An Introduction to Nonlinear Eigenvalue Problems

Richard Hollister
Western Michigan University

17 Feb 2020
at
Colorado Mesa University

Matrix eigenvalue problem

Let A be a square matrix with entries from $\mathbb{C}$. The *standard eigenvalue problem*: find $\lambda \in \mathbb{C}$ such that there is nonzero $\mathrm{v} \in \mathbb{C}^n$ with

$$A\mathrm{v} = \lambda \mathrm{v}.$$

Matrix eigenvalue problem

Let A be a square matrix with entries from $\mathbb{C}$. The *standard eigenvalue problem*: find $\lambda \in \mathbb{C}$ such that there is nonzero $\mathrm{v} \in \mathbb{C}^n$ with

$$A\mathrm{v} = \lambda \mathrm{v}.$$

If we rearrange:

$$(A - \lambda I)\mathrm{v} = 0.$$

Matrix eigenvalue problem

Let A be a square matrix with entries from $\mathbb{C}$. The *standard eigenvalue problem*: find $\lambda \in \mathbb{C}$ such that there is nonzero $\mathrm{v} \in \mathbb{C}^n$ with

$$A\mathrm{v} = \lambda \mathrm{v}.$$

If we rearrange:

$$(A - \lambda I)\mathrm{v} = 0.$$

Definition

An eigenvalue of A is a scalar λ such that $\det(A - \lambda I) = 0$. The vector v is an eigenvector.

Matrix eigenvalue problem

Let A be a square matrix with entries from $\mathbb{C}$. The *standard eigenvalue problem*: find $\lambda \in \mathbb{C}$ such that there is nonzero $\mathrm{v} \in \mathbb{C}^n$ with

$$A\mathrm{v} = \lambda \mathrm{v}.$$

If we rearrange:

$$(A - \lambda I)\mathrm{v} = 0.$$

Definition

An eigenvalue of A is a scalar λ such that $\det(A - \lambda I) = 0$. The vector v is an eigenvector.

Algebraic, geometric, and partial multiplicities

Inverse eigenvalue problem

Standard eigenvalue problem: Given A, find its eigenvalues and eigenvectors.

Many methods to accomplish this.

Inverse eigenvalue problem

Standard eigenvalue problem: Given A, find its eigenvalues and eigenvectors.
Many methods to accomplish this.

Inverse eigenvalue problem: Given a list of eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ together with partial multiplicities:

$$\lambda_i \text{ has multiplicities } n_1^{(i)}, n_2^{(i)}, \ldots n_{\ell_i}^{(i)},$$

does there exist a matrix that realizes it?

Inverse eigenvalue problem

Standard eigenvalue problem: Given A, find its eigenvalues and eigenvectors.
Many methods to accomplish this.

Inverse eigenvalue problem: Given a list of eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ together with partial multiplicities:

$$\lambda_i \text{ has multiplicities } n_1^{(i)}, n_2^{(i)}, \ldots n_{\ell_i}^{(i)},$$

does there exist a matrix that realizes it?

Yes, the Jordan canonical form does the trick.

Example

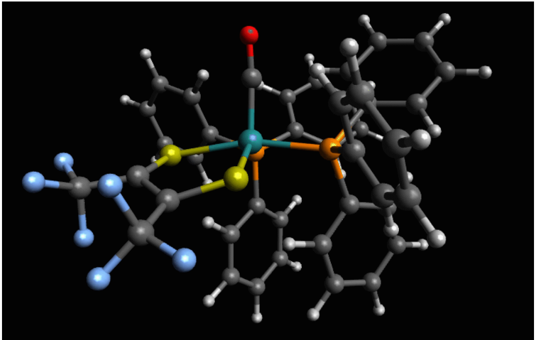
Given e-val $a \in \mathbb{C}$ with partial multiplicites $1, 1, 2, 3$, the Jordan realization is

$$[a] \oplus [a] \oplus \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \oplus \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}.$$

Applications of eigenvalues

Too many to list them all.

- Systems of first-order ODEs.
- Vibrational modes of molecules.


T M Porter et al, Chem. Sci., 2018, DOI: 10.1039/c8sc03258k

- Molecular orbital energies.

