Inverse problems for polynomial and rational matrices

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Rational matrices

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1. poles (finite and infinite),
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2. zeros (finite and infinite),
3. and minimal indices (left and right).

These data satisfy the *rational index sum condition* [Van Dooren 1979]:

$$\sum(\text{pole mult's}) - \sum(\text{zero mult's}) = \sum(\text{min indices})$$

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There is a rational matrix with given data if and only if $\mathcal{L}$ *satisfies the index sum condition.*

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- more than likely dense,
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- more than likely dense,
- and does *not* transparently display the given data.

The solution outlined in this talk corrects these deficiencies.

Rational Product Realization

Our solution takes the form of a five-fold product

$$R := Z_\ell \, D_\ell \, T \, D_r \, Z_r .$$

Each term is full rank, sparse, and the original data of $\mathcal{L}$ is transparently revealed in the factors:

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- Right minimal indices are stored in $Z_r(\lambda)$;
- Finite poles and zeros are in $T(\lambda)$;
- Infinite poles and zeros are revealed by $D_\ell \, T \, D_r$
  (D's are diagonal).

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(c) Finally, adjustments are made to $\widetilde{T}(\lambda, \omega)$, and the ω's are removed to produce $T(\lambda)$.

Step 1

Using poles and zeros (finite and infinite), construct an *extended Smith-McMillan form*.

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Example

$\mathcal{L}$ contains poles and zeros involving $a = \lambda - \alpha$, $b = \lambda - \beta$, and ω:

$$\begin{pmatrix} \text{partial} \\ \text{mult's} \end{pmatrix} \quad \begin{array}{rrrrrrr} a: & -5, & -5, & -4, & -4, & -1, & -1 \\ b: & -1, & -1, & -1, & -1, & 1, & 5 \\ \omega: & -1, & -1, & -1, & 0, & 1, & 4 \end{array}$$

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producing extended S-M form

$$\operatorname{diag}\left\{ \frac{1}{a^5 b\omega}, \frac{1}{a^5 b\omega}, \frac{1}{a^4 b\omega}, \frac{1}{a^4 b}, \frac{b\omega}{a}, \frac{b^5\omega^4}{a} \right\}.$$

Steps 2 through 4

Step 2: Factor out LCM of denominators.

Example

$$\begin{bmatrix} \frac{1}{a^5 b\omega} & & & & & \\ & \frac{1}{a^5 b\omega} & & & & \\ & & \frac{1}{a^4 b\omega} & & & \\ & & & \frac{1}{a^4 b} & & \\ & & & & \frac{b\omega}{a} & \\ & & & & & \frac{b^5\omega^4}{a} \end{bmatrix}$$

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$$\frac{1}{a^5 b\omega}\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & a & & & \\ & & & a\omega & & \\ & & & & a^4 b^2 \omega^2 & \\ & & & & & a^4 b^6 \omega^5 \end{bmatrix}$$

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Steps 2 through 4

Step 2: Factor out LCM of denominators.
Step 3: "Spread out degree" using unimodular transformations and polynomial techniques.

Example

$$\frac{1}{a^5 b\omega}\begin{bmatrix} a^2b^2\omega & b\omega & 1 & & & \\ & a^2b\omega^2 & a^2\omega & & & \\ & & a^2b\omega & a\omega & & \\ & & & a^2\omega^2 & a & \\ & & & & a^2b^2 & 1 \\ & & & & & b^2\omega^2 \end{bmatrix}$$

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$$\begin{bmatrix} \frac{b}{a^3} & \frac{1}{a^5} & \frac{1}{a^5 b\omega} & & & \\ & \frac{\omega}{a^3} & \frac{1}{a^3 b} & & & \\ & & \frac{1}{a^3} & \frac{1}{a^4 b} & & \\ & & & \frac{\omega}{a^3 b} & \frac{1}{a^4 b\omega} & \\ & & & & \frac{b}{a^3\omega} & \frac{1}{a^5 b\omega} \\ & & & & & \frac{b\omega}{a^5} \end{bmatrix}$$

