The Exterior Power of Linear Maps

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An enlightening definition

Definition

Let V be a vector space of dimension n over the field $\mathbb{F}$, and let $k \leq n$ be a positive integer. The k^{th} *exterior power of V* is the vector space

$$\Lambda^k V = \left\langle v_1 \wedge v_2 \wedge \cdots \wedge v_k \mid v_1, v_2, \ldots, v_k \in V \right\rangle.$$

These objects $v_1 \wedge v_2 \wedge \cdots \wedge v_k$ are called simple *alternating k-tensors* or k-forms.

There is a very elegant way to build k-forms from the direct product V^k, but it is long and involved.

Important properties

There are two defining properties of k-forms.

1. They are are multilinear,
2. and they are anti-symmetric.

A fast consequence of the anti-symmetry property is the following lemma.

Lemma

Let $\sigma \in S_k$ be a permutation. Then

$$v_{\sigma(1)} \wedge v_{\sigma(2)} \wedge \cdots \wedge v_{\sigma(k)} = (-1)^{\sigma} \cdot v_1 \wedge v_2 \wedge \cdots \wedge v_k$$

where $(-1)^{\sigma}$ is the sign of the permutation.

What about a basis?

Example

Let $V = \mathbb{R}^3$ with the standard basis $\{e_1, e_2, e_3\}$. Consider the simple 2-form $u \wedge v$ where $u = [u_1, u_2, u_3]^T$ and $v = [v_1, v_2, v_3]^T$.

Notice that bilinearity guarantees

$$u \wedge v = (u_1 e_1 + u_2 e_3 + u_3 e_3) \wedge (v_1 e_1 + v_2 e_2 + v_3 e_3)$$

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\begin{aligned}
u \wedge v &= (u_1 e_1 + u_2 e_3 + u_3 e_3) \wedge (v_1 e_1 + v_2 e_2 + v_3 e_3) \\
&= (u_1 e_1 + u_2 e_3 + u_3 e_3) \wedge v_1 e_1 \\
&\quad + (u_1 e_1 + u_2 e_3 + u_3 e_3) \wedge v_2 e_2 \\
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u \wedge v &= (u_1 e_1 + u_2 e_3 + u_3 e_3) \wedge (v_1 e_1 + v_2 e_2 + v_3 e_3) \\
&= (u_1 e_1 + u_2 e_3 + u_3 e_3) \wedge v_1 e_1 \\
&\quad + (u_1 e_1 + u_2 e_3 + u_3 e_3) \wedge v_2 e_2 \\
&\quad + (u_1 e_1 + u_2 e_3 + u_3 e_3) \wedge v_3 e_3 \\
&= u_1 v_1 (e_1 \wedge e_1) + u_2 v_1 (e_2 \wedge e_1) + u_3 v_1 (e_3 \wedge e_1) \\
&\quad + u_1 v_2 (e_1 \wedge e_2) + u_2 v_2 (e_2 \wedge e_2) + u_3 v_2 (e_3 \wedge e_2) \\
&\quad + u_1 v_3 (e_1 \wedge e_3) + u_2 v_3 (e_2 \wedge e_3) + u_3 v_3 (e_3 \wedge e_3).
\end{aligned}
$$

Example cont.

Before we continue, another easy consequence of anti-symmetry.

Lemma

Any simple k-form with a repeated component is zero.

Example

Now we can apply anti-symmetry to obtain

$$\begin{aligned} u \wedge v = (u_1 v_2 - u_2 v_1) e_1 \wedge e_2 \\ + (u_1 v_3 - u_3 v_1) e_1 \wedge e_3 \\ + (u_2 v_3 - u_3 v_2) e_2 \wedge e_3 \end{aligned}$$

The natural basis

With a little imagination, the previous example gives us a way to find a basis of $\Lambda^k V$ given a basis of V.

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Let $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ be a basis of V (dimension n). Then the following is a basis of $\Lambda^k V$:

$$\Lambda^k \mathcal{B} := \{b_{j_1} \wedge b_{j_2} \wedge \cdots \wedge b_{j_k} \mid 1 \leq j_1 < j_2 < \cdots < j_k \leq n\}.$$

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It will be helpful later to have a concise notation for referring to a single basis form.

Notation

Let $\mathbb{P}(n)$ denote the power set of $\{1, 2, \ldots, n\}$, and let $\mathbb{P}_k(n)$ be the k-element subsets, ordered lexicographically.