Matrix pencils

Eigenvalues problems lead to studying objects of the form $A - \lambda I$ for unknown $\lambda \in \mathbb{C}$ called *matrix pencils*.

Matrix pencils

Eigenvalues problems lead to studying objects of the form $A - \lambda I$ for unknown $\lambda \in \mathbb{C}$ called *matrix pencils*.

Many applications motivate us to generalize:

1. Replace I with B.
2. Relax restriction that A and B are square.

Matrix pencils

Eigenvalues problems lead to studying objects of the form $A - \lambda I$ for unknown $\lambda \in \mathbb{C}$ called *matrix pencils*.

Many applications motivate us to generalize:

1. Replace I with B.
2. Relax restriction that A and B are square.

Definition

A pencil $A - \lambda B$ is regular if it is square and has nonzero determinant, otherwise it is called singular.

Matrix pencils

Eigenvalues problems lead to studying objects of the form $A - \lambda I$ for unknown $\lambda \in \mathbb{C}$ called *matrix pencils*.

Many applications motivate us to generalize:

1. Replace I with B.
2. Relax restriction that A and B are square.

Definition

A pencil $A - \lambda B$ is regular if it is square and has nonzero determinant, otherwise it is called singular.

Matrix pencils are classical objects studied by Weierstrass and Kronecker.

- Entries are degree-1 polynomials: $a_{ij} - \lambda b_{ij}$.

Generalized eigenvalue problem

From now on, λ is a variable.

Generalized eigenvalue problem

From now on, λ is a variable.

Given a pencil $A - \lambda B$, the *generalized e-val problem (GEP)* seeks scalars $\lambda \in \mathbb{C}$ such that

$$\det(A - \lambda B) = 0.$$

Generalized eigenvalue problem

From now on, λ is a variable.

Given a pencil $A - \lambda B$, the *generalized e-val problem (GEP)* seeks scalars $\lambda \in \mathbb{C}$ such that

$$\det(A - \lambda B) = 0.$$

If desired, eigenvectors can be found by solving

$$(A - \lambda B)\mathrm{v} = 0.$$

Generalized eigenvalue problem

From now on, λ is a variable.

Given a pencil $A - \lambda B$, the *generalized e-val problem (GEP)* seeks scalars $\lambda \in \mathbb{C}$ such that

$$\det(A - \lambda B) = 0.$$

If desired, eigenvectors can be found by solving

$$(A - \lambda B)\mathrm{v} = 0.$$

There are many algorithms for solving GEPs. The basic method is

- QZ-algorithm to compute generalized Schur form.

Generalized eigenvalue problem

From now on, λ is a variable.

Given a pencil $A - \lambda B$, the *generalized e-val problem (GEP)* seeks scalars $\lambda \in \mathbb{C}$ such that

$$\det(A - \lambda B) = 0.$$

If desired, eigenvectors can be found by solving

$$(A - \lambda B)\mathrm{v} = 0.$$

There are many algorithms for solving GEPs. The basic method is

- QZ-algorithm to compute generalized Schur form.

GEPs can come from finite elements and can show up in condensed matter physics and many other places.

An interesting surprise

For a regular pencil $L(\lambda) = A - \lambda B$, if B is NOT full-rank, then it makes sense to say $L(\lambda)$ has an eigenvalue at infinity!

An interesting surprise

For a regular pencil $L(\lambda) = A - \lambda B$, if B is NOT full-rank, then it makes sense to say $L(\lambda)$ has an eigenvalue at infinity!

Illustrative example.

Example

$$L_n(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1/n \end{bmatrix} = \begin{bmatrix} 1 - \lambda & 0 \\ 0 & 1 - \lambda/n \end{bmatrix}$$

has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = n$. Now take limit as $n \to \infty$.

An interesting surprise

For a regular pencil $L(\lambda) = A - \lambda B$, if B is NOT full-rank, then it makes sense to say $L(\lambda)$ has an eigenvalue at infinity!

Illustrative example.

Example

$$L_n(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1/n \end{bmatrix} = \begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda/n \end{bmatrix}$$

has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = n$. Now take limit as $n \to \infty$.

In differential algebraic equations, the eigenvalue at infinity determines impulse responses corresponding to discontinuities in the solution.

