Assignment 0

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1 Exercises

1. Find the value x that maximizes $f(x) = -3x^2 + 24x - 30$

In order to find the value x that maximizes $f(x)$, we need to find the first order derivative. The first order derivative of $f(x)$ is $f'(x)$. Here,

$$f'(x) = -6x + 24$$

To find the value of x that maximizes $f(x)$, need to set $f'(x) = 0$.

$$-6x + 24 = 0$$

$$x = 4$$

Therefore, the value of x that maximizes $f(x)$ is 4.

2. Find the partial derivatives of $g(x)$ with respect to x_0 and x_1:

$$g(x) = -3x_0^3 + 2x_0x_1^2 + 4x_1 - 8$$

The partial derivative of $g(x)$ in respect to x_0 is $\frac{\partial g(x)}{\partial x_0} = 9x_0^2 - 2x_1^2$

The partial derivative of $g(x)$ in respect to x_1 is $\frac{\partial g(x)}{\partial x_1} = -4x_0x_1 + 4$

3. What is the value of $AB^T + C^1$, if the following define A, B, and C?

A= $\begin{bmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$ B= $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ C= $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

B^T= $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ AB^T= $\begin{bmatrix} 26 & 62 \\ 44 & 107 \end{bmatrix}$ $C^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$

$AB^T + C^{-1}$= $\begin{bmatrix} 27 & 62 \\ 44 & 107.5 \end{bmatrix}$

For verification my python code: click here

4. Write down the mathematical definitions of the simple Gaussian, multivariate Gaussian, Bernoulli, binomial and exponential distributions

Gaussian Distribution:

$$\phi(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{[\frac{(x-\mu)^2)}{2\sigma^2}]} \tag{1}$$

Here, probability density function ϕ of Gaussian distribution with mean μ , variance σ^2 for $-\infty < x < \infty$. [1]

Multivariate Gaussian Distribution:

A vector-valued random variable $X = [X_1, \ldots, X_n]^T$ is said to have a multivariate normal (or Gaussian) distribution with mean $\mu \in R_n$ and covariance matrix Σ if its probability density function is given by

$$\phi(x;\mu,\Sigma) = \frac{1}{(2\pi)^{\frac{n}{2}}|\Sigma|^{\frac{1}{2}}} e^{-[\frac{(x-\mu)^T\Sigma^{-1}(x-\mu)}{2}]} \tag{2}$$

Bernoulli Distribution:

The Bernoulli distribution is the probability distribution of a random variable X having the probability density function

$$P(X=x) = \begin{cases} \text{p} & \text{x=1} \\ 1-p & \text{x=0} \end{cases}$$

for $0 < p < 1$.This described as single experiment having two outcomes, success is "1" occurring with probability p, and failure is "0" occurrs with probability $1-p$. It is a single trial of a Bernoulli experiment. [3]

Binomial Distribution:

$$P^{(k)} = \tbinom{n}{k} p^k (1-p)^{(n-k)}$$

Here, the random variable S_n has the binomial distribution with parameters n and p, abbreviated $Binom(n,p)$. For $k = 0, 1, ..., n$. The mean and variance are given by np and np(1-p) . [1]

Exponential Distribution:

$$f_X(x) = \begin{cases} \lambda \mathrm{e}^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Here, this is defined for $x \geq 0$, where λ is parameter of the distribution. [3]

5. What is the relationship between the Bernoulli and binomial distributions?

Binomial Distribution is written as, $P_x = \binom{n}{x} p^x q^{n-x}$ where

x = total number of "successes"

p = probability of a success on an individual trial

n = number of trials

q = the probability of failure = 1-p

A Bernoulli distribution is a special case of binomial distribution.When n=1 the binomial distribution becomes Bernoulli distribution.All Bernoulli distributions are binomial distributions, but most binomial distributions are not Bernoulli distributions. If

$$X = \begin{cases} 1 & \text{with probability p,} \\ 0 & \text{with probability 1-p} \end{cases}$$

then the probability distribution of the random variable X is a Bernoulli distribution.

6. Suppose that random variable $X \sim N(1, 3)$. What is its expected value?

This $X \sim N(1, 3)$ is denoted as the Gaussian distribution of random variable X. The parameters are μ and σ^2. When random variable X has normal distribution $X \sim \mathrm{N}(\mu,\sigma^2)$ for some $\mu \in R$, $\sigma \in R > 0$, where N is the Gaussian distribution, then expected value E(X)

$$E(X) = \mu$$

Here $\mu = 1$ therefore, $E(X) = 1$

7.

$$P(Y = y) = \begin{cases} \mathrm{e}^{-y}, & \text{if } y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

a. $\int_{-\infty}^{\infty} P(Y = y)\, dy = \int_{y=-\infty}^{0} \mathrm{P(Y{=}y)}\, dy + \int_{y=0}^{\infty} \mathrm{P(Y{=}y)}\, dy = \int_{y=0}^{\infty} \mathrm{e}^{-y}\ \mathrm{dy} = \left.\dfrac{-1}{e^y}\right|_0^{\infty} = \text{-0+1} = 1$

b. $\mu = E[Y] = \int\limits_{\infty}^{\infty} yP(Y = y)dy$

$= \int\limits_{0}^{\infty} y\mathrm{e}^{-y}\ dy = -ye^{-y}\Big|_0^{\infty} + \int_{y=0}^{\infty} \mathrm{e}^{-y}\ \mathrm{dy} = -ye^{-y}\Big|_0^{\infty} + \mathrm{e}^{-y}\Big|_0^{\infty} = [-ye^{-y} + e^{-y}]\Big|_0^{\infty} = 1$

c. $\sigma^2 = Var[Y] = \int_{-\infty}^{\infty} \mathrm{y^2 P(Y{=}y)\ dy} - \mu^2 = \int_0^{\infty} \mathrm{y}^2 e^{-y}\ \mathrm{dy} - \mu^2 = (-2e^{-y} - 2ye^{-y} - y^2 e^{-y})\Big|_0^{\infty} - \mu^2 = 2 - 1^2 = 1$

d. $E[Y|Y \geq 10] = \int_{10}^{\infty} ye^{-y}\, dy$

$= -ye^{-y}\Big|_{10}^{\infty} + \int_{y=10}^{\infty} \mathrm{e}^{-y}\ \mathrm{dy} = -ye^{-y}\Big|_{10}^{\infty} + (\text{-}\mathrm{e}^{-y})\Big|_{10}^{\infty} = (-ye^{-y} - e^{-y})\Big|_{10}^{\infty} = 4.99 \times 10^{-4}$

References

[1] http://cs.baylor.edu/ hamerly/courses/5325$_{r}esources/haas_{p}rob_{s}tats_{r}efresher.pdf$

[2] https://proofwiki.org/wiki/Expectation$_{o}f_{G}aussian_{D}istribution$

[3] https://nrich.maths.org/6141/solution.