# Recent advances on an inverse problem for rational matrices

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# Rational matrices

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1. poles (finite and infinite),
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2. zeros (finite and infinite),
3. and minimal indices (left and right).

These data satisfy the *rational index sum condition* [Van Dooren 1979]:

$$\sum(\text{pole mult's}) - \sum(\text{zero mult's}) = \sum(\text{min indices})$$

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- more than likely dense,
- and does *not* transparently display the given data.

*The solution outlined in this talk corrects these deficiencies.*

# Rational Product Realization

Our solution takes the form of a five-fold product

$$R := Z_\ell \, D_\ell \, T \, D_r \, Z_r.$$

Each term is full rank, sparse, and the original data of $\mathcal{L}$ is transparently revealed in the factors:

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- Left minimal indices are stored in $Z_\ell(\lambda)$;
- Right minimal indices are stored in $Z_r(\lambda)$;
- Finite poles and zeros are in $T(\lambda)$;
- Infinite poles and zeros are revealed by $D_\ell \, T \, D_r$
  ($D$'s are diagonal).

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(c) Finally, adjustments are made to $\widetilde{T}(\lambda, \omega)$, and the $\omega$'s are removed to produce $T(\lambda)$.

# Getting started

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Example

$\mathcal{L}$ contains poles and zeros involving $a = \lambda - \alpha$, $b = \lambda - \beta$, and $\omega$:

$$\begin{pmatrix} \text{partial} \\ \text{mult's} \end{pmatrix} \quad \begin{array}{rrrrrrr} a: & -5, & -5, & -4, & -4, & -1, & -1 \\ b: & -1, & -1, & -1, & -1, & 1, & 5 \\ \omega: & -1, & -1, & -1, & 0, & 1, & 4 \end{array}$$

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producing extended S-M form

$$\text{diag}\left\{ \frac{1}{a^5 b\omega}, \frac{1}{a^5 b\omega}, \frac{1}{a^4 b\omega}, \frac{1}{a^4 b}, \frac{b\omega}{a}, \frac{b^5\omega^4}{a} \right\}.$$

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## Example

$$\widetilde{T}(\lambda,\omega) = \begin{bmatrix} \frac{b}{a^3} & \frac{1}{a^5} & \frac{1}{a^5 b\omega} & & & \\ & \frac{\omega}{a^3} & \frac{1}{a^3 b} & & & \\ & & \frac{1}{a^3} & \frac{1}{a^4 b} & & \\ & & & \frac{\omega}{a^3 b} & \frac{1}{a^4 b\omega} & \\ & & & & \frac{b}{a^3\omega} & \frac{1}{a^5 b\omega} \\ & & & & & \frac{b\omega}{a^5} \end{bmatrix}$$

# Incorporating minimal indices

Using direct sums of *zig-zag matrices*
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## Example

Suppose $\mathcal{L}$ contains right min indices 6, 3.
Row degs 1, 1, 2, 2, 2, 1, are forced (up to ordering).

$$\widetilde{Z}_r = \begin{bmatrix} \lambda & 1 & & & & & & \\ & \lambda & 1 & & & & & \\ & & \lambda^2 & 1 & & & & \\ & & & \lambda^2 & 1 & & & \\ & & & & 0 & \lambda^2 & 1 & \\ & & & & & & \lambda & 1 \end{bmatrix}.$$

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- $\widetilde{Z}_r$ is full rank with $\sum$(right min indices) $= \sum$(row degs).

# Left minimal indices

Similarly...