Additional structure for singular pencils

When $L(\lambda)$ is singular, there is additional structure:

Additional structure for singular pencils

When $L(\lambda)$ is singular, there is additional structure:

Null spaces of $L(\lambda)$ are vector subspaces of n-tuples of rational functions.

Additional structure for singular pencils

When $L(\lambda)$ is singular, there is additional structure:

Null spaces of $L(\lambda)$ are vector subspaces of n-tuples of rational functions.

Associated with these subspaces are non-negative integers called *minimal indices* (Forney, 1975).

Additional structure for singular pencils

When $L(\lambda)$ is singular, there is additional structure:

Null spaces of $L(\lambda)$ are vector subspaces of n-tuples of rational functions.

Associated with these subspaces are non-negative integers called *minimal indices* (Forney, 1975).

- Right minimal indices for the right null space of $L(\lambda)$,

Additional structure for singular pencils

When $L(\lambda)$ is singular, there is additional structure:

Null spaces of $L(\lambda)$ are vector subspaces of n-tuples of rational functions.

Associated with these subspaces are non-negative integers called *minimal indices* (Forney, 1975).

- Right minimal indices for the right null space of $L(\lambda)$,
- Left minimal indices for the left null space.

Generalized inverse problem

Structural data of a pencil: eigenvalues (finite and infinite) with partial multiplicities, and minimal indices (left and right).

Generalized inverse problem

Structural data of a pencil: eigenvalues (finite and infinite) with partial multiplicities, and minimal indices (left and right).

Inverse problem:

Given a list of structural data $\mathcal{L}$, is there a pencil realizing it?

Generalized inverse problem

Structural data of a pencil: eigenvalues (finite and infinite) with partial multiplicities, and minimal indices (left and right).

Inverse problem:

Given a list of structural data $\mathcal{L}$, is there a pencil realizing it?

Yes! Weierstrass and Kronecker (late 1800s) worked on direct sums of ...

Generalized inverse problem

Structural data of a pencil: eigenvalues (finite and infinite) with partial multiplicities, and minimal indices (left and right).

Inverse problem:

Given a list of structural data $\mathcal{L}$, is there a pencil realizing it?

Yes! Weierstrass and Kronecker (late 1800s) worked on direct sums of ...

$$J_3(\alpha)(\lambda) = \begin{bmatrix} \lambda-\alpha & 1 & \\ & \lambda-\alpha & 1 \\ & & \lambda-\alpha \end{bmatrix}, W_3(\lambda) = \begin{bmatrix} 1 & \lambda & \\ & 1 & \lambda \\ & & 1 \end{bmatrix},$$

$$K_3(\lambda) = \begin{bmatrix} 1 & \lambda & & \\ & 1 & \lambda & \\ & & 1 & \lambda \end{bmatrix}.$$

Quadratic eigenvalue problem

Consider objects of the form $Q(\lambda) = A + B\lambda + C\lambda^2$.

Quadratic eigenvalue problem

Consider objects of the form $Q(\lambda) = A + B\lambda + C\lambda^2$.

The *quadratic eigenvalue problem (QEP)*: given $Q(\lambda)$, find $\lambda \in \mathbb{C}$ such that

$$\det[Q(\lambda)] = 0.$$

Quadratic eigenvalue problem

Consider objects of the form $Q(\lambda) = A + B\lambda + C\lambda^2$.

The *quadratic eigenvalue problem (QEP)*: given $Q(\lambda)$, find $\lambda \in \mathbb{C}$ such that

$$\det[Q(\lambda)] = 0.$$

Eigenvectors are solutions to $Q(\lambda)\mathrm{v} = 0$.

Quadratic eigenvalue problem

Consider objects of the form $Q(\lambda) = A + B\lambda + C\lambda^2$.

The *quadratic eigenvalue problem (QEP)*: given $Q(\lambda)$, find $\lambda \in \mathbb{C}$ such that

$$\det[Q(\lambda)] = 0.$$

Eigenvectors are solutions to $Q(\lambda)\mathrm{v} = 0$.

Same structural data as pencils.

