

The Exterior Power of Linear Maps

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An enlightening definition

Definition

Let V be a vector space of dimension n over the field \mathbb{F} , and let $k \leq n$ be a positive integer. The k^{th} *exterior power of V* is the vector space

$$\Lambda^k V = \langle v_1 \wedge v_2 \wedge \cdots \wedge v_k \mid v_1, v_2, \dots, v_k \in V \rangle.$$

These objects $v_1 \wedge v_2 \wedge \cdots \wedge v_k$ are called simple *alternating k -tensors* or *k -forms*.

There is a very elegant way to build k -forms from the direct product V^k , but it is long and involved.

Important properties

There are two defining properties of k -forms.

1. They are multilinear,
2. and they are anti-symmetric.

A fast consequence of the anti-symmetry property is the following lemma.

Lemma

Let $\sigma \in S_k$ be a permutation. Then

$$v_{\sigma(1)} \wedge v_{\sigma(2)} \wedge \cdots \wedge v_{\sigma(k)} = (-1)^\sigma \cdot v_1 \wedge v_2 \wedge \cdots \wedge v_k$$

where $(-1)^\sigma$ is the sign of the permutation.

What about a basis?

Example

Let $V = \mathbb{R}^3$ with the standard basis $\{e_1, e_2, e_3\}$. Consider the simple 2-form $u \wedge v$ where $u = [u_1, u_2, u_3]^T$ and $v = [v_1, v_2, v_3]^T$.

Notice that bilinearity guarantees

$$u \wedge v = (u_1 e_1 + u_2 e_2 + u_3 e_3) \wedge (v_1 e_1 + v_2 e_2 + v_3 e_3)$$

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$$\begin{aligned} u \wedge v &= (u_1 e_1 + u_2 e_2 + u_3 e_3) \wedge (v_1 e_1 + v_2 e_2 + v_3 e_3) \\ &= (u_1 e_1 + u_2 e_2 + u_3 e_3) \wedge v_1 e_1 \\ &\quad + (u_1 e_1 + u_2 e_2 + u_3 e_3) \wedge v_2 e_2 \\ &\quad + (u_1 e_1 + u_2 e_2 + u_3 e_3) \wedge v_3 e_3 \end{aligned}$$

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Example cont.

Before we continue, another easy consequence of anti-symmetry.

Lemma

Any simple k -form with a repeated component is zero.

Example

Now we can apply anti-symmetry to obtain

$$\begin{aligned}u \wedge v &= (u_1 v_2 - u_2 v_1) e_1 \wedge e_2 \\&\quad + (u_1 v_3 - u_3 v_1) e_1 \wedge e_3 \\&\quad + (u_2 v_3 - u_3 v_2) e_2 \wedge e_3\end{aligned}$$

The natural basis

With a little imagination, the previous example gives us a way to find a basis of $\Lambda^k V$ given a basis of V .

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Let $\mathcal{B} = \{b_1, b_2, \dots, b_n\}$ be a basis of V (dimension n). Then the following is a basis of $\Lambda^k V$:

$$\Lambda^k \mathcal{B} := \{b_{j_1} \wedge b_{j_2} \wedge \cdots \wedge b_{j_k} \mid 1 \leq j_1 < j_2 < \cdots < j_k \leq n\}.$$

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It will be helpful later to have a concise notation for referring to a single basis form.

Notation

Let $\mathbb{P}(n)$ denote the power set of $\{1, 2, \dots, n\}$, and let $\mathbb{P}_k(n)$ be the k -element subsets, ordered lexicographically.

An element $\beta \in \mathbb{P}_k(n)$ is an index set $\beta = \{j_1, j_2, \dots, j_k\}$ such that $1 \leq j_1 < j_2 < \dots < j_k \leq n$. Thus, we will denote

$$\Lambda_{r=1}^k b_{j_r} := b_{j_1} \wedge b_{j_2} \wedge \dots \wedge b_{j_k}.$$

So the basis $\Lambda^k \mathcal{B}$ can be concisely written as $\{\Lambda_{r=1}^k b_{j_r} \mid j_r \in \beta, \beta \in \mathbb{P}_k(n)\}$.

Induced linear maps

Let $T: V \rightarrow W$ be a linear transformation of finite dimensional vector spaces over the field \mathbb{F} .