**Example**

If $\mathcal{L}$ has left minimal indices 5, 1, 1.
Col degs 1, 1, 1, 1, 1, 2, are forced.

$$\widetilde{Z}_\ell = \begin{bmatrix} \lambda & & & & & \\ 1 & 0 & & & & \\ & \lambda & & & & \\ & 1 & \lambda & & & \\ & & 1 & \lambda & & \\ & & & 1 & \lambda & \\ & & & & 1 & \lambda^2 \\ & & & & & \lambda \\ & & & & & 1 \end{bmatrix}$$

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*Multiplying on the left by $\widetilde{Z}_\ell$ or on the right by $\widetilde{Z}_r$ does not change finite spectral structure (truncated unimodular).*

# Factoring out $D_\ell$ and $D_r$

Now factor

$$\widetilde{Z}_r = D_r Z_r = \begin{bmatrix} \lambda & & & & & \\ & \lambda & & & & \\ & & \lambda^2 & & & \\ & & & \lambda^2 & & \\ & & & & \lambda^2 & \\ & & & & & \lambda \end{bmatrix} \begin{bmatrix} 1 & 1/\lambda & & & & & & \\ & 1 & 1/\lambda & & & & & \\ & & 1 & 1/\lambda^2 & & & & \\ & & & 1 & 1/\lambda^2 & & & \\ & & & & 0 & 1 & 1/\lambda^2 & \\ & & & & & & 1 & 1/\lambda \end{bmatrix}$$

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*Multiplying $Z_\ell \, Q \, Z_r$ does not change infinite spectral structure.*

# Design goals for $T$

Recall the form of our product realization $Z_\ell \, D_\ell \, T \, D_r \, Z_r$.

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$$(Z_\ell \, D_\ell) \; T \; (D_r \, Z_r) = \widetilde{Z}_\ell \; T \; \widetilde{Z}_r$$

middle factor reveals finite spectral structure.

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- When parsed as

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  middle factor reveals infinite spectral structure.

We now use the template $\widetilde{T}(\lambda, \omega)$ to build $T(\lambda)$

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Use *neutral factors* to update $\widetilde{T}(\lambda, \omega)$ so that...

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## Example

$$\begin{bmatrix} \frac{b}{a^3} & \frac{1}{a^5} & \frac{1}{a^5 b\omega} & & & \\ & \frac{\omega}{a^3} & \frac{1}{a^3 b} & & & \\ & & \frac{1}{a^3} & \frac{1}{a^4 b} & & \\ & & & \frac{\omega}{a^3 b} & \frac{1}{a^4 b\omega} & \\ & & & & \frac{b}{a^3\omega} & \frac{1}{a^5 b\omega} \\ & & & & & \frac{b\omega}{a^5} \end{bmatrix}$$

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 & \frac{\omega}{a^3} & \frac{c}{a^3 b} & & & \\
 & & \frac{1}{a^3} & \frac{c^2}{a^4 b} & & \\
 & & & \frac{\omega}{a^3 b} & \frac{c^3}{a^4 b\omega} & \\
 & & & & \frac{b}{a^3\omega} & \frac{c^5}{a^5 b\omega} \\
 & & & & & \frac{b\omega}{a^5}
\end{bmatrix}
$$

# Final middle factor

Erase the $\omega$'s from updated $\widetilde{T}(\lambda,\omega)$ to produce $T(\lambda)$:

$$\widetilde{T}(\lambda,\omega) = \begin{bmatrix} \frac{b}{a^3} & \frac{c^3}{a^5} & \frac{c^4}{a^5 b {\color{red}\omega}} & & & \\ & \frac{{\color{red}\omega}}{a^3} & \frac{c}{a^3 b} & & & \\ & & \frac{1}{a^3} & \frac{c^2}{a^4 b} & & \\ & & & \frac{{\color{red}\omega}}{a^3 b} & \frac{c^3}{a^4 b {\color{red}\omega}} & \\ & & & & \frac{b}{a^3 {\color{red}\omega}} & \frac{c^5}{a^5 b {\color{red}\omega}} \\ & & & & & \frac{b{\color{red}\omega}}{a^5} \end{bmatrix}$$

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- and the minimal indices are recovered from $Z_\ell$ and $Z_r$

*without doing any numerical computations*.

The purely combinatorial manipulations are straightforward once you know the techniques, and only require $\mathcal{O}(n)$ work.

Thank you!

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