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Step 2: Factor out LCM of denominators.
Step 3: "Spread out degree" using unimodular transformations
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Example

$$\widetilde{T}(\lambda,\omega) = \begin{bmatrix} \frac{b}{a^3} & \frac{1}{a^5} & \frac{1}{a^5 b\omega} & & & \\ & \frac{\omega}{a^3} & \frac{1}{a^3 b} & & & \\ & & \frac{1}{a^3} & \frac{1}{a^4 b} & & \\ & & & \frac{\omega}{a^3 b} & \frac{1}{a^4 b\omega} & \\ & & & & \frac{b}{a^3\omega} & \frac{1}{a^5 b\omega} \\ & & & & & \frac{b\omega}{a^5} \end{bmatrix}$$

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The next few steps use direct sums of *zig-zag matrices* [De Teran, Dopico, Mackey, Van Dooren 2016].

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- Full rank realization of left or right minimal indices.
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- Sum of left minimal indices equals sum of col degrees.

Example

Suppose $\mathcal{L}$ contains right min indices 6, 3.
Row degs 1, 1, 2, 2, 2, 1, are forced (up to ordering).

$$\widetilde{Z}_r = \begin{bmatrix} \lambda & 1 & & & & & & & \\ & \lambda & 1 & & & & & & \\ & & \lambda^2 & 1 & & & & & \\ & & & \lambda^2 & 1 & & & & \\ & & & & 0 & \lambda^2 & 1 & & \\ & & & & & & \lambda & 1 \end{bmatrix}.$$

Step 5 cont.

Example

If $\mathcal{L}$ has left minimal indices 5, 1, 1.

Col degs 1, 1, 1, 1, 1, 2, are forced.

$$\widetilde{Z}_\ell = \begin{bmatrix} \lambda & & & & & \\ 1 & 0 & & & & \\ & \lambda & & & & \\ & 1 & \lambda & & & \\ & & 1 & \lambda & & \\ & & & 1 & \lambda & \\ & & & & 1 & \lambda^2 \\ & & & & & \lambda \\ & & & & & 1 \end{bmatrix}$$

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Multiplying on the left by $\widetilde{Z}_\ell$ or on the right by $\widetilde{Z}_r$ does not change finite spectral structure (truncated unimodular).

Step 6

Now factor

$$\widetilde{Z}_r = D_r Z_r = \begin{bmatrix} \lambda &&&&& \\ & \lambda &&&& \\ && \lambda^2 &&& \\ &&& \lambda^2 && \\ &&&& \lambda^2 & \\ &&&&& \lambda \end{bmatrix} \begin{bmatrix} 1 & 1/\lambda &&&&&& \\ & 1 & 1/\lambda &&&&& \\ && 1 & 1/\lambda^2 &&&& \\ &&& 1 & 1/\lambda^2 &&& \\ &&&& 0 & 1 & 1/\lambda^2 & \\ &&&&&& 1 & 1/\lambda \end{bmatrix}$$

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Multiplying $Z_\ell \, Q \, Z_r$ does not change infinite spectral structure.

Interlude

Recall the form of our product realization $Z_\ell \, D_\ell \; T \; D_r \, Z_r$.

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$$Z_\ell\, (D_\ell\, T\, D_r)\, Z_r$$

  middle factor reveals infinite spectral structure.

We now use the template $\widetilde{T}(\lambda, \omega)$ to build $T(\lambda)$

Step 7

Multiply $D_\ell(\lambda)\,\widetilde{T}(\lambda,\omega)\,D_r(\lambda)$

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Example

$$
\widetilde{T}(\lambda,\omega) = \begin{bmatrix}
\frac{b}{a^3} & \frac{1}{a^5} & \frac{1}{a^5 b\omega} & & & \\
 & \frac{\omega}{a^3} & \frac{1}{a^3 b} & & & \\
 & & \frac{1}{a^3} & \frac{1}{a^4 b} & & \\
 & & & \frac{\omega}{a^3 b} & \frac{1}{a^4 b\omega} & \\
 & & & & \frac{b}{a^3\omega} & \frac{1}{a^5 b\omega} \\
 & & & & & \frac{b\omega}{a^5}
\end{bmatrix}
$$

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Multiply $D_\ell(\lambda)\,\widetilde{T}(\lambda,\omega)\,D_r(\lambda)$