An element $\beta \in \mathbb{P}_k(n)$ is an index set $\beta = \{j_1, j_2, \ldots, j_k\}$ such that $1 \leq j_1 < j_2 < \cdots < j_k \leq n$. Thus, we will denote

$$\Lambda_{r=1}^{k} b_{j_r} := b_{j_1} \wedge b_{j_2} \wedge \cdots \wedge b_{j_k}.$$

So the basis $\Lambda^k \mathcal{B}$ can be concisely written as $\{\Lambda_{r=1}^{k} b_{j_r} \mid j_r \in \beta, \beta \in \mathbb{P}_k(n)\}$.

Induced linear maps

Let $T\colon V \to W$ be a linear transformation of finite dimensional vector spaces over the field $\mathbb{F}$.

Definition

The k^{th} *exterior power of* T is the map induced by extending

$$\Lambda^k T(v_1 \wedge v_2 \wedge \cdots \wedge v_k) := T(v_1) \wedge T(v_2) \wedge \cdots \wedge T(v_k)$$

linearly to all of $\Lambda^k V$.

We need only define $\Lambda^k T$ on the basis $\Lambda^k \mathcal{B}$ before extending linearly.

Matrix representation

For finite dimensions, we can define a matrix to represent T.

Definition

Given a basis $\{b_1, b_2, \ldots, b_n\}$ of V and a basis $\{c_1, c_2, \ldots, c_m\}$ of W, the *matrix representation of* T has entries a_{ij} given by the coefficient of c_i in the basis expansion of $T(b_j)$.

Example

$T(b_j) = a_{1j}c_1 + a_{2j}c_2 + \cdots + a_{ij}c_i + \cdots + a_{mj}c_m$

Examples

For now, assume $\dim(W) = \dim(V)$. There are two easy cases.

$\boxed{k = 1}$ In this case $\Lambda^1 V = V$ and $\Lambda^1 T = T$. So the matrix representation is just $A = [a_{ij}]$.

$\boxed{k = n}$ Some quick combinatorics tells us that

$$\dim(\Lambda^k V) = |\Lambda^k \mathcal{B}| = \binom{n}{k}.$$

Thus $\Lambda^n V$ is one-dimensional with the only basis form $b_1 \wedge b_2 \wedge \cdots \wedge b_n$.

Examples

$\boxed{k = n \text{ cont.}}$ To figure out what the map $\Lambda^n T$ does, we need only consider what it does to the basis.

$$\Lambda^n T(b_1 \wedge b_2 \wedge \cdots \wedge b_n) = T(b_1) \wedge T(b_2) \wedge \cdots \wedge T(b_n)$$

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$$\begin{aligned}
\Lambda^n T(b_1 \wedge b_2 \wedge \cdots \wedge b_n) &= T(b_1) \wedge T(b_2) \wedge \cdots \wedge T(b_n) \\
&= (a_{11}c_1 + a_{21}c_2 + \cdots + a_{n1}c_n) \\
&\quad \wedge (a_{12}c_1 + a_{22}c_2 + \cdots + a_{n2}c_n) \\
&\qquad\qquad\vdots \\
&\quad \wedge (a_{1n}c_1 + a_{2n}c_2 + \cdots + a_{nn}c_n)
\end{aligned}$$

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&= \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^{n} a_{i\sigma(i)} \cdot c_1 \wedge c_2 \wedge \cdots \wedge c_n
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&\overset{\text{algebra}}{=} \cdots \\
&= \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^{n} a_{i\sigma(i)} \cdot c_1 \wedge c_2 \wedge \cdots \wedge c_n \\
&= \det(A) \cdot c_1 \wedge c_2 \wedge \cdots \wedge c_n
\end{aligned}$$

What about for $1 < k < n$?

It is best to explore some small examples.

Example

Let $\dim(V) = \dim(W) = 3$ and $k = 2$. If the basis for V is $\{b_1, b_2, b_3\}$, then the basis for $\Lambda^2 V = \{b_1 \wedge b_2, b_1 \wedge b_3, b_2 \wedge b_3\}$. Similarly for $\Lambda^2 W$.