- finite and infinite eigenvalues with partial multiplicities
- left and right minimal indices

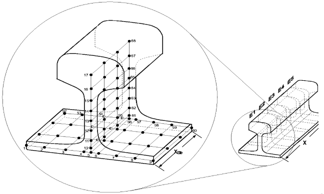
Applications of QEP

Deserves special consideration because of the wide number of applications (Tisseur and Meerbergen, SIAM Review 2001):

Applications of QEP

Deserves special consideration because of the wide number of applications (Tisseur and Meerbergen, SIAM Review 2001):

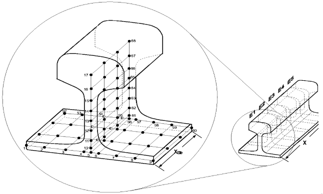
- vibrations of a rigid structure,



Applications of QEP

Deserves special consideration because of the wide number of applications (Tisseur and Meerbergen, SIAM Review 2001):

- vibrations of a rigid structure,

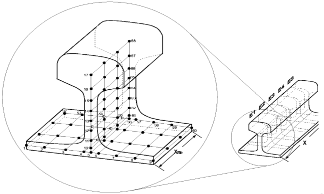


- constrained least-squares,

Applications of QEP

Deserves special consideration because of the wide number of applications (Tisseur and Meerbergen, SIAM Review 2001):

- vibrations of a rigid structure,



- constrained least-squares,
- signal processing.

Structured QEPs

In applications, additional structure often shows up:

$$M\lambda^2 + C\lambda + K$$

where $K = M^T$ and $C = C^T$. Called palindromic QEP.

Structured QEPs

In applications, additional structure often shows up:

$$M\lambda^2 + C\lambda + K$$

where $K = M^T$ and $C = C^T$. Called palindromic QEP.

Other structures that show up include:

Structured QEPs

In applications, additional structure often shows up:

$$M\lambda^2 + C\lambda + K$$

where $K = M^T$ and $C = C^T$. Called palindromic QEP.

Other structures that show up include:

- Hermitian where M, C, K are Hermitian

Structured QEPs

In applications, additional structure often shows up:

$$M\lambda^2 + C\lambda + K$$

where $K = M^T$ and $C = C^T$. Called palindromic QEP.

Other structures that show up include:

- Hermitian where M, C, K are Hermitian
- and Alternating where M, C, K alternate between symmetric and skew symmetric.

Structured QEPs

In applications, additional structure often shows up:

$$M\lambda^2 + C\lambda + K$$

where $K = M^T$ and $C = C^T$. Called palindromic QEP.

Other structures that show up include:

- Hermitian where M, C, K are Hermitian
- and Alternating where M, C, K alternate between symmetric and skew symmetric.

Structure can impose symmetries in the eigenvalues

Structured QEPs

In applications, additional structure often shows up:

$$M\lambda^2 + C\lambda + K$$

where $K = M^T$ and $C = C^T$. Called palindromic QEP.

Other structures that show up include:

- Hermitian where M, C, K are Hermitian
- and Alternating where M, C, K alternate between symmetric and skew symmetric.

Structure can impose symmetries in the eigenvalues

- Palindromic: eigenvalues come in pairs (α, α^{-1}).

Structured QEPs

In applications, additional structure often shows up:

$$M\lambda^2 + C\lambda + K$$

where $K = M^T$ and $C = C^T$. Called palindromic QEP.

Other structures that show up include:

- Hermitian where M, C, K are Hermitian
- and Alternating where M, C, K alternate between symmetric and skew symmetric.

Structure can impose symmetries in the eigenvalues

- Palindromic: eigenvalues come in pairs (α, α^{-1}).
- Hermitian: eigenvalues come in pairs $(\alpha, \bar{\alpha})$.

Structured QEPs

In applications, additional structure often shows up:

$$M\lambda^2 + C\lambda + K$$

where $K = M^T$ and $C = C^T$. Called palindromic QEP.

Other structures that show up include:

- Hermitian where M, C, K are Hermitian
- and Alternating where M, C, K alternate between symmetric and skew symmetric.

Structure can impose symmetries in the eigenvalues

- Palindromic: eigenvalues come in pairs (α, α^{-1}).
- Hermitian: eigenvalues come in pairs $(\alpha, \bar{\alpha})$.
- Alternating: eigenvalues come in pairs $(\alpha, -\alpha)$.

Quadratic inverse problem

More challenging than inverse problem for pencils.

Quadratic inverse problem

More challenging than inverse problem for pencils.

