Scattering of light by circum-stellar dust

Scattering, absorption and extinction

Based on Ariel Goobar's notes. See also [1].

In order to simulate photons propagating through dust we need expressions for the mean free paths for *scattering*, λ_s and *absorption*, λ_a, in terms of quantities that we wish to put into the simulation such as e.g. the observed colour excess, $E(B-V)$ for the $B-V$ colour. The colour excess is defined as the difference between the observed, $(B-V)_o$, and emitted colour, $(B-V)_e$, of an object,

$$E(B-V) = (B-V)_o - (B-V)_e = -\underbrace{\frac{2.5}{\ln 10}}_{\varepsilon}\left(\ln\frac{f_o^B}{f_o^V} - \ln\frac{f_e^B}{f_e^V}\right) = -\varepsilon\ln\left(\frac{f_o^B}{f_e^B}\cdot\frac{f_e^V}{f_o^V}\right), \qquad (1)$$

where f_e and f_o are the emitted and observed fluxes of the object respectively. The observed flux is the difference between the emitted and extincted flux, f_r, which can be calculated over a distance r for the extinction mean free path λ as

$$f_r = f_e\int_0^r \frac{1}{\lambda}e^{-x/\lambda}\,dx = f_e\left(1 - e^{-r/\lambda}\right).$$

giving

$$\begin{aligned} f_o^B &= f_e^B - f_r^B = f_e^B\left(1 - 1 + e^{-r/\lambda_B}\right) = f_e^B e^{-r/\lambda_B}, \\ f_o^V &= f_e^V - f_r^V = f_e^V e^{-r/\lambda_V}. \end{aligned}$$

Inserting these into equation (1), yields

$$E(B-V) = -\varepsilon\ln\left[\exp\left(-\frac{r}{\lambda_B} + \frac{r}{\lambda_V}\right)\right] = \varepsilon\left(\frac{r}{\lambda_B} - \frac{r}{\lambda_V}\right) = \varepsilon n\left(\sigma_B - \sigma_V\right), \qquad (2)$$

where the relation $\lambda = (n\sigma)^{-1}$ between the mean free path, the number density, n, and the cross-section, σ, was used in the last step.

Bruce Draine provides[1] tables for $\alpha = \sigma_a/M$ and the albedo, a,

$$a = \frac{\sigma_s}{\sigma_a + \sigma_s} \quad\Rightarrow\quad \sigma_s = \sigma_a\left(\frac{1}{a} - 1\right)^{-1} \quad\Rightarrow\quad \lambda_s = \left(\frac{1}{a} - 1\right)\lambda_a, \qquad (3)$$

for different dust environments. Here, σ_s and σ_a are the scattering and absorptions cross-sections respectively and M is the molar mass of the dust particles.

The last relation allows us to calculate the mean free path for scattering from the absorption length, but an expression for the latter is still needed. Using these relations the extinction cross-section, σ, which is the sum of both scattering and absorption, can be calculated as

$$\sigma = \sigma_a + \sigma_s = \frac{\sigma_a}{1-a} = \frac{\alpha M}{1-a}.$$

The difference, $\sigma_B - \sigma_V$, between extinction cross section can then be expressed as

$$\sigma_B - \sigma_V = M\left(\frac{\alpha_B}{1-a_B} - \frac{\alpha_V}{1-a_V}\right),$$

[1] http://www.astro.princeton.edu/ draine/dust/dust.diel.html

which together with equation (2) gives the number density as

$$n = \frac{E(B-V)}{\varepsilon r\,(\sigma_B - \sigma_V)} = \frac{E(B-V)}{Mr\varepsilon}\left(\frac{\alpha_B}{1-a_B} - \frac{\alpha_V}{1-a_V}\right)^{-1} ,$$

This relation can now be used to express the mean free path for absorption as

$$\lambda_a = \frac{1}{n\sigma_a} = \frac{1}{nM\alpha} = \frac{rM\varepsilon}{E(B-V)M\alpha}\left(\frac{\alpha_B}{1-a_B} - \frac{\alpha_V}{1-a_V}\right) = \frac{\varepsilon}{\alpha}\left(\frac{\alpha_B}{1-a_B} - \frac{\alpha_V}{1-a_V}\right)\frac{r}{E(B-V)} .$$

The interpretation of the last factor might require some clarification. Here, r, is the thickness of a homogeneous dust layer and $E(B-V)$ is the colour excess due to *extinction* (i.e. neither absorbed nor scattered photons will reach the observer) of a light source observed through the dust layer.

For a circum-stellar dust the colour excess is different, since in this scenario also scattered photons will be observed.