Example

$$D_\ell\,\widetilde{T}\,D_r = \begin{bmatrix} \frac{b\lambda^2}{a^3} & \frac{\lambda^2}{a^5} & \frac{\lambda^3}{a^5 b\omega} & & & \\ & \frac{\lambda^2\omega}{a^3} & \frac{\lambda^3}{a^3 b} & & & \\ & & \frac{\lambda^3}{a^3} & \frac{\lambda^3}{a^4 b} & & \\ & & & \frac{\lambda^3\omega}{a^3 b} & \frac{\lambda^3}{a^4 b\omega} & \\ & & & & \frac{b\lambda^3}{a^3\omega} & \frac{\lambda^2}{a^5 b\omega} \\ & & & & & \frac{b\lambda^3\omega}{a^5} \end{bmatrix}$$

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Notice that

- diagonal entries have rational deg 0 (biproper),

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Notice that

- diagonal entries have rational deg 0 (biproper),
- off-diagonal entries have rational deg ≤ 0 (proper).

Step 8

Fill in off-diagonal entries of $\widetilde{T}(\lambda,\omega)$ with *neutral factors* so that all entries of $D_\ell(\lambda)\widetilde{T}(\lambda,\omega)D_r(\lambda)$ have rat deg 0.

$$\underbrace{\begin{bmatrix} \frac{b\lambda^2}{a^3} & \frac{\lambda^2}{a^5} & \frac{\lambda^3}{a^5 b\omega} & & & & \\ & \frac{\lambda^2\omega}{a^3} & \frac{\lambda^3}{a^3 b} & & & & \\ & & \frac{\lambda^3}{a^3} & \frac{\lambda^3}{a^4 b} & & & \\ & & & \frac{\lambda^3\omega}{a^3 b} & \frac{\lambda^3}{a^4 b\omega} & & \\ & & & & \frac{b\lambda^3}{a^3\omega} & \frac{\lambda^2}{a^5 b\omega} \\ & & & & & \frac{b\lambda^3\omega}{a^5} \end{bmatrix}}_{D_\ell(\lambda)\widetilde{T}(\lambda,\omega)D_r(\lambda)} \qquad \underbrace{\begin{bmatrix} \frac{b}{a^3} & \frac{1}{a^5} & \frac{1}{a^5 b\omega} & & & \\ & \frac{\omega}{a^3} & \frac{1}{a^3 b} & & & \\ & & \frac{1}{a^3} & \frac{1}{a^4 b} & & \\ & & & \frac{\omega}{a^3 b} & \frac{1}{a^4 b\omega} & \\ & & & & \frac{b}{a^3\omega} & \frac{1}{a^5 b\omega} \\ & & & & & \frac{b\omega}{a^5} \end{bmatrix}}_{\widetilde{T}(\lambda,\omega)}$$

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$$\underbrace{\begin{bmatrix} \frac{b\lambda^2}{a^3} & \frac{\lambda^2 c^3}{a^5} & \frac{\lambda^3}{a^5 b\omega} & & & \\ & \frac{\lambda^2\omega}{a^3} & \frac{\lambda^3}{a^3 b} & & & \\ & & \frac{\lambda^3}{a^3} & \frac{\lambda^3}{a^4 b} & & \\ & & & \frac{\lambda^3\omega}{a^3 b} & \frac{\lambda^3}{a^4 b\omega} & \\ & & & & \frac{b\lambda^3}{a^3\omega} & \frac{\lambda^2}{a^5 b\omega} \\ & & & & & \frac{b\lambda^3\omega}{a^5} \end{bmatrix}}_{D_\ell(\lambda)\widetilde{T}(\lambda,\omega)D_r(\lambda)} \qquad \underbrace{\begin{bmatrix} \frac{b}{a^3} & \frac{c^3}{a^5} & \frac{c^4}{a^5 b\omega} & & & \\ & \frac{\omega}{a^3} & \frac{c}{a^3 b} & & & \\ & & \frac{1}{a^3} & \frac{c^2}{a^4 b} & & \\ & & & \frac{\omega}{a^3 b} & \frac{c^3}{a^4 b\omega} & \\ & & & & \frac{b}{a^3\omega} & \frac{c^5}{a^5 b\omega} \\ & & & & & \frac{b\omega}{a^5} \end{bmatrix}}_{\widetilde{T}(\lambda,\omega)}$$