New constraint: (index sum rule)

$$\sum\{\text{e-val multiplicities}\} + \sum\{\text{minimal indices}\} = 2r$$

where r is the rank of the realization.

Quadratic inverse problem

More challenging than inverse problem for pencils.

New constraint: (index sum rule)

$$\sum\{\text{e-val multiplicities}\} + \sum\{\text{minimal indices}\} = 2r$$

where r is the rank of the realization.

Solved in a fashion similar to Kronecker.

- direct sum of blocks
- requires several times as many block types

Quadratic inverse problem

More challenging than inverse problem for pencils.

New constraint: (index sum rule)

$$\sum\{\text{e-val multiplicities}\} + \sum\{\text{minimal indices}\} = 2r$$

where r is the rank of the realization.

Solved in a fashion similar to Kronecker.

- direct sum of blocks
- requires several times as many block types

General	2013	De Terán, Dopico, Mackey
Hermitian	2015	Mackey, Tisseur
Palindromic	2019	De Terán, Dopico, Mackey, Perovič

Polynomial EPs

The *polynomial eigenvalue problem (PEP)* is the natural generalization of the GEP and QEP: given $P(\lambda) = P_0 + P_1\lambda + \cdots + P_k\lambda^k$ find $\lambda \in \mathbb{C}$ such that

$$\det[P(\lambda)] = 0.$$

Polynomial EPs

The *polynomial eigenvalue problem (PEP)* is the natural generalization of the GEP and QEP: given $P(\lambda) = P_0 + P_1\lambda + \cdots + P_k\lambda^k$ find $\lambda \in \mathbb{C}$ such that

$$\det[P(\lambda)] = 0.$$

Like pencils, polynomial matrices have

- eigenvalues (finite and infinite) with partial multiplicities
- and minimal indices (left and right).

Polynomial EPs

The *polynomial eigenvalue problem (PEP)* is the natural generalization of the GEP and QEP: given $P(\lambda) = P_0 + P_1\lambda + \cdots + P_k\lambda^k$ find $\lambda \in \mathbb{C}$ such that

$$\det[P(\lambda)] = 0.$$

Like pencils, polynomial matrices have

- eigenvalues (finite and infinite) with partial multiplicities
- and minimal indices (left and right).

Related by index sum equation (Praagman, 1991)

$$\sum\{\text{e-val multiplicities}\} + \sum\{\text{minimal indices}\} = kr$$

where k is the degree and r is the rank.

Polynomial inverse problem

Solved by Van Dooren et. al. (SIMAX, 2015).

Theorem

A list of structural data $\mathcal{L}$ can be realized by a polynomial of degree k if and only if $\mathcal{L}$ satisfies the index-sum equation.

Polynomial inverse problem

Solved by Van Dooren et. al. (SIMAX, 2015).

Theorem

A list of structural data $\mathcal{L}$ can be realized by a polynomial of degree k if and only if $\mathcal{L}$ satisfies the index-sum equation.

Construction does not mirror the simplicity of Kronecker.

Polynomial inverse problem

Solved by Van Dooren et. al. (SIMAX, 2015).

Theorem

A list of structural data $\mathcal{L}$ can be realized by a polynomial of degree k if and only if $\mathcal{L}$ satisfies the index-sum equation.

Construction does not mirror the simplicity of Kronecker.

Solved in my thesis so that solution

- is "upper triangular",

Polynomial inverse problem

Solved by Van Dooren et. al. (SIMAX, 2015).

Theorem

A list of structural data $\mathcal{L}$ can be realized by a polynomial of degree k if and only if $\mathcal{L}$ satisfies the index-sum equation.

Construction does not mirror the simplicity of Kronecker.

Solved in my thesis so that solution

- is "upper triangular",
- is sparse,

Polynomial inverse problem

Solved by Van Dooren et. al. (SIMAX, 2015).

Theorem

A list of structural data $\mathcal{L}$ can be realized by a polynomial of degree k if and only if $\mathcal{L}$ satisfies the index-sum equation.

Construction does not mirror the simplicity of Kronecker.

Solved in my thesis so that solution

- is "upper triangular",
- is sparse,
- and transparently reveals original data.

Polynomial inverse problem

Solved by Van Dooren et. al. (SIMAX, 2015).