Definition

The k^{th} exterior power of T is the map induced by extending

$$\Lambda^k T(v_1 \wedge v_2 \wedge \cdots \wedge v_k) := T(v_1) \wedge T(v_2) \wedge \cdots \wedge T(v_k)$$

linearly to all of $\Lambda^k V$.

We need only define $\Lambda^k T$ on the basis $\Lambda^k \mathcal{B}$ before extending linearly.

Matrix representation

For finite dimensions, we can define a matrix to represent T .

Definition

Given a basis $\{b_1, b_2, \dots, b_n\}$ of V and a basis $\{c_1, c_2, \dots, c_m\}$ of W , the *matrix representation of T* has entries a_{ij} given by the coefficient of c_i in the basis expansion of $T(b_j)$.

Example

$$T(b_j) = a_{1j}c_1 + a_{2j}c_2 + \cdots + a_{ij}c_i + \cdots + a_{mj}c_m$$

Examples

For now, assume $\dim(W) = \dim(V)$. There are two easy cases.

$k = 1$ In this case $\Lambda^1 V = V$ and $\Lambda^1 T = T$. So the matrix representation is just $A = [a_{ij}]$.

$k = n$ Some quick combinatorics tells us that

$$\dim(\Lambda^k V) = |\Lambda^k \mathcal{B}| = \binom{n}{k}.$$

Thus $\Lambda^n V$ is one-dimensional with the only basis form $b_1 \wedge b_2 \wedge \cdots \wedge b_n$.

Examples

$k = n$ cont. To figure out what the map $\Lambda^n T$ does, we need only consider what it does to the basis.

$$\Lambda^n T(b_1 \wedge b_2 \wedge \cdots \wedge b_n) = T(b_1) \wedge T(b_2) \wedge \cdots \wedge T(b_n)$$

Examples

$k = n$ cont. To figure out what the map $\Lambda^n T$ does, we need only consider what it does to the basis.

$$\begin{aligned}\Lambda^n T(b_1 \wedge b_2 \wedge \cdots \wedge b_n) &= T(b_1) \wedge T(b_2) \wedge \cdots \wedge T(b_n) \\ &= (a_{11}c_1 + a_{21}c_2 + \cdots + a_{n1}c_n) \\ &\quad \wedge (a_{12}c_1 + a_{22}c_2 + \cdots + a_{n2}c_n) \\ &\quad \vdots \\ &\quad \wedge (a_{1n}c_1 + a_{2n}c_2 + \cdots + a_{nn}c_n)\end{aligned}$$

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$k = n$ cont. To figure out what the map $\Lambda^n T$ does, we need only consider what it does to the basis.

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What about for $1 < k < n$?

It is best to explore some small examples.

Example

Let $\dim(V) = \dim(W) = 3$ and $k = 2$. If the basis for V is $\{b_1, b_2, b_3\}$, then the basis for $\Lambda^2 V = \{b_1 \wedge b_2, b_1 \wedge b_3, b_2 \wedge b_3\}$. Similarly for $\Lambda^2 W$.

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Example: $\dim(V) = \dim(W) = 3$, and $k = 2$

We can conclude that the first column of the matrix representation of $\Lambda^2 T$ is

$$\begin{bmatrix} a_{11}a_{22} - a_{21}a_{12} \\ a_{11}a_{32} - a_{31}a_{12} \\ a_{21}a_{32} - a_{31}a_{22} \end{bmatrix} = \begin{bmatrix} \left| \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right| \\ \left| \begin{array}{cc} a_{11} & a_{12} \\ a_{31} & a_{32} \end{array} \right| \\ \left| \begin{array}{cc} a_{21} & a_{22} \\ a_{31} & a_{32} \end{array} \right| \end{bmatrix}$$

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With a little bit of algebra and a lot of perseverance, we find that the second and the third columns are

$$\left[\begin{array}{c|cc} a_{11} & a_{13} \\ a_{21} & a_{23} \end{array} \right] \quad \text{and} \quad \left[\begin{array}{c|cc} a_{12} & a_{13} \\ a_{22} & a_{23} \end{array} \right]$$
$$\left[\begin{array}{c|cc} a_{11} & a_{13} \\ a_{31} & a_{33} \end{array} \right] \quad \text{and} \quad \left[\begin{array}{c|cc} a_{12} & a_{13} \\ a_{32} & a_{33} \end{array} \right]$$
$$\left[\begin{array}{c|cc} a_{21} & a_{23} \\ a_{31} & a_{33} \end{array} \right]$$

respectively.