A spherical dust shell

A simple model of a spherical dust-shell with radius, r, and inner radius, $r_i = R \cdot r$, where $0 < R < 1$ is illustrated in Figure 1. Also shown in the figure is a photon leaving the sphere (solid line). The current position of the photon is defined by the vector, $\bar{r}_0 = (x, y, z)$, and

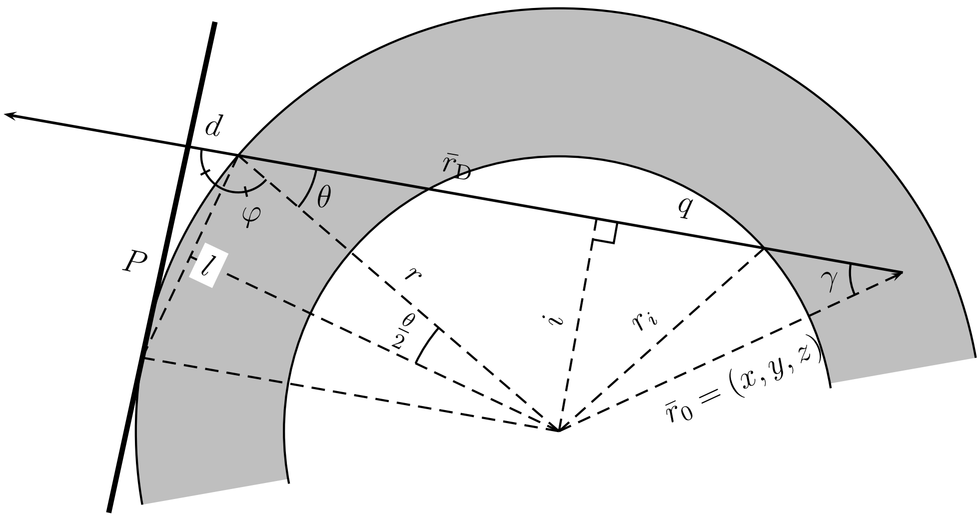

Figure 1: Illustration of a scattered photon (*thin solid line*) propagating through dust shell. The figure shows the intersecting plane of the sphere that is defined by the two vectors $\bar{r}_0$ and $\bar{r}_\mathrm{D}$.

the next position, $\bar{r}_\mathrm{next}$, is given by $\bar{r}_\mathrm{next} = \bar{r}_0 + \bar{r}_\mathrm{D}$, where $\bar{r}_\mathrm{D}$ is the displacement vector $\bar{r}_\mathrm{D} = (dx, dy, dz)$.

Photon travelling inside the shell The impact parameter, i, is the minimum distance to the centre of the sphere for any given photon-path, and can be defined as.

$$i = \begin{cases} r_0 \cdot \sin\gamma & \text{if } 0 \leq \gamma \leq \frac{\pi}{2} , \\ r_0 & \text{if } \frac{\pi}{2} < \gamma . \end{cases}$$

Here, γ is given by

$$\begin{aligned}\bar{r}_0 \cdot \bar{r}_{\mathrm{D}} &= r_0 r_{\mathrm{D}} \cos(\pi - \gamma) = x \cdot dx + y \cdot dy + z \cdot dz \\ \cos\gamma &= -\cos(\pi - \gamma) = -\frac{x \cdot dx + y \cdot dy + z \cdot dz}{r_0 r_{\mathrm{D}}}\end{aligned}$$

If $i < r_i$, the photon will cross the inner radius of the shell and then re-enter the shell. This will in turn extend the mean free path of the photon by the amount $2q = 2\sqrt{r_i^2 + i^2}$, which is the length of the path within r_i.

Path length For each distance between interactions, r_{D} is added to the total distance travelled by the photon before leaving the sphere. For a scattered photon leaving the sphere, the last distance, s, travelled inside the sphere is given by

$$s^2 = r^2 + r_0^2 - 2rr_0 \cos\left(\pi - \theta - \gamma\right) \,,$$

where $\cos\theta = \sin\gamma \cdot r_0/r$. The distance travelled by a scattered photon should be compared to the distance travelled by a non-interacting photon, which means that the distance, d, between the surface of the sphere and the plane (thick, solid line in Figure 1) perpendicular to $\bar{r}_{\mathrm{D}}$ at distance r from the centre must be added to the total distance, where d is given by

$$\begin{aligned}l &= 2r \cdot \sin\frac{\theta}{2} \\ \varphi &= \pi - \frac{\pi}{2} - \frac{\theta}{2} = \frac{\pi - \theta}{2} \\ d &= l \cdot \cos\varphi = 2r \cdot \sin\frac{\theta}{2}\cos\frac{\pi - \theta}{2} = 2r \cdot \sin^2\frac{\theta}{2} \,.\end{aligned}$$

The total distance travelled, D, can then be summarised as

$$D = \begin{cases} r & \text{for non-interacting photons}\,, \\ \sum r_{\mathrm{D}n} + s + d & \text{for scattered photons}\,. \end{cases}$$

References

[1] Goobar, A., *Low R_V from Circumstellar Dust around Supernova*, ApJ, **686**, 2008