Theorem

A list of structural data $\mathcal{L}$ can be realized by a polynomial of degree k if and only if $\mathcal{L}$ satisfies the index-sum equation.

Construction does not mirror the simplicity of Kronecker.

Solved in my thesis so that solution

- is "upper triangular",
- is sparse,
- and transparently reveals original data.

Not a direct sum of blocks, but a product of factors

$$P(\lambda) = L(\lambda)M(\lambda)R(\lambda).$$

Example construction

Data list $\mathcal{L}$ contains eigenvalues α, β, ∞ with partial multiplicities

$$\begin{array}{lllll} \alpha: & 1, & 1, & 4, & 4 \\ \beta: & 2, & 6 & & \\ \infty: & 1, & 2, & 7 & \end{array}.$$

Index sum: $10 + 8 + 10 = 28$.

A degree 7 realization is

$$\begin{bmatrix} a^4 b^2 & a^2 c^4 & 0 & ac^5 \\ & a^2 b^3 & ac^6 & ab^3 c^2 \\ & & ab & 0 \\ & & & a^3 b^2 \end{bmatrix}$$

where $a = \lambda - \alpha$, $b = \lambda - \beta$, and $c = \lambda - \gamma$.

Rational EP

A *rational matrix* over a field $\mathbb{C}$ is a matrix with rational function entries.

Rational EP

A *rational matrix* over a field $\mathbb{C}$ is a matrix with rational function entries.

- Very active area of research.

Rational EP

A *rational matrix* over a field $\mathbb{C}$ is a matrix with rational function entries.

- Very active area of research.
- Have been used for decades in the realm of systems theory.

Rational EP

A *rational matrix* over a field $\mathbb{C}$ is a matrix with rational function entries.

- Very active area of research.
- Have been used for decades in the realm of systems theory.

Rational matrix structural data:

- zeros (finite and infinite) with partial multiplicities,
- poles (finite and infinite) with partial multiplicities,
- minimal indices (left and right).

Rational EP

A *rational matrix* over a field $\mathbb{C}$ is a matrix with rational function entries.

- Very active area of research.
- Have been used for decades in the realm of systems theory.

Rational matrix structural data:

- zeros (finite and infinite) with partial multiplicities,
- poles (finite and infinite) with partial multiplicities,
- minimal indices (left and right).

Rational Index Sum Theorem:

Van Dooren (PhD Thesis, 1979),
Anguas, Dopico, RH, and Mackey (SIMAX, 2019):

$$\sum\{\text{multiplicities of poles}\} - \sum\{\text{multiplicities of zeros}\} = \sum\{\text{minimal indices}\}.$$

Rational inverse problem

Theorem (Anguas, Dopico, RH, Mackey; SIMAX, 2019)

A list of rational structural data can be realized if and only if it satisfies the rational index sum equation.

Rational inverse problem

Theorem (Anguas, Dopico, RH, Mackey; SIMAX, 2019)

A list of rational structural data can be realized if and only if it satisfies the rational index sum equation.

The resulting construction has same short-comings as the 2015 polynomial result of Van Dooren, Dopico, De Terán.

Rational inverse problem

Theorem (Anguas, Dopico, RH, Mackey; SIMAX, 2019)

A list of rational structural data can be realized if and only if it satisfies the rational index sum equation.

The resulting construction has same short-comings as the 2015 polynomial result of Van Dooren, Dopico, De Terán.

PhD Thesis, 2020 (RH):

Built a rational solution that is

- "upper triangular",
- sparse,
- transparently reveals original data.

Rational inverse problem

Theorem (Anguas, Dopico, RH, Mackey; SIMAX, 2019)

A list of rational structural data can be realized if and only if it satisfies the rational index sum equation.

The resulting construction has same short-comings as the 2015 polynomial result of Van Dooren, Dopico, De Terán.

PhD Thesis, 2020 (RH):

Built a rational solution that is

- "upper triangular",
- sparse,
- transparently reveals original data.

This solution is also a product of simple factors (5 of them).