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Putting these columns together, we get the matrix representation of the induced map $\Lambda^2 T$:

$$\begin{bmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{bmatrix}$$

What a curious matrix. Each entry is the determinant of a 2-by-2 submatrix of A .

Compound matrices

The matrix on the last slide has a name: it is the 2nd compound matrix of A .

Definition

The k^{th} compound matrix of $B \in M^{m \times n}$

- ▶ is $\binom{m}{k} \times \binom{n}{k}$,
- ▶ has as entries, the determinants of all the $k \times k$ submatrices,
- ▶ is denoted by $\mathcal{C}_k(B)$,
- ▶ its rows are indexed by $\mathbb{P}_k(m)$, ordered lexicographically,
- ▶ and its columns are indexed by $\mathbb{P}_k(n)$, ordered lexicographically.

For $\alpha \in \mathbb{P}_k(m)$ and $\beta \in \mathbb{P}_k(n)$, the entry of $\mathcal{C}_k(B)$ in the (α, β) -position is $\det(A[\alpha, \beta])$, where $A[\alpha, \beta]$ is the submatrix of A whose row index set is α and column index set is β .

The conjecture

Clearly, $\Lambda^1 T = T$ has representation $A = \mathcal{C}_1(A)$, and we found that $\Lambda^n T$ has representation $\det(A) = \mathcal{C}_n(A)$.

We also saw an example with representation $\mathcal{C}_2(A)$. Will we always get a compound matrix?

Recall our characterization of the basis of $\Lambda^k V$ as

$\{\Lambda_{r=1}^k b_{j_r} \mid j_r \in \beta, \beta \in \mathbb{P}_k(n)\}$, and $\{\Lambda_{r=1}^k c_{i_r} \mid i_r \in \alpha, \alpha \in \mathbb{P}_k(m)\}$ for $\Lambda^k W$.

Once more unto the breach

Let $\beta \in \mathbb{P}_k(n)$, and consider the action of $\Lambda^k T$ on the basis tensor $\Lambda_{r=1}^k b_{j_r}$ for $j_r \in \beta$:

$$\Lambda^k T(\Lambda_{r=1}^k b_{j_r}) = \Lambda_{r=1}^k T(b_{j_r}) = \Lambda_{r=1}^k (a_{1j_r} c_1 + a_{2j_r} c_2 + \cdots + a_{mj_r} c_m).$$

The next couple of steps are a bit “algebraic” and very difficult to write down.

1.

2.

3.

4.

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1. Expand the expression by distributing the wedge product.
- 2.
- 3.
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The next couple of steps are a bit “algebraic” and very difficult to write down.

1. Expand the expression by distributing the wedge product.
2. Get rid of any term with a repeated factor.
- 3.
- 4.

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2. Get rid of any term with a repeated factor.
3. Let $\alpha \in \mathbb{P}_k(m)$ be given and collect all terms of the form $\Lambda_{r=1}^k a_{i_{\sigma(r)} j_r} c_{i_{\sigma(r)}}$.

4.

Once more unto the breach

Let $\beta \in \mathbb{P}_k(n)$, and consider the action of $\Lambda^k T$ on the basis tensor $\Lambda_{r=1}^k b_{j_r}$ for $j_r \in \beta$:

$$\Lambda^k T(\Lambda_{r=1}^k b_{j_r}) = \Lambda_{r=1}^k T(b_{j_r}) = \Lambda_{r=1}^k (a_{1j_r} c_1 + a_{2j_r} c_2 + \cdots + a_{mj_r} c_m).$$

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4. Apply the permutation rule for each of these terms.

$$\Lambda_{r=1}^k a_{i_{\sigma(r)} j_r} c_{i_{\sigma(r)}} = \prod_{r=1}^k a_{i_{\sigma(r)} j_r} \Lambda_{r=1}^k c_{i_{\sigma(r)}} = (-1)^\sigma \prod_{r=1}^k a_{i_{\sigma(r)} j_r} \Lambda_{r=1}^k c_{i_r}$$