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$$
\underbrace{\begin{bmatrix}
\frac{b\lambda^2}{a^3} & \frac{\lambda^2 c^3}{a^5} & \frac{\lambda^3 c^4}{a^5 b\omega} & & & \\
 & \frac{\lambda^2\omega}{a^3} & \frac{\lambda^3 c}{a^3 b} & & & \\
 & & \frac{\lambda^3}{a^3} & \frac{\lambda^3 c^2}{a^4 b} & & \\
 & & & \frac{\lambda^3\omega}{a^3 b} & \frac{\lambda^3 c^3}{a^4 b\omega} & \\
 & & & & \frac{b\lambda^3}{a^3\omega} & \frac{\lambda^2 c^5}{a^5 b\omega} \\
 & & & & & \frac{b\lambda^3\omega}{a^5}
\end{bmatrix}}_{D_\ell(\lambda)\widetilde{T}(\lambda,\omega)D_r(\lambda)}
\qquad
\underbrace{\begin{bmatrix}
\frac{b}{a^3} & \frac{c^3}{a^5} & \frac{c^4}{a^5 b\omega} & & & \\
 & \frac{\omega}{a^3} & \frac{c}{a^3 b} & & & \\
 & & \frac{1}{a^3} & \frac{c^2}{a^4 b} & & \\
 & & & \frac{\omega}{a^3 b} & \frac{c^3}{a^4 b\omega} & \\
 & & & & \frac{b}{a^3\omega} & \frac{c^5}{a^5 b\omega} \\
 & & & & & \frac{b\omega}{a^5}
\end{bmatrix}}_{\widetilde{T}(\lambda,\omega)}
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Step 9

Erase the ω's from updated $\widetilde{T}(\lambda,\omega)$ to produce $T(\lambda)$:

$$\widetilde{T}(\lambda,\omega) = \begin{bmatrix} \frac{b}{a^3} & \frac{c^3}{a^5} & \frac{c^4}{a^5 b {\color{red}\omega}} & & & \\ & \frac{{\color{red}\omega}}{a^3} & \frac{c}{a^3 b} & & & \\ & & \frac{1}{a^3} & \frac{c^2}{a^4 b} & & \\ & & & \frac{{\color{red}\omega}}{a^3 b} & \frac{c^3}{a^4 b {\color{red}\omega}} & \\ & & & & \frac{b}{a^3 {\color{red}\omega}} & \frac{c^5}{a^5 b {\color{red}\omega}} \\ & & & & & \frac{b{\color{red}\omega}}{a^5} \end{bmatrix}$$

Step 9

Erase the ω's from updated $\widetilde{T}(\lambda,\omega)$ to produce $T(\lambda)$:

$$T(\lambda) = \begin{bmatrix} \frac{b}{a^3} & \frac{c^3}{a^5} & \frac{c^4}{a^5 b} & & & \\ & \frac{1}{a^3} & \frac{c}{a^3 b} & & & \\ & & \frac{1}{a^3} & \frac{c^2}{a^4 b} & & \\ & & & \frac{1}{a^3 b} & \frac{c^3}{a^4 b} & \\ & & & & \frac{b}{a^3} & \frac{c^5}{a^5 b} \\ & & & & & \frac{b}{a^5} \end{bmatrix}$$

Is it really a realization?

Let's recap why...

$$R = Z_\ell \, D_\ell \; T \; D_r \, Z_r$$

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  - R has same finite poles and zeros as T

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  - T realizes the finite spectral data by construction (neutral factors do not affect the finite spectrum).

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  - T realizes the finite spectral data by construction (neutral factors do not affect the finite spectrum).
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  - R has same infinite poles and zeros as $D_\ell T D_r$.

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  - $D_\ell T D_r$ realizes infinite spectral data by construction (neutral factors and placeholder).

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  - $D_\ell T D_r$ realizes infinite spectral data by construction (neutral factors and placeholder).
- **Minimal indices:** Each term full rank implies R has same left and right minimal indices as Z_ℓ and Z_r respectively.

What about transparency?

Without going into the details...

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Recall

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  (easy to compute product),
- and the minimal indices are recovered from Z_ℓ and Z_r

without doing any numerical computations.

The purely combinatorial manipulations are straightforward once you know the techniques, and only require $\mathcal{O}(n)$ work.

Thank you!

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