General nonlinear eigenvalue problem

All of these EPs are examples of the *nonlinear eigenvalue problem (NEP)*: given matrix valued function $F(\lambda)$ that is meromorphic on the compact domain $\Omega \subset \mathbb{C}$, find $\lambda \in \Omega$ such that

$$\det[F(\lambda)] = 0.$$

General nonlinear eigenvalue problem

All of these EPs are examples of the *nonlinear eigenvalue problem (NEP)*: given matrix valued function $F(\lambda)$ that is meromorphic on the compact domain $\Omega \subset \mathbb{C}$, find $\lambda \in \Omega$ such that

$$\det[F(\lambda)] = 0.$$

Example

Solving the time-delay differential equation

$$\frac{d}{dt}x(t) = A_0 x(t) + A_1 x(t - \tau_1) + \cdots + A_k x(t - \tau_k)$$

leads to the NEP

$$(A_0 - \lambda I + A_1 e^{-\tau_1 \lambda} + \cdots + A_k e^{-\tau_k \lambda})v = 0.$$

Linearization

For a polynomial matrix $P(\lambda)$, a linearization is a matrix pencil $L(\lambda)$ with *exactly the same* finite spectral structure as $P(\lambda)$.

Linearization

For a polynomial matrix $P(\lambda)$, a linearization is a matrix pencil $L(\lambda)$ with *exactly the same* finite spectral structure as $P(\lambda)$.

If $P(\lambda)$ is $m \times n$ with degree k, then $L(\lambda)$ is necessarily $km \times kn$.

Linearization

For a polynomial matrix $P(\lambda)$, a linearization is a matrix pencil $L(\lambda)$ with *exactly the same* finite spectral structure as $P(\lambda)$.

If $P(\lambda)$ is $m \times n$ with degree k, then $L(\lambda)$ is necessarily $km \times kn$.

Example

One linearization for $P(\lambda) = A_k\lambda^k + A_{k-1}\lambda^{k-1} + \cdots + A_0$ is

$$C_1(\lambda) = \begin{bmatrix} A_k\lambda + A_{k-1} & A_{k-2} & \cdots & A_0 \\ -I & \lambda I & & \\ & \ddots & \ddots & \\ & & -I & \lambda I \end{bmatrix}.$$

Linearization

For a polynomial matrix $P(\lambda)$, a linearization is a matrix pencil $L(\lambda)$ with *exactly the same* finite spectral structure as $P(\lambda)$.

If $P(\lambda)$ is $m \times n$ with degree k, then $L(\lambda)$ is necessarily $km \times kn$.

Example

One linearization for $P(\lambda) = A_k\lambda^k + A_{k-1}\lambda^{k-1} + \cdots + A_0$ is

$$C_1(\lambda) = \begin{bmatrix} A_k\lambda + A_{k-1} & A_{k-2} & \cdots & A_0 \\ -I & \lambda I & & \\ & \ddots & \ddots & \\ & & -I & \lambda I \end{bmatrix}.$$

A linearization for a rational matrix $R(\lambda)$ must also capture the poles as well as zeros.

Linearization

For a polynomial matrix $P(\lambda)$, a linearization is a matrix pencil $L(\lambda)$ with *exactly the same* finite spectral structure as $P(\lambda)$.

If $P(\lambda)$ is $m \times n$ with degree k, then $L(\lambda)$ is necessarily $km \times kn$.

Example

One linearization for $P(\lambda) = A_k\lambda^k + A_{k-1}\lambda^{k-1} + \cdots + A_0$ is

$$C_1(\lambda) = \begin{bmatrix} A_k\lambda + A_{k-1} & A_{k-2} & \cdots & A_0 \\ -I & \lambda I & & \\ & \ddots & \ddots & \\ & & -I & \lambda I \end{bmatrix}.$$

A linearization for a rational matrix $R(\lambda)$ must also capture the poles as well as zeros.

Many ways to build linearizations have been developed in the last decade.

General strategy for solving NEPs

Starting with a NEP for $F(\lambda)$ on Ω

General strategy for solving NEPs

Starting with a NEP for $F(\lambda)$ on Ω

1. Approximate using interpolation:
   - Polynomial or rational where appropriate.
   - Newton, Lagrange, Chebyshev, Padé.

General strategy for solving NEPs

Starting with a NEP for $F(\lambda)$ on Ω

1. Approximate using interpolation:
   - Polynomial or rational where appropriate.
   - Newton, Lagrange, Chebyshev, Padé.
2. Construct a linearization:
   - Preserve any additional structure (palindromic, hermitian, etc.).