More gritty details

Notice that for each $\alpha \in \mathbb{P}_k(m)$, we will have one of these terms for every $\sigma \in S_k$. Now we can combine like terms: for each $\alpha \in \mathbb{P}_k(m)$ we have

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$$\begin{aligned} \left[\sum_{\sigma \in S_k} (-1)^\sigma \prod_{r=1}^k a_{i_{\sigma(r)} j_r} \right] \Lambda_{r=1}^k c_{i_r} \\ = \det(A[\alpha, \beta]) \Lambda_{r=1}^k c_{i_r} \end{aligned}$$

where $A[\alpha, \beta]$ is the $k \times k$ submatrix of A whose row index set is α and whose column index set is β .

Big theorem

We can now conclude that the entry in the (α, β) -position is $\det(A[\alpha, \beta])$. Thus we have the following big theorem:

Theorem

Let $T: V \rightarrow W$ be a linear transformation with matrix representation A . Then the matrix representation of the induced exterior power map

$$\Lambda^k T: \Lambda^k V \rightarrow \Lambda^k W$$

is exactly $C_k(A)$.

More about compounds

The main property is easy to state, but very nasty to prove:

$$\mathcal{C}_k(AB) = \mathcal{C}_k(A)\mathcal{C}_k(B).$$

This property is equivalent to the following formula credited to Cauchy and Binet circa 1812. For perspective, the word “matrix” didn’t exist until Sylvester coined it around 1850.

Let A be $m \times p$ and B be $p \times n$. Then

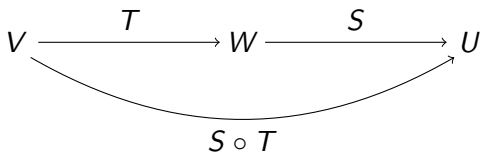
$$\det \left((AB)[\alpha, \beta] \right) = \sum_{\gamma \in \mathbb{P}_k(p)} \det(A[\alpha, \gamma]) \det(B[\gamma, \beta])$$

where $\alpha \in \mathbb{P}_k(m)$ and $\beta \in \mathbb{P}_k(n)$.

Prepare yourselves. Incoming category theory!

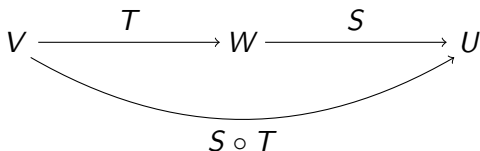
Cauchy-Binet by composition of maps

Let $T: V \rightarrow W$ and $S: W \rightarrow U$ be linear, and fix a basis for each of the spaces V , W , and U . The following diagram commutes.



Cauchy-Binet by composition of maps

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If A represents S and B represents T , then AB represents $S \circ T$.

Cauchy-Binet by composition of maps

Fact: $\Lambda^k(\star)$ is a covariant functor.

This means that $\Lambda^k(\star)$ takes commuting diagrams to commuting diagrams. In particular, the following diagram commutes:

$$\begin{array}{ccccc} \Lambda^k V & \xrightarrow{\Lambda^k T} & \Lambda^k W & \xrightarrow{\Lambda^k S} & \Lambda^k U \\ & \searrow & & \nearrow & \\ & \Lambda^k(S \circ T) & & & \end{array}$$

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By replacing the maps in the above with their matrix representations, we obtain the identity

$$\mathcal{C}_k(AB) = \mathcal{C}_k(A)\mathcal{C}_k(B).$$

Transpose

Another property of compound matrices is

$$\mathcal{C}_k(A^T) = \mathcal{C}_k(A)^T.$$

How does the transpose of a matrix appears as the matrix representation of a linear map. The answer involves dual spaces.

Definition

Let V be a vector space over the field \mathbb{F} . The *dual space* is the vector space

$$V^* := \{\phi: V \rightarrow \mathbb{F} \text{ s.t. } \phi \text{ is linear}\}.$$

The functions $\phi \in V^*$ are called *linear functionals*.