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$$\begin{aligned} \Lambda^2 T(b_1 \wedge b_2) &= T(b_1) \wedge T(b_2) \\ &= (a_{11}c_1 + a_{21}c_2 + a_{31}c_3) \wedge (a_{12}c_1 + a_{22}c_2 + a_{32}c_3) \end{aligned}$$

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&= a_{11} a_{12}(c_1 \wedge c_1) + a_{11} a_{22}(c_1 \wedge c_2) + a_{11} a_{32}(c_1 \wedge c_3)
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&= (a_{11}a_{22} - a_{21}a_{12}) \cdot c_1 \wedge c_2 + (a_{11}a_{32} - a_{31}a_{12}) \cdot c_1 \wedge c_3
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&\quad + a_{21}a_{12}(c_2 \wedge c_1) + a_{21}a_{22}(c_2 \wedge c_2) + a_{21}a_{32}(c_2 \wedge c_3) \\
&\quad + a_{31}a_{12}(c_3 \wedge c_1) + a_{31}a_{22}(c_3 \wedge c_2) + a_{31}a_{32}(c_3 \wedge c_3) \\
&= (a_{11}a_{22} - a_{21}a_{12}) \cdot c_1 \wedge c_2 + (a_{11}a_{32} - a_{31}a_{12}) \cdot c_1 \wedge c_3 \\
&\quad + (a_{21}a_{32} - a_{31}a_{22}) \cdot c_2 \wedge c_3
\end{aligned}$$

Example: $\dim(V) = \dim(W) = 3$, and $k = 2$

We can conclude that the first column of the matrix representation of $\Lambda^2 T$ is

$$\begin{bmatrix} a_{11}a_{22} - a_{21}a_{12} \\ \\ a_{11}a_{32} - a_{31}a_{12} \\ \\ a_{21}a_{32} - a_{31}a_{22} \end{bmatrix} = \begin{bmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \end{bmatrix}$$

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With a little bit of algebra and a lot of perseverance, we find that the second and the third columns are

$$\begin{bmatrix} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{bmatrix}$$

respectively.

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Putting these columns together, we get the matrix representation of the induced map $\Lambda^2 T$:

$$\begin{bmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{bmatrix}$$

What a curious matrix. Each entry is the determinant of a 2-by-2 submatrix of A.

Compound matrices

The matrix on the last slide has a name: it is the 2^{nd} compound matrix of A.

Definition

The k^{th} compound matrix of $B \in M^{m \times n}$

- is $\binom{m}{k} \times \binom{n}{k}$,
- has as entries, the determinants of all the $k \times k$ submatrices,
- is denoted by $\mathcal{C}_k(B)$,
- its rows are indexed by $\mathbb{P}_k(m)$, ordered lexicographically,
- and its columns are indexed by $\mathbb{P}_k(n)$, ordered lexicographically.

For $\alpha \in \mathbb{P}_k(m)$ and $\beta \in \mathbb{P}_k(n)$, the entry of $\mathcal{C}_k(B)$ in the (α, β)-position is $\det\big(A[\alpha, \beta]\big)$, where $A[\alpha, \beta]$ is the submatrix of A whose row index set is α and column index set is β.

The conjecture

Clearly, $\Lambda^1 T = T$ has representation $A = \mathcal{C}_1(A)$, and we found that $\Lambda^n T$ has representation $\det(A) = \mathcal{C}_n(A)$.

We also saw an example with representation $\mathcal{C}_2(A)$. Will we always get a compound matrix?

Recall our characterization of the basis of $\Lambda^k V$ as $\{\Lambda_{r=1}^k b_{j_r} \mid j_r \in \beta, \beta \in \mathbb{P}_k(n)\}$, and $\{\Lambda_{r=1}^k c_{i_r} \mid i_r \in \alpha, \alpha \in \mathbb{P}_k(m)\}$ for $\Lambda^k W$.

Once more unto the breach

Let $\beta \in \mathbb{P}_k(n)$, and consider the action of $\Lambda^k T$ on the basis tensor $\Lambda_{r=1}^k b_{j_r}$ for $j_r \in \beta$:

$$\Lambda^k T(\Lambda_{r=1}^k b_{j_r}) = \Lambda_{r=1}^k T(b_{j_r}) = \Lambda_{r=1}^k (a_{1j_r} c_1 + a_{2j_r} c_2 + \cdots + a_{mj_r} c_m).$$

The next couple of steps are a bit "algebraic" and very difficult to write down.

1.

2.

3.

4.

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1. Expand the expression by distributing the wedge product.
2.
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The next couple of steps are a bit "algebraic" and very difficult to write down.

1. Expand the expression by distributing the wedge product.
2. Get rid of any term with a repeated factor.
3.
4.

Once more unto the breach

Let $\beta \in \mathbb{P}_k(n)$, and consider the action of $\Lambda^k T$ on the basis tensor $\Lambda_{r=1}^k b_{j_r}$ for $j_r \in \beta$:

$$\Lambda^k T(\Lambda_{r=1}^k b_{j_r}) = \Lambda_{r=1}^k T(b_{j_r}) = \Lambda_{r=1}^k (a_{1j_r} c_1 + a_{2j_r} c_2 + \cdots + a_{mj_r} c_m).$$

The next couple of steps are a bit "algebraic" and very difficult to write down.

