tree: The price of the underlying asset either **increases** by a factor of **u** or **decreases** by a factor of **d**. The asset price at node (i,j), where i is the time step, and j is the number of upward movements, is:

$$S_{i,j} = S_0 u^j d^{i-j}$$

3.4 Apply the Barrier Condition

In a barrier option, the payoff depends on whether the underlying asset's price crosses the barrier B.

- **Up and Out Barrier:** The option is invalidated if $S \geq B$ at any node.
- **Down and Out Barrier:** The option is invalidated if $S \leq B$ at any node.

In our use case we'll use the **Up and Out Barrier**. If the barrier is breached, the option value is set to zero:

$$V_{i,j} = 0$$

3.5 Calculate Payoffs at Maturity

At maturity (t = T, i=n), calculate the option payoff for all valid nodes if barrier condition is not violated:

- **Call Option:**

$$Payoff = max(S_{n,j} - K, 0)$$

- **Put Option:**

$$Payoff = max(K - S_{n,j}, 0)$$

In our use case we'll use the **Call Option**

3.6 Backward Induction

Use backward induction to compute the option price at earlier nodes, considering:

3.6.1 Risk-neutral valuation

At each node (i,j), compute:

$$V_{i,j} = e^{-r\Delta t} \cdot [p \cdot V_{i+1,j+1} + (1-p) \cdot V_{i+1,j}]$$

3.6.2 Early Exercise Condition (American Feature)

For an American option, the holder may exercise the option early. At each node, check if early exercise provides a better payoff:

- **Call Option:**

$$V_{i,j} = max(S_{n,j} - K, V_{i,j})$$

- **Put Option:**

$$V_{i,j} = max(K - S_{n,j}, V_{i,j})$$

In our use case we'll use the **Call Option**

3.6.3 Barrier Condition

If the barrier condition is breached, set

$$V_{i,j} = 0$$

3.7 Final Option Price

The value of the option at the root node ($V_{0,0}$) is the price of the American barrier option.

4 Numerical Example

4.1 Inputs

- $S_0 = 100$
- $T = 1$
- $r = 0.05$
- $\sigma = 0.2$
- $N = 5$
- $B = 120$
- Option Type : Call
- Barrier Type : Up and Out

4.2 Tree Parameters Calculation

Time step size (Δt):

$$\Delta t = \frac{T}{n} = \frac{1}{5} = 0.2$$

Upward price factor (u):

$$u = e^{\sigma\sqrt{\Delta t}} = e^{0.2\sqrt{0.2}} = 1.0935$$

Downward price factor (d):

$$d = \frac{1}{u} = \frac{1}{1.0935} = 0.9144$$

Risk-neutral probability (p):

$$p = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.05.0.2} - 0.9144}{1.0935 - 0.9144} = 0.534$$

Discount Factor (disc):

$$disc = e^{-r\Delta t} = e^{-0.05.0.2} = 0.99$$

4.3 Stock Price Tree Construction

We compute stock prices for all nodes $S_{i,j} = S_0 u^j d^{i-j}$, where i is the time step and j is the number of up moves. Any node where $S \geq B$ is invalidated.