Dual space

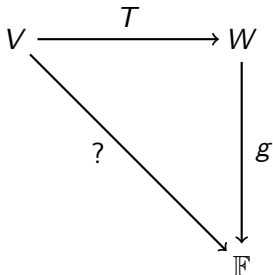
Let $\mathcal{B} = \{b_1, b_2, \dots, b_n\}$ be a basis for V and define the linear functionals

$$b_j^*(v) := \text{coefficient of } b_j \text{ in basis expansion of } v.$$

Fact: the set $\mathcal{B}^* := \{b_1^*, b_2^*, \dots, b_n^*\}$ forms a basis of V^* .

Dual map

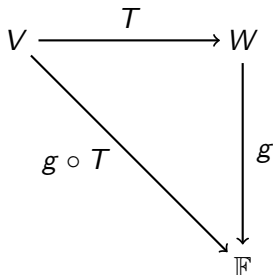
Consider the following diagram.



Question: What function will fill in this diagram so that it commutes?

Dual map

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Answer: $g \circ T$.

Define $T^*: W^* \rightarrow V^*$ by $T^*(g) := g \circ T$. What is the matrix representation?

Adjoint matrix representation

Consider the function $T^*(c_j^*): V \rightarrow \mathbb{F}$. The coefficient of b_i^* in the basis expansion of $T^*(c_j^*)$ is determined by what it does to the vector b_i .

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Thus, the entry in the (i, j) -position is a_{ji} , so the matrix representation of T^* is A^T .

Exterior power of the dual

How does $\Lambda^k(\star)$ interact with dual space? Is $\Lambda^k(V^*) = (\Lambda^k V)^*$?

Fact: f is a linear functional if and only if $f(v) = d^T v$ for some $d \in V$. Thus

$$\Lambda^k f(\omega) = C_k(d^T)\omega.$$

Since d^T is $1 \times n$, $C_k(d^T)$ is $1 \times \binom{n}{k}$. So $\Lambda^k f$ is a linear functional. Therefore,

$$\Lambda^k(V^*) \subseteq (\Lambda^k V)^*.$$

Dual of the exterior power

Define

$$[\Lambda_{r=1}^k b_{j_r}^*](\Lambda_{r=1}^k b_{i_r}) := \sum_{\sigma \in S_k} (-1)^\sigma \prod_{r=1}^k b_{j_r}^*(b_{i_{\sigma(r)}})$$

where $j_r \in \beta$, $i_r \in \alpha$, and $\alpha, \beta \in \mathbb{P}_k(n)$. Extend linearly to define $\Lambda_{r=1}^k b_{j_r}^*$ on all of $\lambda^k V$.

1.

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Conclude that $\Lambda_{r=1}^k b_{j_r}^* = (\Lambda_{r=1}^k b_{j_r})^*$, so

$$(\Lambda^k V)^* \subseteq \Lambda^k(V^*).$$

Compound of the transpose

Recall

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ & \searrow T^*(g) & \downarrow g \\ & & \mathbb{F} \end{array} \qquad W^* \xrightarrow{T^*} V^*$$

and consider

$$\begin{array}{ccc} \Lambda^k V & \xrightarrow{\Lambda^k T} & \Lambda^k W \\ & \searrow \Lambda^k(T^*(g)) & \downarrow \Lambda^k g \\ & & \Lambda^k \mathbb{F} = \mathbb{F} \end{array} \qquad \Lambda^k(W^*) = (\Lambda^k W)^* \xrightarrow{\Lambda^k(T^*)} (\Lambda^k V)^* = \Lambda^k(V^*)$$

More properties

Properties of compound matrices:

1. For conformal A and B , $\mathcal{C}_k(AB) = \mathcal{C}_k(A)\mathcal{C}_k(B)$;
2. $\mathcal{C}_k(A^T) = \mathcal{C}_k(A)^T$;
3. $\mathcal{C}_k(A^*) = \mathcal{C}_k(A)^*$ where $*$ denotes conjugate transpose;
4. $\det(\mathcal{C}_k(A)) = \det(A)^{\binom{n-1}{k-1}}$;
5. $\mathcal{C}_k(I_n) = I_{\binom{n}{k}}$;
6. When A is nonsingular, $\mathcal{C}_k(A^{-1}) = \mathcal{C}_k(A)^{-1}$;
7. When U is unitary, $\mathcal{C}_k(U)$ is also unitary.

There are many more that can be derived from these.

Thank you!