1. Expand the expression by distributing the wedge product.
2. Get rid of any term with a repeated factor.
3. Let $\alpha \in \mathbb{P}_k(m)$ be given and collect all terms of the form $\Lambda_{r=1}^k a_{i_{\sigma(r)} j_r} c_{i_{\sigma(r)}}$.
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4. Apply the permutation rule for each of these terms.

$$\Lambda_{r=1}^k a_{i_{\sigma(r)} j_r} c_{i_{\sigma(r)}} = \prod_{r=1}^k a_{i_{\sigma(r)} j_r} \Lambda_{r=1}^k c_{i_{\sigma(r)}} = (-1)^\sigma \prod_{r=1}^k a_{i_{\sigma(r)} j_r} \Lambda_{r=1}^k c_{i_r}$$

More gritty details

Notice that for each $\alpha \in \mathbb{P}_k(m)$, we will have one of these terms for every $\sigma \in S_k$. Now we can combine like terms: for each $\alpha \in \mathbb{P}_k(m)$ we have

$$\left[\sum_{\sigma \in S_k} (-1)^\sigma \prod_{r=1}^{k} a_{i_{\sigma(r)} j_r}\right] \Lambda_{r=1}^{k} c_{i_r}$$

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$$\begin{aligned} &\left[\sum_{\sigma \in S_k} (-1)^\sigma \prod_{r=1}^{k} a_{i_{\sigma(r)} j_r}\right] \Lambda_{r=1}^{k} c_{i_r} \\ &= \det\left(A[\alpha, \beta]\right) \Lambda_{r=1}^{k} c_{i_r} \end{aligned}$$

where $A[\alpha, \beta]$ is the $k \times k$ submatrix of A whose row index set is α and whose column index set is β.

Big theorem

We can now conclude that the entry in the (α, β)-position is $\det\big(A[\alpha, \beta]\big)$. Thus we have the following big theorem:

Theorem

Let $T\colon V \to W$ be a linear transformation with matrix representation A. Then the matrix representation of the induced exterior power map

$$\Lambda^k T \colon \Lambda^k V \to \Lambda^k W$$

is exactly $\mathcal{C}_k(A)$.

More about compounds

The main property is easy to state, but very nasty to prove:

$$\mathcal{C}_k(AB) = \mathcal{C}_k(A)\mathcal{C}_k(B).$$

This property is equivalent to the following formula credited to Cauchy and Binet circa 1812. For perspective, the word "matrix" didn't exist until Sylvester coined it around 1850.

Let A be $m \times p$ and B be $p \times n$. Then

$$\det\Big((AB)[\alpha,\beta]\Big) = \sum_{\gamma \in \mathbb{P}_k(p)} \det\left(A[\alpha,\gamma]\right) \det\left(B[\gamma,\beta]\right)$$

where $\alpha \in \mathbb{P}_k(m)$ and $\beta \in \mathbb{P}_k(n)$.

Prepare yourselves. Incoming category theory!

Cauchy-Binet by composition of maps

Let $T\colon V \to W$ and $S\colon W \to U$ be linear, and fix a basis for each of the spaces V, W, and U. The following diagram commutes.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[rr, bend right, "S \circ T"'] & W \arrow[r, "S"] & U \end{tikzcd}$$

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\end{tikzcd}
$$

If A represents S and B represents T, then AB represents $S \circ T$.

Cauchy-Binet by composition of maps

Fact: $\Lambda^k(\star)$ is a covariant functor.

This means that $\Lambda^k(\star)$ takes commuting diagrams to commuting diagrams. In particular, the following diagram commutes:

$$\begin{tikzcd}
\Lambda^k V \arrow[r, "\Lambda^k T"] \arrow[rr, bend right, "\Lambda^k(S \circ T)"'] & \Lambda^k W \arrow[r, "\Lambda^k S"] & \Lambda^k U
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\end{tikzcd}$$

By replacing the maps in the above with their matrix representations, we obtain the identity

$$\mathcal{C}_k(AB) = \mathcal{C}_k(A)\mathcal{C}_k(B).$$

Transpose

Another property of compound matrices is

$$\mathcal{C}_k(A^\mathsf{T}) = \mathcal{C}_k(A)^\mathsf{T}.$$

How does the transpose of a matrix appears as the matrix representation of a linear map. The answer involves dual spaces.

Definition

Let V be a vector space over the field $\mathbb{F}$. The *dual space* is the vector space

$$V^* := \{\phi\colon V \to \mathbb{F} \text{ s.t. } \phi \text{ is linear}\}.$$

The functions $\phi \in V^*$ are called *linear functionals*.