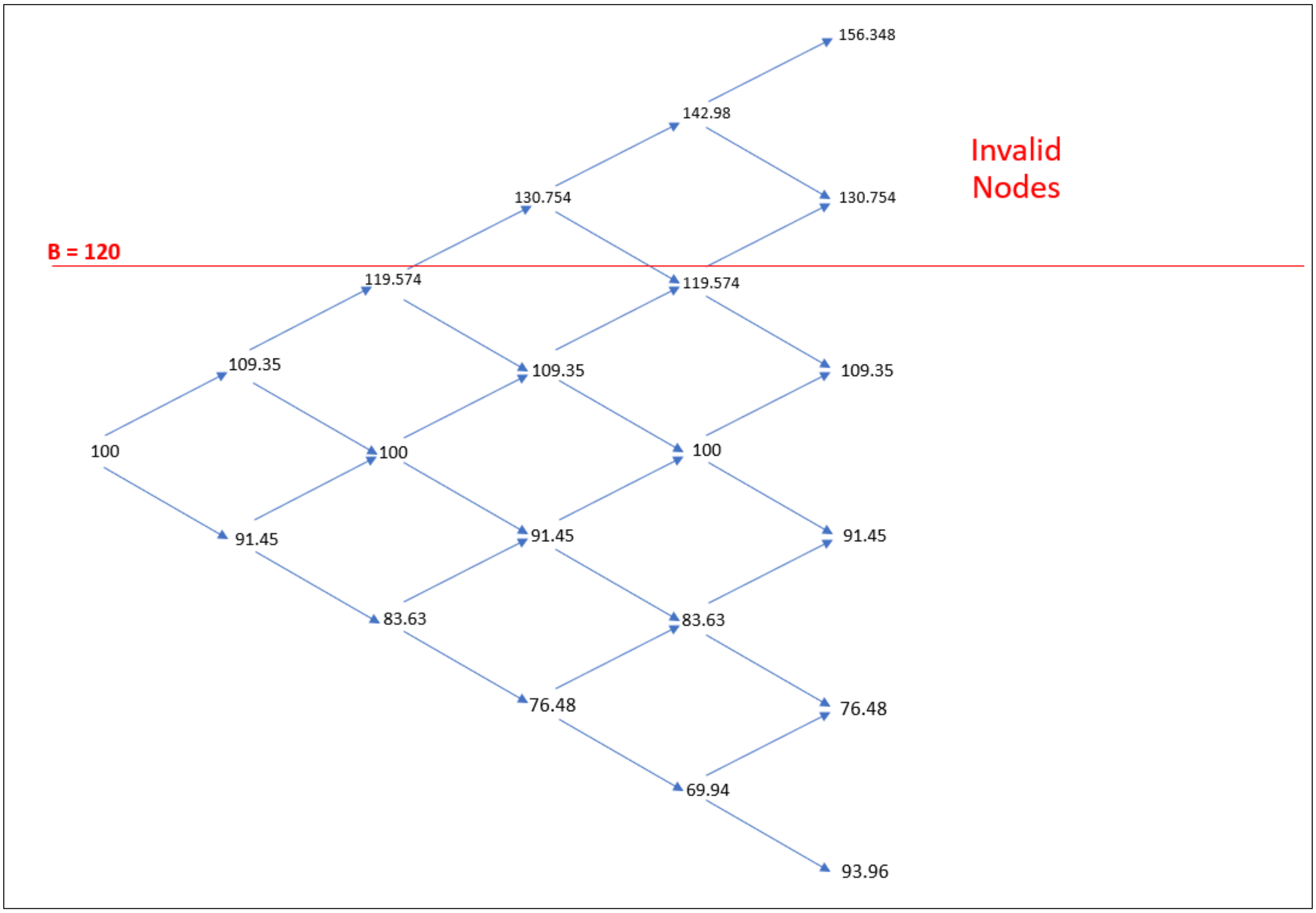

Figure 1: Stock Price Tree

4.4 Payoffs at Maturity and backward induction

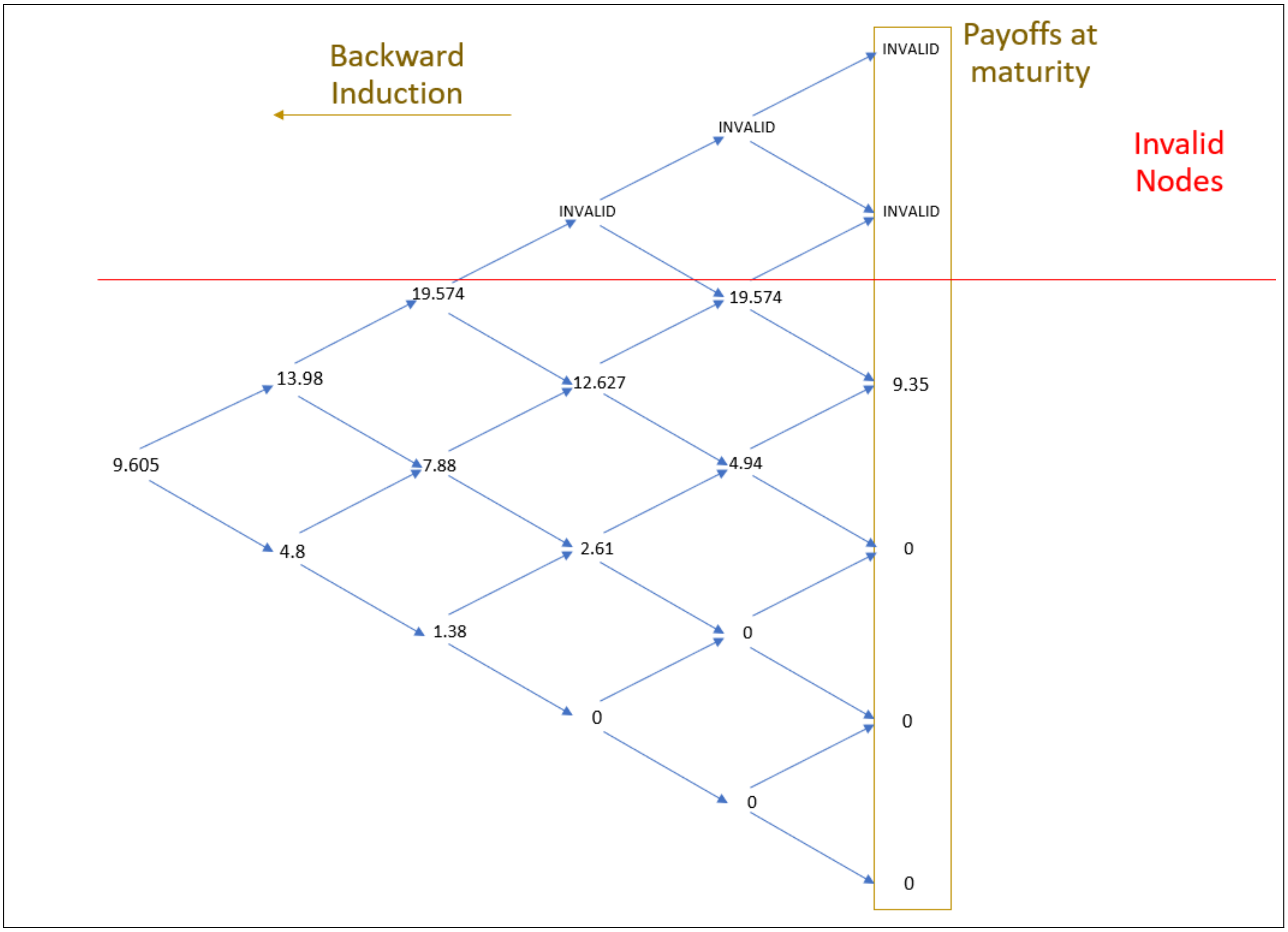

Figure 2: Option Tree

4.5 Final Option Price

The value of the option at the root node is the price of the American barrier option and is equal to **9.605**.

5 Conclusion

The binomial model is a powerful numerical method for pricing American barrier options, which are financial derivatives that combine the early exercise feature of American options with barrier constraints. The model constructs a binomial tree where the underlying asset price evolves over discrete time steps, moving up or down based on pre-calculated factors derived from the asset's volatility and the time to maturity. By assuming a risk-neutral probability, the model ensures no-arbitrage pricing. The backward induction technique is applied to calculate the option's value, starting from the terminal payoff at maturity and discounting it step by step to the present. For barrier options, such as the "up-and-out" type, the model incorporates conditions where the option is invalidated if the underlying asset's price crosses a predefined barrier. This flexibility allows the model to handle complex option structures, making it an essential tool in financial engineering.

6 Future Work

While the binomial model provides a robust framework for pricing American barrier options, several extensions could enhance its practical applicability and accuracy:

- **Improved Computational Efficiency**: Explore advanced numerical techniques, such as the trinomial tree or Monte Carlo simulations, to handle high-dimensional problems and reduce computational time.
- **Stochastic Volatility Models**: Incorporate stochastic volatility to better reflect market dynamics and improve pricing accuracy for assets with highly variable volatility.
- **Integration of Machine Learning**: Develop predictive models using machine learning to estimate key parameters, such as volatility or risk-neutral probabilities, dynamically from historical data.
- **Real-World Constraints**: Extend the model to include transaction costs, liquidity effects, and other market frictions for more realistic pricing scenarios.
- **Exotic Option Types**: Adapt the framework to price other complex derivatives, such as double barrier options, lookback options, or barrier options with rebates.

These directions could pave the way for broader adoption of the model in financial institutions and open new research opportunities in computational finance.

7 References

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