General strategy for solving NEPs

Starting with a NEP for $F(\lambda)$ on Ω

1. Approximate using interpolation:
   - Polynomial or rational where appropriate.
   - Newton, Lagrange, Chebyshev, Padé.
2. Construct a linearization:
   - Preserve any additional structure (palindromic, hermitian, etc.).
3. Solve the bigger GEP.
   - Many algorithms to choose from.
   - Choose an algorithm that exploits additional structure.

General strategy for solving NEPs

Starting with a NEP for $F(\lambda)$ on Ω

1. Approximate using interpolation:
   - Polynomial or rational where appropriate.
   - Newton, Lagrange, Chebyshev, Padé.
2. Construct a linearization:
   - Preserve any additional structure (palindromic, hermitian, etc.).
3. Solve the bigger GEP.
   - Many algorithms to choose from.
   - Choose an algorithm that exploits additional structure.
4. Recover approximate solution to NEP.

General strategy for solving NEPs

Starting with a NEP for $F(\lambda)$ on Ω

1. Approximate using interpolation:
   - Polynomial or rational where appropriate.
   - Newton, Lagrange, Chebyshev, Padé.
2. Construct a linearization:
   - Preserve any additional structure (palindromic, hermitian, etc.).
3. Solve the bigger GEP.
   - Many algorithms to choose from.
   - Choose an algorithm that exploits additional structure.
4. Recover approximate solution to NEP.

There is ongoing research and open problems at each stage in this process.

Future research projects

My research plans for the near future.

Future research projects

My research plans for the near future.

- Inverse problems:
  - Structured inverse problems still open.
  - Revisit direct-sum-of-blocks approach.

Future research projects

My research plans for the near future.

- Inverse problems:
  - Structured inverse problems still open.
  - Revisit direct-sum-of-blocks approach.
- Generalize index sum theorem.

Future research projects

My research plans for the near future.

- Inverse problems:
  - Structured inverse problems still open.
  - Revisit direct-sum-of-blocks approach.
- Generalize index sum theorem.
- Rational linearizations.

Future research projects

My research plans for the near future.

- Inverse problems:
  - Structured inverse problems still open.
  - Revisit direct-sum-of-blocks approach.
- Generalize index sum theorem.
- Rational linearizations.
- Approximate NEP:
  - Methods better than current interpolation.
  - Preserve any known structure in spectrum (node selection).

Future research projects

My research plans for the near future.

- Inverse problems:
  - Structured inverse problems still open.
  - Revisit direct-sum-of-blocks approach.
- Generalize index sum theorem.
- Rational linearizations.
- Approximate NEP:
  - Methods better than current interpolation.
  - Preserve any known structure in spectrum (node selection).

Many of these problems might make interesting senior capstone projects.

Thank you!

G. D. Forney, Jr.,
Minimal Bases of Rational Vector Spaces, with Applications to Multivariable Linear Systems,
SIAM J. Control, 13: 493–520, 1975.

P. Van Dooren,
Eigenstructuur Van Polynome en Rationale Matrices Toepassingen in de Systeemtheorie,
PhD Thesis, KU Leuven, May 1979.

C. Praagman,
Invariants of polynomial matrices,
Proceedings ECC91, Grenoble, 1274-1277, 1991.

F. Tisseur, K. Meerbergen,
The quadratic eigenvalue problem,
SIAM Review, 43 (2): 235-286, 2001.

F. De Terán, F. Dopico, P. Van Dooren,
Matrix polynomials with completely prescribed eigenstructure,
SIAM J. Matrix Anal. Appl., 36: 302-328, 2015.

L. M. Anguas, F. Dopico, R. Hollister, D. S. Mackey,
Van Dooren's index sum theorem and rational matrices with perscribed structural data,
SIAM J. Matrix Anal. & Apps., 40 (2): 720-738, 2019.

F. De Terán, F. Dopico, D. S. Mackey, V. Perović,
Quadratic realizability of palindromic matrix polynomials,
Linear Alg. Apps., 567: 202-262, 2019.

D. S. Mackey, F. Tisseur,
The Hermitian quadratic realizability problem,
In preparation.

F. De Terán, F. Dopico, D. S. Mackey,
A quasi-canonical form for quadratic matrix polynomials,
In preparation.