Dual space

Let $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ be a basis for V and define the linear functionals

$$b_j^*(v) := \text{coefficient of } b_j \text{ in basis expansion of } v.$$

Fact: the set $\mathcal{B}^* := \{b_1^*, b_2^*, \ldots, b_n^*\}$ forms a basis of V^*.

Dual map

Consider the following diagram.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[dr, "?"'] & W \arrow[d, "g"] \\ & \mathbb{F} \end{tikzcd}$$

Question: What function will fill in this diagram so that it commutes?

Dual map

Consider the following diagram.

$$
\begin{tikzcd}
V \arrow[r, "T"] \arrow[dr, "g \circ T"'] & W \arrow[d, "g"] \\
& \mathbb{F}
\end{tikzcd}
$$

Question: What function will fill in this diagram so that it commutes?
Answer: $g \circ T$.
Define $T^* \colon W^* \to V^*$ by $T^*(g) := g \circ T$. What is the matrix representation?

Adjoint matrix representation

Consider the function $T^*(c_j^*) \colon V \to \mathbb{F}$. The coefficient of b_i^* in the basis expansion of $T^*(c_j^*)$ is determined by what it does to the vector b_i.

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Thus, the entry in the (i, j)-position is a_{ji}, so the matrix representation of T^* is A^T.

Exterior power of the dual

How does $\Lambda^k(\star)$ interact with dual space? Is $\Lambda^k(V^*) = \left(\Lambda^k V\right)^*$?

Fact: f is a linear functional if and only if $f(v) = d^\mathsf{T} v$ for some $d \in V$. Thus

$$\Lambda^k f(\omega) = \mathcal{C}_k(d^\mathsf{T})\omega.$$

Since d^T is $1 \times n$, $\mathcal{C}_k(d^\mathsf{T})$ is $1 \times \binom{n}{k}$. So $\Lambda^k f$ is a linear functional. Therefore,

$$\Lambda^k(V^*) \subseteq \left(\Lambda^k V\right)^*.$$

Dual of the exterior power

Define

$$\left[\Lambda_{r=1}^{k} b_{j_r}^{*}\right]\left(\Lambda_{r=1}^{k} b_{i_r}\right) := \sum_{\sigma \in S_k} (-1)^{\sigma} \prod_{r=1}^{k} b_{j_r}^{*}\left(b_{i_{\sigma(r)}}\right)$$

where $j_r \in \beta$, $i_r \in \alpha$, and $\alpha, \beta \in \mathbb{P}_k(n)$. Extend linearly to define $\Lambda_{r=1}^{k} b_{j_r}^{*}$ on all of $\lambda^k V$.

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Conclude that $\Lambda_{r=1}^{k} b_{j_r}^* = \left(\Lambda_{r=1}^{k} b_{j_r}\right)^*$, so

$$\left(\Lambda^k V\right)^* \subseteq \Lambda^k(V^*).$$

Compound of the transpose

Recall

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[dr, "T^*(g)"'] & W \arrow[d, "g"] \\ & \mathbb{F} \end{tikzcd} \qquad W^* \xrightarrow{T^*} V^*$$

and consider

$$\begin{tikzcd} \Lambda^k V \arrow[r, "\Lambda^k T"] \arrow[dr, "\Lambda^k(T^*(g))"'] & \Lambda^k W \arrow[d, "\Lambda^k g"] \\ & \Lambda^k \mathbb{F} = \mathbb{F} \end{tikzcd} \qquad \Lambda^k(W^*) = (\Lambda^k W)^* \xrightarrow{\Lambda^k(T^*)} (\Lambda^k V)^* = \Lambda^k(V^*)$$

More properties

Properties of compound matrices:

1. For conformal A and B, $\mathcal{C}_k(AB) = \mathcal{C}_k(A)\mathcal{C}_k(B)$;
2. $\mathcal{C}_k(A^\mathsf{T}) = \mathcal{C}_k(A)^\mathsf{T}$;
3. $\mathcal{C}_k(A^*) = \mathcal{C}_k(A)^*$ where * denotes conjugate transpose;
4. $\det\left(\mathcal{C}_k(A)\right) = \det(A)^{\binom{n-1}{k-1}}$;
5. $\mathcal{C}_k(I_n) = I_{\binom{n}{k}}$;
6. When A is nonsingular, $\mathcal{C}_k(A^{-1}) = \mathcal{C}_k(A)^{-1}$;
7. When U is unitary, $\mathcal{C}_k(U)$ is also unitary.

There are many more that can be derived from these.

Thank you!